Effects of proximity to an electronic topological transition on normal-state transport properties of the high-T_c superconductors

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Within the time-dependent Ginzburg-Landau theory, the effects of the superconducting fluctuations on the transport properties above the critical temperature are characterized by a nonzero imaginary part of the relaxation rate γ of the order parameter. Here, we evaluate Im γ for an anisotropic dispersion relation typical of the high-T_c cuprate superconductors (HTS's), characterized by a proximity to an electronic topological transition (ETT). We find that Im γ abruptly changes sign at the ETT as a function of doping, in agreement with the universal behavior of the HTS's. We also find that an increase of the in-plane anisotropy, as is given by a nonzero value of the next-nearest to nearest hopping ratio $r = t'/t$, increases the value of $|\mathrm{Im}\,\gamma|$ close to the ETT, as well as its singular behavior at low temperature, therefore enhancing the effect of superconducting fluctuations. Such a result is in qualitative agreement with the available data for the excess Hall conductivity for several cuprates and cuprate superlattices.

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I. INTRODUCTION

The appearance of superconducting fluctuations above the critical temperature T_c leads to precursor effects of the superconducting phase occurring already above T_c. Due to their short coherence length, the discovery of the high-T_c cuprate superconductors (HTS's) made the fluctuation regime experimentally accessible over a relatively wide temperature range above T_c.[1] Superconducting fluctuations manifest themselves in the singular temperature dependence of thermodynamic properties, such as the specific heat and the susceptibility, and of several transport properties (see Refs. 1 and 2 for recent reviews). In particular, the influence of superconducting fluctuations on the Ettinghausen effect,[3] the Nernst effect, the thermopower, the electrical conductivity, and the Hall conductivity [4] has been considered within the time-dependent Ginzburg-Landau (TDGL) theory for a layered superconductor in a magnetic field near T_c. A numerical approach within the fluctuation exchange (FLEX) approximation to the theory of electric transport in the normal state of the high-T_c cuprates has been developed by Yanase *et al.*[5–8]

The effect of fluctuations on the transport properties of the high-T_c superconductors can contribute to a better understanding of the unconventional properties of their normal state. Recent experimental studies of the Nernst effect in underdoped cuprates have demonstrated a sizable Nernst coefficient in the normal state both at high temperature and in high magnetic fields.[9–11] Such findings have been interpreted as an effect of precursor pairing above T_c in the pseudogap region, as well as of quantum superconducting fluctuations.[12]

In the case of the Hall effect, superconducting fluctuations induce a characteristic deviation from the normal-state temperature dependence of the Hall conductivity above T_c (Hall anomaly).[13] In particular, the fluctuation Hall conductivity $\Delta\sigma_{xy}$ has been evaluated within the TDGL theory,[14,4] and it has been shown that a Hall sign reversal takes place below T_c. The value and sign of $\Delta\sigma_{xy}$ strongly depend on the electronic structure of the material under consideration and, in particular, on the topology of its Fermi surface. It is well known that $\Delta\sigma_{xy}$ arises as a result of an electron-hole asymmetry in the band structure.[15] Recently, on the basis of the general requirement of gauge invariance of the TDGL equations, it has been shown that the sign of $\Delta\sigma_{xy}$ is determined by $\partial \ln T_c/\partial \ln \mu$, where μ is the chemical potential.[16] More recently, evidence for a universal behavior of the Hall conductivity as a function of doping has been reported in the cuprate superconductors.[17]

Given the relevance of the electronic structure in establishing the magnitude and sign of the fluctuation Hall effect, it is of obvious interest to study the effect of fluctuations on the transport properties of low-dimensional superconducting materials in the proximity of an electronic topological transition (ETT).[20,18,19] An ETT consists of a change of topology of the Fermi surface and may be induced by doping, as well as by changing the impurity concentration or applying pressure or anisotropic stress. In all such cases, one may introduce a critical parameter z, measuring the proximity to the ETT occurring at $z=0$. In the case of quasi-two-dimensional (quasi-2D) materials, such as the cuprates, the electronic band is locally characterized by a hyperboliclike dispersion relation. Therefore, one is particularly interested in the study of an ETT of the "neck disruption" kind, according to the original classification of Lifshitz.[20]

Some effects of an ETT (namely, the existence of a Van Hove singularity in the density of states) on the superconducting properties of the cuprates are well known.[21–24] Recently, it has been shown also that the effect of the proximity to an ETT is richer than having a Van Hove singularity in the density of states—namely, that the ETT is a specific quantum critical point. This leads to the existence of several quantum critical regimes that can explain the observed anomalous properties of the high-T_c cuprates in the normal state.[25–28] Some of the present authors have recently investigated the dependence of such effects on some specific material properties, such as the next-nearest- to nearest-neighbor hopping ratio[29] and anisotropic stress.[30] Concerning the normal-state transport properties of a superconductor, the effect of the proximity to an ETT has been studied for the thermoelectric power in a quasi-2D metal[31] and for the Nernst and the weak-field Hall effects for both 3D and quasi-2D metals.[32]

In this paper, we will study the anomalous Hall conductivity due to the superconducting fluctuations above T_c for a quasi-2D superconductor close to an ETT. The link between TDGL theory and the microscopic theory is provided by the relaxation rate γ of the fluctuating superconducting order parameter. In particular, a nonzero imaginary part of this quantity gives rise to a fluctuation contribution to the Hall effect. Here, we will study $\mathrm{Im}\,\gamma$ as a function of the ETT parameter z and temperature T, both numerically and analytically, for a realistic band dispersion typical of the high-T_c cuprate compounds. Close to the ETT, $\mathrm{Im}\,\gamma$ is characterized by a steep inflection point, surrounded by a minimum and a maximum, whose height increases with decreasing temperature. In the presence of electron-hole symmetry, we will show that $\mathrm{Im}\,\gamma$ is an odd function of the ETT parameter z and that $\mathrm{Im}\,\gamma$ vanishes and rapidly changes sign at the ETT point. In the cuprate superconductors, electron-hole symmetry is usually destroyed by a nonzero next-nearest- to nearest-neighbor hopping ratio $r=t'/t$.[33] In this case, the peaks in $\mathrm{Im}\,\gamma$ around the ETT point have unequal heights, and we will show that their dependence on the hopping parameter r is in qualitative agreement with the results of several fits against the fluctuation Hall conductivity data of various cuprates and cuprate superlattices.

The paper is organized as follows. In Sec. II we will briefly review the TDGL theory of superconducting fluctuations and the microscopic results for the direct and indirect contributions to the excess Hall conductivity $\Delta\sigma_{xy}$. In Sec. III we will outline the microscopic derivation of γ and explicitly evaluate $\mathrm{Im}\,\gamma$ as a function of the chemical potential and temperature. We will eventually summarize in Sec. IV.

II. EXCESS HALL CONDUCTIVITY

A phenomenological description of the fluctuation effects on the transport properties of a layered superconductor is based on the TDGL equation[1]

$$-\gamma\left(\frac{\partial}{\partial t}+\frac{2ie}{\hbar c}\varphi\right)\psi_\ell(\mathbf{r},t)=\frac{\delta\mathcal{F}}{\delta\psi_\ell^*(\mathbf{r},t)}+\zeta(\mathbf{r},t). \tag{1}$$

Here, $\psi_\ell(\mathbf{r},t)$ is the fluctuating GL order parameter on layer ℓ, φ is the scalar potential of the electic field, and $\zeta(\mathbf{r},t)$ is the Langevin force, taking into account for the order parameter dynamics. In the case of a layered superconductor, the GL functional $\mathcal{F}$ within the Lawrence-Doniach model[34] takes the form

$$\mathcal{F}=\sum_\ell \int d^2\mathbf{r}\left[a|\psi_\ell|^2+\frac{1}{2}b|\psi_\ell|^4+\frac{\hbar^2}{4m}\left|\left(\nabla_\parallel-\frac{2ie}{\hbar c}\mathbf{A}_\parallel\right)\psi_\ell\right|^2 + \mathcal{J}|\psi_{\ell+1}-\psi_\ell|^2\right], \tag{2}$$

where a and b are the usual GL coefficients, $\mathbf{A}_\parallel$ is the vector potential of a magnetic field perpendicular to the layers, and $\mathcal{J}$ characterizes the Josephson coupling between adjacent planes.[1]

In Eq. (1), the complex quantity γ is the relaxation rate of the order parameter within the TDGL theory. A nonzero value of $\mathrm{Re}\gamma$ is at the basis of the phenomenon of paraconductivity.[1] One finds $\mathrm{Re}\gamma=\pi\nu/8T$ at temperature T, where ν is the density of states.

Under complex conjugation and inversion of the magnetic field in Eq. (1), the equation for ψ_ℓ^* would be the same as that for ψ_ℓ, provided that $\mathrm{Im}\,\gamma=0$. Thus, a nonzero value of $\mathrm{Im}\,\gamma$ is associated with a breaking of electron-hole symmetry.[15,16] The condition $\mathrm{Im}\,\gamma\neq 0$ then gives rise to fluctuation effects on the Hall conductivity,[4] the Nernst effect,[4,32,35] and the thermopower.[4,31]

The fluctuation contibution to several transport properties, such as paraconductivity, magnetoconductivity, Nernst effect, and thermopower, have been evaluated under several approximations (see Ref. 1 for a review). From the microscopic point of view, the total fluctuation contributions $\Delta\sigma_{xy}$ to the Hall conductivity σ_{xy} close to T_c can be expressed as the sum of two terms:[36]

$$\Delta\sigma_{xy}^{\mathrm{AL}}=\frac{e^2}{16\hbar d}\,\frac{\sigma_{xy}^N}{\sigma_{xx}^N}\,\beta\,\frac{\pi d}{72\xi_c(0)}\,\frac{1+1/\alpha}{(1+1/2\alpha)^{3/2}}\,\frac{1}{\varepsilon^{3/2}}, \tag{3a}$$

$$\Delta\sigma_{xy}^{\mathrm{MT}}=\frac{e^2}{16\hbar d}\,\frac{\sigma_{xy}^N}{\sigma_{xx}^N}\,\frac{4}{\varepsilon-\delta}\ln\left[\frac{\varepsilon}{\delta}\,\frac{1+\alpha+(1+2\alpha)^{1/2}}{1+\alpha\varepsilon/\delta+(1+2\alpha\varepsilon/\delta)^{1/2}}\right], \tag{3b}$$

respectively, related to the Aslamazov-Larkin[37] (AL) and the Maki-Thomson[38] (MT) contributions. In Eqs. (3), $\varepsilon=\ln(T/T_c)\approx(T-T_c)/T_c$ is the reduced temperature, $\alpha=2\xi_c^2(0)/d^2\varepsilon$, d is the interlayer spacing, $\xi_c(0)$ is the coherence length along the c axis at $T=0$, $\sigma_{\alpha\beta}^N$ refer to the components of the conductivity tensor in the absence of fluctuations, $\delta=\pi\hbar/8k_\mathrm{B}T\tau_\phi$ is the MT pair breaking parameter, with τ_ϕ the phase relaxation time of the quasiparticles, and, finally, $\beta\propto\mathrm{Im}\,\gamma$ (Ref. 4). While $\xi_c(0)$ and δ can be independently determined by fitting analogous (AL+MT) expressions for the paraconductivity,[4] the parameter $\beta\propto\mathrm{Im}\,\gamma$ can be extracted by comparison with experimental data for the excess Hall effect.[36,39–41] Table I lists values of β for several layered cuprate superconductors and HTS superlattices. One can immediately observe that β shows a direct correlation with T_c; i.e., $|\beta|$ increases as T_c increases, which we will discuss in more detail in Sec. III.

TABLE I. Electron-hole asymmetry parameter $\beta \propto \mathrm{Im}\,\gamma$ and critical temperature T_c for several layered cuprates and cuprate superlattices. Tha values of β listed here have been obtained from a fit of the AL+MT corrections to conductivity and Hall conductivity, Eqs. (3), against data for an excess Hall effect.

	T_c [K]	β	
YBCO/PBCO (36 Å/96 Å)	68.68	−0.0003	Ref. 41
YBCO/PBCO (120 Å/96 Å)	86.33	−0.075	Ref. 41
YBCO	88.55	−0.17	Ref. 36
Bi-2223	105.	−0.38	Ref. 40
(Bi,Pb)-2223	109.	−1.	Ref. 39

III. EVALUATION OF Im γ IN THE PRESENCE OF AN ETT

From a microscopic point of view, the TDGL relaxation rate γ in Eq. (1) is related to the static limit of the frequency derivative of the retarded polarization operator as[1]

$$\gamma = i \lim_{\Omega\to 0} \frac{\partial \Pi^{\mathrm{R}}}{\partial \Omega}. \tag{4}$$

Before the analytic continuation, the polarization operator is defined as ($k_{\mathrm{B}}=\hbar=1$)

$$\Pi(\mathbf{k}, i\Omega_m; z, T) = \frac{T}{\mathcal{N}} \sum_{\mathbf{q},\epsilon_n} G^{(0)}(\mathbf{q}, i\epsilon_n + i\Omega_m) \times G^{(0)}(\mathbf{k}-\mathbf{q}, -i\epsilon_n), \tag{5}$$

where ϵ_n [Ω_m] are fermion [boson] Matsubara frequencies,

$$G^{(0)}(\mathbf{k},\epsilon) = \frac{1}{i\epsilon - \xi_{\mathbf{k}}} \tag{6}$$

is the Green's function for free electrons with dispersion relation $\xi_{\mathbf{k}}$, and the outer sum is performed over the $\mathcal{N}$ wave vectors $\mathbf{q}$ in the first Brillouin zone (1BZ). Here, we specifically have in mind the 2D tight-binding dispersion relation

$$\xi_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) + 4t' \cos k_x \cos k_y - \mu, \tag{7}$$

with t, t' being hopping parameters between nearest and next-nearest neighbors of a square lattice, respectively. Equation (7) has been often employed in order to describe the highly anisotropic dispersion relation of the cuprates. For $\mu = \mu_c = -4t'$, the Fermi surface defined by $\xi_{\mathbf{k}}=0$ has a critical form and undergoes an electronic topological transition (see Refs. 26, 28, and 29). Below, we will make use of the parameters $z=(\mu-\mu_c)/4t$, measuring the distance from the ETT ($z=0$), and of the hopping ratio $r=t'/t$ ($0<r<\frac{1}{2}$).[33] A nonzero value of r implies a breaking of electron-hole symmetry, with the electron subband width decreasing, and the hole subband width increasing of an equal amount $8rt$ (Ref. 29). The density of states (DOS) associated with Eq. (7) is characterized by a logarithmic singularity at $z=0$. Such a logarithmic cusp becomes weakly asymmetric around $z=0$ in the case $r\neq 0$ (see Appendix A).

With the help of standard methods,[42] the sum over electronic Matsubara frequencies in Eq. (5) is readily evaluated, and after analytic continuation to the upper complex plane,

$$\Pi^{\mathrm{R}}(\mathbf{k},\Omega) = -\lim_{\delta\to 0^+} \Pi(\mathbf{k}, \Omega + i\delta), \tag{8}$$

one obtains

$$\begin{aligned}\Pi^{\mathrm{R}}(0,\Omega;z,T) &= \frac{i}{2\pi}\frac{1}{\mathcal{N}}\sum_{\mathbf{k}} \frac{1}{\Omega + 2\xi_{\mathbf{k}} + i\delta}\left[\psi\left(\frac{1}{2} + \frac{i(\xi_{\mathbf{k}}+\Omega)}{2\pi T}\right)\right. \\ &\quad - \psi\left(\frac{1}{2} - \frac{i(\xi_{\mathbf{k}}+\Omega)}{2\pi T}\right) + \psi\left(\frac{1}{2} + \frac{i\xi_{\mathbf{k}}}{2\pi T}\right) \\ &\quad \left. - \psi\left(\frac{1}{2} - \frac{i\xi_{\mathbf{k}}}{2\pi T}\right)\right] \\ &= -\frac{1}{2}\frac{1}{\mathcal{N}}\sum_{\mathbf{k}} \frac{1}{\Omega + 2\xi_{\mathbf{k}} + i\delta}\left[\tanh\left(\frac{\xi_{\mathbf{k}}+\Omega}{2T}\right)\right. \\ &\quad \left. + \tanh\left(\frac{\xi_{\mathbf{k}}}{2T}\right)\right],\end{aligned} \tag{9}$$

where $\psi(z)$ here denotes the digamma function[43] and δ is a positive infinitesimal. Performing the frequency derivative and passing to the static limit, as required by Eq. (4), one has

$$\gamma = \frac{i}{8T}\frac{1}{\mathcal{N}}\sum_{\mathbf{k}} \frac{1}{\xi_{\mathbf{k}} + i\delta} F\left(\frac{\xi_{\mathbf{k}}}{2T}\right), \tag{10}$$

where

$$F(y) = \frac{1}{y}\tanh y - \frac{1}{\cosh^2 y}. \tag{11}$$

Figure 1 shows our numerical results for the real and imaginary parts of the TDGL relaxation rate γ as a function of the ETT parameter z over the whole bandwidth, for a representative value of the temperature parameter $\tau=T/4t=0.005$ and hopping ratios $r=0-0.384$. As anticipated, one finds that $\mathrm{Re}\,\gamma \propto \nu(z)$, with a logarithmic singularity at $z=0$ and an asymmetric z dependence in the case $r\neq 0$.

In the electron-hole symmetric case ($r=0$), Im γ is an odd function of the ETT parameter, vanishing at $z=0$—i.e., at the ETT—for all temperatures. Close to the ETT point, Im γ rapidly changes sign, with two symmetric peaks occurring very close to the ETT point. The height of these peaks decreases with increasing temperature (Fig. 2) and eventually diverges as $T\to 0$ [see Eq. (16) below]. Such a behavior, in particular, implies a sign-changing Hall effect as a function of doping and a large Hall effect close to the ETT. Moreover, the result $\mathrm{Im}\,\gamma(z=0)=0$ is consistent with the absence of electron-hole asymmetry.[15] A similar z dependence has been demonstrated also for the thermoelectric power in the proximity of an ETT.[31]

On the other hand, in the electron-hole asymmetric case ($r\neq 0$), one in general has $\mathrm{Im}\,\gamma(z)\neq -\mathrm{Im}\,\gamma(-z)$. However, one still recovers a sign-changing Im γ, with Im γ vanishing very close to the ETT. Moreover, the two peaks around the

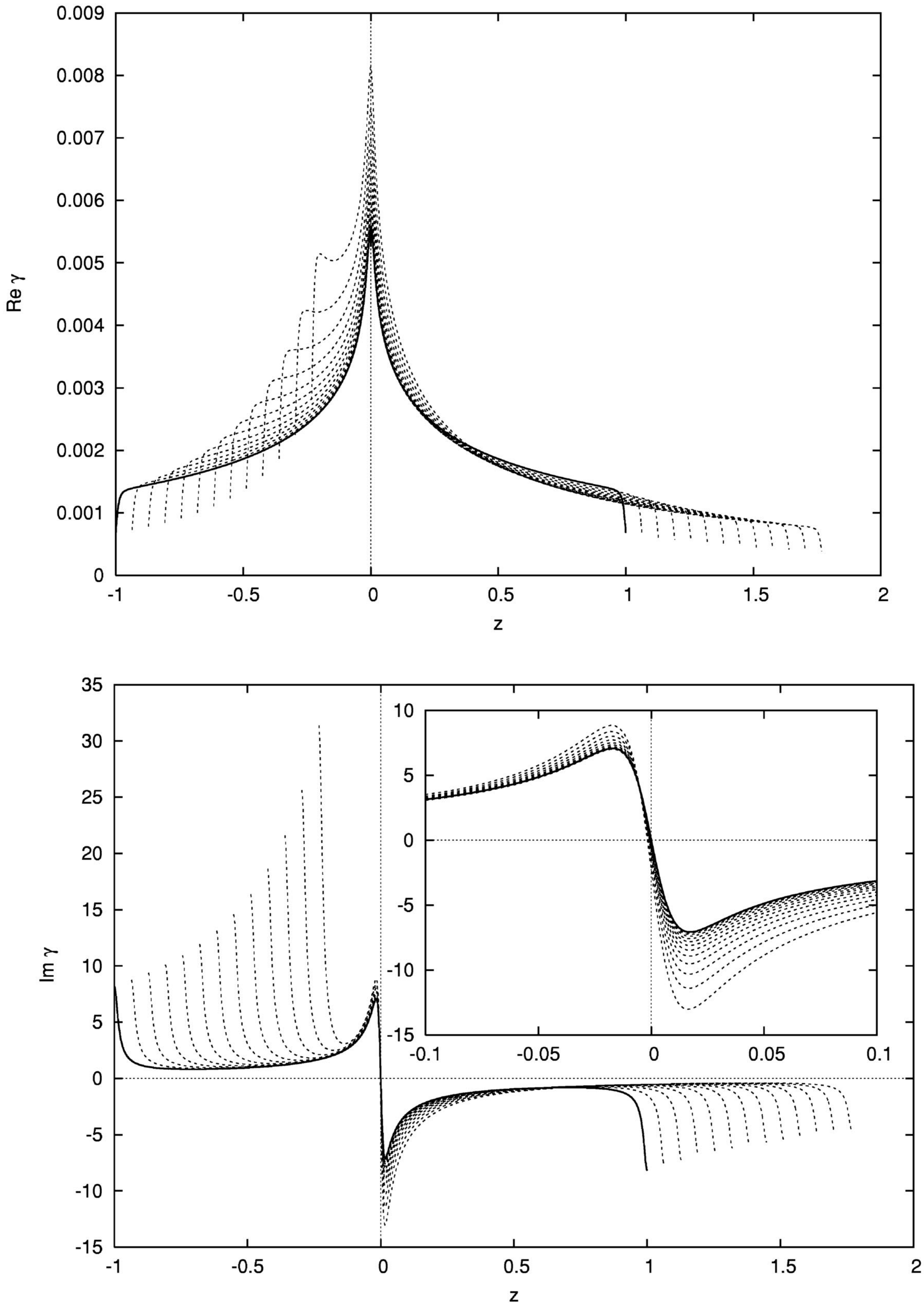

FIG. 1. Real part (top panel) and imaginary part (bottom panel) of TDGL relaxation rate γ, Eq. (10), as a function of the ETT parameter z ranging over the whole bandwidth, for fixed temperature $\tau=0.005$ and hopping ratios $r=0,\ 0.032,\ldots,0.384$, in units such that $4t=1$. Integration in **k** space in Eq. (10) has been performed via the tetrahedra method, with a mesh of 125 751 **k** points in the irreducible wedge of the 1BZ and $\delta=10^{-6}$. The inset in lower panel shows an enlarged view of Im $\gamma(z)$ close to the ETT. In particular, Im $\gamma(z)$ is an odd function of z in the electron-hole symmetric case ($r=0$, solid line), with Im $\gamma(z=0)=0$. Curves corresponding to increasing values of r give rise to more pronounced peaks in Im γ around the ETT point.

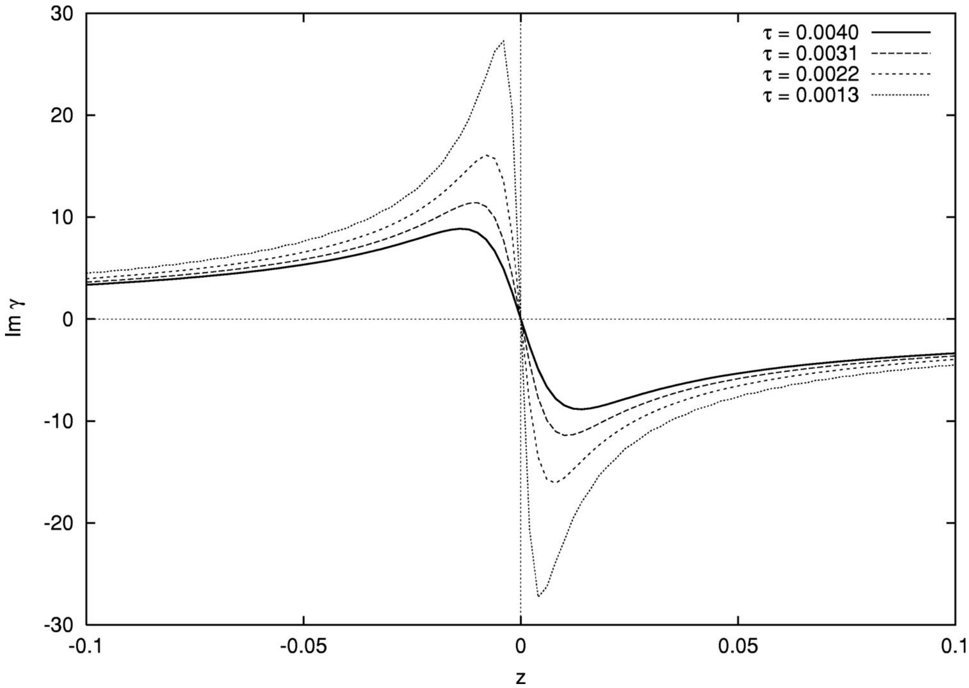

FIG. 2. Im $\gamma(z)$ in the electron-hole symmetric case ($r=0$), for decreasing temperatures τ (in units such that $4t=1$).

ETT have increasing heights with increasing hopping ratio r (Fig. 3). Given that a nonzero value of the hopping ratio r can be associated with structural distortions in the ab plane of the cuprates,[30] one may conclude that in-plane anisotropy enhances the fluctuation effects associated to a nonzero value of Im γ. Moreover, on the basis of the direct correlation existing between $T_{c,\max}$ and the hopping ratio r,[33,29] it follows that the heights of the peaks in Im γ around the ETT increase with increasing $T_{c,\max}$ across different classes of cuprates. Such a result is in agreement with the data listed in Table I for the excess Hall parameter $\beta \propto \mathrm{Im}\,\gamma$.

A further justification of the above numerical results can be drawn from an analysis of the continuum limit $[\mathcal{N}^{-1}\Sigma_{\mathbf{k}} \mapsto \int d^2\mathbf{k}/(2\pi)^2]$ of Eq. (10). Making use of the DOS $\nu(z)$ corresponding to the dispersion relation, Eq. (7) [see Eq. (A1) in Appendix A below], for the imaginary part of the relaxation rate one obtains

$$16t^2 \mathrm{Im}\,\gamma = \frac{1}{8\tau}\int_{-1+2r}^{1+2r} \frac{\nu(x)}{x-z} F\left(\frac{x-z}{2\tau}\right) dx, \qquad (12)$$

where $\tau = T/4t$. Equation (12) confirms that Im γ is an odd function of the ETT parameter z in the electron-hole symmetric case, a source of asymmetry being provided by a non-

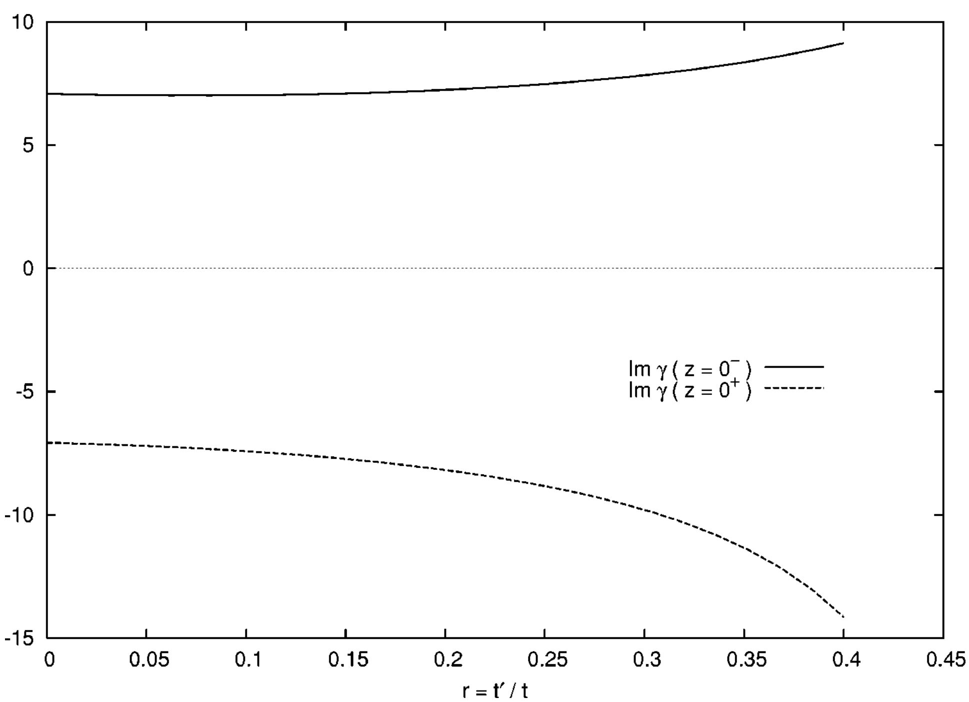

FIG. 3. Peak heights in Im γ around the ETT as a function of the hopping ratio $r=t'/t$, for fixed temperature $\tau=0.005$ (in units such that $4t=1$).

zero value of the hopping ratio r, both through a change of the integration limits and through a change in the DOS (see Appendix A).

Making use of the approximate expression fof the DOS, Eq. (A4), and of the asymptotic expansion of $F(y)$ in Eq. (11),

$$F(y) \simeq \frac{2}{3} y^2, \quad |y| \leqslant d,$$

$$|y|^{-1}, \quad |y| > d, \tag{13}$$

where $d = \sqrt[3]{3/2}$, Eq. (12) can be integrated analytically, yielding the result (in units such that $4t = 1$)

$$\begin{aligned}\mathrm{Im}\,\gamma \simeq & \frac{1}{2\pi^2} \frac{\ln b}{z^2-1} \mathrm{sech}\left(\frac{z-1}{2\tau}\right) \mathrm{sech}\left(\frac{z+1}{2\tau}\right) \\ & \times \left(z \sinh\frac{1}{\tau} - \sinh\frac{z}{\tau}\right) - \frac{1}{2\pi^2 \tau z}\left\{ \ln(1-z^2) \right. \\ & \left. + \frac{z}{2d\tau} \ln\frac{2d\tau+z}{2d\tau-z} + \ln\left[1 - \left(\frac{z}{2d\tau}\right)^2\right]\right\} - \frac{1}{24\pi^2\tau^3} \\ & \times \left[2d\tau z + (4d^2\tau^2 - z^2) \ln \frac{2d\tau+z}{2d\tau-z}\right].\end{aligned} \tag{14}$$

One qualitatively recovers the z dependence shown in Figs. 1 and 2 for $\mathrm{Im}\,\gamma$, with $\mathrm{Im}\,\gamma$ being an odd function of z at any given temperature τ. In particular, $\mathrm{Im}\,\gamma$ vanishes at $z=0$, where it behaves like

$$\mathrm{Im}\,\gamma \simeq -\frac{1}{8\pi^2 d} \frac{z}{\tau^3}, \tag{15}$$

for $|z| \ll 1$, $\tau \ll 1$. $\mathrm{Im}\,\gamma$ is also characterized by two antisymmetric peaks occurring at $z \simeq \pm 2d\tau$. In the particle-hole symmetric case $(r=0)$, the height of such peaks diverge in the limit $T \to 0$ as

$$|\mathrm{Im}\,\gamma_{\mathrm{peak}}| \approx \frac{\ln 2}{2\pi^2 d} \frac{1}{T^2}. \tag{16}$$

The singular behavior of $\mathrm{Im}\,\gamma$ as a function of the ETT critical parameter z in the limit $T \to 0$ is a fingerprint of quantum criticality.[25–28] In the case $r \neq 0$, additional terms arising from Eq. (A4) make this singular behavior asymmetric on the two sides of the ETT, as is hinted numerically by Fig. 1 (bottom panel, inset), thus showing that particle-hole asymmetry enhances the singular behavior of $\mathrm{Im}\,\gamma$ close to the ETT at low temperature.

For the sake of completeness, we also estimated $\mathrm{Im}\,\gamma$ away from the ETT, in the limit $|z|/\tau = |\mu - \mu_c|/T \gg 1$. The derivation of such result is outlined in Appendix B. Here, we just quote the final result

$$\mathrm{Im}\,\gamma \approx \frac{4t\nu_0}{4T(\mu-\mu_c)} \ln\frac{2T}{|\mu-\mu_c|}, \tag{17}$$

where $\nu_0 = (4t\pi^2\sqrt{1-4r^2})^{-1}$ is the density of states in the isotropic limit. Such a result again confirms that $\mathrm{Im}\,\gamma$ is a sign-changing function of doping, with $\mathrm{Im}\,\gamma < 0$ in the hole-like doping range $(z>0)$.

IV. CONCLUSIONS

We have studied the effect of superconducting fluctuations on the Hall conductivity of a quasi-2D layered superconductor close to an electronic topological transition. Within the time-dependent Ginzburg-Landau theory, such an effect is due to a nonzero imaginary part of the relaxation rate γ of the superconducting order parameter. We have evaluated $\mathrm{Im}\,\gamma$ as a function of the ETT parameter z and temperature, both numerically and analitically, for a quasi-2D dispersion relation, typical of the cuprates. Such a dispersion is characterized by a change of topology of the Fermi surface at $z=0$.

In agreement with general theoretical results,[15] we find that $\mathrm{Im}\,\gamma$ is a sign-changing function of the chemical potential, with $\mathrm{Im}\,\gamma = 0$ at $z=0$ in the electron-hole symmetric case. Such a result is in qualitative agreement with the universal behavior exhibited by the Hall anomaly in the cuprates.[17]

As expected, we find that $\mathrm{Im}\,\gamma$ increases with decreasing temperature, with a jumplike structure at the ETT whose height diverges as $T \to 0$ and increases with increasing in-plane anisotropy, given by a nonzero hopping ratio r. On the one hand, the singular behavior developed by $\mathrm{Im}\,\gamma$ at zero temperature is a fingerprint of quantum criticality at $T=0$. On the other hand, the monotonic r dependence of the peak heights in $\mathrm{Im}\,\gamma$ at a finite temperature is in qualitative agreement with the available experimental results for $\mathrm{Im}\,\gamma$ in several cuprate and cuprate superlattices, given the direct correlation between $T_{c,\max}$ and r observed for the cuprates.[33]

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APPENDIX A: DOS CLOSE TO THE ETT IN THE CASE $r \neq 0$

Here, we will derive a useful asymptotic expansion of the density of states close to the ETT in the electron-hole asymmetric case $(r \neq 0)$. In the following, we will employ energy units such that $4t=1$. We start by rewriting the DOS $\nu(z)$ corresponding to the dispersion relation, Eq. (7), for a square lattice (see Ref. 44 and references therein) as a function of the ETT parameter z:

$$\nu(z) = \frac{2}{\pi^2} \frac{1}{\sqrt{1+2r(z-r)}} K\left[\sqrt{\frac{1-(z-2r)^2}{1+2r(z-r)}}\right], \tag{A1}$$

where $K(k)$ denotes the complete elliptic integral of first kind of modulus k (Ref. 45). In Eq. (A1), the ETT parameter ranges as $-1+2r\leqslant z\leqslant 1+2r$, and $\nu(z)$ is characterized by a logarithmic singularity at $z=0$.

In the electron-hole symmetric case ($r=0$), $\nu(z)$ is an even function of z, with $-1\leqslant z\leqslant 1$. When electron-hole symmetry is broken by a nonzero hopping ratio r, the electron subband shrinks, while the hole subband widens of an equal amount $2r$, and the logarithmic cusp loses its symmetry around $z=0$. In order to extract the asymptotic behavior of Eq. (A1) around $z=0$ in the case $r\neq 0$, we introduce the ''electron'' and ''hole'' auxiliary variables $z_1=z/(1-2r)$ and $z_2=-z/(1+2r)$, with

$$(1-2r)z_1+(1+2r)z_2=0. \tag{A2}$$

Clearly, $z_1\to z$ and $z_2\to -z$ in the electron-hole symmetric case ($r=0$). In terms of these variables, the familiar plot of the DOS, Eq. (A1) (see, e.g., Fig. 2 in Ref. 29), can be seen as given by the intersection of Eq. (A2) with the surface plot of

$$\nu(z_1,z_2)=\frac{2}{\pi^2}\frac{1}{\sqrt{1-4r^2}}\frac{1}{\sqrt{1+z_1+z_2}}K\left[\sqrt{1+\frac{z_1z_2}{1+z_1+z_2}}\right]. \tag{A3}$$

While Eq. (A3) is manifestly symmetric under particle-hole conjugation ($z_1\leftrightarrow z_2$), Eq. (A2) is particle-hole symmetric only in the limit $r=0$.

Expanding Eq. (A3) in terms of the auxiliary variables z_1, z_2, and expressing the result back in terms of z, for $|z|\ll 1$ one eventually obtains

$$\nu(z)\approx\frac{2}{\pi^2}\frac{1}{\sqrt{1-4r^2}}(1-az)\ln\frac{b}{|z|}, \tag{A4}$$

which is the desired expansion around the ETT, with $a=2r/(1-4r^2)$ and $b=4\sqrt{1-4r^2}$. A result close to Eq. (A4), although within the context of an excitonic phase, can be found in Ref. 46. From Eq. (A4), one readily sees that, at lowest order in z, the source of asymmetry in the DOS logarithmic cusp at $z=0$ comes only from the prefactor, which is linear in z for $r\neq 0$.

APPENDIX B: BEHAVIOR OF Im γ AWAY FROM THE ETT

In the limit $|z|/\tau\gg 1$, we may forget about the details of the dispersion relation, provided we retain its main topological features. We can therefore expand Eq. (7) around the ETT as

$$\xi_{\mathbf{k}}\approx\frac{p_1^2}{2m_1}-\frac{p_2^2}{2m_2}-z, \tag{B1}$$

where $p_1=k_x$, $p_2=k_y-\pi$, and $m_{1,2}=2/(1\pm 2r)$ are the eigenvalues of the effective mass tensor around the ETT.[29] Here and below, we will make use of energy units such that $4t=1$.

Our starting point will be again the general expression for the retarded polarization operator, Eq. (9). Passing to the new coordinates $x=p_1/\sqrt{4Tm_1}$, $y=p_2/\sqrt{4Tm_2}$, one obtains

$$\Pi^{\mathrm{R}}(0,\omega;z,T)=-\nu_0\int_0^\infty dx \times\int_0^\infty dy\frac{[\tanh(s+\omega)+\tanh(s-\omega)]}{s+i\delta}, \tag{B2}$$

where $s=\omega+x^2-y^2-2\zeta$, $\omega=\Omega/(4T)$, $\zeta=z/(4T)$, and $\nu_0=\sqrt{m_1m_2}/(2\pi^2)=(\pi^2\sqrt{1-4r^2})^{-1}$ is the density of states in the isotropic case. According to Eq. (4), in order to obtain Im γ, we will be eventually interested in ReΠ^{R} in the limit $\omega\to 0$. Since the expansion in ω of the numerator in the integrand of Eq. (B2) gives no contribution linear in ω, we can just consider

$$\mathrm{Re}\Pi^{\mathrm{R}}(0,\omega\to 0;z,T)=-2\nu_0\int_0^\infty dx \times\int_0^\infty dy\frac{\tanh(\omega_1+x^2-y^2)}{\omega_1+x^2-y^2}, \tag{B3}$$

where $\omega_1=\omega-2\zeta$.

Let us now restrict ourselves to the case $z\ll -T$, in which case $\omega_1=\omega+2|\zeta|\gg 1$. The inner integral in Eq. (B3) can then be estimated by introducing a cutoff $\delta\lesssim 1$, splitting the integration range in the intervals $[0,a-\delta]$, $[a-\delta,a+\delta]$, $[a+\delta,\infty[$, with $a^2=x^2+\omega_1$, and replacing the hyperbolic tangent with its appropriate asymptotic expansion in each interval. One eventually obtains

$$\int_0^\infty dy\frac{\tanh(\omega_1+x^2-y^2)}{\omega_1+x^2-y^2}\approx-\frac{1}{2\sqrt{x^2+\omega_1}},\ln\frac{\delta^2}{4(x^2+\omega_1)}. \tag{B4}$$

Making use of such result back in Eq. (B3) and performing the ω derivative as required by Eq. (4), one has

$$\lim_{\omega\to 0}\frac{\partial\mathrm{Re}\Pi^{\mathrm{R}}}{\partial\omega}=-\frac{\nu_0}{2}\int_0^\infty dx\frac{1}{(x^2+2|\zeta|)^{3/2}}\ln\frac{\delta^2e^2}{4(x^2+2|\zeta|)}, \tag{B5}$$

where the last integration is trivial. Repeating an analogous derivation in the case $z\gg T$, one eventually obtains the final result, Eq. (17), to within logarithmic accuracy ($\delta^2e^2/4\approx 1$).

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