

Multicritical phenomena in $O(n_1) \oplus O(n_2)$ -symmetric theoriesPasquale Calabrese,^{1,*} Andrea Pelissetto,^{2,†} and Ettore Vicari^{3,‡}¹*Scuola Normale Superiore and INFN, P.zza Cavalieri 7, I-56126 Pisa, Italy*²*Dip. Fisica dell'Università di Roma "La Sapienza" and INFN, P.le Moro 2, I-00185 Roma, Italy*³*Dip. Fisica dell'Università di Pisa and INFN, V. Buonarroti 2, I-56127 Pisa, Italy*

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We study the multicritical behavior arising from the competition of two distinct types of ordering characterized by $O(n)$ symmetries. For this purpose, we consider the renormalization-group flow for the most general $O(n_1) \oplus O(n_2)$ -symmetric Landau-Ginzburg-Wilson Hamiltonian involving two fields ϕ_1 and ϕ_2 with n_1 and n_2 components, respectively. In particular, we determine in which cases, approaching the multicritical point, one may observe the asymptotic enlargement of the symmetry to $O(N)$ with $N = n_1 + n_2$. By performing a five-loop ϵ -expansion computation we determine the fixed points and their stability. It turns out that for $N = n_1 + n_2 \geq 3$ the $O(N)$ -symmetric fixed point is unstable. For $N = 3$, the multicritical behavior is described by the biconal fixed point with critical exponents that are very close to the Heisenberg ones. For $N \geq 4$ and any n_1, n_2 the critical behavior is controlled by the tetracritical decoupled fixed point. We discuss the relevance of these results for some physically interesting systems, in particular for anisotropic antiferromagnets in the presence of a magnetic field and for high- T_c superconductors. Concerning the $SO(5)$ theory of superconductivity, we show that the bicritical $O(5)$ fixed point is unstable with a significant crossover exponent $\phi_{4,4} \approx 0.15$; this implies that the $O(5)$ symmetry is not effectively realized at the point where the antiferromagnetic and superconducting transition lines meet. The multicritical behavior is either governed by the tetracritical decoupled fixed point or is of first-order type if the system is outside its attraction domain.

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I. INTRODUCTION

The competition of distinct types of ordering gives rise to multicritical behavior. More specifically, a multicritical point (MCP) is observed at the intersection of two critical lines characterized by different order parameters. MCP's arise in several physical contexts. The phase diagram of anisotropic antiferromagnets in a uniform magnetic field H_{\parallel} parallel to the anisotropy axis presents two critical lines in the temperature- H_{\parallel} plane, belonging to the XY and Ising universality classes, which meet at a MCP.¹⁻³ A MCP is also observed in ⁴He. It arises from the competition of crystalline and superfluid ordering in the temperature-pressure phase diagram.⁴ MCP's are also expected in the temperature-doping phase diagram of high- T_c superconductors. Within the $SO(5)$ theory^{5,6} of high- T_c superconductivity, it has been speculated that the antiferromagnetic and superconducting transition lines meet at a MCP in the temperature-doping phase diagram, which is bicritical and shows an effective enlarged $O(5)$ symmetry. On the other hand, the recent experimental evidence of a coexistence region between the antiferromagnetic and superconducting phases is suggestive of a tetracritical behavior.⁷ A MCP should also appear in the temperature baryon-chemical-potential phase diagram of hadronic matter, within the strong-interaction theory with two massless quarks.^{8,9}

Different phase diagrams have been observed close to a MCP. If the transition at the MCP is continuous, one may observe either a bicritical or a tetracritical behavior. A bicritical behavior is characterized by the presence of a first-order line that starts at the MCP and separates the two different ordered low-temperature phases; see Fig. 1. In the tetracritical case, there exists a mixed low-temperature phase in

which both types of ordering coexist and which is bounded by two critical lines meeting at the MCP; see Fig. 2. It is also possible that the transition at the MCP is of first order. A possible phase diagram is sketched in Fig. 3. In this case the two first-order lines, which start at the MCP and separate the disordered phase from the ordered phases, end in tricritical points and then continue as critical lines.

If the order parameters have, respectively, n_1 and n_2 components and the interactions are invariant under $O(n_1)$ and $O(n_2)$, the critical behavior at the MCP can be studied by starting from the most general Landau-Ginzburg-Wilson (LGW) Hamiltonian that is symmetric under $O(n_1) \oplus O(n_2)$ transformations and contains up to quartic terms²:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} [(\partial_{\mu} \phi_1)^2 + (\partial_{\mu} \phi_2)^2] + \frac{1}{2} (r_1 \phi_1^2 + r_2 \phi_2^2) + \frac{1}{4!} [u_1 (\phi_1^2)^2 + u_2 (\phi_2^2)^2 + 2w \phi_1^2 \phi_2^2] \right\}. \quad (1.1)$$

Here, the two fields ϕ_1 and ϕ_2 have n_1 and n_2 components, respectively. The critical behavior at the MCP is determined by the stable fixed point (FP) of the renormalization-group (RG) flow when both r_1 and r_2 are tuned to their critical value. An interesting possibility is that the stable FP has $O(N)$ symmetry, $N = n_1 + n_2$, so that the symmetry gets effectively enlarged when approaching the MCP. This picture has been put forward for the multicritical behavior of anisotropic antiferromagnets in an external magnetic field,^{2,3} for systems with quadratic and cubic anisotropy,¹⁰⁻¹² and for high- T_c superconductors.^{5,6,13,14}

The phase diagram of the model with Hamiltonian (1.1) has been investigated within the mean-field approximation in

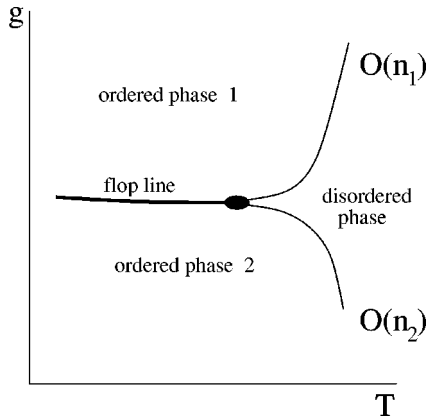


FIG. 1. Phase diagram in the plane T - g presenting a bicritical point. Here, T is the temperature and g a second relevant parameter. The thick line (“flop line”) represents a first-order transition.

Ref. 4 (see also Ref. 15). This analysis predicts the existence of a bicritical or tetracritical point, as observed experimentally. The nature of the MCP depends on the sign of the quantity $\Delta = u_1 u_2 - w^2$, which is relevant in the study of the stability domain of the Hamiltonian (1.1). If $\Delta > 0$, the MCP is tetracritical as in Fig. 2, while for $\Delta < 0$ it is bicritical as in Fig. 1.

The critical behavior of the model has been investigated in the framework of the ϵ expansion.^{2,3} A low-order calculation^{2,3} shows that the isotropic $O(N)$ -symmetric FP ($N \equiv n_1 + n_2$) is stable for $N < N_c = 4 - 2\epsilon + O(\epsilon^2)$. With increasing N , a new FP named biconal FP (BFP), which has only $O(n_1) \oplus O(n_2)$ symmetry, becomes stable. Finally, for large N , the decoupled FP (DFP) is the stable FP. In this case, the two order parameters are effectively uncoupled at the MCP. The extension of these $O(\epsilon)$ results to three dimensions suggests that for $n_1 = 1$ and $n_2 = 2$, the case relevant for anisotropic antiferromagnets, the MCP belongs to the $O(3)$ universality class, while for $n_1 = 2$ and $n_2 = 3$, of relevance for the $SO(5)$ theory of high- T_c superconductivity, the stable FP is the BFP. The $O(\epsilon)$ computations provide useful indications on the RG flow in three dimensions, but a controlled extrapolation to $\epsilon = 1$ requires much longer series and an accurate resummation exploiting their Borel summa-

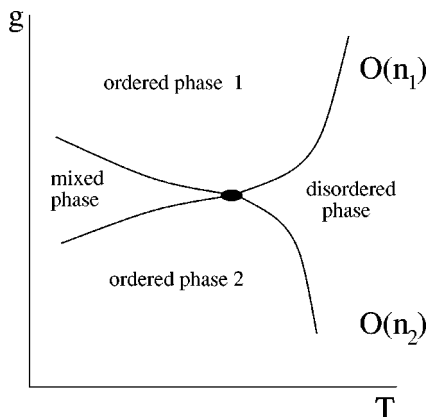


FIG. 2. Phase diagram with a tetracritical point.

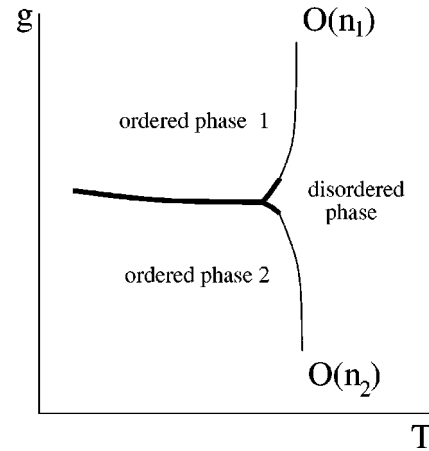


FIG. 3. Phase diagram with a first-order MCP. The thick lines represent first-order transitions.

bility. As we shall see, the above-reported hypotheses on the three-dimensional systems with $n_1 = 1$, $n_2 = 2$ and $n_1 = 2$, $n_2 = 3$ will be both contradicted by a higher-order analysis.

The stability properties of the DFP can be established using nonperturbative arguments,^{16–19} which allow us to compute the RG dimension y_w of the operator $w\phi_1^2\phi_2^2$ at the DFP. The stability of the DFP depends on the sign of y_w : if $y_w < 0$, the DFP is stable. It turns out that in three dimensions $y_w > 0$ for $N \leq 3$ and $y_w < 0$ for $N \geq 4$ for any n_1 and n_2 , showing that the DFP is stable for $N \geq 4$. We should note that the stability of the DFP does not allow us to exclude the existence of another stable FP. This possibility, which is usually considered rather unlikely,¹⁸ has been put forward²⁰ to explain the Monte Carlo results of Refs. 14 and 20, which apparently support the stability of a multicritical $O(5)$ FP.

The phase diagram of the model (1.1) was studied in Refs. 10, 11, and 21. The DFP is expected to be generically tetracritical: indeed, in this case the MCP should correspond to a generic intersection of the two critical lines with $O(n_1)$ and $O(n_2)$ symmetry. The stable $O(N)$ FP—as we shall see, this is the case only for $N = 2$ —can be either bicritical or tetracritical. The possibility of two different phase diagrams for the same FP is due to the presence of a dangerously irrelevant operator.^{10,11} Little is known for the BFP, although a phenomenological extension of the mean-field arguments would predict a tetracritical behavior.³ When the initial parameters of the Hamiltonian are not in the attraction domain of the stable FP, the transition between the disordered and ordered phases should be of first order in the neighborhood of the MCP.^{22–24} However, the transition along the critical lines may become continuous sufficiently far from the MCP.^{25,26} A possible phase diagram is sketched in Fig. 3.

In this paper we extend the analysis of the multicritical RG flow to $O(\epsilon^5)$. The stability of the $O(N)$ FP is also discussed in the framework of fixed-dimension expansion in three dimensions, for which six-loop series have been computed. These calculations allow us to obtain a rather conclusive picture of the multicritical RG flow in three-dimensional systems. In particular, the $O(N)$ FP is stable only for $N = 2$. Therefore, the symmetry enlargement occurs only when the competing order parameters have Ising symmetry. For

$N \geq 3$, the $O(N)$ FP is unstable and therefore the enlargement of the symmetry to $O(N)$ at the MCP requires an additional tuning of the parameters: beside tuning r_1 and r_2 , a third parameter must be properly fixed to decouple the additional relevant interaction. The crossover exponent associated with this RG instability is $\phi_{4,4} \approx 0.01$ for $N=3$, $\phi_{4,4} \approx 0.08$ for $N=4$, $\phi_{4,4} \approx 0.15$ for $N=5$, and $\phi_{4,4} \rightarrow 1$ for $N \rightarrow \infty$. For $N=3$ the stable FP is the BFP. The critical exponents are, however, very close to the Heisenberg ones, so that distinguishing experimentally the $O(3)$ FP and the BFP is a very hard task, taking also into account the very small crossover exponent governing the unstable flow from the $O(3)$ FP. The case $N=5$, $n_1=2$, $n_2=3$ is relevant for the $SO(5)$ theory^{5,6} of high- T_c superconductors, which proposes a description in terms of a three-component antiferromagnetic order parameter and a d -wave superconducting order parameter with $U(1)$ symmetry, with an approximate $O(5)$ symmetry. For $N=5$ the only stable FP is the DFP which predicts, if the transition is continuous, a tetracritical behavior. This may explain a number of recent experiments (see, e.g., Refs. 7 and 27–32) that provided evidence of a coexistence region of the antiferromagnetic and superconducting phases. The $O(5)$ FP is unstable with a crossover exponent $\phi_{4,4} \approx 0.15$, which, although rather small, is nonetheless sufficiently large not to exclude the possibility of observing the RG flow towards the eventual asymptotic behavior for reasonable values of the reduced temperature,³³ even in systems with a moderately small breaking of the $O(5)$ symmetry, for instance in those described by the projected $SO(5)$ model discussed in Refs. 6, 34, and 35. Of course, when the effective Hamiltonian parameters are outside the attraction domain of the stable FP, the transition at the MCP is expected to be of first-order type. Some of the results concerning the stability properties of the $O(N)$ FP were already presented in Ref. 36.

The paper is organized as follows. In Sec. II we present our five-loop calculations in the framework of the ϵ expansion. In Sec. III we discuss the stability of the $O(N)$ -symmetric FP under generic perturbations. The results are then applied to establish the stability properties of the $O(N)$ FP. In Sec. IV the multicritical RG flow is analyzed. In Sec. V we draw our conclusions and discuss their relevance for some physical systems.

II. ϵ EXPANSION OF THE $O(n_1) \oplus O(n_2)$ THEORY

We extended the ϵ expansion of the critical exponents at the different FP's for the $O(n_1) \oplus O(n_2)$ -symmetric theory to $O(\epsilon^5)$. For this purpose, we considered the minimal subtraction (MS) renormalization scheme.³⁷ We computed the divergent part of the irreducible two-point functions of the fields ϕ_1 and ϕ_2 , of the two-point correlation functions with insertions of the quadratic operators ϕ_1^2 and ϕ_2^2 , and of the three independent four-point correlation functions $\langle \phi_1 \cdot \phi_1 \phi_1 \cdot \phi_1 \rangle$, $\langle \phi_1 \cdot \phi_1 \phi_2 \cdot \phi_2 \rangle$, and $\langle \phi_2 \cdot \phi_2 \phi_2 \cdot \phi_2 \rangle$. The diagrams contributing to this calculation are a few hundreds. We handled them with a symbolic manipulation program, which generated the diagrams and computed the symmetry and group factors of each of them. We used the results of Ref. 38,

where the primitive divergent parts of all integrals appearing in our computation are reported. We determined the renormalization constants Z_{ϕ_1} and Z_{ϕ_2} associated with the fields ϕ_1 and ϕ_2 , respectively, the 3×3 renormalization matrix Z_{ij}^g of the quartic couplings defined by $g_{B,i} = \mu^\epsilon Z_{ij}^g g_{R,j}$ where $g_{B,i} \equiv (u_1, u_2, w)$, and the 2×2 renormalization matrix $Z_{ij}^{\phi^2}$ of the quadratic operators ϕ_1^2 and ϕ_2^2 . The β functions $\beta_i(g_{R,j})$ and the RG dimensions γ_{ϕ_1} , γ_{ϕ_2} , and $\gamma_{ij}^{\phi^2}$ are determined using the relations

$$\beta_i(g_{R,i}) = \mu \left. \frac{\partial g_{R,i}}{\partial \mu} \right|_{g_{B,j}}, \quad (2.1)$$

$$\gamma_{\phi_i}(g_{R,i}) = \sum_j \beta_j \frac{\partial Z_{\phi_i}}{\partial g_{R,j}}, \quad (2.2)$$

$$\gamma_{ij}^{\phi^2}(g_{R,i}) = \sum_{kl} \beta_k \frac{\partial Z_{kl}^{\phi^2}}{\partial g_{R,i}} (Z^{\phi^2})_{lj}^{-1}. \quad (2.3)$$

The zeros $g_{R,i}^*$ of the β functions provide the FP's of the theory. In the framework of the ϵ expansion, they are obtained as perturbative expansions in ϵ and are then inserted in the RG functions to determine the ϵ expansion of the critical exponents. The stability of each FP is controlled by the 3×3 matrix

$$\Omega_{ij} = \left. \frac{\partial \beta_i(g_{R,k})}{\partial g_{R,j}} \right|_{g_{R,k} = g_{R,k}^*}. \quad (2.4)$$

The two exponents η_1 and η_2 , related to the short-distance behavior of the two-point functions of the fields ϕ_1 and ϕ_2 , are given by $\eta_1 = \gamma_{\phi_1}(g_{R,i}^*)$ and $\eta_2 = \gamma_{\phi_2}(g_{R,i}^*)$. From the eigenvalues ν_1 and ν_2 of the matrix $\gamma_{ij}^{\phi^2}$, if $\nu_1 > \nu_2$, one obtains $\nu = \nu_1$ and $\phi = \nu_1 / \nu_2$, where ϕ is the crossover exponent associated with the quadratic instability.

We performed several checks of the perturbative series. In particular, the critical-exponent series agree with the existing $O(\epsilon^5)$ ones for the $O(N)$ -symmetric theory^{39,40} in the proper limit. Moreover, as we shall discuss in the following section, we can also compare with some results for the $O(N)$ theory in the presence of cubic anisotropy,⁴¹ finding agreement. Some of the five-loop perturbative series will be reported in the following sections. The complete list of series is available on request.

Since the ϵ expansion is asymptotic, the series must be properly resummed to provide results for three-dimensional systems. We used the Padé-Borel method except for the series at the $O(N)$ FP. In this case, we applied the conformal-mapping method,⁴² which takes into account the known large-order behavior of the expansion. See, e.g., Refs. 43 and 44 for reviews of resummation methods.

III. STABILITY OF THE $O(N)$ FIXED POINT

In this section we discuss the stability of the $O(N)$ FP, where $N = n_1 + n_2$, to establish in which cases the enlargement of the symmetry $O(n_1) \oplus O(n_2)$ to $O(N)$ is realized at

the MCP without the need of further tunings.

Let us consider the general problem of an $O(N)$ -symmetric Hamiltonian in the presence of a perturbation P , i.e.,

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} r \Phi^2 + \frac{1}{4!} u (\Phi^2)^2 + h_p P \right], \quad (3.1)$$

where Φ is an N -component field and h_p an external field coupled to P . Assuming P to be an eigenoperator of the RG transformations, the singular part of the Gibbs free energy for the reduced temperature $t \rightarrow 0$ and $h_p \rightarrow 0$ can be written as

$$\mathcal{F}_{\text{sing}}(t, h_p) \approx |t|^{d\nu} \hat{\mathcal{F}}(h_p |t|^{-\phi_p}), \quad (3.2)$$

where $\phi_p \equiv y_p \nu$ is the crossover exponent associated with the perturbation P , y_p is the RG dimension of P , and $\hat{\mathcal{F}}(x)$ is a scaling function. If $y_p > 0$, the perturbation is relevant and its presence causes a crossover to another critical behavior or to a first-order transition.

In order to discuss the stability of the $O(N)$ FP in general, we must consider any perturbation of the $O(N)$ FP. We shall first consider perturbations that are polynomials of the field Φ^a . Any such perturbation can be written⁴⁵ as a sum of terms $P_{m,l}^{a_1 \dots a_l}$, $m \geq l$, which are homogeneous in Φ^a of degree m and transform as the l -spin representation of the $O(N)$ group. Explicitly, we have

$$P_{m,l}^{a_1 \dots a_l} = (\Phi^2)^{m-l} Q_l^{a_1 \dots a_l}, \quad (3.3)$$

where $Q_l^{a_1 \dots a_l}$ is a homogeneous polynomial of degree l that is symmetric and traceless in the l indices. The lowest-order even polynomials are

$$Q_2^{ab} = \Phi^a \Phi^b - \frac{1}{N} \delta^{ab} \Phi^2, \quad (3.4)$$

$$\begin{aligned} Q_4^{abcd} = & \Phi^a \Phi^b \Phi^c \Phi^d - \frac{1}{N+4} \Phi^2 (\delta^{ab} \Phi^c \Phi^d + \delta^{ac} \Phi^b \Phi^d \\ & + \delta^{ad} \Phi^b \Phi^c + \delta^{bc} \Phi^a \Phi^d + \delta^{bd} \Phi^a \Phi^c + \delta^{cd} \Phi^a \Phi^b) \\ & + \frac{1}{(N+2)(N+4)} (\Phi^2)^2 (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}). \end{aligned} \quad (3.5)$$

The classification in terms of spin values is particularly convenient, since polynomials with different spin do not mix under RG transformations. On the other hand, operators with different m but with the same l do mix under renormalization. At least near four dimensions, we can use standard power counting to verify that the perturbation with indices m, l mixes only with $P_{m',l}$, $m' \leq m$. In particular, $P_{l,l}$ renormalizes multiplicatively and is therefore a RG eigenoperator. Moreover, if $y_{m,l}$ is the RG dimension of the appropriately subtracted $P_{m,l}$, one can verify that for small ϵ , $y_{m,l} < 0$, for

$m, l \geq 5$; i.e., the only relevant operators have $m, l \leq 4$. We will assume this property to hold up to $\epsilon = 1$. We notice that it is certainly incorrect in two dimensions where perturbations are relevant ($N \geq 3$) or marginal ($N = 2$) for all values of l .⁴⁶ In principle, we should also consider terms with derivatives of the field, but again, using power counting, one can show that they are all irrelevant or redundant. Therefore, beside the $O(N)$ -symmetric terms Φ^2 and $(\Phi^2)^2$ there are only three other perturbations that must be considered, $P_{2,2}^{ab}$, $P_{4,2}^{ab}$, and $P_{4,4}^{abcd}$. Note that, according to the above-reported discussion, $P_{2,2}^{ab}$ and $P_{4,4}^{abcd}$ are RG eigenoperators, while $P_{4,2}^{ab}$ must be in general properly subtracted; i.e., the RG eigenoperator is $P_{4,2}^{ab} + z P_{2,2}^{ab}$ for a suitable value of z . The determination of the mixing coefficient z represents a subtle point in the fixed-dimension expansion,⁴⁷ but is trivial in the MS scheme in $4 - \epsilon$ dimensions, in which operators with different dimensions never mix so that $z = 0$.

According to the above-presented general analysis, the stability properties of the $O(N)$ FP can be obtained by determining the RG dimensions of the five operators reported above. Of course, the result does not depend on the specific values of the indices and thus one can consider any particular combination. We now show that such dimensions determine the crossover exponent ϕ and the eigenvalues of the stability matrix Ω at the $O(N)$ FP for the $O(n_1) \oplus O(n_2)$ theory. Starting from the general expressions, one can construct combinations that are invariant under the symmetry group $O(n_1) \oplus O(n_2)$. Explicitly, they are given by

$$\mathcal{P}_{2,0} = \Phi^2, \quad \mathcal{P}_{2,2} = \sum_{a=1}^{n_1} P_{2,2}^{aa} = \phi_1^2 - \frac{n_1}{N} \Phi^2,$$

$$\mathcal{P}_{4,0} = (\Phi^2)^2, \quad \mathcal{P}_{4,2} = \Phi^2 \mathcal{P}_{2,2},$$

$$\begin{aligned} \mathcal{P}_{4,4} = & \sum_{a=1}^{n_1} \sum_{b=n_1+1}^{n_2} P_{4,4}^{aabb} \\ = & \phi_1^2 \phi_2^2 - \frac{\Phi^2 (n_1 \phi_2^2 + n_2 \phi_1^2)}{N+4} + \frac{n_1 n_2 (\Phi^2)^2}{(N+2)(N+4)}. \end{aligned} \quad (3.6)$$

Here Φ is the N -component field (ϕ_1, ϕ_2) . The RG dimensions of $\mathcal{P}_{2,0}$ and of $\mathcal{P}_{4,0}$ are well known and can be computed directly in the $O(N)$ -invariant theory. In particular, $y_{2,0} = 1/\nu$ and $y_{4,0} = -\omega$, where ω is the leading irrelevant exponent in the $O(N)$ -invariant theory. The RG dimension $y_{2,2}$ of $\mathcal{P}_{2,2}$, and therefore of the operator $P_{2,2}^{ab}$, provides the crossover exponent $\phi = y_{2,2} \nu$ at the MCP. We denote such exponent by ϕ_T to stress the fact that it is associated with the tensor quadratic operator. Setting

$$\phi_T = 1 + \sum_{i=1}^2 p_i \epsilon^i, \quad (3.7)$$

we obtain at five loops

$$\begin{aligned}
 p_1 &= \frac{N}{2(N+8)}, & p_2 &= \frac{N(N^2+24N+68)}{4(N+8)^3}, \\
 p_3 &= \frac{N(N^4+48N^3+788N^2+3472N+5024)}{8(N+8)^5} - \frac{6N(5N+22)\zeta(3)}{(N+8)^4}, \\
 p_4 &= \frac{N(N^6+72N^5+2085N^4+28\,412N^3+147\,108N^2+337\,152N+306\,240)}{16(N+8)^7} \\
 &+ \frac{N(-N^4+13N^3-544N^2-4716N-8360)\zeta(3)}{(N+8)^6} - \frac{N(5N+22)\pi^4}{20(N+8)^4} + \frac{20N(2N^2+55N+186)\zeta(5)}{(N+8)^5} \\
 p_5 &= \frac{N(17\,677\,824+28\,388\,096N+19\,390\,624N^2+6\,723\,904N^3+1\,177\,480N^4+95\,668N^5+4154N^6+96N^7+N^8)}{32(8+N)^9} \\
 &- \frac{N(8360+4716N+544N^2-13N^3+N^4)\pi^4}{120(8+N)^6} + \frac{5N(186+55N+2N^2)\pi^6}{189(8+N)^5} \\
 &- \frac{N(554\,064+465\,592N+125\,232N^2+7584N^3-661N^4+9N^5)\zeta(3)}{(8+N)^8} \\
 &+ \frac{2N(24\,528+14\,468N+2028N^2+39N^3+4N^4)\zeta(3)^2}{(8+N)^7} + \frac{N(466\,016+280\,596N+33\,832N^2-2857N^3-230N^4)\zeta(5)}{2(8+N)^7} \\
 &- \frac{441N(526+189N+14N^2)\zeta(7)}{2(8+N)^6}. \tag{3.8}
 \end{aligned}$$

This series extends the three-loop results of Ref. 48 and the four-loop results of Ref. 49. In the appropriate limit, it is in agreement with the $O(N^{-2})$ expression of Ref. 50. In Table I we report the estimates of $y_{2,2}$ and ϕ_T for $N=2,3,4,5$ obtained from the analysis of the five-loop perturbative expansion (3.8). As expected, since $y_{2,2} > 0$ in all cases, the quadratic perturbation $P_{2,2}^{ab}$ is always relevant. The results are compared with the estimates obtained from the analysis of its six-loop fixed-dimension expansion⁵⁴ and of its large- N expansion to $O(1/N^2)$,⁵⁰ and by using high-temperature techniques⁵⁵ and Monte Carlo simulations.¹⁴ We also mention that consistent results were obtained from the analysis of the four-loop series of ϕ_T (Ref. 49) $\phi_T = 1.177$ for $N=2$ and $\phi_T = 1.259$ for $N=3$. Some experimental results for ϕ_T can be found in Ref. 44.

The perturbative expansions of the RG dimensions of the operators $\mathcal{P}_{4,l}$, and therefore of the more general operators $P_{4,l}$, can be obtained from the eigenvalues of the stability matrix Ω at the $O(N)$ FP. For this purpose, it is convenient to perform a change of variables, replacing u_1 , u_2 , and w with g_l , $l=0,2,4$, which are the quartic couplings associated with the operators $\mathcal{P}_{m,l}$ and are explicitly defined by the relation

$$u_1(\phi_1^2)^2 + u_2(\phi_2^2)^2 + 2w\phi_1^2\phi_2^2 = g_0\mathcal{P}_{4,0} + g_2\mathcal{P}_{4,2} + g_4\mathcal{P}_{4,4}. \tag{3.9}$$

In this basis Ω is diagonal and the eigenvalues of Ω are simply given by

$$\omega_l = \left. \frac{\partial \bar{\beta}_l(g_0, g_2, g_4)}{\partial g_l} \right|_{g_0=g_N^*, g_2=0, g_4=0}, \tag{3.10}$$

where $l=0,2,4$, $\bar{\beta}_l$ are the β functions associated with the couplings g_l , and g_N^* is the FP value of the quartic coupling in the $O(N)$ -symmetric theory. The critical exponent ω_0 is the leading irrelevant operator in the $O(N)$ -symmetric theory. Its $O(\epsilon^5)$ expansion can be found in Refs. 39 and 40; several estimates are reported in Refs. 52 and 44. The RG dimension $y_{4,l}$ of the perturbation $P_{4,l}$ is given by

$$y_{4,l} = -\omega_l. \tag{3.11}$$

We report here the five-loop ϵ expansion of $y_{4,2}$ and $y_{4,4}$. Setting

$$y_{4,l} = \sum_{i=1} c_{l,i} \epsilon^i, \tag{3.12}$$

we have

TABLE I. Estimates of the RG dimensions $y_{2,2}$, $y_{4,0}$, $y_{4,2}$, and $y_{4,4}$ and of the crossover exponents $\phi_T \equiv y_{2,2}\nu$ and $\phi_{4,4} \equiv y_{4,4}\nu$ as obtained by various approaches: ϵ expansion (ϵ exp), fixed-dimension expansion ($d=3$ exp), high-temperature expansion (HT exp), Monte Carlo simulations (MC), and $1/N$ expansion ($1/N$ exp). Their values in the large- N limit (see, e.g., Ref. 51) are also reported.

N	Method	$y_{2,2}$	ϕ_T	$y_{4,0}$	$y_{4,2}$	$y_{4,4}$	$\phi_{4,4}$
2	ϵ exp	1.766(6)	1.174(12)	-0.802(18) ^a	-0.624(10)	-0.114(4) ^b	-0.077(3)
	$d=3$ exp		1.184(12) ^c	-0.789(11) ^a		-0.103(8) ^b	-0.069(5)
	HT exp		1.175(15) ^d				
	MC			-0.795(9) ^e		-0.17(2) ^f	
3	ϵ exp	1.790(3)	1.260(11)	-0.794(18) ^a	-0.550(14)	0.003(4) ^b	0.002(3)
	$d=3$ exp		1.27(2) ^c	-0.782(13) ^a		0.013(6) ^b	0.009(4)
	HT exp		1.250(15) ^d				
	MC			-0.773 ^g		-0.0007(29) ^f	
	$1/N$ exp		1.187 ^h				
4	ϵ exp	1.813(6)	1.329(16)	-0.795(30) ^a	-0.493(14)	0.105(6) ^b	0.079(5)
	$d=3$ exp		1.35(4) ^c	-0.774(20) ^a		0.111(4) ^b	0.083(3)
	MC			-0.765 ^g		0.130(24) ^f	
	$1/N$ exp		1.323 ^h				
5	ϵ exp	1.832(8)	1.40(3)	-0.783(26)	-0.441(13)	0.198(11)	0.151(9)
	$d=3$ exp		1.40(4) ^c	-0.790(15)		0.189(10) ⁱ	0.144(8)
	MC		1.387(30) ^j				
	$1/N$ exp		1.422 ^h				
∞		2	2	-1	0	1	1

^aReference 52.
^bReference 53.
^cReference 54.
^dReference 55.
^eReference 56.

^fReference 57.
^gReference 58.
^hReference 50.
ⁱReference 36.
^jReference 14.

$$c_{2,1} = -\frac{8}{N+8}, \quad c_{2,2} = \frac{336+68N+7N^2}{(8+N)^3},$$

$$c_{2,3} = -\frac{76\,544+26\,176N+3264N^2+28N^3-N^4}{4(8+N)^5} - \frac{12(352+82N+7N^2)\zeta(3)}{(8+N)^4}$$

$$c_{2,4} = \frac{(20\,796\,416+10\,251\,520N+2\,207\,744N^2+271\,328N^3+24\,824N^4+820N^5+5N^6)}{16(8+N)^7} - \frac{(352+82N+7N^2)\pi^4}{10(8+N)^4}$$

$$- \frac{2(-92\,928-34\,776N-7544N^2-928N^3-67N^4+N^5)\zeta(3)}{(8+N)^6} + \frac{80(2232+632N+60N^2+N^3)\zeta(5)}{(8+N)^5}$$

$c_{2,5}$

$$= -\frac{6\,019\,366\,912+3\,720\,851\,456N+994\,704\,384N^2+135\,243\,264N^3+6\,891\,584N^4-590\,816N^5-60\,520N^6-1732N^7-13N^8}{64(8+N)^9}$$

$$\begin{aligned}
 & + \frac{(92\,928 + 34\,776N + 7544N^2 + 928N^3 + 67N^4 - N^5)\pi^4}{60(8+N)^6} + \frac{20(2232 + 632N + 60N^2 + N^3)\pi^6}{189(8+N)^5} \\
 & - \frac{(117\,872\,640 + 62\,925\,184N + 14\,334\,912N^2 + 1\,577\,392N^3 + 67\,848N^4 - 1872N^5 + 200N^6 - 7N^7)\zeta(3)}{4(8+N)^8} \\
 & + \frac{8(104\,832 + 100\,312N + 24\,994N^2 + 2571N^3 + 83N^4 + 2N^5)\zeta(3)^2}{(8+N)^7} \\
 & - \frac{(7\,263\,744 + 3\,733\,728N + 1\,095\,516N^2 + 170\,284N^3 + 14\,035N^4 + 322N^5)\zeta(5)}{(8+N)^7} \\
 & - \frac{441(16\,832 + 5590N + 631N^2 + 23N^3)\zeta(7)}{(8+N)^6} \tag{3.13}
 \end{aligned}$$

and

$$\begin{aligned}
 c_{4,1} &= \frac{N-4}{8+N}, \quad c_{4,2} = \frac{152 + 14N + 5N^2}{(8+N)^3}, \\
 c_{4,3} &= -\frac{17\,024 - 1568N - 1464N^2 - 398N^3 - 13N^4}{4(8+N)^5} - \frac{48(46 + 7N + N^2)\zeta(3)}{(8+N)^4}, \\
 c_{4,4} &= -\frac{(2\,995\,712 + 402\,304N + 223\,328N^2 + 112\,856N^3 + 27\,272N^4 + 1516N^5 + 29N^6)}{16(8+N)^7} - \frac{2(46 + 7N + N^2)\pi^4}{5(8+N)^4} \\
 & - \frac{3(-21\,568 + 1664N + 1592N^2 + 256N^3 - 8N^4 + N^5)\zeta(3)}{(8+N)^6} + \frac{120(712 + 130N + 13N^2)\zeta(5)}{(8+N)^5}, \\
 c_{4,5} &= \frac{-365\,813\,760 + 95\,377\,408N + 75\,546\,624N^2 + 35\,042\,816N^3 + 11\,477\,472N^4 + 2\,184\,488N^5 + 148\,600N^6 + 4712N^7 + 61N^8}{64(8+N)^9} \\
 & - \frac{(-21\,568 + 1664N + 1592N^2 + 256N^3 - 8N^4 + N^5)\pi^4}{40(8+N)^6} + \frac{10(712 + 130N + 13N^2)\pi^6}{63(8+N)^5} \\
 & - \frac{(37\,827\,072 + 13\,773\,568N + 3\,633\,344N^2 + 689\,728N^3 + 54\,184N^4 - 3272N^5 + 188N^6 - 5N^7)\zeta(3)}{4(8+N)^8} \\
 & + \frac{12(11\,456 + 24\,112N + 6648N^2 + 790N^3 + 5N^4)\zeta(3)^2}{(8+N)^7} - \frac{2(1\,018\,944 + 128\,152N - 3060N^2 - 9018N^3 + 347N^4 + 12N^5)\zeta(5)}{(8+N)^7} \\
 & - \frac{2646(1268 + 272N + 25N^2 + N^3)\zeta(7)}{(8+N)^6}. \tag{3.14}
 \end{aligned}$$

At one loop, these results agree with those reported in Ref. 45. The results of the analyses of these series are reported in Table I. They show that $y_{4,2}$ is always negative, so that the corresponding spin-2 perturbation $P_{4,2}^{ab}$ is always irrelevant. On the other hand, the sign of $y_{4,4}$ depends on N : it is clearly negative for $N=2$ and positive for $N \geq 4$. For $N=3$ it is

marginally positive, suggesting the instability of the $O(3)$ FP. This fact will be confirmed by the more accurate results discussed below. The corresponding crossover exponents $\phi_{m,l} \equiv y_{m,l} \nu$ can be determined using the following estimates of ν : $\nu=0.67155(27)$ for $N=2$,⁵⁶ $\nu=0.7112(5)$ for $N=3$,⁵⁹ $\nu=0.749(2)$ for $N=4$,⁵⁸ and $\nu=0.762(7)$ for $N=5$.⁶⁰

Other estimates of ν can be found in Ref. 44.

The RG dimension $y_{4,4}$ can also be obtained starting from the cubic-symmetric LGW Hamiltonian

$$\mathcal{H}_c = \int d^d x \left\{ \frac{1}{2} \sum_{i=1}^N [(\partial_\mu \Phi_i)^2 + r \Phi_i^2] + \frac{1}{4!} \left[u \left(\sum_i \Phi_i^2 \right)^2 + v \sum_i \Phi_i^4 \right] \right\} \quad (3.15)$$

(see, e.g., Ref. 16) and in particular from the results for the stability properties of the $O(N)$ FP in the presence of a cubic-symmetric anisotropy. The point is that the cubic-symmetric perturbation is a particular combination of the spin-4 operators $P_{4,4}^{abcd}$ and of the spin-0 term $(\Phi^2)^2$. Indeed, one may rewrite

$$\sum_{i=1}^N \Phi_i^4 = \sum_{a=1}^N P_{4,4}^{aaaa} + \frac{3}{N+2} (\Phi^2)^2. \quad (3.16)$$

Thus, the stability of the $O(N)$ FP against the cubic-symmetric perturbation $\sum_i \Phi_i^4$ is controlled by the RG dimension $y_{4,4}$ of the spin-4 operator $P_{4,4}^{abcd}$.

The RG flow for the cubic-symmetric theory has been investigated by employing field-theoretical methods, based on perturbative expansions^{41,53,61–66} or approximate solutions of continuous RG equations,^{67–69} and lattice techniques, such as Monte Carlo simulations⁵⁷ and high-temperature expansions⁷⁰; see, e.g., Ref. 44 for a recent review. In particular, the RG functions have been computed to five loops in the ϵ expansion⁴¹ and to six loops in a fixed-dimension expansion in powers of the zero-momentum quartic couplings.⁵³ In these perturbative schemes,

$$y_{4,4} = - \left. \frac{\partial \beta_v(u, v)}{\partial v} \right|_{u=g_N^*, v=0}, \quad (3.17)$$

where β_v is the β function associated with the quartic coupling v , and g_N^* is the FP value of the quartic coupling in the $O(N)$ -symmetric theory. This allows us to determine $y_{4,4}$ using the five-loop expansions reported in Ref. 41. We reobtain again Eq. (3.14), confirming the correctness of our calculation. Moreover, using Eq. (3.17) and the results reported in Ref. 53, one can also compute $y_{4,4}$ in the framework of the fixed-dimension expansion to six loops. The resulting estimates, obtained by using the conformal-mapping method, are reported in Table I. They show that the spin-4 perturbation $P_{4,4}$ is relevant for all $N \geq 3$. In Table I the Monte Carlo results of Ref. 57 are also shown; they were obtained by simulating the standard N -vector model and computing the RG dimension of the cubic-symmetric term $\sum_i s_i^4$, where s_i is the N -component spin variable. We may also consider the value N_c such that for $N > N_c$ the cubic-symmetric anisotropy, and therefore the spin-4 perturbation $P_{4,4}^{abcd}$, becomes relevant at the $O(N)$ FP. All studies reported in the literature indicate $N_c \approx 3$ and definitely $N_c < 4$; see, e.g., Refs. 53, 41, 62–64, 66–69, 57, and 70. The most accurate results have been provided by analyses of high-order perturbative

field-theory expansions, which predict $N_c \leq 2.9$ in three dimensions. In particular, different analyses of the six-loop fixed-dimension series yielded the estimates $N_c = 2.89(4)$ (Ref. 53), and $N_c = 2.862(5)$ (Ref. 62); similar results were also obtained from shorter series; see, e.g., Refs. 63 and 64. These results have been confirmed by the analysis of the $O(\epsilon^5)$ series.^{41,66,53} A constrained analysis taking into account the two-dimensional value of N_c , $N_c = 2$, provided the estimate $N_c = 2.87(5)$,⁵³ which makes the evidence supporting $y_{4,4} > 0$ for $N = 3$ stronger than the estimate $y_{4,4} = 0.003(4)$ obtained from the direct analysis of its $O(\epsilon^5)$ series.

In conclusion, these results provide a rather robust evidence that for $N \geq 3$ the $O(N)$ FP is unstable with respect to spin-4 perturbations $P_{4,4}^{abcd}$ and, as a consequence, that the $O(N)$ FP is a unstable MCP for $N \geq 3$.

IV. RG FLOW AT THE MULTICRITICAL POINT

As already shown by the $O(\epsilon)$ computations of Ref. 3, the $O(n_1) \oplus O(n_2)$ theory at the MCP has six FP's. Three of them—i.e., the Gaussian and the $O(n_1)$ and $O(n_2)$ FP's—are always unstable. The other three FP's are the $O(N)$ fixed point, the biconal fixed point, and the decoupled fixed point. The stability of these FP's depends on n_1 and n_2 . In particular, in the preceding section we have established that the $O(N)$ FP is stable for $N = 2$ and unstable for $N \geq 3$, for any n_1 and n_2 .

The stability properties of the DFP can be determined using nonperturbative scaling arguments.^{16–19} At the DFP, the quartic coupling term $w \phi_1^2 \phi_2^2$ scales as the product of two energylike operators, which have RG dimensions $(1 - \alpha_i)/\nu_i$ where α_i and ν_i are the critical exponents of the $O(n_i)$ universality classes. Therefore, the RG dimension related to the w perturbation is given by

$$y_w = \frac{\alpha_1}{2\nu_1} + \frac{\alpha_2}{2\nu_2} = \frac{1}{\nu_1} + \frac{1}{\nu_2} - d. \quad (4.1)$$

Note that this relation is satisfied order by order in the ϵ expansion. Indeed, the ϵ expansion of y_w obtained from the stability matrix Ω at the DFP coincides with the series obtained from the right-hand side of Eq. (4.1), using the five-loop expansions of ν_i for the $O(n_i)$ universality classes. Taking into account that the DFP is stable with respect to the other two RG directions, one can determine the stability properties of the DFP from the sign of y_w . Using the estimates of the critical exponents of the three-dimensional $O(n_i)$ universality classes (see, e.g., Ref. 44 for a review), y_w turns out to be negative for $N \equiv n_1 + n_2 \geq 4$ and positive for $N = 2, 3$.⁷¹ Three-dimensional estimates of y_w for $N \leq 5$ are reported in Table II. These results show that the tetracritical DFP is stable for $N \geq 4$ for any n_1, n_2 .

The results concerning the $O(N)$ FP and the DFP suggest that the stable FP for $N = 3$ is the BFP. This is substantially confirmed by the five-loop analysis of the stability matrix Ω at the BFP. Below we report the expansions of the critical exponents at the BFP for $n_1 = 1$ and $n_2 = 2$. The eigenvalues of the stability matrix Ω are

TABLE II. Estimates of the RG dimension y_w at the DFP. They are obtained by using Eq. (4.1) and the estimates of the $O(N)$ critical exponent ν reported in Refs. 72, 73, 56, 59, and 58.

$N = n_1 + n_2$	n_1	n_2	y_w
2	1	1	0.1740(8)
3	1	2	0.0761(7)
4	1	3	-0.0069(11)
	2	2	-0.0218(12)
5	1	4	-0.078(4)
	2	3	-0.1048(12)

$$\begin{aligned}
\omega_{\text{bi},1} &= \epsilon - 0.579\,364\epsilon^2 + 1.344\,815\epsilon^3 - 4.058\,162\epsilon^4 \\
&\quad + 14.526\,420\epsilon^5 + O(\epsilon^6), \\
\omega_{\text{bi},2} &= 0.491\,105\epsilon - 0.084\,149\epsilon^2 + 0.361\,174\epsilon^3 \\
&\quad - 0.776\,741\epsilon^4 + 2.593\,212\epsilon^5 + O(\epsilon^6), \\
\omega_{\text{bi},3} &= -0.130\,195\epsilon + 0.278\,782\epsilon^2 - 0.379\,711\epsilon^3 \\
&\quad + 0.868\,886\epsilon^4 - 2.656\,984\epsilon^5 + O(\epsilon^6). \quad (4.2)
\end{aligned}$$

The expansions of the critical exponents are

$$\begin{aligned}
\eta_{\text{bi},1} &= 0.020\,830\,6\epsilon^2 + 0.018\,294\,1\epsilon^3 - 0.007\,773\,25\epsilon^4 \\
&\quad + 0.0210\,296\epsilon^5 + O(\epsilon^6), \\
\eta_{\text{bi},2} &= 0.0200\,806\epsilon^2 + 0.0195\,184\epsilon^3 - 0.00848\,020\epsilon^4 \\
&\quad + 0.0236\,398\epsilon^5 + O(\epsilon^6), \\
\nu_{\text{bi},1} &= \frac{1}{2} + 0.1114\,875\epsilon + 0.0667\,684\epsilon^2 - 0.00616\,190\epsilon^3 \\
&\quad + 0.0779\,498\epsilon^4 - 0.193\,367\epsilon^5 + O(\epsilon^6), \\
\nu_{\text{bi},2} &= \frac{1}{2} + 0.0234\,143\epsilon + 0.0289\,670\epsilon^2 - 0.00547\,548\epsilon^3 \\
&\quad + 0.0381\,483\epsilon^4 - 0.106\,076\epsilon^5 + O(\epsilon^6), \\
\phi_{\text{bi}} &= \frac{\nu_{\text{bi},1}}{\nu_{\text{bi},2}} = 1 + 0.176\,147\epsilon + 0.0673\,541\epsilon^2 - 0.0147\,318\epsilon^3 \\
&\quad + 0.0783\,198\epsilon^4 - 0.190\,099\epsilon^5 + O(\epsilon^6). \quad (4.3)
\end{aligned}$$

We analyzed these series using the Padé-Borel resummation method. The estimates of the eigenvalues of the stability matrix are $\omega_{\text{bi},1} = 0.79(2)$, $\omega_{\text{bi},2} = 0.57(4)$, and $\omega_{\text{bi},3} = 0.01(1)$. They are all positive, supporting the stability of the BFP, although the result for $\omega_{\text{bi},3}$ is not sufficiently precise to definitely exclude the opposite sign. Concerning the critical exponents, we obtained $\eta_{\text{bi},1} = 0.037(5)$, $\eta_{\text{bi},2} = 0.037(5)$, $\nu_{\text{bi}} = \nu_{\text{bi},1} = 0.70(3)$, and $\phi_{\text{bi}} = 1.25(1)$. Note first that $\eta_{\text{bi},1} \approx \eta_{\text{bi},2}$, as can be directly guessed by looking at the coefficients of their expansions. A direct analysis of their difference gives the bound $|\eta_{\text{bi},1} - \eta_{\text{bi},2}| \leq 0.0005$. Second, note that, within the errors, the BFP exponents are very close to

the Heisenberg ones, whose best estimates are $\eta_H = 0.0375(5)$, $\nu_H = 0.7112(5)$, and $\phi_H = 1.250(15)$ from high-temperature techniques,^{59,55} $\eta_H = 0.0355(25)$, $\nu_H = 0.7073(35)$, and $\phi_H = 1.27(2)$ from the six-loop fixed-dimension expansion,^{52,54} and $\eta_H = 0.0375(45)$, $\nu_H = 0.7045(55)$, and $\phi_H = 1.260(11)$ from the five-loop ϵ expansion.⁵² Rather stringent bounds on the differences between the biconal and Heisenberg exponents can be obtained by considering the expansions of their differences, which have much smaller coefficients. Their analysis yields

$$\begin{aligned}
|\eta_{\text{bi},1} - \eta_H| &\leq 0.0005, \\
|\eta_{\text{bi},2} - \eta_H| &\leq 0.0001, \\
|\nu_{\text{bi}} - \nu_H| &\leq 0.001, \\
|\phi_{\text{bi}} - \phi_H| &\leq 0.005. \quad (4.4)
\end{aligned}$$

We have also studied the stability of the BFP for larger values of N . For $N=4$, and in both cases $n=1$, $n_2=3$ and $n_1=n_2=2$, the five-loop calculation gives the expansions of the critical exponents at the BFP only to $O(\epsilon^4)$, because of the additional degeneracy of the $O(4)$ FP and of the BFP at $O(\epsilon)$. In particular, for the smallest eigenvalue we obtain

$$\begin{aligned}
\omega_{\text{bi},3}(n_1=1, n_2=3) &= \frac{1}{6}\epsilon^2 - 0.3306\,439\epsilon^3 \\
&\quad + 0.7376\,491\epsilon^4 + O(\epsilon^5), \\
\omega_{\text{bi},3}(n_1=2, n_2=2) &= \frac{1}{6}\epsilon^2 - 0.319\,872\epsilon^3 \\
&\quad + 0.696\,458\epsilon^4 + O(\epsilon^5). \quad (4.5)
\end{aligned}$$

It is difficult to extract reliable estimates from these series. In both cases, we find that $\omega_{\text{bi},3}$ is small, but we are unable to determine reliably its sign.

For $N \geq 5$ we find that the BFP is unstable for all values of n_1 and n_2 . In particular, for $N=5$, $n_1=2$, $n_2=3$, and for the smallest eigenvalue we obtain

$$\begin{aligned}
\omega_{\text{bi},3} &= 0.052\,584\epsilon + 0.0331\,401\epsilon^2 - 0.242\,179\epsilon^3 \\
&\quad + 0.358\,964\epsilon^4 - 1.242\,100\epsilon^5 + O(\epsilon^6), \quad (4.6)
\end{aligned}$$

which gives $\omega_{\text{bi},3} = -0.07(5)$.

V. CONCLUSIONS AND DISCUSSION

We have studied the multicritical behavior at a MCP, where two critical lines with $O(n_1)$ and $O(n_2)$ symmetry meet. It has been determined by studying the RG flow of the most general $O(n_1) \oplus O(n_2)$ -symmetric LGW Hamiltonian involving two fields ϕ_1 and ϕ_2 with n_1 and n_2 components, respectively. We have extended the ϵ expansion of the critical exponents and of the stability matrix of the FP's, previously known to one-loop order, to five loops. The stability of the $O(N)$ FP has also been discussed in the framework of the fixed-dimension expansion in three dimensions to six loops.

The main properties of the RG flow of the $O(n_1)$

$\oplus O(n_2)$ -symmetric system at the MCP can be summarized as follows.

- (i) The $O(N)$ FP is stable only for $N=2$, i.e., when two Ising-like critical lines meet. It is unstable in all other cases, i.e., for all n_1 and n_2 such that $n_1+n_2=N \geq 3$. Beside being unstable with respect to the spin-0 and spin-2 quadratic perturbations, for $N \geq 3$ the $O(N)$ FP is also unstable with respect to quartic perturbations belonging to the spin-4 representation of the $O(N)$ group; cf. Eq. (3.5). This implies that for $N \geq 3$ the enlargement of the symmetry $O(n_1) \oplus O(n_2)$ to $O(N)$ requires an additional parameter to be tuned, beside those associated with the quadratic perturbations r_1 and r_2 in the LGW Hamiltonian. The associated crossover exponents $\phi_{4,4} \equiv y_{4,4} \nu$ are $\phi_{4,4} \approx 0.01$ for $N=3$, $\phi_{4,4} \approx 0.08$ for $N=4$, $\phi_{4,4} \approx 0.15$ for $N=5$, and $\phi_{4,4} \rightarrow 1$ for $N \rightarrow \infty$ (see Table I).
- (ii) For $N=3$, i.e., for $n_1=1$ and $n_2=2$, the critical behavior at the MCP is described by the BFP, whose critical exponents turn out to be very close to those of the Heisenberg universality class; see Eq. (4.4).
- (iii) For $N \geq 4$ and for any $n_1 \geq 1$ and $n_2 \geq 1$, the tetracritical DFP is stable. This has been inferred using non-perturbative arguments¹⁶⁻¹⁹ that allow us to write the relevant stability eigenvalue y_w in terms of the critical exponents of the $O(n_i)$ universality classes; cf. Eq. (4.1). The ϵ -expansion analysis shows that the BFP is unstable for all cases with $N \geq 5$, while it is not conclusive for the cases with $N=4$.
- (iv) When the initial parameters of the Hamiltonian are not in the attraction domain of the stable FP, the transition between the disordered and ordered phases should be of first order in the neighborhood of the MCP. In this case, a possible phase diagram is given in Fig. 3. Close to the MCP all transition lines are first-order ones. However, far from the MCP, the high-temperature transitions may become continuous, belonging to the $O(n_1)$ and $O(n_2)$ universality classes.

As already mentioned in the Introduction, a multicritical behavior has been observed in several systems.

Anisotropic antiferromagnets in a uniform magnetic field H_{\parallel} parallel to the anisotropy axis present a MCP in the T - H_{\parallel} phase diagram, where two critical lines belonging to the XY and Ising universality classes meet.^{2,3} The results presented above predict a multicritical BFP. The mean-field approximation assigns a tetracritical behavior to the MCP,³ but a more rigorous characterization, which requires the computation of the corresponding scaling free energy, is needed to draw a definite conclusion. Experimentally, the MCP appears to be bicritical (see, e.g., the experimental results of Refs. 74 and 75); numerical Monte Carlo results hint at the same behavior, although with much less confidence.⁷⁶ Our results contradict the $O(\epsilon)$ calculations of Refs. 2 and 3, suggesting the stability of the $O(3)$ FP. Notice that it is very hard to distinguish the biconal from the $O(3)$ critical behavior. For instance, the

correlation-length exponent ν differs by less than 0.001 in the two cases. However, one may still hope to distinguish the two FP's by measuring some universal amplitude ratio that varies more significantly in the two cases. The crossover exponent describing the crossover from the $O(3)$ critical behavior is very small, i.e., $\phi_{4,4} \approx 0.01$, so that systems with a small effective breaking of the $O(3)$ symmetry cross very slowly towards the biconal critical behavior or, if the system is outside the attraction domain of the BFP, towards a first-order transition; thus, they may show the eventual asymptotic behavior only for very small values of the reduced temperature.

Isotropic antiferromagnets in a magnetic field are quite a special case. Indeed, the critical transition at $H=0$ is a MCP with $O(3)$ symmetry, as observed experimentally; see, e.g., Ref. 77. As we discussed, for $H \neq 0$, two relevant perturbations are switched on, and they can give rise in principle to a more complex phase diagram. Finally, it should be noted that in real antiferromagnets additional nonisotropic interactions are present, giving rise to lower-symmetry MCP's. In Ref. 78 the magnetic phase diagram of $\text{NiCl}_2 \cdot 4\text{H}_2\text{O}$ was studied. The orthorhombic symmetry of the crystal gives rise to Ising transition lines both for small and large H_{\parallel} , so that $n_1 = n_2 = 1$. As predicted by the theory, the MCP is a bicritical XY point. A similar experiment is reported in Ref. 79. A tetracritical XY MCP is observed in anisotropic antiferromagnets when the magnetic field is perpendicular to the symmetry axis; see, e.g., Ref. 74 for an experimental study.

High- T_c superconductors are other interesting physical systems in which MCP's may arise from the competition of different order parameters. At low temperatures these materials exhibit superconductivity and antiferromagnetism depending on doping. The $SO(5)$ theory^{5,6} attempts to provide a unified description of these two phenomena, involving a three-component antiferromagnetic order parameter and a d -wave superconducting order parameter with $U(1)$ symmetry, with an approximate $O(5)$ symmetry. This theory predicts a MCP arising from the competition of these two order parameters when the corresponding critical lines meet in the temperature-doping phase diagram. Neglecting the fluctuations of the magnetic field and the quenched randomness introduced by doping (see, e.g., Ref. 19 for a critical discussion of this point), one may consider the $O(3) \oplus O(2)$ -symmetric LGW Hamiltonian to infer the critical behavior at the MCP (see, e.g., Refs. 80,34,13,26,14, and 15). In particular, the analysis of Ref. 34, which uses the projected $SO(5)$ model⁸¹ as a starting point, shows that one can use Eq. (1.1) as an effective Hamiltonian. Different scenarios have been proposed for the critical behavior at the MCP. In Refs. 5,13, and 14, it was speculated that the MCP is a bicritical point where the $O(5)$ symmetry is asymptotically realized. On the other hand, on the basis of the $O(\epsilon)$ results of Refs. 2 and 3, Refs. 80 and 26 predicted a tetracritical behavior governed by the BFP. However, since it was expected that the BFP is close to the $O(5)$ FP, it was suggested that at the MCP the critical exponents were in any case close to the $O(5)$ ones.

The $O(5)$ -symmetric scenario would require the stability of the $O(5)$ FP. Evidence in favor of this picture has been

recently claimed using Monte Carlo simulations for a five-component $O(3) \oplus O(2)$ -symmetric spin model.^{14,20} The numerical results show that, within the parameter ranges considered, the scaling behavior at the MCP is consistent with an $O(5)$ -symmetric critical behavior. Similar results have been obtained in Ref. 35 by a quantum Monte Carlo study of the quantum projected $SO(5)$ model in three dimensions. On the other hand, the interpretation of these numerical results as an evidence for the stability of the $O(5)$ FP (Refs. 14 and 20) is untenable, because the results discussed in this paper definitely show that the $O(5)$ FP is unstable, and that the asymptotic approach to the MCP is characterized by a decoupled critical behavior or by a first-order transition. The $O(5)$ symmetry can be asymptotically realized only by tuning a further relevant parameter, beside the double tuning required to approach the MCP. We note that the crossover exponent $\phi_{4,4}$, related to the spin-4 perturbation of the $O(5)$ FP, $\phi_{4,4} \approx 0.15$, is much larger than its $O(\epsilon)$ approximation, i.e., $\phi_{4,4} \approx \frac{1}{26} \epsilon$ from which one would obtain $\phi_{4,4} \approx 0.04$ setting $\epsilon = 1$. It is of the same order of the crossover exponent appearing in many other physical systems. For instance, in randomly dilute uniaxial magnetic materials—a class of systems whose asymptotic critical behavior has been precisely ob-

served both numerically and experimentally (see, e.g. Refs. 44 and 82 for reviews)—the pure Ising fixed point is unstable with crossover exponent $\phi \approx 0.11$, which is even smaller than the above-reported estimate for the $O(5)$ case. Therefore, contrary to some recent claims,³⁵ it cannot be excluded *a priori* that experiments are able to observe the unstable flow out of the $O(5)$ fixed point, even in those systems with a moderately small breaking of the $O(5)$ symmetry, such as the projected $SO(5)$ model. Evidence in favor of a tetracritical behavior has been recently provided by a number of experiments (see, e.g., Refs. 27–32) which seem to show the existence of a coexistence region of the antiferromagnetic and superconductivity phases. The possible coexistence of the two phases has been discussed in Refs. 7,15, and 83.

Finally, we mention that a multicritical behavior with two XY order parameters is expected in liquid crystals, at the nematic–smectic-A–smectic-C multicritical point¹⁷ and in the presence of ferromagnetic and nematic interactions.⁸⁴

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- nated by the O(5) FP and that it is practically impossible to observe both experimentally and numerically the flow out of the symmetric O(5) FP. In order to obtain the above-reported estimate of t_{cross} , they computed the crossover exponent from its $O(\epsilon)$ approximation (Ref. 3), i.e., $y_{4,4} = \frac{1}{13}\epsilon + O(\epsilon^2)$ and $\phi_{4,4} = \frac{1}{26}\epsilon + O(\epsilon^2)$, thus $\phi_{4,4} \approx \frac{1}{26}$ setting $\epsilon = 1$. Unfortunately, the actual value of $\phi_{4,4}$ is much larger: $\phi_{4,4} \approx 0.15$. Taking for granted the arguments leading to the estimate $t_{\text{cross}} \approx 10^{-10}$, one can easily see that, using the actual value of $\phi_{4,4}$, the crossover reduced temperature t_{cross} would change from 10^{-10} to 10^{-x} where $x = 10/(26 \times 0.15) \approx 2.6$. Taking into account that this is a very rough estimate, which may easily miss one order of magnitude, one cannot exclude the possibility of observing experimentally and numerically the unstable flow towards the eventual critical behavior, even in systems with a moderately small breaking of the O(5) symmetry at the multicritical point.
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