XY ring-exchange model on the triangular lattice

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We study ring-exchange models for bosons or *XY* spins on a triangular lattice. A four-spin exchange leads to a manifold of ground states with gapless excitations and critical power-law correlations. With a nearest-neighbor exchange, fluctuations select a four-fold ferrimagnetically ordered ground state with a small spin (superfluid) stiffness which breaks the global U(1) and translational symmetry. We explore consequences for phase transitions at finite temperature and in an in-plane magnetic field.

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I. INTRODUCTION

Multispin-exchange models incorporating ring-exchange processes have been of interest since the early studies of magnetism in solid helium-3.[1,2] Ring-exchange processes could also play an important role in Wigner crystals near the melting density[3,4] and in Mott insulators which retain a fair degree of local charge fluctuations.[5] Indeed, neutron scattering experiments[6] on insulating La_2CuO_4 have shown that aspects of the spin-wave dispersion in the antiferromagnet may be understood by invoking ring-exchange terms. Another reason for the interest in such models is that they may support spin-liquid phases which are translationally invariant Mott insulators with no magnetic order, as indicated from numerics on triangular and kagomé lattices.[7] This has been established analytically in some models with U(1) symmetry,[8,9] which may also be viewed as boson models. Finally, many models such as the quantum dimer model on a triangular lattice,[10] the easy-axis version of a generalized Heisenberg model on the kagomé lattice,[8] and the easy-axis antiferromagnet on the pyrochlore lattice[11] may be mapped onto effective ring-exchange models in their low-energy subspace of states. Thus, understanding the phases and phase transitions in ring-exchange models is important.

In this paper, we will focus on an *XY* (''easy-plane'') ring-exchange model on the triangular lattice with both four-spin and two-spin exchange terms (with strengths K and J, respectively) in the regime $J \ll K$. This model is interesting from several points of view. First, several systems such as Wigner crystals and certain organic Mott insulators form a triangular lattice of spins with possibly appreciable ring-exchange processes, and the *XY* spin problem is one tractable limit of such models. Second, easy-plane magnets on triangular lattices are known to exist (though mostly as stacked layers forming a three-dimensional system) and this could be of some relevance to them—in fact, these systems have motivated several studies of easy-plane triangular quantum Heisenberg models.[12] Finally, earlier work by some of us has shown that related models on the kagomé lattice are fractionalized,[8] whereas they support a critical phase (the ''exciton Bose liquid'') on the square lattice.[13] This was essentially shown by perturbing around the pure ring-exchange limit ($J=0$) where the kagomé and square lattices, respectively, have $\mathcal{O}(L^2)$ and $\mathcal{O}(L)$ conserved quantities. It is thus worthwhile to examine cases where such extensive symmetries are absent from the outset, even in the pure ring-exchange model, as is the case with the triangular lattice.

The principal results of this paper are the following: (i) We show that the U(1) four-spin exchange model on the triangular lattice (with $S_z=0$ or half-filling for the bosons) has a manifold of degenerate ground states and the correlation functions are critical in the ground states. We identify the appropriate symmetries and conserved quantities which give rise to this. (ii) Perturbing in the nearest-neighbor exchange (boson hopping) we find that fluctuations select a fourfold set of states from the degenerate manifold and the system develops long-range order at zero temperature. The ordered states break the global U(1) symmetry as well as translational symmetry. (iii) In the ordered ground states, the spin (superfluid) stiffness $\sim J^2/K$ and is very small for small J/K. At any small nonzero temperature, the U(1) symmetry is restored and we are in a phase with power-law spin (phase) order, which gives way to a disordered phase via a Berezinskii-Kosterlitz-Thouless[14] (BKT) transition for $T > T_{BKT} \sim J^2/K$. However, the discrete symmetry of the broken translations is not destroyed until a higher temperature $T_c \sim J$. We analyze phase transitions at finite temperature, and also in the presence of an in-plane magnetic field, within a Landau theory, and discuss the phase diagram and experimental consequences.

The outline of the paper is as follows. In Sec. II, we will present the Hamiltonian. In Sec. III we analyze its symmetries and conserved quantities, and define the dual representation of the model which allows us to analyze the instabilities of the system towards charge and energy ordering. We present arguments and numerical results to show that the model with $J=0$ has critical power-law correlations at $T=0$. In Sec. IV, we perturbatively analyze effects on introducing the nearest-neighbor exchange (boson hopping) and find a phase-ordered state. We also discuss the phase diagram within Landau theory as a function of temperature and in-plane magnetic-field. We close with a discussion and speculations for three-dimensional generalizations and SU(2)-invariant models in Sec. V.

II. MODEL

We define the model in rotor variables as

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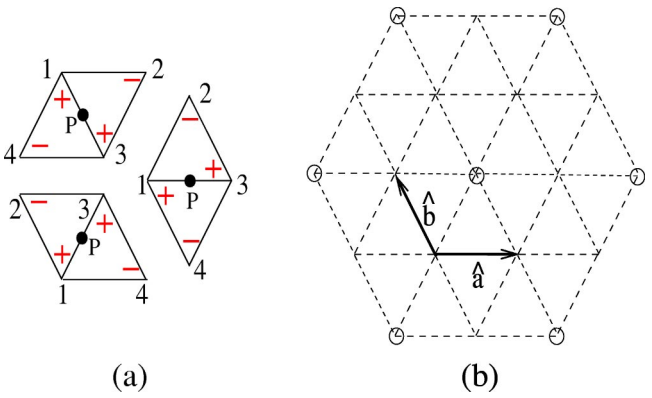

FIG. 1. (a) The three kinds of plaquettes on the triangular lattice. The ring-exchange process involves the spins located at the points 1–4 of each plaquette. The labeling is chosen to coincide with that appropriate for the definition dual variables in the dual model discussed in Sec. II C, and the sites P form the sites of the dual kagomé lattice. (b) Rotating $\varphi \to \varphi + \pi$ on the sites indicated by open circles (which form one of four possible triangular sublattices) changes the sign of the ring-exchange term K in the Hamiltonian. Also shown are the basis vectors $\hat{a}, \hat{b}$ for the triangular lattice.

$$H = K \sum_{P} \cos(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4) + J \sum_{\langle i,j \rangle} \cos(\varphi_i - \varphi_j) + \frac{U}{2} \sum_{i} (n_i - \bar{n})^2, \tag{1}$$

where φ_i is the phase of the boson variable ($\varphi_i = 0$ and $\varphi_i = 2\pi$ are identified) and n_i is the the canonically conjugate boson number [respectively the angle and angular momentum of a U(1) rotor], satisfying the commutation relation $[\varphi_i, n_j] = i\delta_{i,j}$. The terms $\varphi_1, \ldots, \varphi_4$ in the first term of H denote angles around four-site plaquettes of the triangular lattice—there are three such kinds of plaquettes as shown in Fig. 1(a). The J term denotes nearest-neighbor hopping of bosons while U denotes a repulsive interaction between bosons. With $U/K \to \infty$, we can identify $S_z(\mathbf{r}) = n_{\mathbf{r}} - 1/2$, $S^{\pm}(\mathbf{r}) = \exp(\pm i\varphi_{\mathbf{r}})$, and this model reduces to an $S = 1/2$ XY quantum spin model with $\bar{n} = 1/2$ corresponding to total $S_z = 0$.

With $J = 0$, changing $\varphi_{\mathbf{r}} \to \phi_{\mathbf{r}} = \varphi_{\mathbf{r}} + \pi$ on the sites indicated by open circles in Fig. 1(b) changes the sign of K (there are four such choices of a triangular sublattice on which to make this transformation, as is clear from the figure). Thus, for $J = 0$, the ground-state energy is independent of the sign of K. For nonzero J, however, the sign of K is important.

For $K < 0$, the leading perturbative corrections on including a nonzero J are $\mathcal{O}(J)$, and the energy and phases depend on the sign of J—for $J < 0$ we get a ferromagnetic phase, while $J > 0$ leads to a $\sqrt{3} \times \sqrt{3}$ Néel order, which are also the ground states for large $|J/K|$ (where we may ignore the four-spin term). Thus there are no phase transitions at any nonzero J and we only obtain the well-studied phases.

For $K > 0$, as appropriate for, say, spin degrees of freedom in an electronic Mott insulator, it turns out that the leading perturbative corrections to the free energy are $\mathcal{O}(J^2)$. To this order, the free energy is independent of the sign of J and we obtain a ferrimagnetic phase which breaks the global U(1) invariance as well as fourfold translational symmetry. The physics of this phase and the phase transitions out of it will be the focus of this paper. For large $|J|/K$ we of course recover the conventional phases mentioned above.

To make an estimate of the coupling constants in one case, let us imagine starting from the Hubbard model for electrons (with nearest-neighbor hopping) at half-filling on a triangular lattice and perturbing in t/U to derive an effective spin model with ring-exchange terms in the insulator. This takes the form

$$H_{\text{spin}} = K \sum_{\square} [(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_3 \cdot \mathbf{S}_4) + (\mathbf{S}_1 \cdot \mathbf{S}_4)(\mathbf{S}_2 \cdot \mathbf{S}_3) - (\mathbf{S}_1 \cdot \mathbf{S}_3)(\mathbf{S}_2 \cdot \mathbf{S}_4)] + \sum_{i,j} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{2}$$

where the first term involves all four-site plaquettes on the triangular lattice. Adapting results from Ref. 5 to this case, we find $J = 4t^2/U - 28t^4/U^3$, $J' = J'' = 4t^4/U^3$, and $K = 80t^4/U^3$. For $U/t = 6$, numerical results[15] show that the model is in an insulating phase though still close to the metal-insulator boundary; in this case $J/t = 0.53$, $J'/t = J''/t = 0.02$, and $K/t = 0.37$. Clearly, it seems that one can describe the spin degrees of freedom in the insulator by setting the further-neighbor couplings $J' = J'' = 0$ and retaining only nonzero J, K. Indeed, exact-diagonalization studies[7] find a spin-liquid phase for a closely related model in qualitative agreement with the Monte Carlo results on the Hubbard model.[15] It thus seems profitable to understand the above model for $J/K \ll 1$ as a starting point to analyze this full problem. For $U/K \to \infty$, the Hamiltonian in Eq. (1) is precisely the XY limit of the above model obtained by setting terms containing S_z to zero.

III. MODEL WITH $J = 0$

A. Symmetries

For $J = 0$, we can change the sign of K by shifting $\varphi_{\mathbf{r}} \to \phi_{\mathbf{r}} = \varphi_{\mathbf{r}} + \pi$ on any one of four sublattices of the triangular lattice indicated earlier (we will work with these ''rotated'' variables $\phi_{\mathbf{r}}$ and a ferromagnetic four-spin term for convenience). This means shifting $\phi_{\mathbf{r}} \to \phi_{\mathbf{r}} + \pi$ on any two such sublattices leaves the action invariant and is a symmetry operation. We can identify this with a conservation of total boson number modulo two on alternate rows of the triangular lattice. These rows can run in any of the three symmetry directions of the lattice; this corresponds to a total of four symmetry operations including the identity.

The rotor Hamiltonian above describes bosons hopping on plaquettes of the triangular lattice. For $J = 0$, the dynamics conserves the center of mass of the bosons, since the bosons hop equal distances in opposite directions on any plaquette. In other words the center of mass operator $\hat{\mathbf{R}}_{\text{c.m.}} = \Sigma_{\mathbf{r}} \mathbf{r} \hat{n}_{\mathbf{r}}$ satisfies $[\hat{\mathbf{R}}_{\text{c.m.}}, H] = 0$ and is a constant of motion.

Thus, in deriving the imaginary-time path integral for the partition function in terms of ϕ, we may insert factors of $\exp(i\mathbf{q}\cdot\hat{\mathbf{R}}_{\text{c.m.}})$ with an arbitrary vector $\mathbf{q}$, since $\hat{\mathbf{R}}_{\text{c.m.}}$ commutes with H. This corresponds to $\phi_{\mathbf{r}}\rightarrow\phi_{\mathbf{r}}+\mathbf{q}\cdot\mathbf{r}$ being a symmetry of the phase action.

If we identify $\phi=0$ with $\phi=2\pi$ or keep track of the fact that the charge is quantized, the first symmetry operation discussed above is a special case of the second and corresponds to choosing $\mathbf{q}=\mathbf{Q}_i$ $(i=1,2,3)$ where the momenta $\mathbf{Q}_i$ are at the center of the edges of the Brillouin zone, indicated in Fig. 4(a), below. The above symmetries for $J=0$ imply we cannot have terms which depend on $(\nabla\phi)$ in the phase action (in the path integral for the partition function). The lowest allowed gradient term on which the action can depend is of the form $(\nabla^2\phi)$. We explicitly confirm this in a spin-wave calculation in Sec. III C, where we show the collective mode disperses as $\omega^2(\mathbf{k})\sim|\mathbf{k}|^4$ for small $|\mathbf{k}|$ and calculate its consequences for correlation functions. The presence of a nonzero J explicitly breaks these symmetries; in fact $J=0$ will turn out to be a critical point of the model.

B. Plaquette duality and dual action

For $J=0$, it is useful to define a dual representation for the model which permits us to obtain spin-wave theory as a well-defined limit of the model and to analyze instabilities of the spin-wave phase towards charge and energy ordering. For this purpose, we use a plaquette duality transformation[13] and work with dual variables θ_P and N_P which reside at the centers of each of the four-site plaquettes of the triangular lattice—these sites (labeled by P) form a kagomé lattice with nearest-neighbor spacing 1/2 in units of the lattice spacing of the original triangular lattice. Alternatively, we may label sites on the dual lattice as $(\mathbf{r},\alpha)$, viewing the kagomé lattice as a triangular lattice three sites (labeled $\alpha=1,2,3$) per unit cell, and we will use this notation whenever convenient. Thus, there are 3 times as many sites (and hence degrees of freedom) on the dual lattice compared to the original, and this is reflected in a redundancy in the description which we will discuss shortly.

Define

$$\pi N_P=(\phi_1-\phi_2+\phi_3-\phi_4), \tag{3}$$

$$\pi n_{\mathbf{r}}=\sum_{\text{hex}}\theta_P-\sum_{\text{star}}\theta_P, \tag{4}$$

where the angles $\phi_1,\ldots,\phi_4$ in Eq. (3) are around plaquette P of the lattice with the convention indicated in Fig. 1(a). The sites hex and star lie on the 12-site unit of the kagomé lattice around the site $\mathbf{r}$ as shown in Fig. 2(a) (marked by "+" and "−," respectively), while the center of each kagomé unit lies on the triangular lattice (and is labeled $\mathbf{r}$). In these variables, the dual Hamiltonian on the kagomé lattice is

$$H_{\text{dual}}=K\sum_P\cos(\pi N_P)+\frac{U}{2\pi^2}\sum_{\mathbf{r}}\left(\sum_{\text{hex}}\theta_P-\sum_{\text{star}}\theta_P-\pi\overline{n}\right)^2 \tag{5}$$

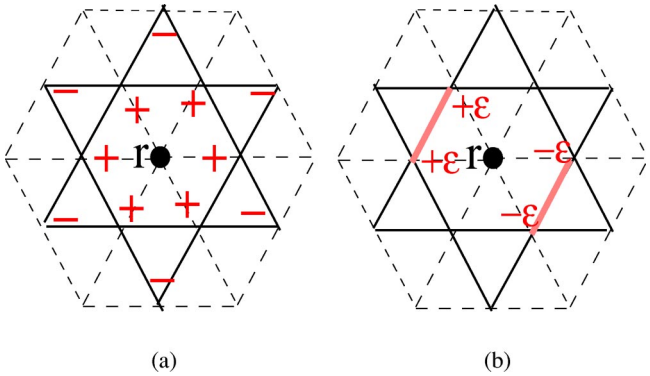

FIG. 2. (a) The original triangular lattice (dashed lines) and the dual kagomé lattice (bold lines). The sites "+" and "−" indicate the sites of the kagomé unit around the site $\mathbf{r}$ of the triangular lattice and are referred to as "hex" and "star," respectively in the paper. (b) The "gauge" transformation on the dual lattice—shifting θ by ε on the indicated sites as shown, with ε/π being an integer—is an exact symmetry of the dual Hamiltonian.

and the dual variables satisfy $[N_P,\theta_{P'}]=i\delta_{P,P'}$, with θ_P/π having integer eigenvalues, while N_P has a continuous spectrum.

We pause to elaborate further upon the redundancy of the θ variables. Since only the combination $n_{\mathbf{r}}$ of the θ_P variables is physical, the Hamiltonian will be invariant under any integer multiple of π shift of the θ_P that leaves all the $n_{\mathbf{r}}$ invariant. This can be done locally as follows. Pick any hexagon of the dual kagomé lattice, and choose two neighboring sites on this hexagon. Shift $\theta_P\rightarrow\theta_P+\pi$ on those sites, and simultaneously take $\theta_P\rightarrow\theta_P-\pi$ on the sites on the opposite side of this hexagon, leaving the remaining two site of the hexagon (and all the other sites not on this hexagon) untouched as shown in Fig. 2(b). Since this leaves all physical properties invariant, this should be regarded as a local gauge symmetry. Such transformations are generated by the unitary operators

$$G_{\mathbf{r}}(\mathbf{K}_\alpha)=\exp\left(i\mathbf{K}_\alpha\cdot\sum_{\text{hex}}(\mathbf{r}_P-\mathbf{r})N_P\right), \tag{6}$$

where $\mathbf{K}_\alpha$ with $\alpha=1,2,3$ are the three reciprocal lattice vectors for the triangular lattice. In the spirit of a gauge theory, we could restrict our consideration to physical gauge-invariant states which satisfy $G_{\mathbf{r}}(\mathbf{K}_\alpha)=1$. This can be rewritten suggestively as

$$\sum_{\text{hex}}(\mathbf{r}_P-\mathbf{r})N_P=0(\text{mod}\,\mathbf{a}), \tag{7}$$

where $\mathbf{a}$ is an arbitrary Bravais lattice vector for the triangular lattice. As in Refs. 8 and 13, the operator N_P may be regarded as the local "vortex number" (modulo 2) on the site P of the dual lattice. Thus the condition of Eq. (7) requires that the local *vortex center of mass* on each hexagonal plaquette of the kagomé lattice vanish—the ambiguity by a Bravais lattice vector corresponding exactly to the ambiguity

in N_P by a shift of 2. Thus the gauge invariance of the dual description is related to the immobility of ''vortices'' in this model.

Using a Trotter decomposition, we may write the partition function as a discretized imaginary-time path integral in the standard manner. This leads to $Z=\int \mathcal{D}\theta(\mathbf{r},\tau)\exp\{-S[\theta(\mathbf{r},\tau)]\}$ with the action

$$S_{\rm dual}=\frac{1}{\pi^2}\ln\left(\frac{2}{\epsilon_\tau K}\right)\sum_{\mathbf{r},\alpha,\tau}(\theta_{\mathbf{r},\alpha,\tau}-\theta_{\mathbf{r},\alpha,\tau+1})^2 + \frac{\epsilon_\tau U}{2\pi^2}\sum_{\mathbf{r}}\left(\sum_{\rm hex}\theta_P-\sum_{\rm star}\theta_P-\pi\overline{n}\right)^2, \quad (8)$$

where the field $\theta_{\mathbf{r},\alpha,\tau}/\pi$ is an integer-valued field, $\epsilon_\tau=\beta/N_\tau$ with $\beta=1/T$ is the inverse temperature, and N_τ is the number of imaginary-time slices. The quantum problem at a temperature T corresponds to the continuum limit $\epsilon_\tau\to 0, N_\tau\to\infty$ with fixed $1/\epsilon_\tau N_\tau=T$. Thus, we reduce the quantum model to an anisotropic $(2+1)$-dimensional classical model—since there is no sign problem, this proves useful for carrying out Monte Carlo simulations in the dual representation to numerically test for charge ordering and plaquette-energy ordering in the ground state at arbitrary U/K.

C. Effective theory and spin-wave approximation

We may rewrite the dual action as

$$S_{\rm dual}=\frac{1}{\pi^2}\ln\left(\frac{2}{\epsilon_\tau K}\right)\sum_{\mathbf{r},\alpha,\tau}(\theta_{\mathbf{r},\alpha,\tau}-\theta_{\mathbf{r},\alpha,\tau+1})^2 + \frac{\epsilon_\tau U}{2\pi^2}\sum_{\mathbf{r}}\left(\sum_{\rm hex}\theta_P-\sum_{\rm star}\theta_P-\pi\overline{n}\right)^2 - \sum_{q,\mathbf{r},\tau,\alpha}v_{2q}^0\cos(2q\theta_{\mathbf{r},\alpha,\tau}), \quad (9)$$

where the bare couplings v_{2q}^0 are chosen to enforce the integer constraint on θ and may be singular. We will now write down a low-energy effective action in terms of $\vartheta_{\mathbf{r},\alpha,\tau}=[\theta]_f-(Qmn)/2$. Here $[\theta]_f$ symbolically denotes an integration over the fast (high-energy) degrees of freedom while retaining the spatial structure of the lattice, $Q=2\pi\overline{n}$, and the site $\mathbf{r}\equiv m\hat{a}+n\hat{b}$ in terms of the basis vectors $\hat{a},\hat{b}$ of the triangular lattice shown in Fig. 1(b). The factor we have subtracted out eliminates the mean density $\overline{n}$ from the second (boson repulsion) term in the Hamiltonian.

If we ignore the terms with v_{2q}^0, we obtain a Gaussian action in terms of ϑ. Studying this action motivates us to guess that the effective low-energy action may be of the form $S_{\rm dual}^{\rm eff}=S_{\rm dual}^{(0)}+S_{\rm dual}^{(1)}$ with

$$S_{\rm dual}^{(0)}=\int\sum_{\omega\mathbf{k},\alpha,\beta}\vartheta_{-\mathbf{k},\alpha,-\omega}\left(\frac{\omega^2\delta_{\alpha\beta}}{2\pi^2\mathcal{K}(\mathbf{k})}+\frac{\mathcal{U}(\mathbf{k})}{2\pi^2}G_{\alpha\beta}(\mathbf{k})\right)\vartheta_{\mathbf{k},\beta,\omega} \quad (10)$$

and

$$S_{\rm dual}^{(1)}=\int\sum_{\tau q,\mathbf{r},\alpha}v_{2q}\cos(2q\vartheta_{\mathbf{r},\alpha,\tau}+Qqmn)+S_{\rm higher}, \quad (11)$$

where we have defined renormalized couplings $\mathcal{U}(\mathbf{k}),\mathcal{K}(\mathbf{k})$ (which are smooth nonzero functions of $\mathbf{k}$) and v_{2q}, the site $\mathbf{r}\equiv(m,n)$ as before, and the 3×3 matrix $G_{\alpha,\beta}(\mathbf{k})=A_\alpha^*(\mathbf{k})A_\beta(\mathbf{k})$ with

$$A_1(\mathbf{k})=1+e^{-ik_a}-e^{-ik_a-ik_b}-e^{ik_b},$$
$$A_2(\mathbf{k})=1+e^{-ik_a-ik_b}-e^{-ik_a}-e^{-ik_b},$$
$$A_3(\mathbf{k})=1+e^{-ik_b}-e^{ik_a}-e^{-ik_a-ik_b}, \quad (12)$$

where $k_a=\mathbf{k}\cdot\hat{a}$ and $k_b=\mathbf{k}\cdot\hat{b}$. Here $S_{\rm higher}$ denotes other terms which might be generated in deriving the low-energy action. For the present purposes, we will not write down the explicit form of these terms but we will discuss them later.

If we set $v_{2q}=0$ and ignore $S_{\rm higher}$, we can obtain the eigenmodes of the Gaussian action $S_{\rm dual}^{(0)}$ by diagonalizing $G_{\alpha\beta}(\mathbf{k})$. We find there is one nonzero eigenvalue $\lambda(\mathbf{k})$ and two zero eigenvalues. This corresponds to one dispersing mode (Θ_1) and two nondispersive modes ($\Theta_{2,3}$) with zero energy (flat bands), and the action at this level takes the form

$$S_{\rm dual}^{(0)}=\int_\omega\sum_{\mathbf{k}}\left(\frac{\omega^2+\mathcal{E}^2(\mathbf{k})}{2\pi^2\mathcal{K}(\mathbf{k})}\right)|\Theta_1(\mathbf{k},\omega)|^2 + \int_\omega\sum_{\mathbf{k},\alpha=2,3}\left(\frac{\omega^2}{2\pi^2\mathcal{K}(\mathbf{k})}\right)|\Theta_\alpha(\mathbf{k},\omega)|^2, \quad (13)$$

where we have set $\lambda(\mathbf{k})=\mathcal{E}^2(\mathbf{k})/\mathcal{U}(\mathbf{k})\mathcal{K}(\mathbf{k})$ so that $\mathcal{E}(\mathbf{k})$ appears as the excitation energy of the collective mode Θ_1. Explicitly,

$$\lambda(\mathbf{k})=2[6-3\cos k_a-3\cos k_b-3\cos(k_a+k_b)+\cos(k_a-k_b)+\cos(2k_a+k_b)+\cos(2k_b+k_a)]; \quad (14)$$

thus the dispersing mode Θ_1 has one gapless point at $\mathbf{k}=(0,0)$. The leading dispersion away from this point is $\mathcal{E}^2(\mathbf{k}\to 0)=(9/16)\mathcal{U}(0)\mathcal{K}(0)|\mathbf{k}|^4$—as stated in Sec. III A, this dispersion is dictated by the symmetry of conservation of center of mass of the bosons which forbids a term $\sim|\mathbf{k}|^2$.

We can also now see that the extra degrees of freedom (two per site of the triangular lattice) introduced in going over to the dual description manifest themselves as zero-energy modes corresponding to unphysical fluctuations, while the dispersing mode corresponds to physical fluctuations. In fact, by Fourier transforming a general superposition $f_2(\mathbf{k})\Theta_2(\mathbf{k})+f_3(\mathbf{k})\Theta_3(\mathbf{k})$, we can show that the zero modes indeed arise from the local gauge symmetry generated by Eq. (6) in real space. More precisely, in the coarse-grained quadratic low-energy theory, these discrete symmetry operations are promoted to continuous shifts (with arbitrary $|\mathbf{K}_\alpha|$). Similarly, Fourier transforming $f_1(\mathbf{k})\Theta_1(\mathbf{k})$, we find that the physical fluctuations are composed precisely of the gauge-invariant boson densities.

At this stage, we may reintroduce the boson phase fields by a Hubbard–Stratonovitch decoupling of the time derivative term of the physical fluctuation Θ_1 (as in Ref. 13), namely,

$$\exp\left(-\frac{\omega^2}{2\pi^2\mathcal{K}(\mathbf{k})}|\Theta_1(\mathbf{k},\omega)|^2\right)$$

$$=\int D\phi\phi^*\exp\left(-\frac{\mathcal{E}^2(\mathbf{k})}{2\mathcal{U}(\mathbf{k})}|\phi(\mathbf{k},\omega)|^2\right)$$

$$\times\exp\left(\frac{\omega\mathcal{E}(\mathbf{k})}{2\pi\sqrt{\mathcal{U}(\mathbf{k})\mathcal{K}(\mathbf{k})}}[\phi(\mathbf{k})\Theta_1^*(\mathbf{k})-\phi^*(\mathbf{k})\Theta_1(\mathbf{k})]\right). \tag{15}$$

Provided v_{2q} and S_{higher} are irrelevant and can be ignored, we can integrating out the field Θ_1 which leads to a Gaussian action for the boson field ϕ,

$$S_\phi^{(0)}=\int_\omega\sum_{\mathbf{k}}\left(\frac{\omega^2+\mathcal{E}^2(\mathbf{k})}{2\mathcal{U}(\mathbf{k})}\right)|\phi(\mathbf{k})|^2. \tag{16}$$

This corresponds to a spin-wave (harmonic) approximation and may be used to evaluate boson correlation functions to zeroth order in v_{2q}.

D. Boson correlation functions in spin-wave theory

To characterize the Gaussian theory $S_\phi^{(0)}$, we evaluate the space-time correlation functions for the boson creation operator $\exp(i\phi_{\mathbf{r}\tau})$. At long times or large separations, these correlation functions reduce to

$$\langle e^{i\phi_{\mathbf{r},0}}e^{-i\phi_{\mathbf{0},0}}\rangle\sim|\mathbf{r}|^{(-1/2\pi\sqrt{3})\sqrt{\mathcal{U}(0)/\mathcal{K}(0)}}$$

$$\langle e^{i\phi_{\mathbf{0},\tau}}e^{-i\phi_{\mathbf{0},0}}\rangle\sim|\tau|^{(-1/4\pi\sqrt{3})\sqrt{\mathcal{U}(0)/\mathcal{K}(0)}}. \tag{17}$$

Clearly, the Gaussian theory describes a critical liquid of the bosons with power-law correlations arising from the gapless excitations dispersing as $\mathcal{E}(\mathbf{k})\sim|\mathbf{k}|^2$. From the long-time behavior of the two-point correlation in a finite-size system, we may evaluate the finite-size scaling of the gap to adding a particle to be just $\Delta(L)\simeq\mathcal{U}(0)/L^2$. In the thermodynamic system, the low-energy density of states per unit volume of collective excitations is given by

$$N(\omega)=\frac{1}{V}\sum_{\mathbf{k}}\delta(\omega-\mathcal{E}(\mathbf{k}))\underset{\omega\to0}{\simeq}\frac{1}{2\pi\sqrt{3\mathcal{U}(0)\mathcal{K}(0)}}, \tag{18}$$

which is a constant depending on the interactions. Thus, all low-energy long-wavelength properties of the liquid are determined in terms of the $\mathbf{k}\to0$ behavior of the functions $\mathcal{U}(\mathbf{k}),\mathcal{K}(\mathbf{k})$. We next present arguments and numerical evidence that this Gaussian description of a critical liquid in the model with $J=0$ may be valid even in the presence of terms v_{2q} and S_{higher}.

E. Argument for irrelevance of higher-order terms

To see whether the term v_{2q} is relevant, we can evaluate the correlation functions of this operator, and we find it is local in space and exponentially decaying in time,

$$\langle\cos 2\vartheta_{\mathbf{r},\alpha,\tau}\cos 2\vartheta_{\mathbf{0},\beta,0}\rangle\sim\delta_{\mathbf{r},0}\delta_{\alpha,\beta}\ \exp(-\gamma|\tau|), \tag{19}$$

with a constant $\gamma>0$. This is not hard to understand—it arises from the fact that the variable ϑ is a combination of the eigenmodes $\Theta_{1,2,3}$ and $\cos 2\vartheta_{\mathbf{r},\alpha,\tau}$ is not ''gauge invariant.'' As a result its spatial correlations are local, while the fluctuations of the zero modes, $\Theta_{2,3}$, determine the exponential temporal decay. Clearly, v_{2q} is irrelevant.

Consider the possible forms we can obtain for S_{higher}. These terms would be of the form of cosines involving multiple ϑ's at different space-time points. However, similar arguments as above apply to these operators. The only operators which can have nonzero correlations at nonzero separation would be gauge-invariant combinations of the ϑ's. As we have shown, these are the local boson densities. Thus, if we admit only slow fluctuations of Θ_1, these would take the form of weak density-density interactions within a perturbative treatment and could renormalize the coefficients of the Gaussian action, but not cause an instability. Of course, such density-density interactions might lead to a charge-ordered state at strong couplings, but our arguments show that the Gaussian theory is perturbatively stable.[16]

F. Numerical results

To confirm the above arguments we have carried out Monte Carlo calculations on the dual model using a Metropolis algorithm, for lattice sizes up to 432 spatial sites and 48 time slices, and periodic boundary conditions. We find no evidence of any charge ordering for $\bar{n}=1/2$, even for large U/K. We have compared the density-density correlations and find agreement with what we expect from a Gaussian theory $S_{\text{dual}}^{(0)}$. In particular, we show results for the compressibility κ defined through $\kappa=\chi_{nn}(\mathbf{k}\to0,\omega_n=0)$ where

$$\chi_{nn}(\mathbf{k},\omega_n)=\sum_{\mathbf{r},\tau}e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_n\tau}\langle n(\mathbf{r},\tau)n(\mathbf{0},\tau)\rangle. \tag{20}$$

Within the Gaussian theory, $\kappa=1/\epsilon_\tau\mathcal{U}(0)$. In Fig. 3 we plot the scaled compressibility $(\epsilon_\tau K)\kappa$ which is expected to scale as $K/\mathcal{U}(0)$. We find that that it is nonzero at all U/K in the quantum (τ-continuum) limit, with $\mathcal{U}(0)/K\simeq5$ as $U/K\to\infty$. We have also checked and found no evidence for energy ordering on the plaquettes of the triangular lattice. Thus, we believe that for $\bar{n}=1/2$, the ring-exchange model with $J=0$ is well described by the Gaussian fixed-point action $S_{\text{dual}}^{(0)}$ where terms with v_{2q} and all other terms S_{higher} are irrelevant.

IV. PERTURBING IN NEAREST-NEIGHBOR COUPLING J

A. Ordered ground state

To perturb in the nearest-neighbor exchange J, it is convenient to work in the phase representation which we have

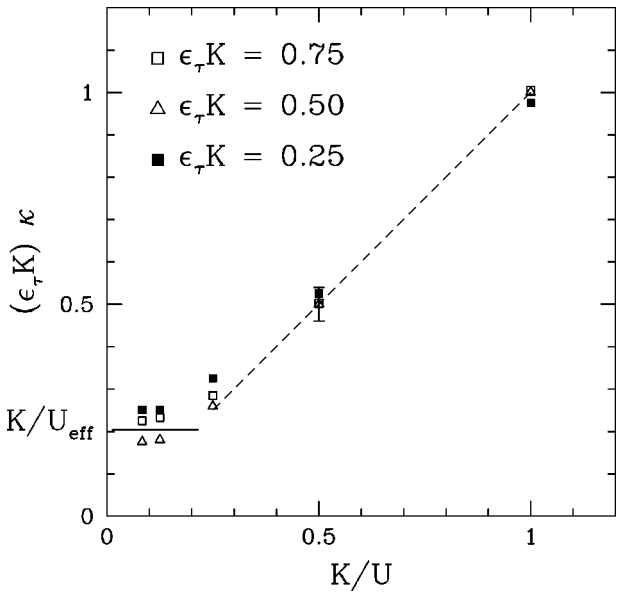

FIG. 3. The dependence of the scaled compressibility $(\epsilon_\tau K)\kappa$ on the bare boson repulsion U, for various $\epsilon_\tau K$. Within spin-wave theory, we expect $(\epsilon_\tau K)\kappa = K/\mathcal{U}(0)$ as discussed in the text. From the figure we see that $K/\mathcal{U}(0)$ coincides with K/U for small U (as indicated by the dashed line), but deviates and tends to a constant $K/U_{\rm eff} \sim 0.2$ as the bare $U \to \infty$ (as shown by the solid line drawn as a guide to the eye). Typical error bars are indicated at one point. We thus expect the $S=1/2$ XY spin model with $J=0$ to be compressible, with $\mathcal{U}(0) = U_{\rm eff}$.

argued remains a valid ''fixed-point'' description of the pure ring-exchange model for all values of the bare coupling U/K. In this, we make the approximation of completely ignoring irrelevant operators—this seems reasonable in retrospect since we will find that the system acquires long-range phase order at $T=0$.

The nearest-neighbor perturbation we add to the Gaussian phase action takes the form

$$S_J = J \sum_{m,n} \int d\tau [(-1)^n \cos(\phi_{m,n,\tau} - \phi_{m+1,n,\tau}) + (-1)^m \cos(\phi_{m,n,\tau} - \phi_{m,n+1,\tau}) + (-1)^{m+n} \cos(\phi_{m,n,\tau} - \phi_{m+1,n+1,\tau})], \quad (21)$$

where we have chosen to label sites on the triangular lattice as $\mathbf{r} = m\hat{a} + n\hat{b}$, and we are working in rotated variables $\phi_{\mathbf{r}}$ for which the ring-exchange term is ferromagnetic.

To carry out a systematic expansion, we define $\phi_{\mathbf{r}} = \tilde{\phi}_{\mathbf{r}} + \mathbf{Q}\cdot\mathbf{r}$ with an arbitrary vector $\mathbf{Q}$. This allows us to examine at the effect of perturbing in J in any of the ground states of the degenerate manifold to see if fluctuations select any state. Perturbing in S_J around the pure ring-exchange Gaussian theory, we find that the corrections to the free energy vanish at leading order in J. At $\mathcal{O}(J^2)$, we find a correction to the free energy density,

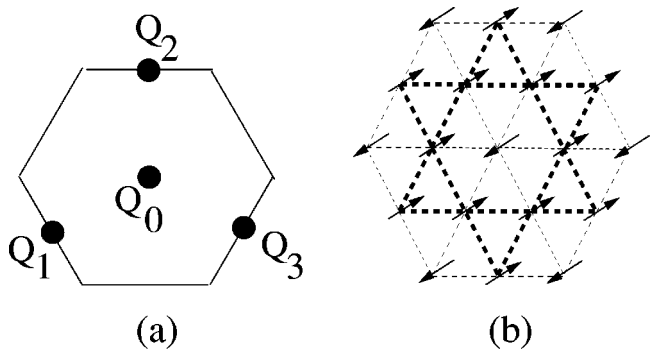

FIG. 4. (a) The four ordering wave vectors $\mathbf{Q}_\alpha$ $(\alpha = 1, \ldots, 4)$ obtained by perturbing in the nearest-neighbor coupling J, with the hexagon indicating the first Brillouin zone of the triangular lattice. (b) The phase-spin order in the original variables $\varphi_{\mathbf{r}}$ in one of the four broken-symmetry ground states. The order may be most easily thought of as alternating rows of ferromagnetic and antiferromagnetic spins. The bold dashed lines indicate bonds on which the spins point in the same direction, while the other bonds have antiferromagnetic spin orientation. As discussed in the text, these correspond to energy ordering in the ground state with an associated fourfold broken discrete symmetry.

$$\delta f = -\frac{1}{2}\beta J^2 [\cos^2 Q_a + \cos^2 Q_b + \cos^2(Q_a + Q_b)] \times \sum_{m,n} \int_\tau (-1)^n \langle \cos(\tilde{\phi}_{0,0,0} - \tilde{\phi}_{1,0,0} + \tilde{\phi}_{m,n,\tau} + \tilde{\phi}_{m+1,n,\tau}) \rangle_0, \quad (22)$$

where $\langle \cdots \rangle_0$ is the expectation value evaluated in the Gaussian theory, $Q_a \equiv \mathbf{Q}\cdot\hat{a}$, $Q_b \equiv \mathbf{Q}\cdot\hat{b}$, and we have used the sixfold symmetry of the triangular lattice to simplify certain intermediate expressions. The expectation value may be analytically evaluated at small U/K by expanding the cosine term since its argument is small in the Gaussian theory, and we find that the sum over m,n is positive. Thus, (i) the corrections to the free energy are independent of the sign of J and (ii) minimizing δf with respect to variations of $\mathbf{Q}$ corresponds to maximizing $[\cos^2 Q_a + \cos^2 Q_b + \cos^2(Q_a + Q_b)]$ which leads to ordering at four wave vectors $\mathbf{Q}_\alpha$ $(\alpha = 1, \ldots, 4)$ which are shown in Fig. 4(a). We expect this order to persist even at large effective (U/K) (recall that the bare $U/K \to \infty$ in the $S=1/2$ spin limit). To obtain the ordered states in the original phase variables, we rotate back $\phi(\mathbf{r}) \to \varphi(\mathbf{r}) = \phi(\mathbf{r}) - \pi$ on the triangular sublattice and we then find, as shown in Fig. 4(b), that the resulting ordered states break (i) global U(1) spin rotational invariance and (ii) fourfold translational invariance. For small J/K, such states have been shown to occur at finite magnetic fields for the classical version of a related SU(2)-invariant ring-exchange model,[17] and this ordering has been called $uuud$ (for ''up-up-up-$d$own'' which is the spin order on a four-site plaquette, but in the plane in our case). Here, we find such an ordered state is stabilized by quantum fluctuations in the XY limit even in zero magnetic field.

B. Fluctuations in the ordered state

As shown above, in the presence of a nonzero J, the model has an ordered ground state. This order breaks U(1) invariance. At the same time, it also breaks translational symmetry. In the boson language, this state may be viewed as a superfluid coexisting with broken translational order. This broken discrete symmetry is also evident from our implicit choice of a state $\mathbf{Q}_\alpha$ to minimize the free energy. Thus, we expect two kinds of excitations at low energies about the ordered state—(i) phase fluctuations and (ii) domain walls in the discrete order.

It is easy to see that with $\phi_{\mathbf{r}}=\mathbf{Q}\cdot\mathbf{r}+\tilde{\phi}_{\mathbf{r}}$, there is an energy cost $\sim(\mathbf{Q}-\mathbf{Q}_\alpha)^2$ for small deviations $|(\mathbf{Q}-\mathbf{Q}_\alpha)|\ll 1$ where $\mathbf{Q}_\alpha$ denotes any one of the ordering vectors. This implies that the long wavelength effective theory of phase fluctations around each of these ordered states is described by an action of the form

$$S=\int_{\mathbf{r},\tau}\left(\frac{1}{U_{\rm eff}}(\partial_\tau\varphi)^2+J_s(\nabla\varphi)^2+K_{\rm eff}(\nabla^2\varphi)^2\right), \quad (23)$$

with a nonzero J_s, and effective couplings $U_{\rm eff}$ and $K_{\rm eff}$. For small $U/K, J/K$, we find $U_{\rm eff}\sim U$, $K_{\rm eff}\sim K$, and $J_s\sim J^2/K\sqrt{U/K}$. Thus, phase fluctuations are controlled by the phase stiffness J_s.

A sharp domain wall separating regions with different $\mathbf{Q}_\alpha$ order would cost an energy $\sim K$ per unit length. However, since the model is spin disordered for $J=0$, we expect that a smooth deformation of the spin configuration by making a domain wall with nonzero width ξ_D, would cost less energy. Indeed, using the above action one can estimate the energy per unit length of a straight domain wall as $e_{\rm dom}=\gamma_1 K_{\rm eff}/\xi_D+\gamma_2 J_s\xi_D$, where $\gamma_{1,2}\sim 1$ are constants. Minimizing the energy cost with respect to ξ_D, we find that $\xi_D^*\sim\sqrt{K_{\rm eff}/J_s}$, and thus domain wall excitations cost an energy $e^*_{\rm dom}\sim\sqrt{K_{\rm eff}J_s}$ per unit length. For small U/K, $e^*_{\rm dom}\sim J(U/K)^{1/4}$.

C. Phase transitions at finite temperature

At high temperature, we expect the model to be in a fully disordered phase where the U(1) symmetry as well as the discrete broken symmetry is restored. To study these phase transitions, we appeal to the above estimates of the spin stiffness and domain wall energies. For $U/K\sim 1$, since the spin stiffness $J_s\sim J^2/K$, while the domain wall energy per unit length $e_{\rm dom}\gtrsim J\gg J_s$, we expect the U(1) symmetry to be restored via a BKT transition[14] at a temperature $T_{\rm BKT}\sim J_s$ once vortices are included in the effective theory, while the discrete symmetry is not restored until a much higher temperature $T_c\sim J$. Thus there is an intermediate phase with exponentially decaying spin correlations, but with the discrete order still present. This bears resemblance to proposed scenarios of chiral ordering in the *XY* antiferromagnet on a triangular lattice as well as the SU(2) multiple-spin exchange model. However, in contrast to the *XY* model where a single scale—namely, the two-spin exchange coupling—sets the scale for both transition temperatures, the transition temperatures are governed by distinct energy scales in the present case, and the discrete symmetry corresponds to broken translational invariance.

In order to better understand the discrete transition, let us identify the appropriate order parameter and construct a Landau theory. Since the discrete symmetry is easily identified with translations in the original variables $\varphi_{\mathbf{r}}$, we will discuss it in terms of that. As shown in Fig. 4(b), the ordered state at $T=0$ has broken translational invariance associated with ordering of $\cos(\varphi_i-\varphi_j)$ on nearest-neighbor bonds. Since there is a nearest-neighbor spin exchange term J, this leads to ordering in the energy density. We identify the broken discrete symmetry with energy ordering on the bonds. To identify the appropriate order parameter for this transition, note that we may write $K_{\mathbf{r},i}=\langle\mathbf{S}_{\mathbf{r}}\cdot\mathbf{S}_{\mathbf{r}+\hat{a}_i}\rangle$ where $\hat{a}_i$ $(i=1,2,3)$ are the unit lattice vectors making an angle $2\pi/3$ with each other [specifically, $\hat{a}_1=\hat{a}$, $\hat{a}_2=\hat{b}$, and $\hat{a}_3=-(\hat{a}+\hat{b})$]. Labeling the four ordered states by $\mu=0,\ldots,4$ and with $a_0=0$, we can write the expectation value in any of these ordered states as $K^\mu_{\mathbf{r},i}=\cos(\mathbf{Q}_i\cdot\mathbf{r})\cos(\mathbf{Q}_i\cdot\hat{a}_\mu)$. For a general superposition,

$$K_{\mathbf{r},i}=\sum_{\mu=0\ldots 4}B_\mu\cos(\mathbf{Q}_i\cdot\mathbf{r})\cos(\mathbf{Q}_i\cdot\hat{a}_\mu), \quad (24)$$

and we identify B_μ as the appropriate order parameter for the Landau theory. Under shifts of $B_\mu\to B_\mu+\lambda$, the physical correlation $K_{\mathbf{r},i}$ is unchanged. Thus we require the Landau functional to be invariant under such shifts. Further, studying the transformation of $K_{\mathbf{r},i}$ under symmetry operations of the lattice (unit translations, $\pi/3$ rotations, and reflections about the three mirror planes), we find that such symmetry operations correspond to all possible permutations of the set $\{B_\mu\}$. For the Landau theory we find, beyond quadratic order, one cubic invariant and two quartic invariants. It appears likely that the finite-temperature phase transition restoring this discrete symmetry is in the same universality class as a four-state Potts model.

To summarize, we expect the system to exhibit with increasing temperature a BKT transition with $T_{\rm BKT}\sim J^2/K$ from a state with power-law spin correlations to a state with exponentially decaying spin correlations. The discrete broken symmetry associated with energy ordering on the bonds of the lattice is expected to be restored above a temperature $T_c\sim J$, with the transition being in the universality class of the four-state Potts model. In a weak applied in-plane magnetic field, the U(1) invariance is lost, and the BKT transition would become rounded but the discrete transition would survive as a finite-temperature transition. At large enough fields, we expect a first-order transition to a state where all the spins point in the same direction and the translation invariance is restored.

V. DISCUSSION AND CONCLUSIONS

Studying an *XY* model with four-spin couplings and small two-spin exchange, we have obtained a ferrimagnetically ordered ground state and analyzed phase transitions out of this phase. Since the ground state is phase ordered, we expect our

result to be stable to small out-of-plane couplings involving S_z. Similarly, introducing weak couplings between such ordered two-dimensional planes would lead to a three-dimensionally ordered state,[18] where the spin (phase) order would persist to finite temperature. We thus expect this state might be of some interest for Mott insulators on a triangular lattice. If such an ordered state exists, it could be detected in neutron diffraction studies. The spin-disordered state with persisting discrete broken symmetry may be indirectly observable through lattice distortions if the spins couple to the lattice.

We can also analyze the opposite limit from the one studied in the paper, where we assume ''easy axis'' anisotropy in the model in Eq. (2) and weak J/K. In this Ising limit, we find eight degenerate ground states for the spin model, again of the $uuud$ type, with the spins aligned along S_z axis. These states correspond to charge-ordered incompressible states for the bosons at density $\bar{n}=1/4$ or $\bar{n}=3/4$ (four ground states at either density). It may be that any anisotropy which takes us away from the SU(2)-invariant model leads to ordered ground states, while the SU(2)-invariant model seems to be a uniform spin liquid.[7,15] It would be interesting to further explore this possibility.

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