Scaling of the conductance distribution near the Anderson transition

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The scaling hypothesis is the foundation of our understanding of the Anderson transition. We present a direct numerical demonstration of the scaling of the conductance distribution of a disordered system in the critical regime. This complements a previous demonstration of the scaling of certain averages of the conductance distribution [K. Slevin *et al.*, Phys. Rev. Lett. **86**, 3594 (2001)].

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I. INTRODUCTION

The single parameter scaling hypothesis of Abrahams *et al.* is the basis of our understanding of the Anderson metal-insulator transition in disordered systems.[1] In Ref. 1 it was proposed that the zero-temperature conductance $G=(e^2/h)g$, as measured by the ''Thouless number,'' obeys a single parameter scaling law. However, the large sample-to-sample fluctuations in the conductance of disordered systems were not explicitly considered. In the critical and localized regimes the fluctuations are of the same order as the mean conductance. (The relation of the mean conductance to the Thouless number is discussed in Ref. 2.) This led to suggestions that the scaling hypothesis should be reformulated in terms of the typical conductance,[3] or perhaps the distribution of conductance.[4,5]

For a disordered system of size L in $d=2+\epsilon$ dimensions, Altshuler *et al.* estimated the cumulants c_n of the conductance distribution using a field-theoretic method.[6] At the mobility edge they found that

$$c_n(L)=\begin{cases}\epsilon^{n-2} & n\leqslant n_0\approx 1/\epsilon,\\ (L/l)^{\epsilon n^2-2n} & n>n_0.\end{cases} \tag{1}$$

If single parameter scaling holds, the only relevant length should be the correlation length ξ, and the appearance of the mean free path l in the expression for the higher cumulants is unexpected. (In the insulating regime ξ is the localization length, while in the metallic regime it is the correlation length.) Shapiro reconstructed the critical conductance distribution from Eq. (1) and showed that, appearances to the contrary, these cumulants are consistent with a single parameter scaling of the distribution.[7]

To apply results for $d=2+\epsilon$ to three dimensions we must make a questionable extrapolation to $\epsilon=1$. While the Anderson transition occurs at weak disorder $k_F l\gg 1$ when $\epsilon\ll 1$, it occurs at strong disorder $k_F l\approx 1$ in three dimensions. (Here k_F is the Fermi wave number.) Comparison of the distribution obtained by Shapiro with numerical results shows that the behavior of the conductance distribution at large g, and also the nonuniversal behavior of the higher cumulants in Eq. (1), are qualitatively incorrect.[8,9]

To overcome the limitation to weak disorder Cohen *et al.* used a Migdal-Kadanoff type real-space renormalization scheme.[10] They found that the scaling of the conductance distribution is described by two parameters; only in the limit of weak disorder is single parameter scaling recovered. However, and as pointed out by Cohen *et al.*, the Migdal-Kadanoff scheme involves an uncontrolled approximation and some of the results obtained with it are known to be incorrect. For example, in the metallic regime the predicted conductance fluctuations are too large and in disagreement with the theoretically well-established and experimentally verified phenomena of universal conductance fluctuations.[11,12] They concluded that while the Migdal-Kadanoff scheme may be exact for hierarchical lattices, in three dimensions it is primarily of pedagogical value.[13]

Recent work on Anderson localization in one dimension has highlighted the importance of a second length scale l_s. Deych *et al.*[14] demonstrated the existence of a crossover between single parameter and two parameter scaling regimes dependent on the ratio of l_s to the localization length ξ. Single parameter scaling is observed when $\xi>l_s$, and two parameter scaling when $\xi<l_s$. The implications of this result for Anderson localization in higher dimensions are not yet clear.

Numerical studies of the Anderson model have demonstrated single parameter scaling of the localization length of electrons in quasi-one-dimensional systems,[15] and also of the mean resistance, mean conductance, and typical conductance near the Anderson transition in three dimensions.[16] The observation of scaling for any one of these averages does not rule out a two parameter scaling of the conductance distribution because the fluctuations of the relevant quantity might scale quite differently. This is precisely what happens in the Migdal-Kadanoff scheme in which the mean of the logarithm of resistance obeys a single parameter scaling law while at the same time the conductance distribution obeys a two parameter scaling law. This scenario seems less likely in light of the consistent scaling of the three different averages ob-

served in Ref. 16. Perhaps even more telling is the demonstration in numerical studies of the existence of a universal size-independent critical distribution that is accessible by varying only a single parameter.[8] This observation makes two parameter scaling unlikely since two parameter scaling should require that both parameters be varied simultaneously to access the critical point. Nevertheless, given the crucial role that the scaling hypothesis plays in the theory of transport in disordered systems, we feel that a very direct and clear cut demonstration of the scaling of the conductance distribution is called for. Such a demonstration is presented below.

An interesting aspect of the method we use to demonstrate scaling is that it is not necessary to have any analytic approximation for the form of the conductance distribution or to identify a single quantity which parametrizes the distribution. Instead we establish scaling by analyzing the percentiles of the distribution. This provides a much more direct and robust demonstration of scaling than can be archived by any analysis based on average quantities such as the moments of the distribution.

II. METHOD

Following Ref. 5 a single parameter scaling law for the conductance distribution $p_L(g)$ of a three-dimensional system of linear dimension L can be mathematically formulated as follows:

$$p_L(g) \simeq F(g;X), \tag{2}$$

where X is a parameter which must obey the single parameter scaling law

$$\frac{\mathrm{d}\ln X}{\mathrm{d}\ln L} = \beta(X). \tag{3}$$

A limiting process is implicit in Eq. (2); we refer the reader to Ref. 5 for a detailed discussion. The parameter X need not be one of the moments of the distribution.

At first sight it appears that we must know the functional form of the function F in Eq. (2) in order to verify single parameter scaling of the distribution numerically. In fact, this is not so. The procedure we have adopted is to analyze the scaling of the percentiles of the distribution. The precise definition of the percentile g_q is

$$q = \int_0^{g_q} p_L(g)\mathrm{d}g, \tag{4}$$

where $0 \leqslant q \leqslant 1$. By establishing single parameter scaling for a representative set of percentiles we indirectly establish Eqs. (2) and (3), *provided* that the scaling of different percentiles are consistent. When considering the percentiles it is not necessary to distinguish g, $\ln g$, or $1/g$ as it is when considering average quantities.

We have analyzed the conductance distribution of the Anderson model numerically. The motion of the electrons is described by

$$H = V\sum_{\langle i,j\rangle} C_i^\dagger C_j + \sum_i W_i C_i^\dagger C_i, \tag{5}$$

where $C_i^\dagger$ (C_i) is the creation (annihilation) operator of an electron at the site i of a three-dimensional cubic lattice. The amplitude of the random potential at site i is W_i. Hopping is restricted to nearest neighbors and its amplitude was taken as the unit of energy, $V=1$. We assumed a box distribution with each W_i uniformly distributed on the interval $[-W/2,W/2]$. In what follows we refer to the strength of the potential fluctuations W as the disorder. The numerical method used is described in Ref. 17. The two terminal zero-temperature conductance was evaluated using the Landauer formula

$$g = 2\,\mathrm{tr}tt^\dagger, \tag{6}$$

where t is the transmission matrix describing the propagation of electrons from one contact to the other.[18,19]

The conductance distribution depends on the system size L, the disorder W, the Fermi energy E_F, and the boundary conditions. We set $E_F=0.5$ and imposed fixed boundary conditions in the transverse directions. We accumulated data for the disorder range $15 \leqslant W \leqslant 18$ and system sizes $6 \leqslant L \leqslant 18$. At the extremes of the disorder range the localization (correlation) length is of the same order as the system size,[20] so that our data covers the critical regime.

To estimate g_q we simulated 1 000 000 realizations of the random potential and calculated the conductance for each realization. (For $L=18$ the number of realizations was approximately 500 000.) We sorted the data into ascending order and our estimate of g_q was then the $n=[qN_d]$th datum in this list, where $[x]$ is the integer part of x. When fitting the numerical data it is also necessary to have an estimate of the accuracy of the percentiles. Following the standard method we used the binomial distribution to estimate the likely accuracy of the percentile. We defined

$$\Delta n = \sqrt{N_d q(1-q)}, \tag{7}$$

located the $(n+\Delta n)$th and $(n-\Delta n+1)$th data in the list, and calculated the differences with g_q. Our estimate of the accuracy is then the largest of these two differences. In practice, we found that the accuracy of all the percentiles were comparable, being of the order of 0.2%. The data were then fitted with the finite-size scaling forms below by minimizing the χ^2 statistic in the usual way.

To fit the system size and disorder dependence of the percentile we supposed a single parameter scaling law but allowed for deviations from scaling due to an irrelevant scaling variable and nonlinearity of the scaling variables.[15] We fitted the data to

$$\ln g_q = F(\psi,\phi), \tag{8}$$

where ψ is the relevant scaling variable and ϕ is the irrelevant scaling variable. We approximated this scaling function by its first-order expansion in the irrelevant scaling variable

$$\ln g_q = F_0(\psi) + \phi F_1(\psi). \tag{9}$$

We expanded each scaling function as a power series,

$$F_0(x) = \ln(g_q)_c + x + a_2x^2 + \cdots + a_{n_0}x^{n_0}, \quad (10)$$

$$F_1(x) = 1 + b_1x + b_2x^2 + \cdots + b_{n_1}x^{n_1}. \quad (11)$$

Here $(g_q)_c$ is the critical value of the percentile. The scaling variables were approximated by expansions in terms of the dimensionless disorder

$$w = (W_c - W)/W_c, \quad (12)$$

where W_c is the critical disorder separating the insulating and metallic phases

$$\psi = L^{1/\nu}(\psi_1 w + \psi_2 w^2 + \cdots + \psi_{n_\psi} w^{n_\psi}), \quad (13)$$

$$\phi = L^y(\phi_0 + \phi_1 w + \phi_2 w^2 + \cdots + \phi_{n_\phi} w^{n_\phi}). \quad (14)$$

The critical exponent ν describes the divergence of the localization (correlation) length as the transition is approached.

$$\xi = \xi_\pm |\psi_1 w + \psi_2 w^2 + \cdots + \psi_{n_\psi} w^{n_\psi}|^{-\nu}. \quad (15)$$

The constants $\xi_\pm$, and hence the absolute scale of the localization (correlation) length ξ, cannot be determined from the fit. The decay of the irrelevant scaling variable with system size is described by the exponent $y<0$. Redundancy in the definition of the fitting parameters between the coefficients in the expansions of F_0 and F_1 and the expansions of ψ and ϕ are eliminated by setting some of the expansion coefficients of F_0 and F_1 to unity as shown. This choice is also necessary if F_0 and F_1 are to be universal. The total number of parameters is $N_p = n_0 + n_1 + n_\psi + n_\phi + 4$.

The minimum of χ^2 was found using the DRNLIN routine of the IMSL numerical library. The starting values of the fitting parameters supplied to DRNLIN are in the region $\nu \approx 1.6$, $W_c \approx 16.5$, $y \approx -3$, and $\psi_1 \approx 1$. We set the initial value of $\ln(g_q)_c$ to a value close to its best-fit value by visual inspection of the raw data, and all other parameters were initially zero. The results of the fitting procedure are not especially sensitive to the choice of the starting values. A number of fits corresponding to different choices of n_0, n_1, n_ψ, and n_ϕ are possible and a selection criterion is necessary. We set a cutoff for the goodness-of-fit probability Q at $Q=0.1$ and searched for the fit that required the fewest parameters to satisfy this. Broadly speaking all sensible choices of n_0, n_1, n_ψ and n_ϕ lead to consistent estimates of the critical parameters. The goodness of fit and the accuracy of the fitted parameters were estimated using Monte Carlo simulations of synthetic data sets.[21]

TABLE I. The estimated critical values for each percentile and 95% confidence intervals.

q	W_c	$\ln(g_q)_c$	ν
0.025	16.48±.02	−4.14±.03	1.56±.03
0.17	16.47±.01	−2.55±.01	1.56±.01
0.5	16.48±.04	−1.08±.03	1.59±.03
0.83	16.46±.05	0.13±.03	1.60±.04

TABLE II. The details of the fit. N_d is the number of data.

q	n_0	n_ψ	n_1	n_ϕ	N_p	N_d	χ^2	Q
0.025	3	2	1	0	10	179	159	0.7
0.17	2	2	1	0	9	179	155	0.8
0.5	3	2	1	0	10	179	172	0.4
0.83	3	2	1	0	10	179	180	0.3

III. RESULTS

Results for the $q=0.025$, 0.17, 0.5, and 0.83 percentiles are shown in Table I. (For a normal distribution with mean μ and variance σ^2, these choices would correspond to the points $\mu-2\sigma, \mu-\sigma, \mu$, and $\mu+\sigma$ in the distribution.) Precise details of the fits are given in Table II. The estimates of the irrelevant exponent are consistent with those obtained in Ref. 16 and are not shown again here.

Data for the $q=0.17$ percentile are plotted in Fig. 1. In Fig. 2 the same data, after subtraction of corrections to scaling, are replotted to demonstrate single parameter scaling. This is done by plotting the corrected data as a function of the ratio of the systems size L to the localization (correlation) length ξ. When displayed in this way the data fall on two different curves corresponding to the localized (lower curve) and the delocalized (upper curve) regimes. The two curves are described by two scaling functions F_+ and F_- derived from F_0,

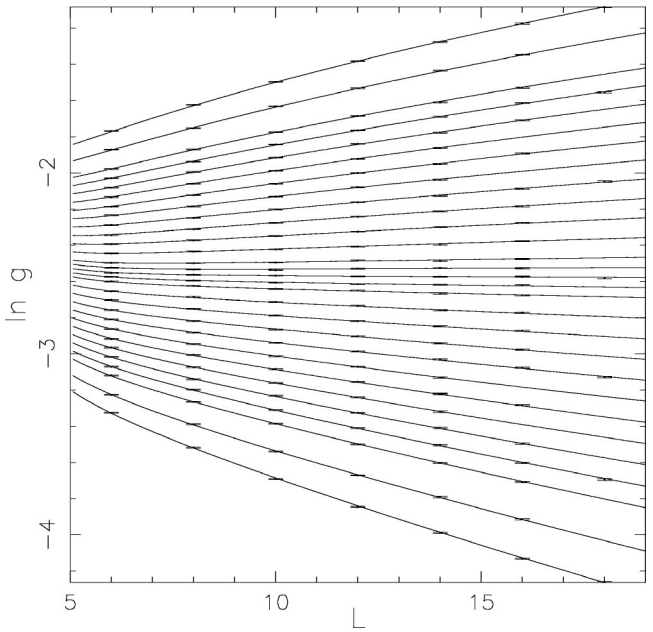

FIG. 1. The $q=0.17$ percentile of the conductance distribution for disordered $L\times L\times L$ systems versus system size L for disorder W in the range (Refs. 15 and 18). The lines are the fit of Eq. (8).

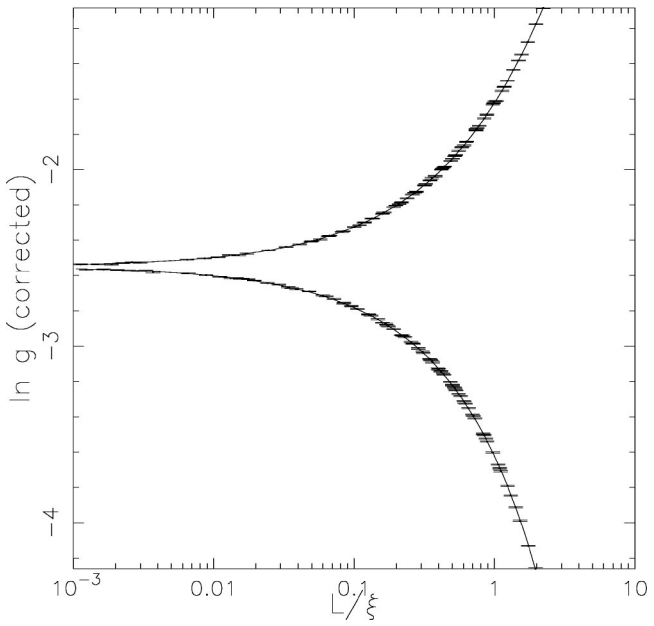

FIG. 2. The data of Fig. 1, after subtraction of corrections to scaling, replotted as function of the ratio L/ξ to display single parameter scaling. The lines are the scaling functions (16) and (17) described in the text.

$$F_+(x) = \ln(g_q)_c + x^{1/\nu} + \cdots + a_{n_0} x^{n_0/\nu}, \tag{16}$$

$$F_-(x) = \ln(g_q)_c - x^{1/\nu} + \cdots + (-1)^{n_0} a_{n_0} x^{n_0/\nu}. \tag{17}$$

Data on the metallic branch follow

$$\ln g_q = F_+\left(\frac{L}{\xi_+}\right), \tag{18}$$

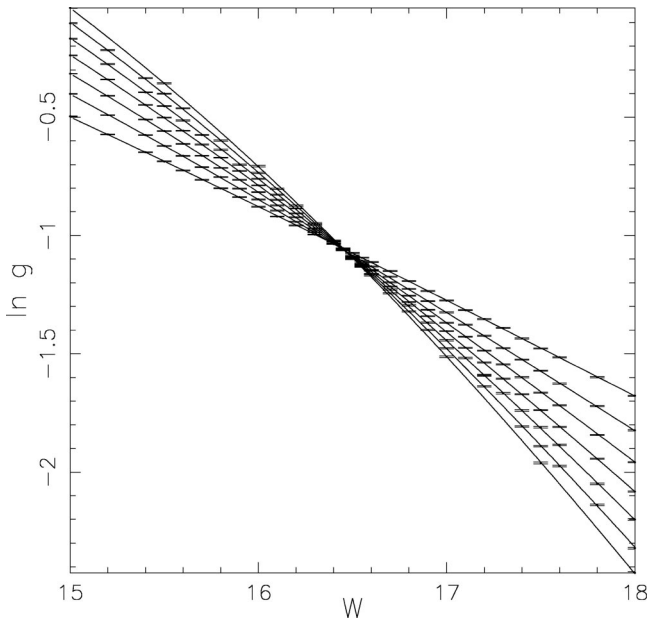

FIG. 3. Data for the median conductance ($q=0.5$ percentile), together with the best fit of Eq. (8), as a function of disorder W for systems sizes in the range $L=6-18$.

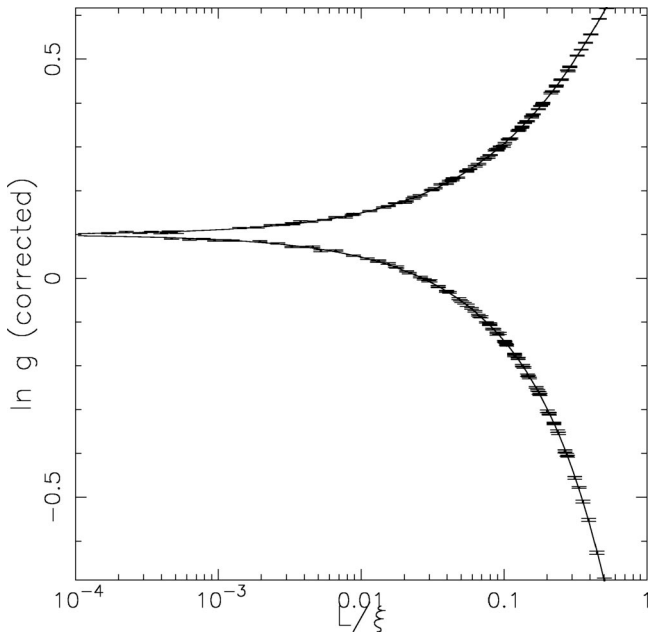

FIG. 4. The data for the $q=0.83$ percentile after corrections to scaling are subtracted and plotted as function of the ratio L/ξ to make single parameter scaling evident. The lines are the scaling functions (16) and (17) described in the text.

while data on the the insulating branch follow

$$\ln g_q = F_-\left(\frac{L}{\xi_-}\right). \tag{19}$$

For completeness some representative data for the median ($q=0.5$) and $q=0.87$ percentiles appear in Figs. 3 and 4, respectively.

We also analyzed the $q=0.975$ percentile of the distribution but were unable to convincingly fit its systems' size and disorder dependence. The origin of the difficulties may be the large corrections to scaling encountered for this percentile. Larger systems sizes will probably be needed for a definitive analysis of the high-conductance tail of the distribution.

For the percentiles analyzed the estimates of the critical disorder and the critical exponent obtained from the scaling of different percentiles are consistent as required. The estimates of the critical exponent in Table I are consistent with our previous estimates based on the scaling of the localization length in quasi-one-dimensional systems,[15] scaling of higher Lyapunov exponents,[22,23] and scaling of the mean conductance, mean resistance, and typical conductance.[16] The estimates are also consistent with numerical estimates reported by other authors.[24–26]

IV. CONCLUSION

Our numerical results demonstrate single parameter scaling of the zero-temperature conductance distribution in the critical regime of the Anderson transition in three dimensions. This result complements a previous demonstration of

the scaling of the mean conductance, typical conductance, and mean resistance.[16]

A two parameter scaling of the conductance distribution, similar to that found by Deych *et al.*[14] for one-dimensional systems, might be recovered in the strongly localized regime. The localization length diverges at the critical point while l_s, which is related to the integrated density of states, is always finite. Thus, on approaching the critical point we should always find $\xi > l_s$ and single parameter scaling should be observed. Far from the critical point, if ξ becomes less than l_s, a two parameter scaling might appear. It remains to be seen, however, if the results of Deych *et al.* carry over to higher dimensions.

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