Multisite-interaction Ising model approach to the solid ^{3}He system on a triangular lattice

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We consider the Ising model with multiple-spin interactions on a recursive lattice of special type (the Bethe lattice of square plaquettes with additional inner link) as some approximation to the two-dimensional multiple-spin exchange model with two-, three-, and four-spin exchanges. This statistical system can be regarded as an approach to solid ^{3}He films which have the structure of a regular triangular two-dimensional lattice. The corresponding recursion relations for the partition function and magnetization per site are obtained. Using the dynamical method, we plot the diagrams of magnetization versus the external magnetic field for various values of exchange parameters and finite temperatures. The system exhibits different magnetic behaviors, depending on the values of the exchange parameters. The plots, corresponding to a purely ferromagnetic situation as well as more complex plots containing magnetization plateaus at $m=0$ and $m/m_{sat}=1/2$, are obtained. We also show that on some recursive lattices plateaus can appear with Ising pair interactions solely. A diagram with one bifurcation loop is also obtained. As usual, the bifurcation points are identified with second-order phase transitions. An attempt is made to elucidate the microscopic structure of the emerging novel ordered phases.

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I. INTRODUCTION

Solid helium, due to the physical conditions providing its existence—low temperatures and high pressures—and due to the light mass of helium atoms, is a unique example of a purely quantum crystal.[1] Moreover, properties of solid ^{3}He and solid ^{4}He are steeply different from each other as, in contrast to ^{4}He nuclei, the nuclei of ^{3}He are fermions with spin 1/2. Thus, solid ^{3}He can be regarded as a system of almost localized identical fermions. The microscopic theory of magnetism for such systems is based on the concept of the permutation of particles.[2,3]

Recently, solid ^{3}He films adsorbed on the surface of graphite[4] have attracted extensive attention, since it is a typical example of a two-dimensional frustrated quantum-spin system.[5] Particularly, the magnetically active second layer exhibits a large variety of interesting features, which are not still completely understood.

In these films the nuclei of ^{3}He form a system of quantum one-half spins on a triangular lattice. Many experimental[6–8] and theoretical[9–11] studies suggest that the exchanges of more than two particles are dominant in this system. The change from ferromagnetic behavior to antiferromagnetic takes place when the coverage of ^{3}He atoms decreases. This phenomenon can be explained in terms of "multiple-spin exchange" (MSE): in a fully packed system, three-spin exchange is dominant and, according to general principles,[3] is ferromagnetic. In loosely packed systems four- and six-spin exchanges become important and lead to a frustrated antiferromagnetic system. So multiple-spin exchanges produce frustration by themselves and, moreover, strong competition between odd- and even-particle exchanges is also responsible for frustration.[12] Thus, solid ^{3}He films, described in terms of the multiple-spin exchange model, exhibit very interesting and complex magnetic behavior including the appearance of various ordered phases[13] and magnetization plateaus.

The magnetization plateau is quite a new member, which has joined a large family of nontrivial quantum phenomena in one- and two-dimensional spin systems. These kinds of plateaus are not associated with saturation phenomena. Actually, beginning at some intermediate value of the external magnetic field $H_{1_C}<H_S$ (saturation field) the system ceases to respond to the increase of the field and adsorbs energy without any change of magnetization. This takes place until H reaches some other value $H_{2_C}<H_S$ when magnetization recovers its normal behavior and begins to change its value again. Thus, some horizontal region appears on the magnetization versus external magnetic field curve between the values H_{1_C} and H_{2_C}. Typical examples of such systems are different kinds of $S=1/2$ and $S=3/2$ Heisenberg spin chains with next-nearest-neighbor and alternating next-nearest-neighbor interactions, bond-alternating quantum-spin chains and -spin ladders.[14] The values of the magnetization at which plateaus occur are restricted to be rational numbers. In quantum-spin chains at zero temperature, general criteria for these restrictions have been established by Oshikawa *et al.*[15]

The appearance of magnetization plateaus in two-dimensional systems has been established both theoretically and experimentally. On the triangular lattice a magnetization plateau was observed at $m/m_{sat}=1/3$ for compounds like C_6Eu (Ref. 16), and $CsCuCl_3$ (Ref. 17) where the plateau originates from a three-sublattice "up-up-down" (*uud*) ordered structure.

Theoretical studies of antiferromagnets (with two-spin exchanges) show that quantum corrections to the magnetization plateau at $m/m_{sat}=1/3$ are also vanishing.[18] A similar magnetization plateau was also discovered in a multiple-spin exchange model using exact diagonalizations of finite-size clusters in a model with four-spin exchange on a square lattice[19] and in a model with two-, three-, four-, five-, and six-spin exchanges on the triangular lattice.[20] Further investigation of this magnetization plateau was made in Ref. 21. It originates from the appearance of a four-sublattice ''up-up-up-down'' (*uuud*) structure due to four-spin exchange. Thermal effects were considered using Monte Carlo simulations. The appearance of a magnetization plateau in a spin ladder with four-spin exchange was established in Ref. 22.

The approach we have developed in the present paper is based on several approximations.[23] The key point of our approach is the so-called Bethe approximation (recursive lattice),[24] which is a very powerful tool in investigating many theoretical problems in statistical mechanics, condensed matter physics, gauge models, macromolecule physics, etc.[25,26] Moreover, it has been argued recently[27] that, in some cases, Bethe lattice calculations are more reliable than mean-field calculations. Actually there are a few different ways of construction of recursive lattices. One of them is realized via connection through sites between successive generations and the other by links. We assume that the latter case may be relevant in obtaining a variety of modulated phases, which are typical for spin systems with competing interactions.[28]

However, the specificity of the calculations, which we implement, strictly requires us to use Ising spin variables instead of Heisenberg ones. Undoubtedly this approximation is rather rough, since there are no reasonable conditions in solid ^{3}He under which we could neglect the nondiagonal part of the Heisenberg or four-spin interaction. Nevertheless, there is now clear evidence of the colinear nature of the plateau phase in the MSE model on a triangular lattice.[13,21] If the spins points preferentially along the direction of the magnetic field, we expect that a large magnetic field reduces the transverse fluctuations. In that case, an Ising model with spins along the direction of the magnetic field should catch the essence of the physical behavior.

The present approach allows us to obtain magnetization diagrams for arbitrary finite temperatures and any set of exchange parameters, using the general advantages of recursive lattices or dynamical system method.

We propose a special type of recursive lattice, which is actually a Bethe-type lattice consisting of square plaquettes with additional inner links, and obtain the corresponding recursion relations for the partition function and magnetization. The dynamical method allows us, using the technique of simple iteration, to plot the magnetization diagrams for our model for arbitrary finite temperature and various values of exchange parameters.

The paper is organized in the following way. Section II is devoted to the main ideas of multiple-spin exchange models in relation to solid ^{3}He. In Sec. III we construct the recursive lattice and formulate the problem of multiple-spin exchange models with two-, three-, and four-spin exchanges on it. In Sec. IV we derive the recursion relations for the partition function and magnetization per site. Some details about the numerical calculations and plots of magnetization curves versus external magnetic field are presented in Sec. VI. Sec. VII contains the conclusion.

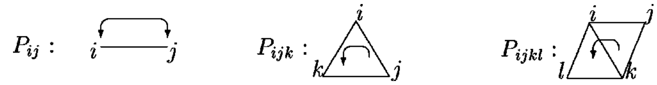

FIG. 1. Two-, three-, and four-spin exchange operators acting on the triangular lattice.

II. EXCHANGE HAMILTONIAN FOR SOLID ^{3}He

It is known that the Hamiltonian describing solid ^{3}He can be written as

$$\mathcal{H}=\mathcal{H}_{Ph}+\mathcal{H}_{ex}+\mathcal{H}_Z . \tag{1}$$

The first term $\mathcal{H}_{Ph}$ is the phonon contribution and is not essential for our considerations because it is not coupled to the exchange $\mathcal{H}_{ex}$ and Zeeman $\mathcal{H}_Z$ terms which are responsible for magnetism in solid ^{3}He.

The most general expression for the spin-exchange Hamiltonian is[1–3]

$$\mathcal{H}_{ex}=-\sum_{n,\alpha} J_{n\alpha}(-1)^p P_n . \tag{2}$$

Here the summation runs over all permutations of particles, P_n is the permutation operator of n particles, $J_{n\alpha}$ is the corresponding exchange energie (positive) (α distinguishes topologically inequivalent cycles), and p is the parity as defined in permutation group theory, i.e., it is odd (even) if the decomposition of the permutation into a product of pair transpositions involves an odd (even) number of transpositions.

It is obvious from Eq. (2) that exchanges with odd and even numbers of particles contribute with different signs. In our case there is no need for index α in Eq. (2), since on the regular two-dimensional lattice all three- and four-spin cycles formed with nearest neighbors are equivalent.

Thus one can write down the exchange Hamiltonian for planar solid ^{3}He in the following way:

$$\mathcal{H}_{ex}=J_2\sum_{pairs}(P_2+P_2^{-1})-J_3\sum_{triangles}(P_3+P_3^{-1}) + J_4\sum_{rectangles}(P_4+P_4^{-1})+\cdots, \tag{3}$$

where the sum in the first term is going over all pairs of particles, in the second term over all triangles, and in the third term over all parallelograms consisting of two triangles and so on (see Fig. 1).

The expression of a pair transposition operator P_{ij} has been given by Dirac,[2]

$$P_{ij}=\frac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \tag{4}$$

where $\boldsymbol{\sigma}_i$ is the Pauli matrix, acting on the spin at the ith site. It is easy to see that $P_2^{-1}=P_2$ which is not valid for $n>2$.

Other P_n operators can be expressed in terms of pair transposition operator P_2. For the three-spin permutation operator we have

$$P_{ijk}=P_{ij}\cdot P_{ik}=\frac{1}{4}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k). \qquad (5)$$

Using the identity

$$(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)=\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_i\cdot[\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_k], \qquad (6)$$

one can write the former expression as

$$P_{ijk}=\frac{1}{4}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_i\cdot[\boldsymbol{\sigma}_j\times\boldsymbol{\sigma}_k]) \qquad (7)$$

and, hence,

$$P_{ijk}+(P_{ijk})^{-1}=\frac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_i). \qquad (8)$$

In the same way one can obtain an expression for the four-spin permutation operators:

$$P_{ijkl}=P_{ijk}\cdot P_{il}, \qquad (9)$$

$$P_{ijkl}+(P_{ijkl})^{-1}=\frac{1}{4}\left(1+\sum_{\mu<\nu}(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_\nu)+G_{ijkl}\right), \qquad (10)$$

where the sum is taken over six distinct pairs $(\mu\nu)$ among the four particles (*ijkl*), and

$$G_{ijkl}=(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_l\cdot\boldsymbol{\sigma}_k)+(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_l)(\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k)-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_l). \qquad (11)$$

Thus the spin-exchange Hamiltonian for planar solid ^{3}He up to four-spin exchange is, in terms of Pauli matrices,

$$\mathcal{H}_{ex}=\frac{J_2}{2}\sum_{\langle i,j\rangle}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)-\frac{J_3}{2}\sum_{\langle i,j,k\rangle}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_i)+\frac{J_4}{4}\sum_{\langle i,j,k,l\rangle}[1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_l+\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_l+\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l+(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_l\cdot\boldsymbol{\sigma}_k)+(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_l)(\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_k)-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)(\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_l)]. \qquad (12)$$

The first sum is taken over all pairs of nearest-neighbor sites, the second sum over all triangles, and the third sum over all diamond plaquettes (see Fig. 1). Omitting the constant term for clarity in Eq. (12) one can see that this is nothing else than the generalized Heisenberg model with a multiparticle (four-spin) interaction which is too complicated for further analysis and thus it is the object of some approximations and simplifications. The first of them is the classical counterpart of Eq. (12) when we have to replace the Pauli matrices by classical three-dimensional vectors (u_i^x,u_i^y,u_i^z) of unit length $(|u_i^x|^2+|u_i^y|^2+|u_i^z|^2=1)$ [it is the so called $O(3)$ model]. Using this approach Kubo and Momoi[13] have found a variety of ground states of planar solid ^{3}He at $T=0$: the perfect ferromagnetic state, the tetrahedral state with four-sublattice structure and zero magnetization, the so-called 120° state, and the *uuud* state. The two latter phases appear due to the four-spin exchange interaction.

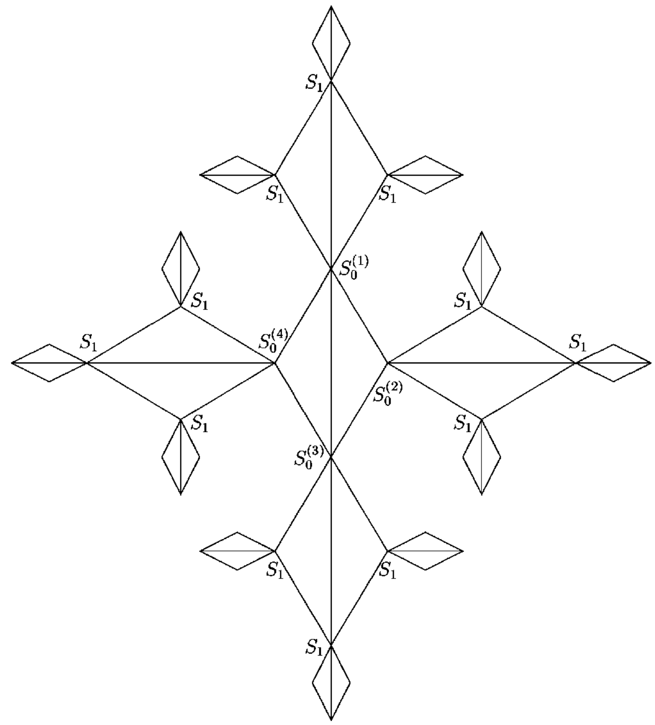

FIG. 2. The recursive Bethe-type lattice of four-polygons with an additional inner bound. $S_0^{(i)}$ are the spin variables of the zeroth shell, S_1 of the first shell.

We have made a further simplification based on the observation that in a strong magnetic field—say, in the z direction—the main contribution will be from the z component of classical spin variables, which can effectively take the values ± 1. Thus we actually have an Ising model with discrete spin variables instead of the $O(3)$ model.

So our further considerations will concern the treatment of the emergent multisite interaction Ising model (MII model) which we propose as some approximation for planar solid ^{3}He. This approach is expected to become rather robust in a strong external magnetic field.

III. SPECIAL-TYPE RECURSIVE LATTICE

Now let us consider instead of the periodic triangular lattice a recursive one of special type. The lattice is constructed in the following way: at zeroth generation, spin variables are placed at the sites of a central square plaquette with an additional bond between two sites, so spins $s_0^{(1)}$ and $s_0^{(3)}$ are regarded as neighbor ones and $s_0^{(2)}$ and $s_0^{(4)}$ are not (see Fig. 2).

Then we attach to each site of the central square plaquette the new one so that the additional inner bond of each new plaquette connects each site of the zeroth generation with the corresponding site of the first-generation plaquette. Spin variables of the first generation are placed at the 12 sites of the first shell consisting of four plaquettes. Carrying out this procedure successively for each new shell we can obtain a recursive lattice which actually is a Bethe-type lattice with square plaquettes and additional inner links. It is evident that there are two sites of each plaquette with coordination number 6, the two others having coordination number 5. If four-particle exchange is important, the use of such a lattice is a relevant approach to the underlying triangular periodic lattice, taking into account that we restrict ourselves to the exchanges of no more than four particles.

Now let us put the system described by Hamiltonian (12) onto our recursive lattice and introduce an interaction with the external magnetic field, which is given by the Zeeman Hamiltonian

$$\mathcal{H}_Z=-\sum_i \frac{\gamma}{2}\hbar \boldsymbol{B}\cdot\boldsymbol{\sigma}_i, \tag{13}$$

where γ is the gyromagnetic ratio of the ^{3}He nucleus.

After some algebra, we have

$$\begin{aligned}-\beta\mathcal{H}=\sum_{\square}\{&\alpha_1(\boldsymbol{S}_i\cdot\boldsymbol{S}_j+\boldsymbol{S}_j\cdot\boldsymbol{S}_k+\boldsymbol{S}_k\cdot\boldsymbol{S}_l+\boldsymbol{S}_l\cdot\boldsymbol{S}_i)+\alpha_2(\boldsymbol{S}_i\cdot\boldsymbol{S}_k)\\&+\alpha_3[\boldsymbol{S}_j\cdot\boldsymbol{S}_l+(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)(\boldsymbol{S}_k\cdot\boldsymbol{S}_l)+(\boldsymbol{S}_l\cdot\boldsymbol{S}_i)(\boldsymbol{S}_j\cdot\boldsymbol{S}_k)\\&-(\boldsymbol{S}_i\cdot\boldsymbol{S}_k)(\boldsymbol{S}_l\cdot\boldsymbol{S}_j)]+h(S_i^z+S_j^z+S_k^z+S_l^z)\}.\end{aligned} \tag{14}$$

Here the sum goes over all square plaquettes of our recursive lattice, the external magnetic field is assumed to be in the z direction, and the parameters are

$$\alpha_1=\beta\left(\frac{J_3}{2}-\frac{J_4}{4}-\frac{J_2}{2}\right),$$

$$\alpha_2=\beta\left(J_3-\frac{J_4}{4}-\frac{J_2}{2}\right),$$

$$\alpha_3=-\frac{\beta J_4}{h},$$

$$h=\beta\frac{\gamma\hbar B}{2}. \tag{15}$$

We have written down Eq. (14) in general terms; one may consider the variables $\boldsymbol{S}_i$ as classical vectors as well as Pauli matrices. Now, if we use the multisite interaction Ising model, Eq. (14) takes the following form:

$$\begin{aligned}-\beta\mathcal{H}=\sum_{\square}\{&\alpha_1(s_is_j+s_js_k+s_ks_l+s_ls_i)+\alpha_2 s_is_k\\&+\alpha_3(s_js_l+s_is_js_ks_l)+h(s_i+s_j+s_k+s_l)\}.\end{aligned} \tag{16}$$

Here s_i takes the values ± 1.

IV. RECURSION RELATION AND MAGNETIZATION FUNCTION

Here we proceed to the derivation of the exact recursion relations for our model. When the lattice is cut apart from the central plaquette, it separates into four identical branches. So we can first realize a summation over all spin configurations on each branch, getting the same result every time, and then to sum over spins of the central plaquette.

After this procedure, the partition function

$$\begin{aligned}Z=\sum_{\{S\}}\prod_{\square}\exp\{&\alpha_1(s_is_j+s_js_k+s_ks_l+s_ls_i)+\alpha_2 s_is_k\\&+\alpha_3(s_js_l+s_is_js_ks_l)+h(s_i+s_j+s_k+s_l)\}\end{aligned} \tag{17}$$

takes the form

$$\begin{aligned}Z=\sum_{\{S_0\}}\exp\{&\alpha_1(s_0^{(1)}s_0^{(2)}+s_0^{(2)}s_0^{(3)}+s_0^{(3)}s_0^{(4)}+s_0^{(4)}s_0^{(1)})\\&+\alpha_2 s_0^{(1)}s_0^{(3)}+\alpha_3(s_0^{(2)}s_0^{(4)}+s_0^{(1)}s_0^{(2)}s_0^{(3)}s_0^{(4)})+h(s_0^{(1)}\\&+s_0^{(2)}+s_0^{(3)}+s_0^{(4)})\}g_N(s_0^{(1)})g_N(s_0^{(2)})g_N(s_0^{(3)})g_N(s_0^{(4)}),\end{aligned} \tag{18}$$

where $s_0^{(a)}$ are spins of the central plaquette, $g_N(s_0^{(a)})$ denotes the contribution of a branch at the ath site of the central plaquette, and N is the number of generations ($N\to\infty$ corresponds to the thermodynamic limit, neglecting surface effects[24–26])

In the same manner we can obtain the equation for one branch, cutting it along any site of the first generation which is nearest to the central plaquette. For instance,

$$\begin{aligned}g_N(s_0^{(1)})=\sum_{\{s_1\}}\exp\{&\alpha_1(s_0^{(1)}s_1^{(4)}+s_1^{(4)}s_1^{(1)}+s_1^{(1)}s_1^{(2)}+s_1^{(2)}s_0^{(1)})\\&+\alpha_2 s_1^{(1)}s_0^{(1)}+\alpha_3(s_1^{(2)}s_1^{(4)}+s_0^{(1)}s_1^{(1)}s_1^{(2)}s_1^{(4)})\\&+h(s_1^{(1)}+s_1^{(2)}+s_1^{(4)})\}\\&\times g_{N-1}(s_1^{(1)})g_{N-1}(s_1^{(2)})g_{N-1}(s_1^{(4)}).\end{aligned} \tag{19}$$

The sum is taken over $s_1^{(1)}$, $s_1^{(2)}$, and $s_1^{(3)}$ variables. One can easily calculate Eq. (19) for both values (± 1) of $s_0^{(1)}$,

$$\begin{aligned}g_N(+)=&a^4bc^2d^3g_{N-1}^3(+)+2bc^{-2}dg_{N-1}^2(+)g_{N-1}(-)\\&+b^{-1}dg_{N-1}^2(+)g_{N-1}(-)+a^{-4}bc^2d^{-1}g_{N-1}\\&\times(+)g_{N-1}^2(-)+2b^{-1}d^{-1}g_{N-1}(+)g_{N-1}^2(-)\\&+b^{-1}d^{-3}g_{N-1}^3(-),\end{aligned} \tag{20}$$

$$\begin{aligned}g_N(-)=&b^{-1}d^3g_{N-1}^3(+)+2b^{-1}dg_{N-1}^2(+)g_{N-1}(-)\\&+a^{-4}bc^2dg_{N-1}^2(+)g_{N-1}(-)+b^{-1}d^{-1}g_{N-1}\\&\times(+)g_{N-1}^2(-)+2bc^{-2}d^{-1}g_{N-1}(+)g_{N-1}^2(-)\\&+a^4bc^2d^{-3}g_{N-1}^3(-),\end{aligned} \tag{21}$$

where the following notation has been introduced:

$$a=\exp\alpha_1,\quad b=\exp\alpha_2,\quad c=\exp\alpha_3,\quad d=\exp h. \tag{22}$$

Taking Eq. (20) over Eq. (21) one can obtain the recursion relation for the variable $x_N=g_N(+)/g_N(-)$:

$$x_N=f(x_{N-1}),$$

$$f(x)=\frac{A\mu^3x^3+(2B+1)\mu^2x^2+(C+2)\mu x+1}{\mu^3x^3+(C+2)\mu^2x^2+(2B+1)\mu x+A}. \tag{23}$$

Here

$$A=\exp[\beta(4J_3-2J_4-3J_2)],$$

$$B=\exp[\beta(2J_3-J_2)],$$

$$C=\exp(\beta J_2),$$

$$\mu=\exp 2h=\exp(\beta\gamma\hbar B). \tag{24}$$

Such a recursion relation plays a crucial role in our further investigations, because one can obtain all thermodynamic data of the system, using the technique of dynamical system theory, applied to the one-dimensional (1D) map (23). Particularly, we will deal with the magnetization per site, which can be obtained in the same way as the recursion relation (23). For an homogeneous lattice the magnetization function is given by the formula

$$m=\frac{\sum_{(s)} s_i e^{-\beta\mathcal{H}}}{\sum_{(s)} e^{-\beta\mathcal{H}}}, \tag{25}$$

where s_i is the spin on an arbitrary site. In our case, however, the lattice is not homogeneous and actually there are two types of sites which have different coordination numbers 6 and 5, respectively. Thus, the full magnetization function in the model under consideration consists of two parts. The first one m_1 can be obtained as the thermodynamical average of the spin at an arbitrary site of coordination number 6 (say, $S_0^{(1)}$ in Fig. 2); the second part m_2 can be obtained in the same way for a site with coordination number 5 ($S_0^{(2)}$ in Fig. 2). The total magnetization is

$$m=\frac{1}{2}(m_1+m_2), \tag{26}$$

$$m_1=\langle s_1\rangle=\frac{\sum_{s_1,s_2,s_3,s_4} s_1\Omega(s_1,s_2,s_3,s_4)\prod_{i=1}^{4} g_n(s_i)}{\sum_{s_1,s_2,s_3,s_4} \Omega(s_1,s_2,s_3,s_4)\prod_{i=1}^{4} g_n(s_i)}, \tag{27}$$

$$m_2=\langle s_2\rangle=\frac{\sum_{s_1,s_2,s_3,s_4} s_2\Omega(s_1,s_2,s_3,s_4)\prod_{i=1}^{4} g_n(s_i)}{\sum_{s_1,s_2,s_3,s_4} \Omega(s_1,s_2,s_3,s_4)\prod_{i=1}^{4} g_n(s_i)}, \tag{28}$$

where $\Omega(s_1,s_2,s_3,s_4)=\exp\{\alpha_1(s_1s_2+s_2s_3+s_3s_4+s_4s_1)+\alpha_2s_1s_3+\alpha_3(s_2s_4+s_1s_2s_3s_4)+h(s_1+s_2+s_3+s_4)\}$ is the statistical weight of the central plaquette. Carrying out calculations analogous to Eqs. (20)–(23) we obtain the following expressions for m as a function of x:

$$m_1=\frac{A\mu^4x^4+2B\mu^3x^3-2B\mu x-A}{A\mu^4x^4+2(B+1)\mu^3x^3+2(C+2)\mu^2x^2+2(B+1)\mu x+A}, \tag{29}$$

$$m_2=\frac{A\mu^4x^4+2\mu^3x^3-2\mu x-A}{A\mu^4x^4+2(B+1)\mu^3x^3+2(C+2)\mu^2x^2+2(B+1)\mu x+A}, \tag{30}$$

$$m=\frac{A\mu^4x^4+(B+1)\mu^3x^3-(B+1)\mu x-A}{A\mu^4x^4+2(B+1)\mu^3x^3+2(C+2)\mu^2x^2+2(B+1)\mu x+A}. \tag{31}$$

The coefficients A, B, and C are defined in Eqs. (24).

Having these dynamical expressions one can draw the plots of magnetization versus external magnetic field for various temperatures. For this purpose one has to fix the value of the dimensionless magnetic field h (for a given temperature and exchange parameters) and implement the simple iteration of Eq. (23), beginning with some initial x_0. The amount of iterations must be large enough to achieve the thermodynamical limit. In the general case the recursion sequence $\{x_n\}$ either converges to a stable fixed point or to a stable two-cycle. Substituting the final result into Eq. (31) one can obtain the magnetization versus the external magnetic field. The case of a stable two-cycle corresponds to a structure with two sublattices.

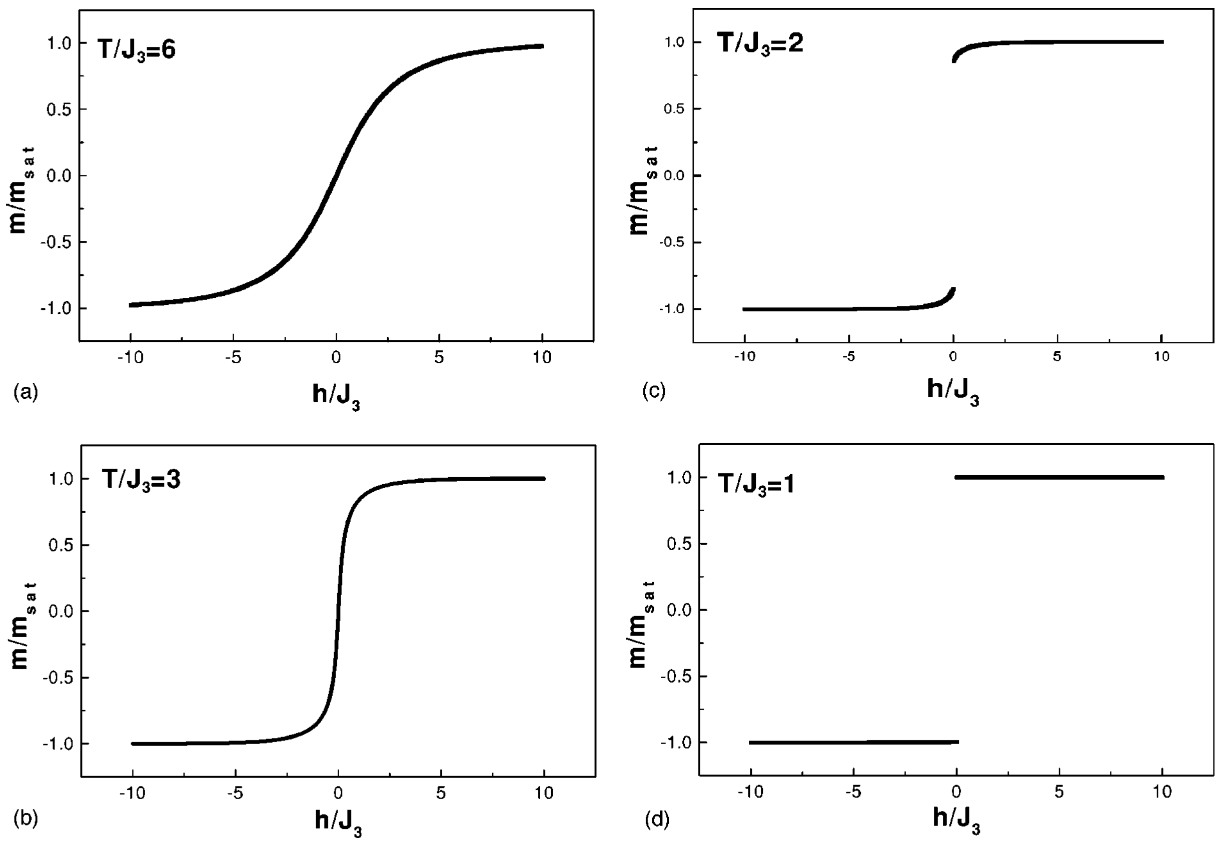

FIG. 3. The plots of magnetization processes for the case when only J_3 is nonzero: (a) a typical high-temperature behavior at $T=6J_3$; (b) $T=3J_3$, the curve still remains continuous; (c) the curve with jump and at $T=2J_3$; (d) $T=J_3$ ferromagnetic ground state at low temperatures.

V. MAIN RESULTS

This section is devoted to the analysis of the magnetic behavior of our model in a strong magnetic field. For the 2D systems, recent experimental measurements[8] as well as theoretical calculations[11] predict the following relations between the exchange energies J_n on the regular triangular lattice:

$$J_3>J_2>J_4\geqslant J_6\geqslant J_5. \tag{32}$$

It is worth noticing that the values of pure J_n are not obtainable in the experimental measurements, because each n-spin exchange makes also a contribution to a few $(n-1)$-spin exchanges, and thus there are some effective exchange parameters, which are certain combinations of J_n (such as $J=J_2-2J_3$ and $K=J_4-2J_5$) and can be directly obtained from experiments. So the multiple-spin exchange model with two-, three-, and four-spin exchanges is usually described by two parameters

$$J=J_2-2J_3 \quad \text{and} \quad K=J_4 \tag{33}$$

The calculations, based on the WKB approximation,[10] show that all exchange parameters vary in a rather wide range depending on the particle density. It is also obvious that the values of the exchange parameters depend not only on the particle density but also on the type of lattice, particularly on its coordination number and dimensionality. So for the 2D triangular lattice it was found that at high densities, the exchange J is dominant, mainly due to the three-spin exchange, but the ratio $|K/J|$ increase rapidly with the lowering of the particle density, and below some value of the particle density four-spin exchange K becomes important. The magnetic properties of the system change from ferromagnetic to antiferromagnetic depending on whether three- or four-spin exchange is dominant at the present value of the particle density.

From the arguments stated above one can conclude that there is a large freedom in the choice of concrete values of the exchange parameters J_2, J_3, and J_4. Moreover, it is quite difficult to identify our model, having so many assumptions, with some concrete value of the particle density.

Since the main features of the resulting magnetic behavior of the system under consideration are caused by the interplay between ferromagnetic (J_3) and antiferromagnetic (J_2 and J_4) interactions, it is of interest first to consider the three above-mentioned cases, each taken separately.

The simplest case of pure ferromagnetic behavior ($J_3=1$, $J_2=J_4=0$) is presented in Fig. 3. At relatively high temperatures the magnetization curve has a smooth monotone form of Langevin type [Fig. 3(a)] with a rather large value of the saturation field. Decreasing the temperature, the curve becomes more steep and the value of the saturation field decreases [Fig. 3(b)]. Further decreasing the tempera-

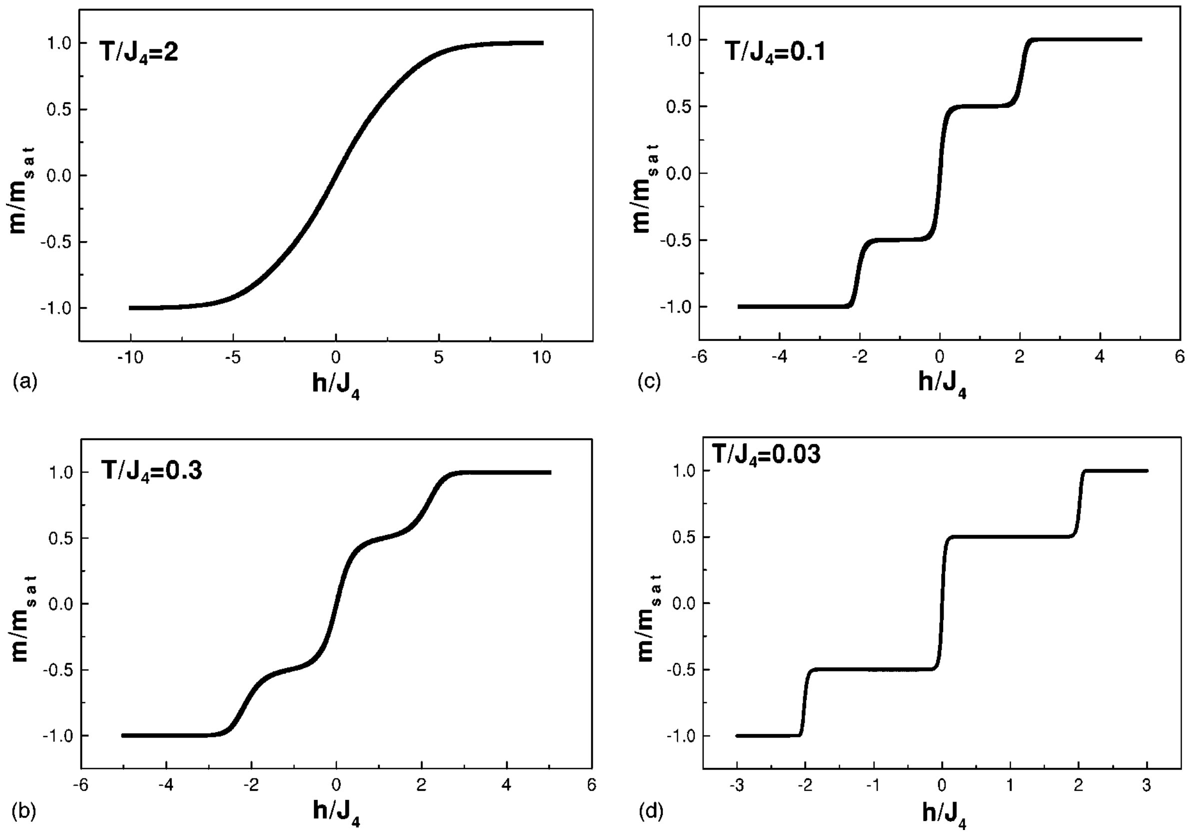

FIG. 4. Plots of magnetization for the case when only J_4 is nonzero: (a) $T=2J_4$, the curve is of Langevin type; (b) $T=0.3J_4$, the first indication of a future plateau; (c) $T=0.1J_4$, the curve with magnetization plateau at $m/m_{sat}=1/2$; (d) the curve with a wide magnetization plateau which appears already in very low external field at extremely low temperature $T=0.03J_4$.

ture leads to the magnetization diagram, presented in Fig. 3(c). Here the jump of magnetization takes place at an arbitrary low value of the applied magnetic field. And finally, when $T=J_3$ the ground state of the model becomes the ordered ferromagnetic one. The magnetization in zero field is equal to its maximal value, which corresponds to the ferromagnetic phase with all spins pointed in the same direction. Under the effect of an arbitrary weak magnetic field the twofold degeneracy of this phase is removed by orienting all spins along the field. The corresponding diagram is presented in Fig. 3(d).

The plots, obtained for the pure antiferromagnetic four-spin interaction $(J_2=J_3=0,\ J_4=1)$, are presented in Fig. 4. As usual the high-temperature magnetization curve is of Langevin type [Fig. 4(a)]. And the susceptibility increases at low field with decreasing the temperature. But meanwhile a slowing down in the course of the magnetization function occurs at some intermediate values of the external field [Fig. 4(b)]. With further decreasing of the temperature this region of the curve transforms to a horizontal line, the magnetization plateau at the half of the saturation magnetization [Fig. 4(c)]. The appearance of a magnetization plateau at $m/m_{sat}=1/2$ in the MSE model on a two-dimensional triangular lattice[21] is associated with the *uuud* ordered phase. It is very natural to suggest that in our case of the Ising model the situation is the same; i.e., for each plaquette one of the spins is oriented opposite to the field and the three others are along the field. This ordered *uuud* phase becomes stable at low temperatures. The length of the plateau increases with decreasing temperature. In Fig. 4(d) one can see the plot for $T=0.03J_4$ where the length of the plateau is more than $2J_4$.

The case corresponding to the common antiferromagnetic Ising model, when only pair interactions are included $(J_2=1,\ J_3=J_4=0)$, turns out to be highly nontrivial on our hierarchical lattice of square plaquettes with an additional inner link represented in Fig. 2. The corresponding plots are presented in Fig. 5. The high-temperature behavior is analogous to that of the previous cases; the curve has a monotonous form similar to the Langevin function [Fig. 5(a)]. The plot for $T=0.2J_2$ [Fig. 5(b)] shows that the susceptibility begins to decrease in the regions at zero magnetization and at $m/m_{sat}=1/2$. The appearance of plateaus in the corresponding regions is expected. In contrast to the case of a purely four-spin interaction, here we have two plateaus for $m=0$ and $m/m_{sat}=1/2$.

The plateau at $m=0$ arises from the so-called *uudd* phase. But due to the nonuniform nature of the lattice, the *uudd* configuration of spins on a single plaquette contributes with different energies, depending on whether the spins, connected by the additional inner link, are in the same position or not. One can easily calculate that the lowest energy $E=-3/2J_2$ corresponds to the configuration depicted in Fig. 6(a), in which spins, connected by an inner link, are in the same position, while the configuration in which the spins,

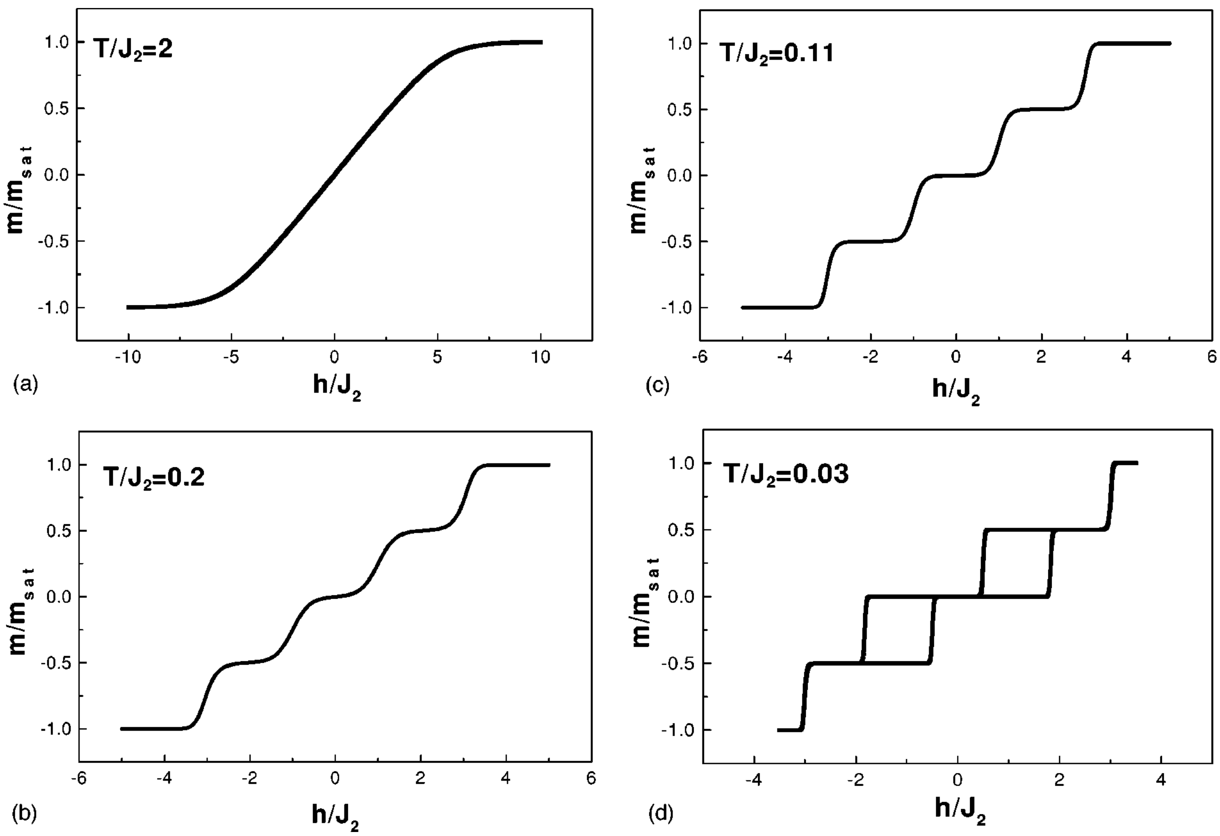

FIG. 5. The plots of magnetization processes for the case when only simple pair interactions are included: (a) standard high-temperature curve of Langevin type; (b) $T=0.2J_2$, susceptibility begins to increase in two different regions of the curve; (c) $T=0.11J_2$, magnetization plateaus have formed at $m=0$ and $m/m_{sat}=1/2$; (d) the magnetization diagram with one bifurcation loop at $T=0.03J_2$, the region between two bifurcation points corresponds to a novel ordered phase with two sublattice structure.

connected by an inner link, aligned in the opposite directions, has energy $E=-1/2J_2$ [Fig. 6(b)]. In what follows we will use $(uudd)_-$ and $(uudd)_+$ notation in order to distinguish these two states and *uudd* for both of them. Thus the zero-temperature ground state in this case is ordered in the *uudd* configuration with spins connected by an inner link in the same position [the $(uudd)_-$ phase].

In Fig. 5(c) one can see the plot of the magnetization process for $T=0.11J_2$ where the above-mentioned plateaus are shown evidently. At low but still finite temperatures the transitions $(uudd)_-\leftrightarrow(uudd)_+$ caused by thermal fluctuations are possible. So we cannot be sure that the origin of the plateau at $m=0$ in Fig. 5(c) is static and due to the $(uudd)_-$ phase. Apparently here we are dealing with the mixture of $(uudd)_-$ and $(uudd)_+$ plaquettes.

The width of the plateau is not so long because with increasing of the external field the system gradually develops

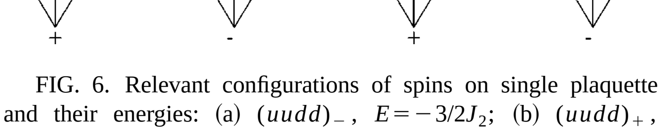

FIG. 6. Relevant configurations of spins on single plaquette and their energies: (a) $(uudd)_-$, $E=-3/2J_2$; (b) $(uudd)_+$, $E=-1/2J_2$; (c) $(uuud)_-$, $E=-1/2J_2+h$; (d) $(uuud)_-$, $E=1/2J_2+h$.

to another ordered state, which leads to the next plateau at $m/m_{sat}=1/2$. This state has also a four-sublattice structure with three spins aligned "up" and one "down" on each plaquette. This state is completely analogous to that observed in the case of a purely four-spin interaction. But in contrast to the previous case where the position of the spin aligned opposite to the other three did not affect the energy of state, here one has to distinguish, fully analogous to the *uudd* situation, two cases: spins, connected by an inner link, are parallel $[(uuud)_-]$ and spins, connected by an inner link, are antiparallel $[(uuud)_+]$, with energies $-J_2/2+h$ and $J_2/2+h$, respectively.

So if we restrict ourselves to rough calculations for one plaquette and $T=0$, the magnetization process will be the following: a plateau at $m=0$, corresponding to the $(uudd)_-$ ground state, an abrupt jump at $H=1/2J_2$ to the plateau at $m/m_{sat}=1/2$, corresponding to the $(uuud)_-$ state, which at this value of the external field becomes more favorable, and, finally, the last jump to the saturation magnetization at some large value of H.

But our further calculation for low temperature ($T=0.03J_2$) indicates that there is a novel complex ordered phase between *uudd* and *uuud*. In the Fig. 5(d) one can see the bifurcations which are, as usual, identified with second-order phase transition points.[23–26] The first bifurcation appears at $H=1/2J_2$ when the $(uudd)_-$ phase becomes unstable and the system acquires a new phase with long-range order. When the external magnetic field reaches a value $H = 2J_2$ another second-order phase transition leaves the place to the *uuud* phase. The nature of the novel ordered phase which appears between *uudd* and *uuud* phases needs a detailed analysis.

First of all it is obvious from Fig. 5(d) that between $H = 1/2J_2$ and $H = 2J_2$ we have an ordered structure with two sublattices of plaquettes with average magnetizations $m = 0$ and $m/m_{sat} = 1/2$, respectively. The separation into these two sublattice is shellwise; i.e., the shells, having $m = 0$ and $m/m_{sat} = 1/2$, are alternating. But nothing can be said about the arrangement of spins inside a certain shell. All we know from Fig. 5(d) is just that we have an ordered state which very roughly is the remote analog of the antiferromagnetic state on Husimi tree.[23] But taking into account the extremely low value of the temperature $T = 0.03J_2$ we can neglect the role of thermal fluctuations and use a simple argument of the minimum of energy in order to make an assumption about the microscopic structure of the intermediate phase between *uudd* and *uuud*. It is very natural to suggest that this phase is a certain ordered mixture of the *uudd* and *uuud* plaquettes with the global properties presented in Fig. 5(d). In order to determine rigorously the disposition of spins we took the few first generations of the recursive lattice and examined all possible combinations of ''up'' and ''down'' spins which lead the picture, depicted in Fig. 5(d), following the general principles of energy minimization. As a result we conclude that a possible structure of the intermediate mixed phase is the following: for those shells which have $m = 0$ magnetization, 3/4 of the plaquettes are in the *uudd* state while the other 1/4 consist of *uuud* plaquettes. On the contrary, for those shells which have magnetization $m/m_{sat} = 1/2$ the ratio is inverse: 3/4 of all plaquettes are in the *uuud* state whereas the other 1/4 are in the *uudd* state. It is obvious that the situation inside the shells is not the static one. Due to the presence of frustration,[5,23] certain transformations are possible, which do not destroy the general symmetry and shellwise ordering of the mixed phase. And this is the reason why the mixed phase survives in a rather large interval of the external magnetic field from $H = 1/2J_2$ until $H = 2J_2$. Further increasing of the magnetic field leads to the *uuud* [apparently $(uuud)_-$] phase.

The magnetization processes in the general case when all kinds of interactions are included show no qualitatively different behavior from those obtained previously. Roughly speaking, the plots for certain values of the exchange parameters J_2, J_3, J_4, depending on which one of them dominates, have exactly all those features, which are typical for the case when only this kind of exchange is present. So the combination of J_2 and J_3 leads either to a picture with magnetization plateaus at $m = 0$ and $m/m_{sat} = 1/2$ and one bifurcation loop or to the plots analogous to Fig. 4. The effect of the interplay is absent in this case, because in the MSE model permutations of three spins reduce to the effective pair exchanges, though in the case of our lattice the value of the effective pair exchange constant depends on the type of link. Each inner link is involved in two three-spin permutations, whereas the external link is just involved in one. But this circumstance does not change the general picture. The effect of the interplay is also absent between two- and four-spin interactions except the phenomenon of enlargement of the plateau at $m/m_{sat} = 1/2$.

Finally, we have tried to take those values of the exchange parameters (J_2, J_3, J_4) which have been estimated from susceptibility and specific-heat data in the low-density region:[8]

$$J = J_2 - 2J_3 = -3.07 \text{ mK}, \quad K = J_4 \cong 1.873 \text{ mK}. \tag{34}$$

For one of the possible sets of J_n, which is in agreement with Eqs. (32) and (34), $J_2 = 2, J_3 = 2.535, J_4 = 1.873$, we obtained the plots presented in Fig. 7. The standard high-temperature curve [Fig. 7(a)] is depicted here for comparison. In the next plot, obtained for $T = 0.5$ mK, a steep increase of the susceptibility in the vicinity of $h = 0$ appears. As one can see in Fig. 7(b) this increase soon is changed for the less steep region of the curve. With the further decreasing of the temperature a jump of magnetization takes place at $h = 0$ [Fig. 7(c)] from $m = 0$ to the $m/m_{sat} = 1/2$ value with the formation of a small magnetization plateau. Thus, here we have a first-order phase transition from the antiferromagnetic ground state to the *uuud* state. It is worth noticing that in Ref. 8 the effective constant of four-spin exchange $K = J_4 - 2J_5$ was estimated. For the antiferromagnetic region it has the value $K = 0.562$ mK. If in the calculation presented above one takes $J_4 = 0.562$ mK, the resulting plots will have only quantitative differences with respect to the latter case. Namely, in order to obtain the magnetization plateau at $m/m_{sat} = 1/2$ the value of J_2 must be increased (see Fig. 8); the value of the saturation field as well as the length of the plateau and all temperatures will be less than in the case presented in Fig. 7. The further increasing of J_2 and, following from $J = J_2 - 2J_3 = -3.07$ mK, increasing of J_3 at constant J_4 lead to the disappearance of the jump of magnetization at $h = 0$. It changes to the plateau, the length of which increases with increasing J_2. In Fig. 9 we presented the plot for $J_2 = 7$. Thus, the model under consideration can exhibit antiferromagnetic properties at the estimated values of the parameters J and K only if J_2 becomes sufficiently large (for $K = 0.562$ mK, $J_2 \simeq 4$ mK is enough). Further analysis has shown that all cases presented above exhaust the variety of the magnetic behavior for the system under consideration.

VI. RELEVANCE OF OUR THEORETICAL MODEL AS AN APPROXIMATION TO REAL SYSTEMS

A. Nearest-neighbor antiferromagnetic models

We have obtained an unexpectedly rich behavior of the magnetization in our nearest-neighbor antiferromagnetic Ising model on the Bethe lattice of plaquettes with additional inner links represented in Fig. 2, with two plateaus at $m = 0$ and $m = 1/2$. We have also been able to interpret these plateaus in terms of the occurrence of colinear *uuud* and *uudd* phases. These phases are obviously related to the topology of the four-site plaquettes. This peculiar Bethe lattice of plaquettes has been chosen as the most suitable hierarchical-lattice approximation to a triangular lattice with preponderant four-spin interactions. When the Hamiltonian is restricted

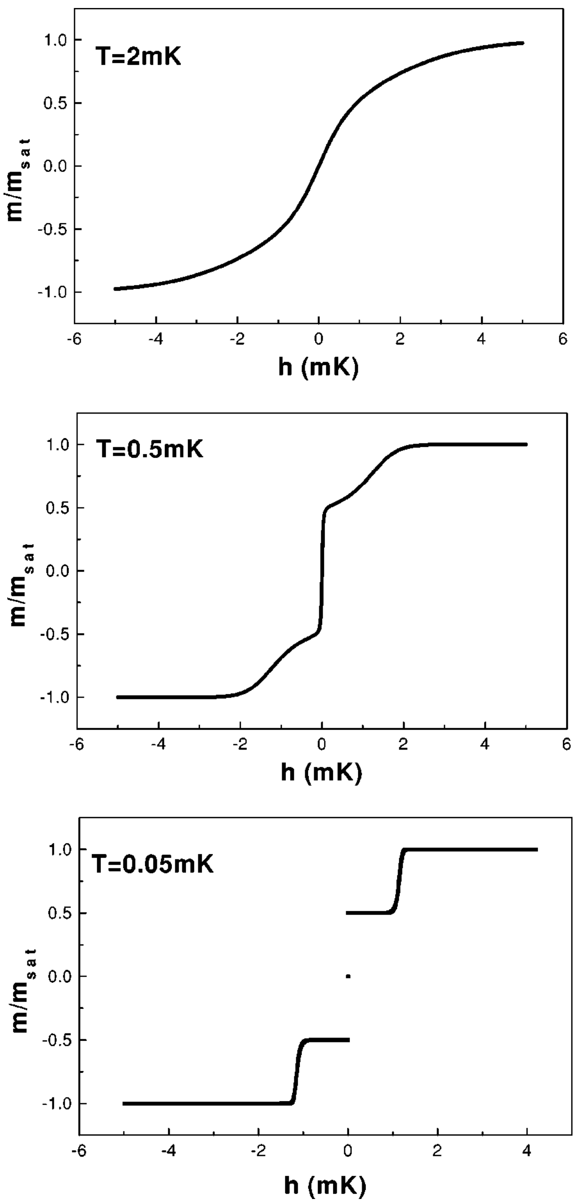

FIG. 7. Magnetization processes for the values of exchange constants estimated for the low-density region in Ref. 8, $J_2=2$ mK, $J_3=2.535$ mK, $J_4=1.873$ mK: (a) the standard Langevin-type curve for $T=2$ mK; (b) $T=0.5$ mK, indication of the forthcoming plateau at $m/m_{sat}=1/2$ and a steep increase of the susceptibility in the vicinity of $h=0$; (c) $T=0.05$ mK, the jump of the magnetization from the nonmagnetic state with $m=0$ to the *uuud* state.

to the nearest-neighbor pair interactions, this hierarchical lattice is certainly a better approximation to the square lattice than to the triangular lattice. For $J_3=J_2/2$ and $J_4=0$, we obtain $\alpha_2=\alpha_3=0$ and the only remaining interactions in Eq. (16) are two-spin Ising terms along the outer links of each plaquette. Exact diagonalisations on finite clusters have also put in evidence an $m=0$ magnetization plateau (i.e., a spin gap) for the Ising model on the square lattice. This $m=0$ plateau disappears in the isotropic Heisenberg model.[30] Of course, the arguments we have given in the Introduction for an approximation of the Heisenberg model by its Ising part are only valid at high fields. At zero field we have a complete

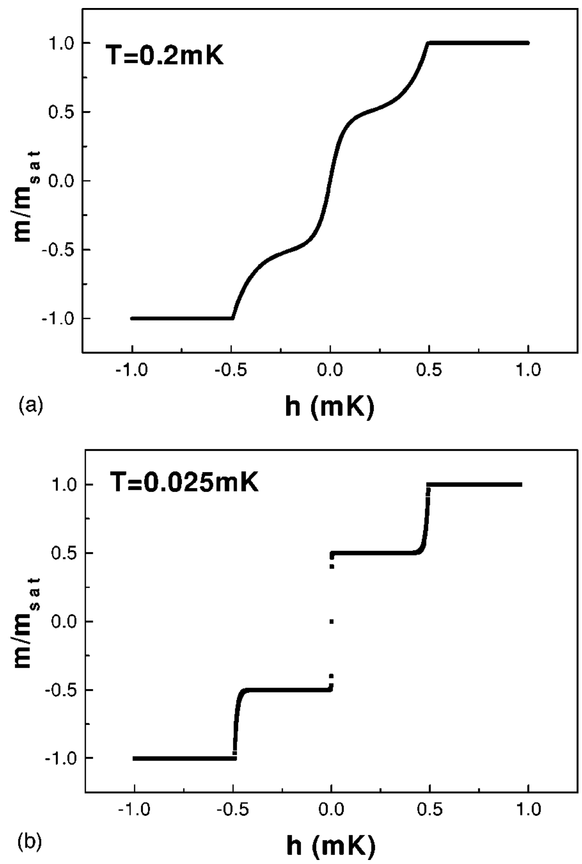

FIG. 8. Magnetization processes for case, analogous to Fig. 7, but taking into account the effective constant of four-spin exchange, $J_4=K=0.562$ mK, $J_2=3$ mK, and $J_3=3.035$ mK: (a) $T=0.2$ mK, the plot is qualitatively similar to that of Fig. 7(b), but with a narrower plateau; (b) $T=0.025$ mK, the jump of magnetization as in Fig. 7(c).

invariance of the Heisenberg Hamiltonian with respect to spin rotations and there is no realistic way to map it on a Ising model.

It is interesting to compare those results to that obtained with the same nearest-neighbor Ising model on a Bethe lattice of triangles [see Fig. 10(a)]. Applying the same method, we obtain at $T=0.2J$ the magnetization curve shown in Fig.

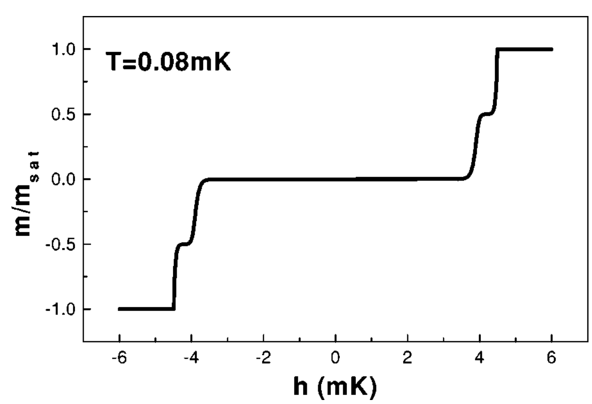

FIG. 9. Magnetization plateau at $m=0$ for $J_2=7$ mK, $J_3=5.035$ mK, and $J_4=0.562$ mK.

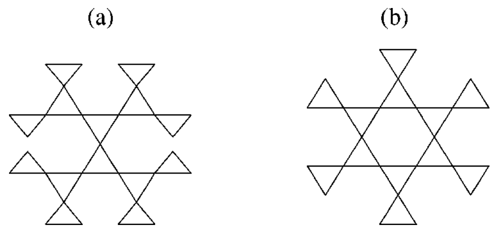

FIG. 10. A hierarchical-lattice approximation (a) to the kagomé lattice (b).

11. This curve presents a magnetization plateau at $m=1/3$ which is due to the occurrence of a colinear *uud* phase. The two-dimensional Bravais lattice having the closest topology to this hierarchical lattice is certainly the kagomé lattice [Fig. 10(b)]. A magnetization plateau at $m=1/3$ has been recently put in evidence through exact diagonalizations on finite clusters with up to $N=33$ spins in the nearest-neighbor antiferromagnetic Heisenberg model on the kagomé lattice.[31] Similar diagonalizations on a finite-size clusters have also proved the occurrence of a magnetization plateau at $m=1/3$ in both Ising and Heisenberg antiferromagnets on a triangular lattice.[29,30]

B. Four-spin interaction Hamiltonian

Our exact four-spin Ising model on the hierarchical Bethe lattice of square plaquettes (Fig. 2) has put in evidence the occurrence of a magnetization plateau at $m=1/2$. To deduce that this result has some relevance to the four-spin exchange Hamiltonian on a triangular lattice which describes the physics of solid ^{3}He, the following drastic approximations have to be made.

(i) The rotationally invariant four-spin exchange Hamiltonian has to be mapped on to its Ising counterpart. The magnetic field at which the $m=1/2$ plateau appears is relatively high. Since the corresponding magnetic state can be described as a colinear state, we expect spins to orient along the $\hat{z}$ direction of the magnetic field, reducing the transverse $(\hat{x},\hat{y})$ fluctuation. The robustness of the $m=1/3$ plateau observed in earlier exact-diagonalization studies for the Ising as well as for the Heisenberg model on the triangular lattice gives us some confidence in such an approximation.

(ii) The triangular lattice has to be mapped on the hierarchical Bethe lattice of square plaquettes (Fig. 2). Once again, the robustness of the $m=1/3$ plateau obtained in the kagomé lattice as well as in his hierarchical-lattice approximation gives us some confidence in the fact that the hierarchical-lattice approximation catch the main features of the physical behavior.

We conclude that our hierarchical-lattice approximation confirms through a completely different approach the previous results on the existence of a $m=1/2$ plateau in the presence of large four-spin exchange.

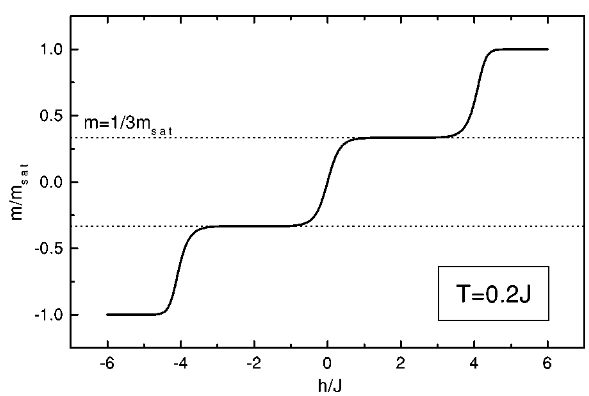

FIG. 11. Magnetization plateau at $m=1/3$ for the nearest-neighbor antiferromagnetic Ising model on the hierarchical lattice represented in Fig. 10(a) $(T=0.2J)$.

VII. CONCLUSIONS

In the present paper we have made an attempt to apply the general ideas of the Bethe approximation with the use of recurrent lattice calculations to solid ^{3}He films, described by a multiple-spin exchange model with two-, three-, and four-spin exchanges. For this purpose we have proposed a recursive lattice, which reflects the coordination properties of the underlying periodic triangular lattice in a more proper way but in contrast to the latter is inhomogeneous. In order to realize this program some assumptions were made. Instead of a model based on the Heisenberg Hamiltonian we have considered the multispin-interaction Ising model with four-site interactions.

The developed dynamical approach has allowed us to obtain plots of the magnetization per site versus external magnetic field for arbitrary finite temperatures. In spite of some roughness of our approximations, the results, at least on the qualitative level, are rather impressive. With large four-spin interactions, we have obtained magnetization curves with a plateau at $m=1/2$, which confirm earlier predictions for the two-dimensional MSE model through different approximations.[21] We gave some simple arguments concerning the structure of the ordered phases which corresponds to these magnetization plateaus. Fully analogous to the case of the MSE model studied in Ref. 21, magnetization plateaus in our case appear to be due to the stability of *uudd* and *uuud* spin configurations.

With dominant nearest-neighbor pair interactions, our hierarchical Bethe lattice of square plaquettes shows two plateaus at $m=0$ and $m=1/2$. The plateau at $m=0$, which corresponds to a spin-gapped ground state, is a feature of the Ising model and will probably disappear with a Heisenberg Hamiltonian. Our approach allowed us not only to point out the appearance of magnetization plateaus in rather simple models with an Ising interaction, but also to show that there are systems in which nearest-neighbor interactions can lead to the formation of complex ordered structures. We suppose that in our case a crucial role is played by additional inner links into the plaquettes which make the lattice inhomogeneous. Moreover, the appearance of magnetization plateaus in the Ising models is very important by itself. This circumstance could serve for the finding of other more simple explanations for the phenomenon of magnetization plateaus in lattice systems.

Surely on a quantitative level our model is still far from being in admissible correspondence with real solid ^{3}He films, especially in a weak external field, but taking into account the specificity of our approach (recursive lattice approximation, reduction to the Ising model) the qualitative picture is reflected rather well. The intermediate mixed ordered phase which was found to exist between *uudd* and *uuud* phases is also a novel phenomenon and it requires further detailed investigations.

In conclusion, we hope that the confirmation by our completely different approach of the theoretical prediction by Ref. 21 of a plateau at $m=1/2$ in the magnetization of solid ^{3}He films adsorbed on graphite will encourage further experimental investigations at high fields. We also point out the direction of further calculations which can lead to a more reliable approach to the solid ^{3}He films at finite temperatures using the general idea of a recursive lattice approximation. One can consider a recursive lattice constructed of plaquettes connected by bonds. This kind of recursive lattice can provide a stronger competition between ferromagnetic and antiferromagnetic interactions and, as a consequence, a stronger frustration. Another feature of a bond-connected lattice is the possibility of more than one recursive relations appearing. The latter case is a way to obtain complex eigenvalues of the Jacobian which indicate the appearance of commensurate and incommensurate modulated phases[28] (in particular *uudd* and *uuud* configurations). A more relevant approach to the underlying MSE Hamiltonian can be provided by the discrete counterpart of the classical Heisenberg model [$O(3)$ model]: the so-called face-cubic models,[32] in which a system of two recursion relations appears.[33]

ACKNOWLEDGMENTS

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