Spinons in more than one dimension: Resonance valence bond state stabilized by frustration

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For two spatially anisotropic, SU(2)-invariant models of frustrated magnets in arbitrary space dimension we present a nonperturbative proof of the existence of neutral spin-1/2 excitations (spinons). In one model the frustration is static and based on fine tuning of the coupling constants, whereas in the other it is dynamic and does not require adjusting of the model parameters. For both models we derive a low-energy effective action that does not contain any constraints. Though our models admit the standard gauge theory treatment, we follow an alternative approach based on Abelian and non-Abelian bosonization. We prove the existence of propagating spin-1/2 excitations (spinons) and consider in detail certain exactly solvable limits. A qualitative discussion of the most general case is also presented.

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I. INTRODUCTION

Excitations with fractional quantum numbers in general and neutral excitations with spin 1/2 in particular constitute an essential ingredient of many theoretical approaches aimed to explain the non-Fermi-liquid behavior of quasi-two-dimensional copper oxide materials. These approaches try to generalize for higher dimensions mechanisms such as spin-charge separation and quantum number fractionalization, phenomena well understood and adequately described for one-dimensional systems (see recent review articles,[1–3] and references therein).

As a part of the general program to construct a theory of copper oxides, there have been many attempts to find neutral spin-1/2 excitations (spinons) in purely magnetic systems, that is, to find higher-dimensional analogs of the one-dimensional (1D) spin $S=1/2$ Heisenberg antiferromagnet. Such a generalization was qualitatively outlined by Anderson[4] in the form of the famous resonant valence bond (RVB) state. This is a *spin-liquid* state that breaks neither translational nor spin rotational symmetry. Building on this proposal, Kivelson, Rokhsar, and Sethna showed[5] that such a state, if it exists, must support neutral spin-1/2 excitations.

Despite more than a decade of strong efforts, no realization of a RVB state supporting excitations with fractional spin in $D>1$ Heisenberg magnets with short-range exchange interactions has been found in models with interactions of the *Heisenberg* type. We exclude from the consideration models of *quantum dimers*, where such excitations have been shown to exist,[6] and other recently suggested models with fractionalized excitations that are not microscopic electronic models (see, for example Refs. 7 and 8), as well as generalizations for symmetries higher than SU(2). We also do not consider magnets with *incommensurate* ordering (such as a spiral one) that also support spin-1/2 excitations.[9,10] Strong qualitative arguments against the RVB scenario have been presented by Read and Sachdev,[11] who argued that the most likely mechanism for disordering an antiferromagnetic state is the *spin-Peierls* one. Here, although spin rotational symmetry is preserved, translational symmetry is broken since the spin system undergoes an explicit dimerization *at finite temperature*. That makes the existence of deconfined spinons *in the low-T phase* impossible.

An interesting byproduct of the RVB scenario is the strong coupling compact quantum electrodynamics (cQED) field theory. This gauge theory describes the so-called π-flux RVB state in the continuum limit[12] (see also Refs. 13 and 14), under the assumption that the system does not develop any spin-Peierls ordering. The gauge theory has infinite bare coupling and therefore is difficult to study. All existing analysis relies on the results obtained for the SU(N) or Sp($2N$) generalizations of this theory based on the $1/N$ expansion, as in Refs. 11, 12, 15, and 16.

In this paper we describe two D-dimensional ($D>1$) models that exhibit neutral spin-1/2 excitations (spinons) and apparently represent realizations of the π-flux RVB state. The paper is organized as follows. In Sec. II we introduce the models. They are spatially anisotropic and actually represent collections of weakly coupled chains. In all of these models, the naively strongest interactions between the chains, namely those that couple the antiferromagnetic (with momentum close to π) fluctuations on neighboring chains, are frustrated and so can be set to zero in the Hamiltonian. In this case, the physics is governed by a *marginally* relevant interaction between the long-wavelength magnetic fluctuations. In Sec. III we discuss results we managed to obtain for infinite number of chains. These can be summarized as follows:

(1) we give a proof of the existence of spin-1/2 excitations in the limit of infinite number of chains;

(2) we demonstrate that in the low-energy limit the effective spin Hamiltonian decouples into two commuting parts describing sectors with different parity;

(3) we show that the models are stable against an antiferromagnetic phase transition;

(4) for the models in question we demonstrate existence of a zero-temperature phase transition (spontaneous transverse dimerization).

(5) we show that the ground state is degenerate; for periodic transverse boundary conditions the degeneracy is equal to 2^{2N-1}, where $2N$ is the number of chains.

To achieve a more detailed understanding of the model, its spectrum, and correlation functions, we study some limiting cases. Thus in Sec. IV we discuss two exactly solvable cases: two and four chains with periodic boundary conditions in the transverse direction. In Sec. V we consider another solvable case: the model of three chains with open boundary conditions. In all these cases magnetic excitations carry spin 1/2, in agreement with the general result. Systems with more than two chains also display nonmagnetic (singlet) modes and a topological order. We show that this order is similar to the one that exists in the 1D quantum Ising model.

Furthermore, in Sec. V we introduce a two-dimensional model with a hierarchy of interactions. We use this apparently artificial but solvable model as a substitute for the original (spatially uniform) model with a hope to extract more detailed information about the structure of the Hilbert space. The hierarchical model is solved by a cluster expansion. The results indicate that about half of the Hilbert space is occupied by singlet modes. The ground state remains massively degenerate with the ground state entropy proportional to the number of chains.

Having in view future numerical simulations for our models, we discuss in Sec. VI finite-size effects for systems of a finite number of chains. In Sec. VII we use the acquired knowledge to conjecture the properties of correlation functions in the thermodynamic limit. Our conclusions are summarized in Sec. VIII.

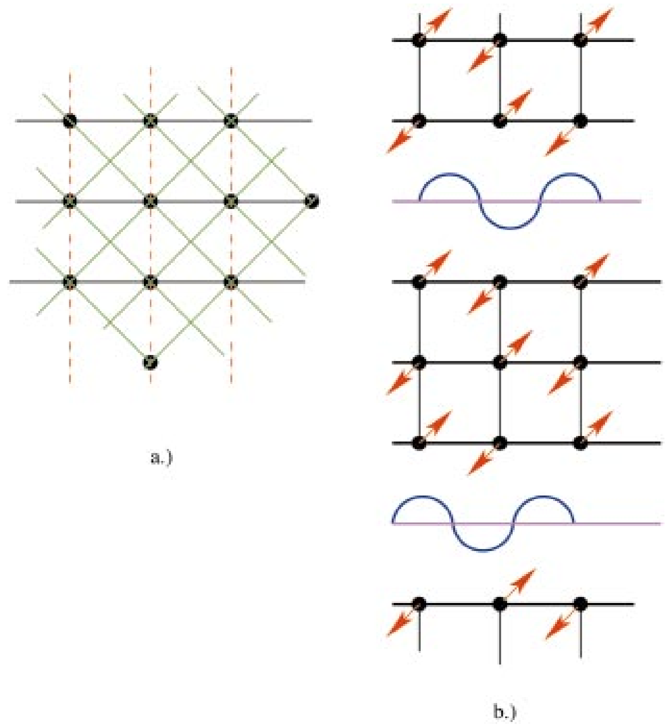

FIG. 1. (Color) (a) Exchange interactions pattern for the model; the red lines correspond to J_r and the green ones to J_d. (b) Lattice of stripes for the Kondo-Heisenberg model.

II. THE MODELS

As in the original suggestion by Anderson,[4] the element most essential for our construction is frustration. Another ingredient of our approach is strong spatial anisotropy: so far we have only been able to deal with models where the exchange interaction in one direction is much greater than in the others. Thus it is proper to describe our models as assemblies of weakly coupled chains. Our results also suggest an alternative approach to strong coupling cQED. We will not elaborate on this analogy, though, postponing this interesting question for future studies.

The first model is a spin-1/2 Heisenberg magnet on an anisotropic lattice. In what follows, we will be working with a two-dimensional version of this model. The interaction pattern is depicted in Fig. 1(a); one can easily generalize this construction to three dimensions. The model Hamiltonian is given by

$$H_{ACB}=\sum_{j,n}\left\{J_{\|}\mathbf{S}_{j,n}\cdot\mathbf{S}_{j+1,n}+\sum_{\mu=\pm1}[J_r\mathbf{S}_{j,n}+J_d(\mathbf{S}_{j+1,n}+\mathbf{S}_{j-1,n})]\cdot\mathbf{S}_{j,n+\mu}\right\}, \qquad (1)$$

where $\mathbf{S}_{j,n}$ are spin-1/2 operators, and $J_{\|},J_r,J_d>0$. Since the interchain couplings (J_r,J_d) are much smaller than the exchange along the chains $(J_{\|})$ it is legitimate to adopt a continuum description of individual chains. In this description, the local spin densities are represented as sums of the smooth and staggered parts:

$$\mathbf{S}_{j,n}/a_0\rightarrow\mathbf{S}_n(x)=\mathbf{M}_n(x)+(-1)^j\mathbf{N}_n(x), \quad x=ja_0 \qquad (2)$$

a_0 being the lattice spacing in the chain direction. The low-energy dynamics of the spin-1/2 Heisenberg antiferromagnet

$$H_{1D}=J_{\|}\sum_j(\mathbf{S}_j\mathbf{S}_{j+1}) \qquad (3)$$

is described by the $SU_1(2)$ Wess-Zumino-Novikov-Witten model. The latter Hamiltonian can be written in terms of the so-called chiral vector *current* operators, $\mathbf{J}$ and $\overline{\mathbf{J}}$, satisfying the level $k=1$ Kac-Moody algebra (this approach has been described in a vast number of publications; see for a review Ref. 17 or 18):

$$H_{1D}\rightarrow\frac{2\pi v}{3}\int dx[:(\mathbf{J}\cdot\mathbf{J}):+:(\overline{\mathbf{J}}\cdot\overline{\mathbf{J}}):]+\cdots, \qquad (4)$$

where $v=\pi J_{\|}a_0/2$ is the spin velocity and the dots stand for a marginally irrelevant perturbation that will be discarded in what follows. It is remarkable that the smooth part of magnetization

$$\mathbf{M}=\mathbf{J}+\overline{\mathbf{J}} \qquad (5)$$

and the spin current

$$\mathbf{j}=v(\mathbf{J}-\overline{\mathbf{J}}) \qquad (6)$$

are locally expressed in terms of the chiral currents.

In the model, the exchange is frustrated in the direction perpendicular to the chains and can be fine tuned to make its Fourier transform vanish at $q_{\|}=\pi$. This is achieved when the rung and plaquette-diagonal coupling constants satisfy the relation $J_r=2J_d$, in which case the direct interchain interaction between staggered magnetizations, $\mathbf{N}_n(x)\cdot\mathbf{N}_{n+1}(x)$, is completely eliminated. The absence of this strongly relevant perturbation is the most important property of our model. The triangular lattice is excluded from the consideration. It was demonstrated in Ref. 19 that the coupling of the staggered parts of magnetization cannot be removed completely in that case: parity-breaking (''twist'') terms surviving the continuum limit make the analysis very complicated.

Apart from the collective spin excitations, the second model also involves charge degrees of freedom. It is a quasi-one-dimensional Kondo-Heisenberg model [see Fig. 1(b)], which has been already discussed in the context of theory of stripes[20,21] (see also Ref. 22, where the two chain case is discussed). It is assumed that neighboring chains have different band filling such that metallic chains are surrounded by insulating spin-1/2 magnetic chains. In order to eliminate the electron tunneling across the magnetic chain, one needs to have magnetic stripes consisting of at least three chains. At low energies one can consider such a stripe as an effective single spin-1/2 chain. Here the frustration is dynamical: the electrons on metallic chains do not experience backscattering on magnetic excitations due to incommensurability of the Fermi wave vectors: $2k_F\neq\pi$. It is easy to demonstrate (see, for instance, Ref. 20) that the charge excitations in this model decouple and remain one-dimensional (this is true at least in the first approximation when one discards various virtual processes). On the other hand, the spin sector is described by the same effective Hamiltonian as in the model.

Using the continuum description of individual chains in the interchain exchange interaction, based on the asymptotic representation (2) of the spin operators, we arrive at the following effective Hamiltonian:

$$H=\sum_{n=1}^{2N}\left[H_{1\mathrm{D},n}+\frac{\gamma}{2}\sum_{\mu=\pm1}(\mathbf{J}+\bar{\mathbf{J}})_n\cdot(\mathbf{J}+\bar{\mathbf{J}})_{n+\mu}\right], \quad (7)$$

where $H_{1\mathrm{D}}$ is given by Eq. (4). Here γ is determined by the lattice Hamiltonian; in the ACB model with interactions chosen as in Fig. 1(a) we have $\gamma=2J_r a_0$. For the Kondo-Heisenberg model shown in Fig. 1(b), Eq. (4) is modified in such a way that even (metallic) and odd chains (effective spin-1/2 chains representing a stripe of three coupled Heisenberg chains) have different velocities. This detail, however, does not lead to any qualitative changes.

The model (7) is closely related to the gauge theories extensively studied in the context of strongly correlated systems. This analogy is briefly discussed in Appendix A. In the rest of this paper, however, we will follow a different route. Namely, we will return to Eqs. (4) and (7) and employ the Abelian bosonization, which proves particularly useful in revealing the topological nature of elementary excitations and determining their quantum numbers.

III. PROOF OF THE EXISTENCE OF THE SPIN-1/2 EXCITATIONS: INFINITE NUMBER OF CHAINS

It is well known (see, e.g., Ref. 18) that the chiral $SU_1(2)$ currents can be faithfully represented in terms of holomorphic (left) and antiholomorphic (right) scalar fields, φ and $\bar{\varphi}$:

$$J^3_n=\frac{i}{\sqrt{2\pi}}\partial\varphi_n,\quad J^{\pm}_n=\frac{1}{2\pi\alpha}e^{\pm i\sqrt{8\pi}\varphi_n}, \quad (8)$$

$$\bar{J}^3_n=-\frac{i}{\sqrt{2\pi}}\bar{\partial}\bar{\varphi}_n,\quad \bar{J}^{\pm}_n=\frac{1}{2\pi\alpha}e^{\mp i\sqrt{8\pi}\bar{\varphi}_n}. \quad (9)$$

Here $\partial,\bar{\partial}=\frac{1}{2}(v^{-1}\partial_\tau\mp i\partial_x)$, α is the ultraviolet cutoff of the bosonic theory, and the fields φ and $\bar{\varphi}$ are governed by the *chiral* Gaussian actions[23,24] (the latter work used this form of bosonization in the context of the theory of *edge states* in quantum Hall effect). The bosonized low-energy effective action can then be suitably represented as

$$S=\sum_{n=1}^{N}\int d\tau\,dx[\mathcal{L}^+_n+\mathcal{L}^-_n+\mathcal{L}^{\mathrm{int}}_n], \quad (10)$$

$$\begin{aligned}\mathcal{L}^+_n={}&\partial_x\varphi_{2n}(\partial_x\varphi_{2n}-iv^{-1}\partial_\tau\varphi_{2n})+\partial_x\bar{\varphi}_{2n+1}(\partial_x\bar{\varphi}_{2n+1}\\&+iv^{-1}\partial_\tau\bar{\varphi}_{2n+1})+\gamma\sum_{\mu=\pm1}\Big\{(2\pi\alpha)^{-2}\cos[\sqrt{8\pi}(\varphi_{2n}\\&+\bar{\varphi}_{2n+\mu})]+\frac{1}{2\pi}\partial_x\varphi_{2n}\partial_x\bar{\varphi}_{2n+\mu}\Big\},\end{aligned} \quad (11)$$

$$\mathcal{L}^-_n=\mathcal{L}^+_n(\varphi\to\bar{\varphi}), \quad (12)$$

$$\begin{aligned}\mathcal{L}^{\mathrm{int}}_n={}&\gamma\sum_{\mu=\pm1}\Big\{(2\pi\alpha)^{-2}\cos[\sqrt{8\pi}(\varphi_{2n}-\varphi_{2n+\mu})]\\&+\frac{1}{2\pi}\partial_x\varphi_{2n}\partial_x\varphi_{2n+\mu}\Big\}+(\varphi\to\bar{\varphi}).\end{aligned} \quad (13)$$

This is the form of the action in which the requirement of vanishing charge current ($J^\mu_c=0$), imposed in the gauge field RVB approach, is explicitly resolved.

According to the definitions (8) and (9), the total spin projection S^z is equal to

$$S^z=\frac{1}{\sqrt{2\pi}}\sum_n\int_{-\infty}^{\infty}dx(\partial_x\varphi_n+\partial_x\bar{\varphi}_n). \quad (14)$$

This is a general definition that does not assume that the fields φ and $\bar{\varphi}$ are independent. In fact, the interchain interaction that flows to strong coupling in the low-energy limit freezes certain combinations of the fields with opposite chiralities and thus makes them coupled.

In the first loops the renormalization group equations for the model of infinite number of chains coincide with the equations for two chains. At $\gamma>0$ the interaction of currents with different chirality in Eqs. (11) and (12) is marginally relevant and reaches the strong coupling at an energy scale

$$\Delta=C(v\gamma)^{1/2}\exp(-\pi v/\gamma), \quad (15)$$

where C is a number. At the same time, the (Lorentz-noninvariant) perturbation $\mathcal{L}^{\mathrm{int}}$, which is responsible for the velocity renormalization, is irrelevant and flows only in the presence of the other interaction. Thus it is likely that this interaction remains weak even when the relevant one reaches the strong coupling. For this reason, in the situation when the bare constant $\gamma \ll 1$, we deem it possible to neglect $\mathcal{L}^{\rm int}$. This results in an approximate Lorentz invariance in (1+1)-dimensional space-time, which plays an important role in the analysis that follows. Without $\mathcal{L}^{\rm int}$, the action splits into two independent, chirally asymmetric sectors, $S=S^{+}+S^{-}$. The two sectors are mapped onto each other under the reversal of the chiralities, $\varphi_n \leftrightarrow \bar{\varphi}_n$, or a shift by one lattice spacing in the transverse direction. Hence the total symmetry becomes $[\mathrm{SU}(2)]_{+} \otimes [\mathrm{SU}(2)]_{-}$ and the excitations in this model carry two quantum numbers: spin and parity.

Notice that the multichain effective action (10) features a local Z_2 symmetry already mentioned for the two-chain problem:[25] as opposed to the original lattice model that displays global translational invariance, the effective Hamiltonian of the continuum theory (7) is fully expressed in terms of the vector currents and, therefore, is invariant under *independent* translations along the chains by one lattice spacing. These translations are generated by the shifts

$$\varphi_n \to \varphi_n \pm \sqrt{\frac{\pi}{2}}, \quad \bar{\varphi}_n \to \bar{\varphi}_n \tag{16}$$

or

$$\varphi_n \to \varphi_n, \quad \bar{\varphi}_n \to \bar{\varphi}_n \pm \sqrt{\frac{\pi}{2}}. \tag{17}$$

The development of the strong-coupling regime in the low-energy limit leads to a spontaneous breakdown of this symmetry. Indeed, in the ground state the field combinations $\varphi_n + \bar{\varphi}_{n+\mu}$ get frozen at the values (see Appendix B for the details)

$$\varphi_n + \bar{\varphi}_{n+\mu} = \sqrt{\frac{\pi}{8}} + \sqrt{\frac{\pi}{2}} m_n^{(\mu)}, \quad m_n^{(\mu)} = 0, \pm 1, \pm 2, \ldots . \tag{18}$$

For an even number of chains and with periodic boundary conditions in the transverse direction the integers $m_n^{(\mu)}$ satisfy

$$\sum_n m_n^{(+)} = \sum_n m_n^{(-)}. \tag{19}$$

The sets of integers $\{m_n^{(\mu)}\}$ subject to the condition (19) label different degenerate ground states, each of them breaking the local translational symmetry (16) and (17). The existence of stable degenerate minima of the potential implies the existence of massive topological excitations, solitons, interpolating between adjacent vacua. Using Eqs. (14) and (18) one can represent the spin of a given state as

$$S^z = \frac{1}{2} \sum_n \left[m_n^{(+)}(\infty) - m_n^{(+)}(-\infty) \right]. \tag{20}$$

Thus, the minimal nonzero value of the spin is 1/2. This is the spin carried by the topological solitons.

We emphasize that this statement is based on the exact symmetries of the multichain Hamiltonian (10) and does not depend on the adopted approximations. Within the approximation that neglects $\mathcal{L}^{\rm int}$ there are solitons and antisolitons with a well-defined (+) and (–) parity.

We would like to briefly comment on the above results. The factor $\sqrt{8\pi}$ that appears in the arguments of the cosines in Eqs. (11) and (12), representing a marginal current-current interchain interaction, is crucial. Recalling the bosonization rules for the staggered magnetization of a single spin-1/2 chain (see, e.g., Ref. 18)

$$\mathbf{N}_n \sim (\cos \sqrt{2\pi}(\varphi_n - \bar{\varphi}_n), \sin \sqrt{2\pi}(\varphi_n - \bar{\varphi}_n), -\sin \sqrt{2\pi}(\varphi_n + \bar{\varphi}_n)), \tag{21}$$

we realize that interchain coupling between staggered magnetizations would give rise to a strongly relevant perturbation containing cosines with $\sqrt{2\pi}$. If such terms were present in the action, the period of the potential and hence the topological quantum numbers of the excitations would be doubled as compared to our case and thus the spinons would be confined. However, by construction these terms do not appear in our models, and this is why the interchain interactions do not lead to spinon confinement. This argument does not require neglecting $L_{\rm int}$ because the latter term also contains only $\sqrt{8\pi}$ cosines.

Thus we are confident that in the models in question spin-1/2 excitations do exist. We also may rest assured that these excitations do not remain confined to individual chains, as it happens, for instance in the so-called ''sliding'' Luttinger liquid phase in arrays of crossed chains.[26,27] In that latter case, the spinons survive due to irrelevance of the interchain interactions. Contrary to that scenario, in our case the interchain coulings remain relevant making the system truly multidimensional.

A. A failure of the ''most obvious'' solution

The following calculation suggests that freezing of the fields does not imply that one can expand around the vacuum configurations. Naively, one would adopt a self-consistent harmonic approximation (SCHA) in which the cosine term is expanded around the minimum:

$$\cos[\sqrt{8\pi}(\varphi_n + \bar{\varphi}_{n+\mu})] \approx \text{const} + \frac{1}{2}\rho[\delta(\varphi_n + \bar{\varphi}_{n+\mu})]^2. \tag{22}$$

Here δ denotes a deviation from the vacuum value. SCHA assumes that the transverse stiffness ρ should be determined self-consistently:

$$\rho=\frac{2\gamma}{\pi\alpha^2}\langle\exp[i\sqrt{8\pi}(\varphi_n+\bar{\varphi}_{n+\mu})]\rangle. \tag{23}$$

We shall demonstrate, however, that such a procedure yields zero ρ.

The effective action for fluctuations around the minima is (here we set $v=1$)

$$S=\sum_p(\alpha_p^*,\beta_p^*)\begin{pmatrix} q(q+i\omega)+\rho & \rho\cos k \\ \rho\cos k & q(q-i\omega)+\rho\end{pmatrix}\begin{pmatrix}\alpha_p\\ \beta_p\end{pmatrix}, \tag{24}$$

where $\alpha=\delta\varphi$, $\beta=\delta\bar{\varphi}$, $p=(\omega,q,k)$, q and k being the longitudinal and transverse momenta, respectively. The self-consistency condition (23) becomes

$$\begin{aligned}\ln\rho &\sim -[G_{\alpha\alpha}+G_{\beta\beta}+2G_{\alpha\beta}](\tau=0,x=0)\\ &\sim\int d\omega\, dq\, dk\frac{q^2+\rho(1-\cos k)}{\omega^2q^2+(q^2+\rho)^2-\rho^2\cos^2k}\sim\\ &-\int\frac{dk\,dq}{q}\frac{q^2+\rho(1-\cos k)}{\sqrt{(q^2+\rho)^2-\rho^2\cos^2k}}.\end{aligned} \tag{25}$$

Since the latter integral diverges, the suggested naive solution is not self-consistent. Its failure originates from the chiral structure of the Gaussian part of action (11), which includes the first power of the time gradient. For the conventional (nonchiral) Gaussian model a similar solution would be self-consistent. One immediate consequence would be the appearance of an average staggered magnetization. Thus the present calculation demonstrates stability against antifferomagnetism.

B. Order parameters

As we have already seen, the fact that the combinations of the fields $\varphi_{2n}+\bar{\varphi}_{2n+\mu}$ and $\bar{\varphi}_{2n}+\varphi_{2n+\mu}$ get locked at $\sqrt{\pi/8}+\sqrt{\pi/2}n$ ($n\in Z$) implies that the ground state of the system is characterized by a spontaneously broken local translational invariance (16) and (17). The *most obvious* order parameters (OP) in the two sectors of the model (10), whose structure directly follows from the form of the interaction, are given by

$$\begin{aligned}\mathcal{T}_\mu^{(+)}&=i\langle\exp[i\sqrt{2\pi}(\varphi_{2n}+\bar{\varphi}_{2n+\mu})]\rangle,\\ \mathcal{T}_\mu^{(-)}&=i\langle\exp[i\sqrt{2\pi}(\bar{\varphi}_{2n}+\varphi_{2n+\mu})]\rangle.\end{aligned} \tag{26}$$

The above OP's are real [see Eq. (18)], defined on the links of the lattice and are nonlocal in terms of original spins. Moreover, since the OP's include chiral fields, they are expressed (nonlocally!) in terms of magnetization density and *spin currents*. As is shown in Appendix B, for different pairs of neighboring chains, the signs of the order parameters are uncorrelated. This leads to a massive gound state degeneracy 2^{2N-1} (recall that $2N$ is the total number of chains).

The nonlocality of the order parameter suggests a possibility of *topological* order. This supports the original idea by Wiegmann that topological order exists in RVB states.[29,30] Characteristic features of such order is nonlocality of the order parameter, ground state degeneracy (in a closed system), and the existence of edge states (in an open system). The ground state degeneracy depends on the topology of the manifold (hence the label "topological"). The relative stability of the state with respect to perturbations is protected by the spectral gap. One well-known example of topologically ordered states is provided by incompressible fractional quantum Hall states (see the discussion in Ref. 31 and references therein). In our case the degeneracy of the ground state is exponential in the number of chains. This fact alone indicates that these models have a topological order of a different type. This order is closely related to that existing in 2D Ising model, multichannel Kondo models and models with scattering matrices of the RSOS type.

The first examples of nonlocal order parameters are given by the disorder field of the 2D Ising model[32] and the dual field exponentials in the 2D XY model.[33,28] These fields acquire nonzero expectation values in disordered phases of the corresponding models. According to Refs. 34–37 and 18, such order parameters (also called "string" order parameters) exist in various massive phases of 1D spin liquids. All these liquids are related to either spin $S=1$ Heisenberg chain or two-leg spin-1/2 Heisenberg ladders. As was demonstrated in Refs. 37–39 (see also Ref. 18), after the Wick rotation these systems can be described as a collection of 2D Ising models. The evidence that the order is really topological was presented by Affleck and co-workers,[40] who proved that 1D spin liquids have edge, or rather, end states: open chains have free spin-1/2 states located at the chain's ends. We postpone a more detailed discussion of the issue of topological nature of the order until sections.

Order parameters (26) change their sign on soliton configurations and therefore vanish at finite temperatures when the densities of solitons and antisolitons are finite and *equal*. This mechanism is sufficient (though not necessary) to destroy the order parameter at finite T. The sign changes may also originate from fluctuations in nonmagnetic channels. Later we will discuss an exactly solvable example of four coupled chains in which the latter scenario is realized. The temperature dependence of the correlation length depends on what mechanism is realized. If the order parameter is destroyed only by (anti)solitons, the correlation length is exponentially large in $1/T$. This is because the topological excitations have a finite spectral gap and therefore their density is exponentially small: $n\sim\exp(-\Delta/T)$. In this case, the correlation length is defined as the average spacing between the solitons: $\xi\sim n^{-1/D}$ (D is the space dimensionality).

As we shall see, the singlet excitations either have a very small gap or even no gap at all (we do not know whether the latter is true). Then the order parameter is governed by another correlation length quite distinct from the above. The density of nonmagnetic excitations above their gap depends on T as a power law. Such a situation would correspond to a quantum critical point with a power law correlation length.

Turning to physical fields, we note that the products of the staggered magnetizations on neighboring chains representing the transverse dimerization order parameters (see Appendix B) are expressed as bilinear combinations of the true order parameters:

$$\langle N^z_{2n}N^z_{2n+\mu}\rangle = \frac{1}{8\pi a_0}\langle\cos\{\sqrt{2\pi}(\varphi_{2n}+\bar{\varphi}_{2n+\mu}+\bar{\varphi}_{2n}+\varphi_{2n+\mu})\}\rangle = \frac{1}{4}[\mathcal{T}^{(+)}_\mu\mathcal{T}^{(-)}_\mu+\text{c.c.}], \quad (27)$$

$$\langle N^x_{2n}N^x_{2n+\mu}\rangle = \langle N^y_{2n}N^y_{2n+\mu}\rangle = \frac{1}{8\pi a_0}\langle\exp[i\sqrt{2\pi}(\varphi_{2n}+\bar{\varphi}_{2n+\mu})]\exp[-i\sqrt{2\pi}(\bar{\varphi}_{2n}+\varphi_{2n+\mu})]\rangle = \frac{1}{4}\{\mathcal{T}^{(+)}_\mu[\mathcal{T}^{(-)}_\mu]^*+\text{c.c.}\}. \quad (28)$$

Since $\mathcal{T}^*=\mathcal{T}$ we have

$$\langle N^a_n N^b_{n+\mu}\rangle \sim \delta^{ab},$$

implying that the SU(2) symmetry remains unbroken in the ground state.

Now let us consider the magnetic correlation functions. The order parameter field of 1D spin-1/2 chain is not just a staggered magnetization; it is a 2×2 SU(2) matrix $\hat{g}$ whose entries include not only vector components of the magnetization n^a $(a=x,y,z)$, but also the dimerization field $\epsilon=(-1)^n(\mathbf{S}_n\mathbf{S}_{n+1})$:

$$\hat{g}(x)=\epsilon(x)I+i\sigma^a N^a(x). \quad (29)$$

It can also be represented in a factorized bosonic form:

$$\hat{g}_{\sigma\sigma'}=\frac{1}{\sqrt{2}}C_{\sigma\sigma'}:\exp[-i\sqrt{2\pi}(\sigma\varphi+\sigma'\bar{\varphi})]:\equiv C_{\sigma\sigma'}z_\sigma\bar{z}_{\sigma'},$$

$$C_{\sigma\sigma'}=e^{i\pi(1-\sigma\sigma')/4}, \quad (30)$$

where

$$z_\sigma=\exp[i\sigma\sqrt{2\pi}\varphi], \quad \bar{z}_\sigma=\exp[-i\sigma\sqrt{2\pi}\bar{\varphi}] \quad (\sigma=\pm1). \quad (31)$$

As follows from the factorization of the low-energy effective action, if we consider correlation functions on chains with the same parity (for instance, both even), z correlates only with z's and $\bar{z}$ with $\bar{z}$. For Green's functions on chains with different parity, z correlates with $\bar{z}$ and not with z. Therefore the correlation functions of the staggered magnetizations and energy densities factorize:

$$\langle g_{ab}(\tau,x;2n)g^+_{cd}(0,0;2m)\rangle = \langle z_a(\tau,x;2n)z^*_d(0,0;2m)\rangle \times\langle\bar{z}_b(\tau,x;2n)\bar{z}^*_c(0,0;2m)\rangle \equiv \delta_{ad}\delta_{bc}G_{nm}(\tau,x)G_{nm}(\tau,-x), \quad (32)$$

$$\langle g_{ab}(\tau,x;2n)g^+_{cd}(0,0;2m+1)\rangle = \langle z_a(\tau,x;2n)\bar{z}^*_c(0,0;2m)\rangle\langle\bar{z}_b(\tau,x;2n)z^*_d(0,0;2m+1)\rangle \equiv \delta_{ac}\delta_{bd}D_{nm}(\tau,x)D_{nm}(-\tau,x). \quad (33)$$

Various components of the z's are conveniently assembled into vectors and covectors:

$$(z^*_{\sigma,2n},\bar{z}^*_{\sigma,2n+1}), \quad \begin{pmatrix} z_{\sigma,2n} \\ \bar{z}_{\sigma,2n+1}\end{pmatrix} \quad (34)$$

for the (+) sector and

$$(z^*_{\sigma,2n+1},\bar{z}^*_{\sigma,2n}), \quad \begin{pmatrix} z_{\sigma,2n+1} \\ \bar{z}_{\sigma,2n}\end{pmatrix} \quad (35)$$

for the (–) sector. Then the corresponding correlation functions become matrix elements of the 2×2 Green's function:

$$\hat{G}=\begin{pmatrix} G(\omega,k^z;k_\perp) & D(\omega,k^z;k_\perp) \\ D(-\omega,k^z;k_\perp) & G(\omega,-k^z,k_\perp)\end{pmatrix}. \quad (36)$$

An important fact is that both the function D and $k_\perp$ dependence of G are nonperturbative effects: they do not appear in any order of perturbation theory in γ and, as we shall see later, are proportional to $\exp(-\pi v/\gamma)$.

IV. EXACTLY SOLVABLE CASES: TWO AND FOUR CHAINS

From now on we will discuss the (+) and (–) sectors independently. To distinguish between the amplitude of the RVB order parameter and various mass gaps, we shall denote spectral gaps by the letter m.

To check the ideas outlined in the preceding section, we shall discuss some exactly solvable cases. Consider a two-dimensional array of an even number $(2N)$ of chains with periodic boundary conditions. In this case we can arrange chains in pairs. Exact solution is possible in two cases: $N=1$ and $N=2$.

Throughout this section we shall use the so-called form-factor approach to calculate leading asymptotics of the correlation functions. This method has been widely applied to massive integrable field theories and there are excellent review articles[42] and even books[43] about it. The form-factor approach uses information extracted from the scattering theory and symmetry properties of physical operators to calculate matrix elements of these operators between various exact eigenstates. To estimate the correlation functions one uses the Lehmann expansion. This approach is well defined for theories with massive excitations. In one dimension, for reasons of kinematics, matrix elements of multiparticle states become smaller and smaller when the number of excited particles increases. This feature greatly improves the convergence of the Lehmann expansion and allows one to obtain an accurate description of the correlation functions with only a few (sometimes even one) matrix elements involved.

A. $N=1$ (two chains)

Though the number of chains is really small, this case contains certain interesting features that may shed light on more general cases. For $N=1$ each sector with a given parity is equivalent to the isospin sector of the SU(2) invariant Thirring model, as it was shown in Ref. 25. The excitations are massive solitons with spin 1/2. There are no other modes (singlet modes), as it must be for the case of large N. For this reason this case is not representative, as we have already said. The massive modes correspond to amplitude fluctuations of Δ.

The correlation functions of chiral bosonic exponents for the Thirring model can be extracted from the recent paper by Lukyanov and Zamolodchikov[44] (see also Ref. 41). In particular, for the correlation functions of spinon fields (36) and for the correlators of staggered magnetizations we have

$$G(\tau,x)=\langle e^{i\sqrt{2\pi}\varphi(\tau,x)}e^{-i\sqrt{2\pi}\varphi(0,0)}\rangle\sim\frac{1}{\sqrt{iv\tau-x}}\exp(-m\rho_{12}),$$

$$\langle e^{i\sqrt{2\pi}\Phi_1(\tau,x)}e^{-i\sqrt{2\pi}\Phi_1(0,0)}\rangle\sim\frac{1}{\rho_{12}}\exp(-2m\rho_{12}),$$

$$D(\tau,x)=\langle e^{i\sqrt{2\pi}\varphi(\tau,x)}e^{i\sqrt{2\pi}\bar{\varphi}(0,0)}\rangle\sim m^{1/2}K_0(m\rho_{12}), \tag{37}$$

$$\langle e^{i\sqrt{2\pi}\Phi_1(\tau,x)}e^{i\sqrt{2\pi}\Phi_2(0,0)}\rangle=Z_0mK_0^2(m\rho_{12}),$$

where $\rho=\sqrt{\tau^2+(x/v)^2}$ and the mass gap $m=\Delta$ given by Eq. (15). In writing down these formulas we used the single-mode approximation for the spinon operators (for the details we again refer the reader to Ref. 44):

$$z_\sigma^+(\tau,x)=Z_0^{1/2}m^{1/4}\int d\theta\, e^{\theta/4}e^{-ixm\sinh\theta}[e^{-m\tau\cosh\theta}\hat{Z}_\sigma^+(\theta)+e^{m\tau\cosh\theta}\hat{Z}_{-\sigma}(\theta)], \tag{38}$$

$$\overline{z_\sigma}(\tau,x)=Z_0^{1/2}m^{1/4}\int d\theta\, e^{-\theta/4}e^{-ixm\sinh\theta}[e^{-m\tau\cosh\theta}\hat{Z}_\sigma^+(\theta)+e^{m\tau\cosh\theta}\hat{Z}_{-\sigma}(\theta)], \tag{39}$$

where $Z_\pm$ ($Z_\pm^+$) are annihilation (creation) operators of solitons and antisolitons and Z_0 is a numerical constant. Approximation (39) fails at small distances; inclusion of terms with multiple soliton production removes the logarithmic singularity of K_0.

The two-point correlation function of currents were calculated in Ref. 45. Equation (37) supports the earlier claim made at the end of the preceding section, that the interchain correlation function of the staggered magnetization is nonanalytic in γ and thus vanishes in all orders of perturbation theory.

B. $N=2$ (four chains)

In the $N=2$ case the interaction can be written as

$$(\mathbf{J}_1+\mathbf{J}_3)(\overline{\mathbf{J}}_2+\overline{\mathbf{J}}_4)+(\mathbf{J}_2+\mathbf{J}_4)(\overline{\mathbf{J}}_1+\overline{\mathbf{J}}_3). \tag{40}$$

The sum of two $k=1$ SU(2) currents is the $k=2$ current; moreover, according to Ref. 46 the sum of two $SU_1(2)$ WZNW models (the central charge 2) can be represented as the $SU_2(2)$ WZNW model with central charge 3/2 and plus one massless Majorana fermion (a critical Ising model) with central charge 1/2. Using the results of Ref. 46 we rewrite the entire Hamiltonian density (4) as follows [here only the (+)-parity part is written]:

$$\mathcal{H}_+=\mathcal{H}_{\text{massless}}+\mathcal{H}_{\text{massive}},$$

$$\mathcal{H}_{\text{massless}}=-\frac{iv}{2}\chi_0\partial_x\chi_0+\frac{iv}{2}\bar{\chi}_0\partial_x\bar{\chi}_0, \tag{41}$$

$$\mathcal{H}_{\text{massive}}=\frac{\pi v}{2}(:\mathbf{I}\cdot\mathbf{I}:+:\overline{\mathbf{I}}\cdot\overline{\mathbf{I}}:)+\gamma\mathbf{I}\cdot\overline{\mathbf{I}}$$
$$=\frac{iv}{2}(-\chi^a\partial_x\chi^a+\bar{\chi}^a\partial_x\bar{\chi}^a)-\frac{\gamma}{2}(\chi^a\bar{\chi}^a)^2, \tag{42}$$

where $a=1,2,3$ and

$$\mathbf{I}=\mathbf{J}_1+\mathbf{J}_3,\quad \overline{\mathbf{I}}=\overline{\mathbf{J}}_2+\overline{\mathbf{J}}_4,$$

$$\mathbf{J}_{1,3}=\frac{i}{2}\left\{\pm\chi_0\boldsymbol{\chi}+\frac{1}{2}[\boldsymbol{\chi}\times\boldsymbol{\chi}]\right\},$$

$$\overline{\mathbf{J}}_{2,4}=\frac{i}{2}\left\{\pm\bar{\chi}_0\bar{\boldsymbol{\chi}}+\frac{1}{2}[\bar{\boldsymbol{\chi}}\times\bar{\boldsymbol{\chi}}]\right\}. \tag{43}$$

The fields χ and $\bar{\chi}$ denote real (Majorana) fermions.

Equation (41) describes a critical Ising model; the corresponding excitations are gapless and nonmagnetic; they appear in the sectors with both parities. The criticality is an artifact of the periodic boundary conditions in the transverse direction.

The massive sector can be described using three equivalent representations. One of those is the $SU_2(2)$ WZNW model perturbed by a current-current interaction (the ultraviolet central charge $C_{\text{uv}}=3/2$). Here the fundamental field is the SU(2) matrix $\hat{g}$. Another representation is the O(3) Gross-Neveu (GN) model where fundamental fields are Majorana fermions (42). The third representation is three critical Ising models coupled by the energy density operators ϵ_i:

$$S=\sum_{i=1}^{3} S_{\rm Ising}(\sigma_i)-\gamma\sum_{i>j}\int d\tau\, dx\, \epsilon_i\epsilon_j . \tag{44}$$

In the latter case the fundamental fields are order parameter fields of the Ising models or, via the duality relationships, the corresponding disorder parameter fields.

An external magnetic field couples to the sum of all currents and therefore only to the fermions of the O(3) GN model:

$$\mathcal{H}_{\rm magn}=i\mathbf{h}\cdot([\boldsymbol{\chi}\times\boldsymbol{\chi}]+[\bar{\boldsymbol{\chi}}\times\bar{\boldsymbol{\chi}}]). \tag{45}$$

Since this model has a spectral gap, the net magnetization does not appear until the magnitude of the field reaches the value of the gap.

The thermodynamic Bethe ansatz equations for the O(3) GN model were obtained by one of the present authors;[47] some of the correlation functions were calculated by Smirnov.[48] A very illuminating discussion of the symmetries and the excitation spectrum of this model can be found in the recent paper by Fendley and Saleur.[49]

The massive magnetic excitations are particles with non-Abelian statistics, as in the Pfaffian state of quantum Hall effect.[50] An unusual fact about the O(3) GN model is that its ground state is doubly degenerate. This degeneracy is of *topological* nature, being related to properties of *zero modes* formed on solitons. The picture simplifies drastically if one takes into account that the O(3) GN model is equivalent to the supersymmetric sine-Gordon model[47] (SUSY SG)

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\Phi)^2+\frac{i}{2}\bar{\chi}\gamma_\mu\partial_\mu\chi+M^2\sin^2(\beta\Phi)+iM\bar{\chi}\chi\cos(\beta\Phi), \tag{46}$$

with $\beta^2=4\pi$. For the SUSY SG model there is a semiclassical limit $\beta^2\ll 1$, where details are easier to grasp. In this limit the spectrum is determined by the minima of the bosonic part of the potential. There are two sets of minima: one where $\cos(\beta\Phi)=1$ and one where it is -1. This sign difference does not affect the energy spectrum of the Majorana fermion but affects expectation values of the Ising model order and disorder parameters. Therefore vacua with different sign of $\cos(\beta\Phi)$ are physically distinguishable. It is clear that the fact of this degeneracy is related to (i) the odd number of Majorana fermions species present and (ii) the existence of solitons. Both these features are very general and are likely to survive in a finite parameter domain. The described situation should be opposite to what happens in the conventional sine-Gordon model, where the potential is just $\cos(\beta\Phi)$ and the degeneracy between its minima is lifted in the ground state.

There are two descriptions of the spectrum. The first was suggested by Zamolodchikov.[51] This description abandons the notion of asymptotic particle states and therefore leads to difficulties when being applied to the correlation functions. The second approach was suggested in Ref. 52 and used in Ref. 48 to calculate the correlation functions. In this approach solitons are treated as particles carrying two quantum numbers—an SU(2) spin $\sigma=\uparrow,\downarrow$ and an isotopic number $p=\pm$. The isotopic part of the K-soliton Hilbert space is truncated, that is, not all combinatorically possible 2^K states are allowed. The number of allowed states is $2^{[K/2]}$; in fact, the isotopic space of K solitons is isomorphic to the irreducible representation of a K-dimensional Clifford algebra (in other words, one may say that each soliton carries a γ matrix—a zero mode of the Majorana fermion). In particular, the two-soliton wave function must be an isotopic singlet.

This approach allows one to use the conventional scattering matrix description at the end, projecting out unwanted states. The two-particle S matrix is given by the tensor product

$$S(\theta_{12})=S_{\rm ITM}(\theta_{12})\otimes S_{4\pi}(\theta_{12}), \tag{47}$$

where the first S matrix corresponds to the SU(2) invariant (isotropic) Thirring model (ITM) and second is the S matrix of solitons of the sine-Gordon model with the particular anisotropy $\xi=4\pi$ [in Smirnov's notations $\xi=\pi\beta^2/(8\pi-\beta^2)$].

The order parameters are local operators both in the Ising and the WZNW representations:

$$\begin{aligned}\mathcal{T}_{12}&=\exp[i\sqrt{2\pi}(\varphi_1+\bar{\varphi}_2)]=e^{i\sqrt{\pi}\Phi_+}e^{i\sqrt{\pi}\Phi_-}\\&=(\sigma_0\sigma_3+i\mu_0\mu_3)(\sigma_1\sigma_2+i\mu_1\mu_2)=\sigma_0{\rm Tr}\hat{g}+i\mu_0{\rm Tr}\hat{g}^+,\\ \mathcal{T}_{14}&=\exp[i\sqrt{2\pi}(\varphi_1+\bar{\varphi}_4)]=e^{i\sqrt{\pi}\Phi_+}e^{i\sqrt{\pi}\Theta_-}\\&=(\sigma_0\mu_3+i\mu_0\sigma_3)(\sigma_1\sigma_2+i\mu_1\mu_2),\end{aligned} \tag{48}$$

where σ_i and μ_i are order and disorder parameters of the corresponding Ising models. The spontaneous mass generation freezes σ_i $(i=1,2,3)$, but leaves σ_0 and μ_0 critical. Therefore the most singular parts of the above order parameters

$$\mathcal{T}_{12}\sim\sigma_0,\quad \mathcal{T}_{14}\sim\mu_0 \tag{49}$$

still have power law correlations at $T=0$. The correlation length at $T\neq 0$ is $\xi\sim 1/T$ and is not determined by the solitons—a possibility we have mentioned in Sec. III B.

The chiral exponents entering in the expressions for the staggered magnetization are nonlocal:

$$\exp[i\sqrt{2\pi}\varphi_1]=C_{ab}\sigma^a(z)g^b(z,\bar{z}), \tag{50}$$

where σ^a are chiral *vertex operators* of the critical Ising model (see Ref. 53 for details). The operators g in the ultraviolet limit become vertex operators of the $S=1/2$ tensor operators of $k=2$ WZNW model.

In order to understand the subsequent calculations of the correlation functions the reader must keep in mind two principal facts about excitations in integrable models. The first fact is that these excitations cannot be unambiguously classified as fermions, bosons, semions, etc. Their commutation relations are determined by the S matrix and therefore depend on the momenta of the particles (see, for example, Ref. 56). The second fact is that the creation and annihilation operators of elementary excitations in integrable theories are usually strongly nonlocal in terms of the bare creation and annihilation operators. Hence these excitations are extended objects and therefore do not belong to any particular representation of the Lorentz group. Therefore their Lorentz spin is not fixed and the same operator can represent physical fields with different Lorentz spin depending on circumstances.

1. Correlation functions of the currents

According to Eq. (43) there are two parts in a given current operator, J_- and I. The first, containing a product of the gapless fermion χ_0 and the O(3) GN model fermion, is not conserved. A GN fermion is a convolution of two solitons, each entering with Lorentz spin 1/4 to make the total spin 1/2. Superficially it may appear that the second term (the conserved current) is a convolution of four solitons, but this is incorrect: the minimal matrix element contains two solitons that enter with Lorentz spin 1/2. Since χ_0 remains massless, the leading asymptotics of the correlation function of nonconserved currents $\langle J^- J^- \rangle$ with the threshold at $s=2m$ $[s^2=\omega^2-(vq)^2]$ exists only for the currents with the same chirality (here we set $v=1$):

$$\langle J_-^a(\tau,x)J_-^b(0,0)\rangle = \delta_{ab}(\tau+ix)^{-1}\int d\theta_1 d\theta_2 |F_1(\theta_{12})|^2 e^{(\theta_1+\theta_2)/2} \times \exp[-m\tau(\cosh\theta_1+\cosh\theta_2) + imx(\sinh\theta_1+\sinh\theta_2)], \quad (51)$$

which gives

$$\frac{(\tau-ix)}{(\tau+ix)\sqrt{\tau^2+x^2}}\int d\theta |F_1(\theta)|^2 K_1(2m\rho\cosh\theta), \quad (52)$$

where F_1 can be calculated. The Fourier transform

$$\mathrm{Im}\langle J^- J^-\rangle_{(\omega,q)} = \left(\frac{\omega+q}{\omega-q}\right)^2 L(s^2),$$

$$\mathrm{Im}\langle \bar{J}^- \bar{J}^-\rangle_{(\omega,q)} = \left(\frac{\omega-q}{\omega+q}\right)^2 L(s^2), \quad (53)$$

$$L(s^2)=s^2\int \frac{d\theta}{\cosh^3\theta}\theta_H(s^2-4m^2\cosh^2\theta)|F_1(\theta)|^2$$

where $\theta_H(x)$ is the Heaviside function.

Now let us consider the pair correlation function of the conserved O(3) currents $I=J_1+J_3, \bar{I}=\bar{J}_2+\bar{J}_4$. Their matrix elements were calculated in Ref. 48. In the scheme adopted in that paper a soliton carries two quantum numbers—an SU(2) spin $\sigma=\uparrow,\downarrow$ and an isotopic number $p=\pm$. The isotopic part of the multisoliton Hilbert space is truncated such that the two soliton wave function must be an isotopic singlet. Due to the SU(2) symmetry, it is sufficient to have an expression for one component of the current, for example, I^3. The matrix element into a state of two solitons with rapidities θ_1 and θ_2 and quantum numbers $(\sigma,p)_1$ and $(\sigma,p)_2$ is given by

$$\langle \theta_1,(\sigma_1,p_1);\theta_2,(\sigma_2,p_2)|I^z|0\rangle = d e^{(\theta_1+\theta_2)/2}\frac{\coth(\theta_{12}/2)\zeta_\infty(\theta_{12})\zeta_4(\theta_{12})}{(\theta_{12}+i\pi)\cosh\left[\frac{1}{8}(\theta_{12}+i\pi)\right]}[|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle] \otimes[|+-\rangle-|-+\rangle], \quad (54)$$

where

$$\zeta_k(\theta)=\sinh(\theta/2)\tilde{\zeta}_k(\theta),$$

$$\tilde{\zeta}_k(\theta)=\exp\left[\int_0^\infty dx\frac{\sin^2[x(i\pi+\theta)]\sinh[\pi(1-k)x/2]}{x\sinh\pi x\cosh(\pi x/2)\sinh(\pi k x/2)}\right], \quad (55)$$

and

$$d=\frac{1}{64\pi^3\zeta_\infty(i\pi)\zeta_4(i\pi)}.$$

Thus the leading asymptotics of $\langle I^a I^b\rangle$ and $\langle I^a \bar{I}^b\rangle$ are

$$\langle I^a(\tau,x)I^b(0,0)\rangle = 2d^2\delta_{ab}\int d\theta_1 d\theta_2|F_2(\theta_{12})|^2 e^{(\theta_1+\theta_2)} \times\exp[-m\tau(\cosh\theta_1+\cosh\theta_2) + imx(\sinh\theta_1+\sinh\theta_2)], \quad (56)$$

$$\langle I^a(\tau,x)\bar{I}^b(0,0)\rangle = \delta_{ab}2d^2\int d\theta_1 d\theta_2|F_2(\theta_{12})|^2 \times\exp[-m\tau(\cosh\theta_1+\cosh\theta_2) + imx(\sinh\theta_1+\sinh\theta_2)], \quad (57)$$

where

$$|F_2(\theta)|^2=\frac{\sinh^2\theta}{[\cosh(\theta/4)+\cos(\pi/4)](\theta^2+\pi^2)}|\tilde{\zeta}_\infty(\theta)\tilde{\zeta}_4(\theta)|^2. \quad (58)$$

For the Fourier transform we have

$$\langle I^a_\mu I^b_\nu\rangle=\delta_{ab}(\delta_{\mu\nu}-q_\mu q_\nu/q^2)D(s^2), \quad (59)$$

$$\mathrm{Im}\,D(s^2)=\frac{4d^2}{\sqrt{s^2-4m^2}}|F_2(\theta)|^2, \quad \cosh^2(\theta/2)=s^2/4m^2$$

such that at the threshold we have the same behavior as for $N=1$:

$$\mathrm{Im}\,D(s^2)\sim\sqrt{s^2-4m^2} \quad (60)$$

2. Single-electron Green's function for the model of stripes

All correlation functions discussed so far exist for both models depicted in Fig. 1. Now we are going to discuss the single-electron function that exists only for the model of stripes [Fig. 1(b)]. For $N=1,2$ we can calculate the spinon Green's function G and find the following expression for the spectral function:

$$A_R(\omega,q)\sim\frac{1}{\omega-vq}\left(\frac{m^2}{\omega^2-(vq)^2-m^2}\right)^{3/8} \tag{61}$$

and for the tunneling density of states:

$$\rho(\omega)\sim\int_0^{\cosh^{-1}(\omega/m)}\frac{d\theta\cosh(3\theta/8)}{(\omega/m-\cosh\theta)^{3/8}}. \tag{62}$$

3. Correlation functions of the staggered magnetizations

As we have stated above, the correlation functions of staggered magnetizations factorize into a product of two kinds of correlation functions: G and D [see Eqs. (33) and (36)]. The function D is essentially nonperturbative and therefore especially interesting. We have demonstrated that for $N=1$ this function is nonzero. For $N=2$ with periodic boundary conditions this function vanishes. This is related to the fact that for periodic boundary conditions the singlet sector is critical. The operator $\exp[i\sqrt{2\pi}\varphi]$ contains vertex operators σ_0; at criticality correlation functions of such operators with vertex operators of different chirality always vanish: $\langle\sigma_0\bar{\sigma}_0\rangle=0$.

To understand this better it is convenient to write the staggered magnetizations in the appropriate basis. Thus we have

$$\begin{aligned}\mathbf{N}_{1,3}&=\sigma_0^{(1)}\mathrm{Tr}[\boldsymbol{\sigma}(g+g^+)]\pm i\mu_0^{(1)}\mathrm{Tr}[\boldsymbol{\sigma}(g-g^+)],\\ \mathbf{N}_{2,4}&=\sigma_0^{(2)}\mathrm{Tr}[\boldsymbol{\sigma}(G+G^+)]\pm i\mu_0^{(2)}\mathrm{Tr}[\boldsymbol{\sigma}(G-G^+)],\end{aligned} \tag{63}$$

where g and G are the SU(2) matrices. The fields σ_0 and μ_0 remain critical. The fields g and G have to be decomposed into their vertex operators and the interaction involves sectors with different chirality.

From Eqs. (63) we have

$$\begin{aligned}\langle\langle\mathbf{N}_1\mathbf{N}_1\rangle\rangle&=(\langle\langle\sigma_0\sigma_0\rangle\rangle+\langle\langle\mu_0\mu_0\rangle\rangle)\langle\langle\mathrm{Tr}[\boldsymbol{\sigma}g]\mathrm{Tr}[\boldsymbol{\sigma}g^+]\rangle\rangle,\\ \langle\langle\mathbf{N}_1\mathbf{N}_3\rangle\rangle&=(\langle\langle\sigma_0\sigma_0\rangle\rangle+\langle\langle\mu_0\mu_0\rangle\rangle)\{\langle\langle\mathrm{Tr}[\boldsymbol{\sigma}g]\mathrm{Tr}[\boldsymbol{\sigma}g]\rangle\rangle\\ &\quad+\langle\langle\mathrm{Tr}[\boldsymbol{\sigma}g^+]\mathrm{Tr}[\boldsymbol{\sigma}g^+]\rangle\rangle\},\end{aligned} \tag{64}$$

where $\langle\langle\sigma_0\sigma_0\rangle\rangle=\langle\langle\mu_0\mu_0\rangle\rangle\sim|(v\tau)^2+x^2|^{1/8}$.

There are no reasons to think that correlation functions on the chains with the same parity vanish. Here solitons enter with Lorentz spin 3/16. The asymptotics of the correlation function is given by

$$\begin{aligned}\chi(\tau,x)&=\langle e^{i\sqrt{2\pi}\Phi_1(\tau,x)}e^{-i\sqrt{2\pi}\Phi_1(0,0)}\rangle\\ &\sim\langle e^{i\sqrt{2\pi}\Phi_1(\tau,x)}e^{-i\sqrt{2\pi}\Phi_3(0,0)}\rangle\\ &\sim\rho^{-1/4}\left|\int d\theta\, e^{3\theta/8}e^{-m\tau\cosh\theta+ixm\sinh\theta}\right|^2\\ &=2\rho^{-1/4}\int d\theta\, K_{3/4}(2m\rho\cosh\theta),\end{aligned} \tag{65}$$

where $\rho^2=(v\tau)^2+x^2$. The Fourier transform is $[s^2=\omega_n^2+(vq)^2]$

$$\begin{aligned}&4\pi\int d\rho\,\rho^{3/4}J_0(s\rho)\int d\theta\, K_{3/4}(2m\rho\cosh\theta)\\ &\quad\sim\int d\theta(\cosh\theta)^{-7/4}F(5/4,1/2,1;-s^2/4m^2\cosh^2\theta).\end{aligned} \tag{66}$$

After the analytic continuation $i\omega=\omega+i0$ we get the imaginary part [now $s^2=\omega^2-(vq)^2>4m^2$]

$$\begin{aligned}\chi''(\omega,q)&\sim s^{-3/4}\int_{(2/s)}^1\frac{dx\,x^{3/4}}{(1-x^2)^{3/4}(s^2x^2/4m^2-1)^{1/2}}\\ &\quad\times F(1/2,1/2,1/4;1-x^2).\end{aligned} \tag{67}$$

Close to the threshold $s=2m$ we have

$$\chi''(\omega,q)\sim(s^2-4m^2)^{-1/4}. \tag{68}$$

At $s^2\gg m^2$ Eq. (67) gives the correct asymptotics $\chi''\sim s^{-1}$, which indicates that the two-soliton approximation may give a reasonable description throughout the entire range of energies. Recall that for $N=1$ the power was $-1/2$. It is tempting to speculate that the threshold singularity further diminishes with an increase of N.

The case of four chains introduces some new features. Some of them, as we believe, are accidental and some are generic. The generic feature that persists for a higher number of chains (see above) is the presence of singlet degrees of freedom. Here they appear in the form of the critical Ising model. The accidental features are the criticality of the latter model and the non-Abelian statistics of the massive kinks. These two properties are closely related and are unstable with respect to a change in boundary conditions. In the most general setting, as we shall see in the next section, the gap in the Ising model sector will be finite, but much smaller than the magnetic gap.

V. TOWARDS AN INFINITE NUMBER OF CHAINS: A TRIADIC CLUSTER EXPANSION

As the reader probably understands, the problem in question is difficult, naive attempts to develop a self-consistent expansion schemes fail (see Sec. III A), and we have to resort to some other methods. The exact results for two and four chains discussed in the preceding section give a glimpse of the complexity of the problem. Unfortunately, for a higher number of chains there are no exact solutions. However, a nonperturbative analysis can be imagined, though for a somewhat modified model. In this modification the interactions are set up in such a way that the chains are assembled into clusters of three (triads), nine (enneads), 27 chains, etc. (see Fig. 2).

The interaction inside a cluster of a given size is supposed to be stronger than interactions between the clusters of larger sizes. We believe that the study of such a modified model sheds light on classification of the excitations and establishes the structure of the effective action for the collective modes below the soliton mass gap.

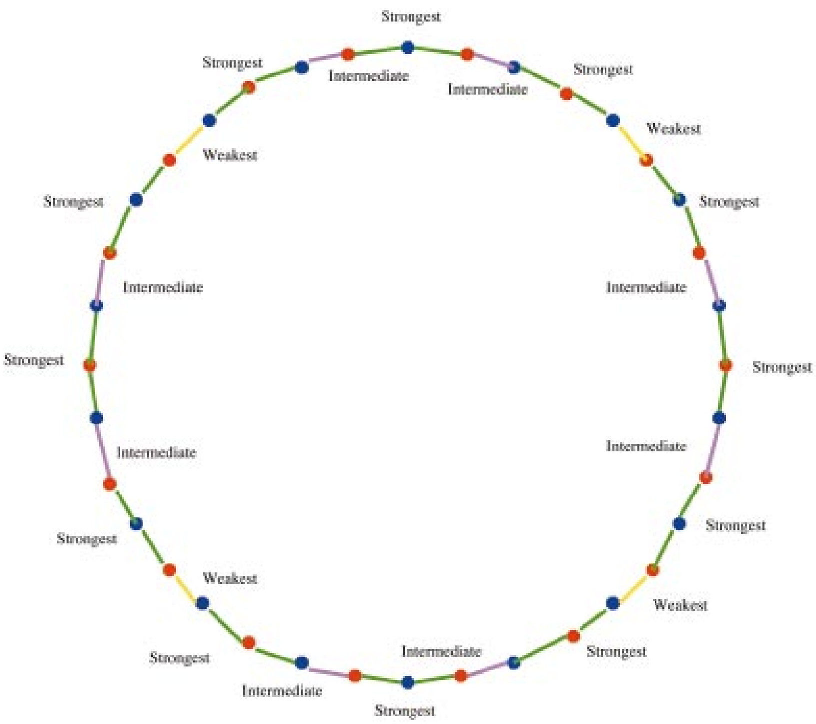

FIG. 2. (Color) The hierarchical model for 36 chains. Differently colored circles denote currents with different chirality. The green bonds are the strongest, the magenta ones are of intermediate strength, and the yellow ones are the weakest.

A. The fundamental triad (three chains)

The solution for two chains with one chirality (let it be the left one) coupled with the current-current interaction to a chain with opposite chirality was described in Refs. 57 and 58 (see also Ref. 59). The Hamiltonian density is

$$\mathcal{H}=\left[\frac{iv}{2}\bar{\chi}_0\partial_x\bar{\chi}_0+\frac{\pi v}{2}:\bar{\mathbf{I}}\bar{\mathbf{I}}:\right]+\frac{2\pi v}{3}:\mathbf{J}\mathbf{J}:+\gamma\bar{\mathbf{I}}\mathbf{J}, \tag{69}$$

where $\bar{I}^a$ and J^a are right $SU_2(2)$ and left $SU_1(2)$ currents, respectively. The expression in the square brackets describes a sum of two chiral $SU_1(2)$ WZNW models. Such a representation was already discussed in Sec. IV B. The right-moving Majorana fermion does not participate in the interaction. In the ultraviolet regime the model is a sum of three chiral conformal field theories: the $k=2$ SU(2) WZNW model with the right central charge $C_R=3/2$, the right-moving free Majorana field with $C_R=1/2$, and the $SU_1(2)$ WZNW model with the left central charge $C_L=1$. The current-current interaction generates a massless renormalization group (RG) flow to the infrared critical point. The theory in the infrared is represented by two free Majorana fermions with opposite chirality $\bar{\chi}_0$ and χ_0 and the right-moving sector of the $SU_1(2)$ WZNW model:

$$\mathcal{H}_{IR}=\frac{2\pi v}{3}:\bar{\mathbf{j}}\,\bar{\mathbf{j}}:+\frac{iv}{2}(\bar{\chi}_0\partial_x\bar{\chi}_0-\chi_0\partial_x\chi_0). \tag{70}$$

Under the RG flow the fields transmute in the following way:

$$\bar{\mathbf{I}}\rightarrow 2\bar{\mathbf{j}}+\cdots,\quad i\bar{\chi}_0\bar{\boldsymbol{\chi}}\rightarrow i\bar{\chi}_0\chi_0\bar{\mathbf{j}}+\cdots, \tag{71}$$

where the dots stand for less relevant operators.

B. The first ennead (nine chains)

Let us consider the case of nine chains with open boundary conditions, arranging them into three triads in such a way that the bare coupling constant inside each triad γ_0 is greater than the bare coupling γ_1 between the triads (see Fig. 3). Then we can have a situation when the ir fixed point for a given triad is already achieved [the corresponding energy scale is $\Delta\sim J_{\parallel}\gamma\exp(-\pi/\gamma)$], well before the renormalized

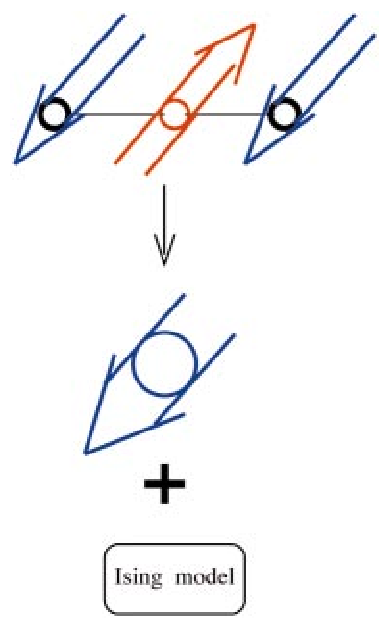

FIG. 3. (Color) The renormalization scheme for three chains. In the infrared limit the system is equivalent to a single chiral $C=1$ chain and a critical (nonchiral) Ising model. The circles with different thicknesses denote currents with different chirality.

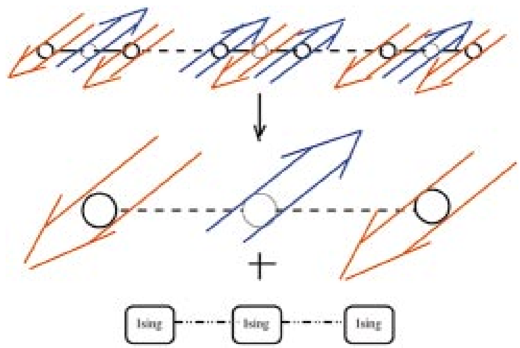

FIG. 4. (Color) A schematic picture of renormalization of nine chains.

coupling for the triad-triad interaction becomes of the order of 1.

According to Eq. (71), the renormalized coupling between any neighboring triads (denoted 1 and 2; recall also that they always have different chirality) in the infrared is

$$\gamma_1(\Delta)\mathbf{j}_1\bar{\mathbf{j}}_2+\gamma_1(\Delta)(i\bar{\chi}_0\chi_0)_1(i\bar{\chi}_0\chi_0)_2\mathbf{j}_1\bar{\mathbf{j}}_2+\cdots. \quad (72)$$

The first term reproduces the original interaction (7). Its presence in the Hamiltonian also converts the second term into a marginal operator:

$$\gamma_1(\Delta)(i\bar{\chi}_0\chi_0)_1(i\bar{\chi}_0\chi_0)_2\mathbf{j}_1\bar{\mathbf{j}}_2=\tilde{\gamma}_1(i\bar{\chi}_0\chi_0)_1(i\bar{\chi}_0\chi_0)_2+\gamma_1(\Delta)\times(i\bar{\chi}_0\chi_0)_1(i\bar{\chi}_0\chi_0)_2:\mathbf{j}_1\bar{\mathbf{j}}_2:,$$

$$\tilde{\gamma}_1=\gamma_1(\Delta)\langle\mathbf{j}_1\bar{\mathbf{j}}_2\rangle. \quad (73)$$

Thus by integrating out the high-energy degrees of freedom in the ennead of chains one generates at energies smaller than Δ (the new uv cutoff) the effective action that contains the original action for a triad of chains with a renormalized coupling constant γ_1 plus the action for three critical Ising models coupled by the products of energy density operators:

$$\mathcal{H}_{\text{Ising}}=\sum_{r=1}^{3}\left[\frac{i}{2}(\bar{\chi}\partial_x\bar{\chi}-\chi\partial_x\chi)_r\right]+\tilde{\gamma}_1(\bar{\chi}\chi)_2[(\bar{\chi}\chi)_1+(\bar{\chi}\chi)_3]. \quad (74)$$

The Ising subsystem decouples from the magnetic one. The resulting model is similar to the O(3) GN model discussed

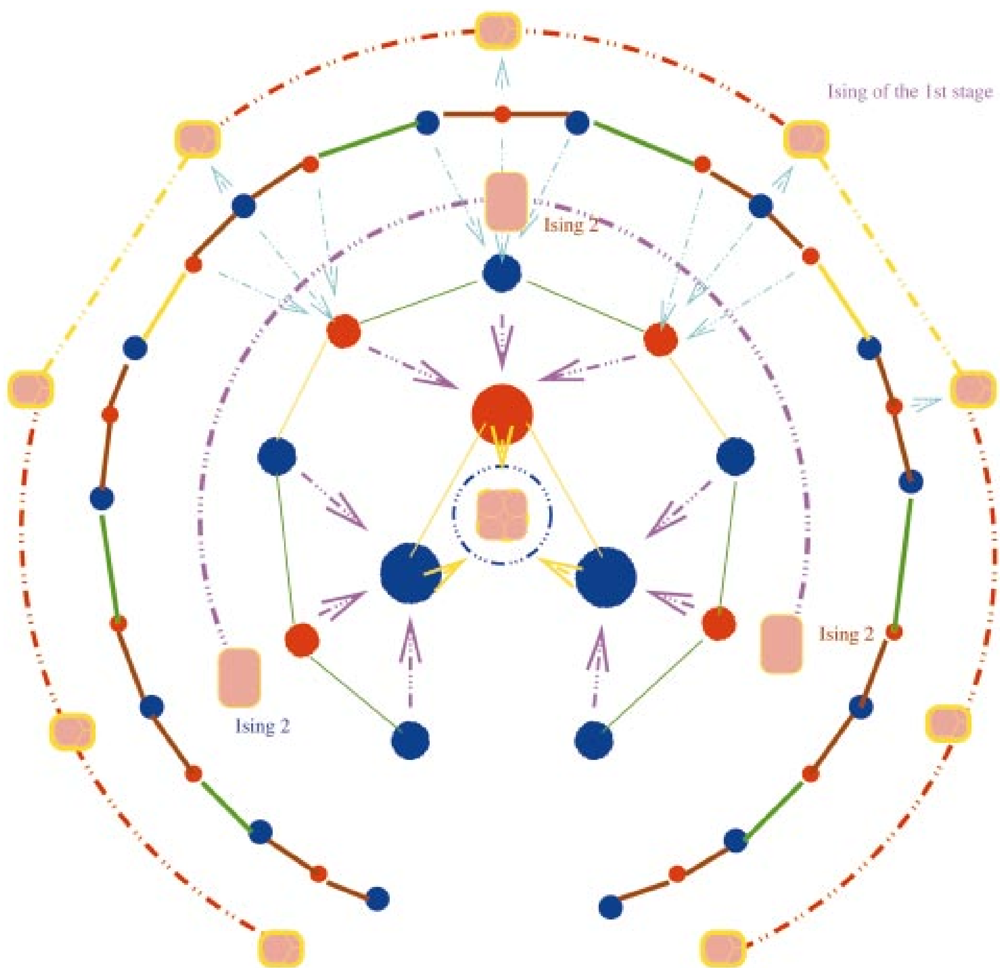

FIG. 5. (Color) The hierarchy of interactions and the renormalization process for 27 chains. The strongest bonds are brown, the intermediate are green and the weakest ones are yellow. The Ising variables are shown as pink rectangles. The red and blue circles of various sizes depict spin currents of different chirality. Their size increases with every step of the RG process.

above [see the discussion around Eq. (47)]. The only difference is that the symmetry is broken down to U(1). The interaction can be written in terms of the currents

$$\tilde{\gamma}_1[(\bar{\chi}\chi)_1(\bar{\chi}\chi)_2+(\bar{\chi}\chi)_2(\bar{\chi}\chi)_3]\sim\gamma_\perp[I^x\bar{I}^x+I^y\bar{I}^y]+\gamma_\parallel I^z\bar{I}^z, \tag{75}$$

which makes the model somewhat similar to the anisotropic Thirring model [I^a are $SU_2(2)$ currents made of χ_r's and the coupling constants $\gamma_\perp\sim\tilde{\gamma}_1$ and $\gamma_\parallel=0$ on the bare level]. We believe (the details will be given in a separate publication) that this particular type of anisotropy vanishes in the strong coupling regime in the same way as it does in the C sector of the anisotropic Thirring model.[54,55] In other words, in the strong coupling regime the model (74) is equivalent to the O(3) GN model.

Thus on each step of the triadic real space RG we generate the O(3) GN models. Their number is equal to the number of clusters on the given level. For example, if the initial number of chains is 3^N and the first GN model appears for a cluster of 9 chains, the number of copies of GN models on the first level is 3^{N-2}. Then the next levels contain 3^{N-3}, 3^{N-4}, etc., copies such that the total number of such models in both parity sectors is 3^{N-1}. Taking into account that each of them has a twofold degenerate ground state, we get the ground state degeneracy $2^{\mathcal{N}/3}$, where $\mathcal{N}=3^N$ is the total number of chains. The resulting ground state entropy is three times smaller than the one that was obtained for the uniform case.

Now we can discuss the thermodynamics. In our cluster expansion we have the following energy scales. First, there is a sequence of crossover scales $\Delta_0>\Delta_1>\cdots>\Delta_{N-1}$ (magnetic gaps) corresponding to crossovers inside of each cluster. Then there are energy gaps of the interacting Ising models $M_2, M_3, \ldots,$ which are formed by clusters of 9, 27, 81, etc., chains (see Figs. 4 and 5). The first scale in this sequence is of the order of

$$M_2\sim\Delta_1\exp[-\pi/\tilde{\gamma}_1]. \tag{76}$$

Since $\tilde{\gamma}_1\sim\gamma_1^2$ we deem these energy gaps to be much smaller than the magnetic gaps. We conjecture that this difference survives even in the limit when all interactions become equal.

All these crossovers affect the temperature behavior of the specific heat (see Fig. 6). The first crossover occurs at temperature $T\sim\Delta_0$, which is the crossover temperature of a fundamental triad. Above this temperature we have a bunch of noninteracting chiral Heisenberg chains, each having the heat capacity linear in T:

$$C(T>\Delta_0)=S\mathcal{C},\quad \mathcal{C}=\frac{\pi T}{6v}, \tag{77}$$

where $S=3^N L_\parallel$ is the total area occupied by the system. At $\Delta_0>T>\Delta_1$ we effectively have 3^{N-1} critical Ising chains with central charge $C=1/2$ and the same amount of noninteracting chiral Heisenberg chains. As a result the slope of the *specific heat* drops by the factor of 2. When the entire

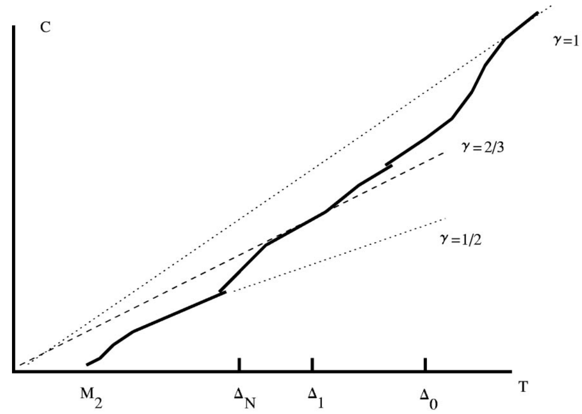

FIG. 6. A schematic picture of the temperature dependence of the specific heat.

sequence of magnetic crossovers is passed, that is, at $\Delta_{N-1}>T>M_2$, the specific heat is given only by the Ising models:

$$\mathcal{C}=\frac{\pi T}{6v}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)=\frac{\pi T}{12v}, \tag{78}$$

that is, remains the same as after the first crossover. Thus we can say that in the thermodynamic limit the singlets occupy half of the original Hilbert space.

When the temperature falls below the first singlet gap M_2, one may say to the first approximation that the Ising modes with the largest gaps cease to contribute. After a certain crossover the linear slope in the specific heat drops by a factor of 9. The specific heat remains linear in T until M_3 is reached, where the slope falls by another factor of 3, etc. In our scheme where the distribution of coupling constants between the clusters of different size is rather arbitrary, it makes little sense to do more detailed calculations. The only thing we can say is that $C\sim T$ above a certain temperature and then experiences a fast decrease. In our cluster expansion there are two areas with linear specific heat characterized by slopes differing by the factor of 2. This due to the fact that within this approach the gap for magnetic excitations Δ is different from the gaps for singlet excitations. Whether this feature will survive in the limit of uniformly coupled chains is open for debate.

VI. REMAINING CASES

It would make a lot of sense to study the models described in this paper numerically. Since numerical calculations will probably be performed for systems with a restricted number of chains, we decided to describe certain tractable cases in more detail.

A. An approximate solution for $N=3$ (six chains)

In this case the magnetic subsystem is equivalent to $N=1$ [that is, to the SU(2) Thirring model] and the Ising subsystem contains two Ising models with the $\epsilon_1\epsilon_2$ interaction. The latter is equivalent to the spinless Thirring model with a

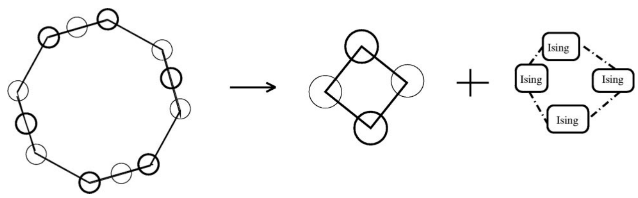

FIG. 7. A schematic picture of twelve chains. The thick lines correspond to larger exchange interactions. The circles of different sizes denote chains with different chirality.

truly marginal interaction. Thus the Ising sector is critical and is described by the $C=1$ Gaussian model.

B. An approximate solution for $N=6$ (twelve chains)

Applying the procedure outlined above to the system of twelve chains with periodic boundary conditions (see Fig. 3), we obtain the $N=2$ model in the magnetic sector plus the model (74) with four chains (see Fig. 7).

Thus the excitation spectrum includes the GN O(3) soliton mode, one gapless Majorana fermion, and massive excitations from the Ising sector.

VII. SINGLE-ELECTRON GREEN'S FUNCTION FOR THE MODEL OF STRIPES AND THE SPINON PROPAGATOR FOR $N=\infty$

Now we shall try to use the wisdom accumulated in the exact solution of the $N=2$ case to obtain results for the case of an infinite number of chains. Though our ultimate goal is to calculate the correlation function of staggered magnetizations, the path to it lies through the single-electron Green's function, which exists only for the model of stripes [Fig. 1(b)]. The reason for this will become clear in the process of calculation.

Let conduction electrons belong to even chains. Then the electron creation operator is

$$R_{\sigma,2n}=e^{i\sqrt{2\pi}\varphi_c(2n)}e^{\sigma i\sqrt{2\pi}\varphi(2n)},$$

$$L_{\sigma,2n}=e^{-i\sqrt{2\pi}\bar{\varphi}_c(2n)}e^{-\sigma i\sqrt{2\pi}\bar{\varphi}(2n)}, \tag{79}$$

where v_c is the Fermi velocity. Though, in general, v_c may be quite different from the spin velocity, we shall not consider this possibility. With $v_c=v_s$ the model is (1+1)-dimensional Lorentz invariant. The charge fields φ_c and $\bar{\varphi}_c$ are free Gaussian fields and hence there are no correlations between the fields belonging to different chains,

$$G_{RR}(\tau,x)=\frac{1}{(\tau v_c+ix)^{1/2}}G_{2n,2n}(\tau,x),$$

$$G_{LL}(\tau,x)=\frac{1}{(\tau v_c-ix)^{1/2}}G_{2n,2n}(\tau,-x), \tag{80}$$

where the function G was defined in Eq. (32) and can be called the spinon Green's function. Since the electron Green's function must be a single-valued function of x at $\tau\to 0$ and a single-valued function of τ at $x=0$, we conclude that the spinon function G must be a double-valued function to compensate for the double-valuedness of $(\tau v_c\pm ix)^{-1/2}$. This suggests that the spinons are semions, as it was suggested by Laughlin.[60]

Now we shall use these facts to determine the asymptotics of the single-fermion Green's function in a similar way as was done for the (1+1)-dimensional Thirring model.[41,44] Namely, we shall combine these arguments with the Lehmann expansion for the Green's function. In this expansion we shall take into account only terms with an emission of a single massive soliton. The spectrum of this soliton is

$$E(k,k_\perp)=\sqrt{(vk)^2+M^2(k_\perp)}, \tag{81}$$

where $M(k_\perp)\sim\exp[-\pi v/\gamma]$ is as yet an unknown function (the soliton gap). We believe that this gap always remains finite for any momentum. A convenient parametrization of the energy and the momentum component along the chain direction is

$$E=M(k_\perp)\cosh\theta,\quad vk_x=M(k_\perp)\sinh\theta. \tag{82}$$

In this notation a Lorentz rotation on an "angle" γ corresponds to the shift of rapidity $\theta\to\theta+\gamma$ since an operator with Lorentz spin S transforms under such Lorentz transformation as

$$A_S\to e^{\gamma S}A_S.$$

We generalize for $N=\infty$ the formulas obtained for the $N=1$ and $N=2$ cases, treating the vanishing of the D function is the $N=2$ as an artifact of periodic boundary conditions:

$$G\sim(v\tau-ix)^{-1/2}Z(k_\perp)\exp[-M(k_\perp)\sqrt{\tau^2+(x/v)^2}],$$

$$D\sim f(k_\perp)M^{1/2}(k_\perp)K_0(M(k_\perp)\sqrt{\tau^2+(x/v)^2}), \tag{83}$$

where Z and f are yet unknown functions. The f function may vanish at some point in momentum space, as suggested by the example of four chains. Notice that both $k_\perp$ dependence of G and the very existence of D are effects exponential in $1/\gamma$. These equations dictate the following asymptotic form for the correlation function of the staggered magnetizations:

$$\chi_1(\tau,x;q_\perp)=\rho^{-1}\int\frac{dk_\perp}{(2\pi)}Z(k_\perp)Z(q_\perp+k_\perp)\exp\{-[M(k_\perp)+M(q_\perp+k_\perp)]\rho\}, \tag{84}$$

$$\chi_2(\tau,x;q_\perp)=\int\frac{dk_\perp}{(2\pi)}f(k_\perp)f(k_\perp+q_\perp)\times[M(k_\perp)M(q_\perp)]^{1/2}K_0(M(k_\perp)\rho)K_0(M(k_\perp+q_\perp)\rho),$$

where $\rho^2=\tau^2+(x/v)^2$ and χ_1 is a correlation function between the chains with the same and χ_2 with different parities.

VIII. CONCLUSIONS

Let us make a summary of our results.

(1) We have found models of magnets with a short-range Heisenberg exchange that have neither magnetic nor spin-Peierls order.

(2) We have given a formal proof that these models possess excitations with spin 1/2.

(3) We established the existence of $T=0$ critical point and a massive ground state degeneracy. The ground state entropy is proportional to the number of chains.

(4) We have also established that the low-energy Hamiltonian separates into two weakly interacting parts describing sectors with different parity.

(5) We established that the correlation functions of staggered magnetizations can be written as real space products of correlation functions of nonlocal operators belonging to the sectors with different parity [see Eqs. (32) and (33)].

(6) We have solved the problem exactly for the case of two and four coupled chains (the latter one with periodic boundary conditions). The results are consistent with the statements made for the case of infinite number of chains. The spectrum contains a massive amount of singlet excitations. In the language of gauge theory these excitations describe dynamics of Wilson loops.

A natural question is whether the models considered in our work can be realized. We believe that the the answer is positive. The model of stripes may well be relevant in a strongly underdoped regime of copper oxides.

We have already mentioned in the Introduction that there are other models of fractionalization, such as suggested in Refs. 7 and 8 together with the corresponding dimer models mentioned earlier. Though these works use Hamiltonians not based on any microscopic electron models, one might hope that they capture some features of the solution. In particular, it would be interesting to study in detail how singlet excitations, which in our model are associated with the O(3) GN models emerging on each triad of chains, are related to Z_2 vortices (visons) introduced in Refs. 7 and 61 (see also the recent work,[62] which essentially clarifies the concept of vison). We leave this question, as many others, to future research.

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APPENDIX A: GAUGE THEORY FORMULATION

The relationship with the gauge theories can be inferred from the fact that the level $k=1$ SU(2) Kac-Moody currents can be identified with (iso)spin currents of massless Dirac fermions:

$$J^a=\frac{1}{2}R^\dagger_\alpha\sigma^a_{\alpha\beta}R_\beta, \quad \bar{J}^a=\frac{1}{2}L^\dagger_\alpha\sigma^a_{\alpha\beta}L_\beta, \tag{A1}$$

σ^a being Pauli matrices. The Hamiltonian (4) and (7) becomes

$$H=\sum_n \int dx\Big[iv(-R^\dagger_\alpha\partial_x R_\alpha+L^\dagger_\alpha\partial_x L_\alpha)_n+\frac{\gamma}{8}\sum_\mu (R^\dagger\boldsymbol{\sigma}R + L^\dagger\boldsymbol{\sigma}L)_n\cdot(R^\dagger\boldsymbol{\sigma}R+L^\dagger\boldsymbol{\sigma}L)_{n+\mu}\Big]. \tag{A2}$$

As gauge theories of the RVB state, this theory possesses redundant charge degrees of freedom that do not participate in the interactions. Since every approximation violates this subtle property, dealing with the charge sector would become the same awkward problem as it is in the standard approach to the RVB gauge theory, once one decides to adopt this fermionic representation. Let us, however, follow the well-trodden path for a while just to make sure that the model we are discussing does fall in the category of RVB liquids. To this end, we use the identity

$$\sigma^a_1\sigma^a_2=2P_{12}-1,$$

where P_{12} is the permutation operator, and apply the Hubbard-Stratonovich transformation to rewrite the interaction term in Eq. (A2) as

$$\sum_{\mu,n}\int dx\left[\frac{|\Delta_{\mu+n,n}(x)|^2}{2\gamma}+(\Delta_{n,n+\mu}R^\dagger_{\alpha,n}L_{\alpha,n+\mu}+\text{H.c.})\right]+\cdots, \tag{A3}$$

where the dots stand for the terms we deem irrelevant. The procedure essentially coincides with the conventional decoupling scheme in the RVB approach. The fact that such decoupling here is done only in one lattice direction is not important provided one can justify that fluctuations of $|\Delta|$ may be neglected. As we shall see, excitations associated with breaking of singlets carry the largest spectral gap. This justifies the assumption about small fluctuations of $|\Delta|$. Once the amplitude $|\Delta|\sim J_\parallel\exp(-\pi v/\gamma)$ is frozen, one is left with the compact U(1) lattice gauge theory in the strong coupling limit (indeed, the gauge field has no bare $F^2_{\mu\nu}$ term, which corresponds to infinite bare charge). The vector potential A_y is represented by the phase of $\Delta_{n,n+\mu}$. Since by omitting the term with $R^\dagger_\alpha R_\alpha L^\dagger_\alpha L_\alpha$ we violate the decoupling of the charge degrees of freedom, we have to enforce the constraint on the absence of charge fluctuations by introducing the time component A_0 of the gauge field. In the mean field approximation (that is, when fluctuations of the gauge fields are neglected) we obtain the spectrum of the π-flux state:[12]

$$E(p)=\sqrt{(vk_{\|})^2+4|\Delta|^2\cos^2 k_{\perp}}, \tag{A4}$$

which has two Dirac-like conical singularities in the Brillouin zone. From what we have done in this paper it appears very dubious that this result will survive inclusion of fluctuations.

APPENDIX B: THE STRUCTURE OF STRONG COUPLING REGIME AND THE GROUND-STATE DEGENERACY

Let us assume that the system is two-dimensional and the number of chains is even, $2N$. We will drop the Lorentz-noninvariant part, $\mathcal{L}_n^{\rm int}$, in which case one has a decomposition $\mathcal{L}_n=\mathcal{L}_n^+ + \mathcal{L}_n^-$.

Let us first consider $\Sigma_n\mathcal{L}_n^+$. In the strong-coupling ground state, the following combinations of the chiral fields are locked:

$$\varphi_{2n}+\bar{\varphi}_{2n+\mu}=\sqrt{\frac{\pi}{8}}[1+2m_{2n}^{(\mu)}], \quad \mu=\pm 1,$$

$$m_{2n}^{(\mu)}=0,\pm 1,\pm 2,\dots . \tag{B1}$$

Assuming that periodic boundary conditions are imposed in the transverse direction, the set of equations (B1) can be rewritten as

$$\varphi_0+\bar{\varphi}_{2N-1}=\sqrt{\frac{\pi}{8}}[1+2m_0^{(-)}],$$

$$\varphi_0+\bar{\varphi}_1=\sqrt{\frac{\pi}{8}}[1+2m_0^{(+)}],$$

$$\varphi_2+\bar{\varphi}_1=\sqrt{\frac{\pi}{8}}[1+2m_2^{(-)}],$$

$$\varphi_2+\bar{\varphi}_3=\sqrt{\frac{\pi}{8}}[1+2m_2^{(+)}],$$

$$\dots$$

$$\varphi_{2N-2}+\bar{\varphi}_{2N-3}=\sqrt{\frac{\pi}{8}}[1+2m_{2N-2}^{(-)}],$$

$$\varphi_{2N-2}+\bar{\varphi}_{2N-1}=\sqrt{\frac{\pi}{8}}[1+2m_{2N-2}^{(+)}]. \tag{B2}$$

Considering pairs of neighboring equations in Eq. (B2), one finds that the integers $m_{2n}^{(\mu)}$ satisfy the following condition:

$$\sum_{k=0}^{N-1} m_{2k}^{(+)}=\sum_{k=0}^{N-1} m_{2k}^{(-)}. \tag{B3}$$

With this constraint, the number of independent fields, locked in the ground state, is $2N-1$. To select these fields we first define N scalar fields

$$\chi_n=\varphi_{2n}+\bar{\varphi}_{2n+1}, \quad n=0,1,2,\dots,N-1 \tag{B4}$$

together with their N dual counterparts

$$\vartheta_n=\varphi_{2n}-\bar{\varphi}_{2n+1}. \tag{B5}$$

According to Eq. (B1), the fields χ_n are locked at

$$[\chi_n]_{\rm vac}=\sqrt{\frac{\pi}{8}}[1+2m_{2n}^{(+)}]. \tag{B6}$$

On the other hand, there are N more combinations, $\varphi_{2n}+\bar{\varphi}_{2n-1}$, that get frozen, again according to Eq. (B1). We can express them in terms of χ_n and ϑ_n:

$$\varphi_{2n}+\bar{\varphi}_{2n-1}=\frac{1}{2}(\chi_n+\vartheta_n+\chi_{n-1}-\vartheta_{n-1})$$

$$=\sqrt{\frac{\pi}{8}}[1+2m_{2n}^{(-)}]. \tag{B7}$$

Thus N relative dual fields

$$\theta_n^{(-)}=\frac{1}{\sqrt{2}}(\vartheta_n-\vartheta_{n+1}), \quad n=0,1,2,\dots,N-1 \tag{B8}$$

are also locked:

$$[\theta_n^{(-)}]_{\rm vac}=\frac{\sqrt{\pi}}{2}[m_{2n}^{(+)}+m_{2n+2}^{(+)}-2m_{2n+2}^{(-)}]. \tag{B9}$$

Notice that transverse periodic boundary conditions imply that

$$\sum_n \theta_n^{(-)}=0$$

[which is actually the same condition as Eq. (B3)], and so the number of independent relative fields is $N-1$. We choose them to be

$$\theta_0^{(-)},\theta_1^{(-)},\dots \theta_{N-2}^{(-)} \tag{B10}$$

The total field

$$\theta^*=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\vartheta_n \tag{B11}$$

remains unlocked and, hence, disordered.

Thus the N fields χ_n, Eq. (B4), are locked while N their dual counterparts ϑ_n are *not*; only their $N-1$ combinations (B10) are. The frozen values of these $2N-1$ independent fields characterize the vacuum state of $\Sigma_n\mathcal{L}_n^+$.

The same can be done for $\Sigma_n\mathcal{L}_n^-$: we introduce

$$\psi_n=\bar{\varphi}_{2n}+\varphi_{2n+1},$$
$$\omega_n=-\bar{\varphi}_{2n}+\varphi_{2n+1} \tag{B12}$$

and select N fields ψ_n and $N-1$ relative duals

$$\omega_0^{(-)},\omega_1^{(-)},\dots,\omega_{N-2}^{(-)}, \tag{B13}$$

where

$$\omega_n^{(-)}=\frac{\omega_n-\omega_{n+1}}{\sqrt{2}}.$$

The total field

$$\omega^*=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\omega_n$$

remains disordered. The ψ_n fields are frozen at

$$[\psi_n]_{\rm vac}=\sqrt{\frac{\pi}{8}}[1+2m_{2n+1}^{(-)}], \tag{B14}$$

while $\omega_n^{(-)}$ at

$$[\omega_n^{(-)}]_{\rm vac}=\frac{\sqrt{\pi}}{2}[2m_{2n+1}^{(+)}-m_{2n+1}^{(-)}-m_{2n+3}^{(-)}]. \tag{B15}$$

We can now express the physical fields of individual chains in terms of the locked fields and two extra fields that remain disordered. We have

$$\begin{aligned}\Phi_{2n}&=\frac{1}{2}(\chi_n+\psi_n+\vartheta_n-\omega_n),\\ \Theta_{2n}&=\frac{1}{2}(\chi_n-\psi_n+\vartheta_n+\omega_n),\\ \Phi_{2n+1}&=\frac{1}{2}(\chi_n+\psi_n-\vartheta_n+\omega_n),\\ \Theta_{2n+1}&=\frac{1}{2}(-\chi_n+\psi_n+\vartheta_n+\omega_n).\end{aligned} \tag{B16}$$

The elements of the Wess-Zumino matrix field $\hat{g}_n(x)$ contain exponents $e^{\pm i\sqrt{2\pi}\Phi_n}$ and $e^{\pm i\sqrt{2\pi}\Theta_n}$. According to Eqs. (B16), in the strong coupling phase these exponents are proportional to

$$\exp\left(i\sqrt{\frac{2\pi}{N}}\theta^*\right),\quad \exp\left(i\sqrt{\frac{2\pi}{N}}\omega^*\right),$$

respectively, and thus will have vanishing expectation values. This effect, however, disappears in the thermodynamic limit.

Transverse dimerization and degeneracy of the ground state

Spontaneous transverse dimerization has been identified for the case of two chains in Ref. 25. In the continuum limit, the corresponding order parameter is given by a simple expression: $\mathbf{N}_n(x)\cdot\mathbf{N}_{n+1}(x)$. For the $2N$-chain model

$$\begin{aligned}\mathbf{N}_{2n}\cdot\mathbf{N}_{2n+1}&\sim\cos\sqrt{2\pi}(\Theta_{2n}-\Theta_{2n+1})+\frac{1}{2}[\cos\sqrt{2\pi}(\Phi_{2n}\\ &\quad-\Phi_{2n+1})-\cos\sqrt{2\pi}(\Phi_{2n}+\Phi_{2n+1})]\\ &=\cos\sqrt{2\pi}(\chi_n-\psi_n)+\frac{1}{2}[\cos\sqrt{2\pi}(\vartheta_n-\omega_n)\\ &\quad-\cos\sqrt{2\pi}(\chi_n+\psi_n)].\end{aligned}$$

Using Eqs. (B6) and (B14) when averaging over the ground state, one finds that

$$\begin{aligned}\eta_{2n}&\equiv\langle\mathbf{N}_{2n}\cdot\mathbf{N}_{2n+1}\rangle\\ &\sim\langle\cos\sqrt{2\pi}(\chi_n-\psi_n)\rangle-\frac{1}{2}\langle\cos\sqrt{2\pi}(\chi_n+\psi_n)\rangle\\ &\propto\frac{3}{2}(-1)^{m_{2n}^{+}-m_{2n+1}^{-}}.\end{aligned} \tag{B17}$$

We observe the existence of two, doubly degenerate values of this local order parameter; its sign depends on the parity of the integer $m_{2n}^{+}-m_{2n+1}^{-}$.

Similarly

$$\begin{aligned}\mathbf{N}_{2n+1}\cdot\mathbf{N}_{2n+2}&\sim\cos\sqrt{2\pi}(\Theta_{2n+1}-\Theta_{2n+2})\\ &\quad+\frac{1}{2}[\cos\sqrt{2\pi}(\Phi_{2n+1}-\Phi_{2n+2})\\ &\quad-\cos\sqrt{2\pi}(\Phi_{2n+1}+\Phi_{2n+2})]\\ &\sim\cos\sqrt{\frac{\pi}{2}}[(\psi_n+\psi_{n+1})-(\chi_n+\chi_{n+1})\\ &\quad+\sqrt{2}(\theta_n^-+\omega_n^-)]+\frac{1}{2}\cos\sqrt{\frac{\pi}{2}}[(\psi_n-\psi_{n+1})\\ &\quad+(\chi_n-\chi_{n+1})+(\omega_n+\omega_{n+1})-(\theta_n+\theta_{n+1})]\\ &\quad-\frac{1}{2}\cos\sqrt{\frac{\pi}{2}}[(\psi_n+\psi_{n+1})+(\chi_n+\chi_{n+1})\\ &\quad-\sqrt{2}(\theta_n^--\omega_n^-)].]\end{aligned} \tag{B18}$$

The first and third terms in the right-hand side of (B18) have nonzero expectation values. Using the locked values of the corresponding fields given above, we find that arguments of these two cosines are $\pi[m_{2n+1}^{(+)}-m_{2n+2}^{(-)}]$ and $\pi[m_{2n+1}^{(+)}+m_{2n+2}^{(-)}+1]$, respectively. Therefore

$$\eta_{2n+1}=\langle\mathbf{N}_{2n+1}\cdot\mathbf{N}_{2n+2}\rangle\propto\frac{3}{2}(-1)^{m_{2n+1}^{(+)}-m_{2n+2}^{(-)}}. \tag{B19}$$

Inspecting Eqs. (B17) and (B19) we observe that the two signs of local order parameters η_m reflect a spontaneously broken Z_2 symmetry. This is the symmetry related to independent translations by one lattice spacing along the chains. Notice, however, that for different pairs of chains the signs of the order parameters are *uncorrelated*. The only condition imposed on the η's is

$$\prod_{n=0}^{2N} \eta_n \propto (-1)^Q = 1, \tag{B20}$$

where the integer

$$Q = \sum_{\mu=\pm} \sum_{m=0}^{N-1} \mu [m_{2m}^{\mu} - m_{2m+1}^{\mu}]$$

vanishes according to relations (B3) in the (+) sector and their counterparts in the (–) sector. Equation (B20) is not a restrictive condition; it simply says that the numbers of positive and negative η's should be even. Thus the ground state of the system exhibits a huge degree of degeneracy. Taking the constraint (B20) into account, the number of the degenerate ground-state configurations in a system of $2N$ chains is estimated as

$$\frac{1}{2} \sum_{n=0}^{2N} C_{2N}^{n} = \frac{1}{2} \sum_{n=0}^{2N} \frac{(2N)!}{n!(2N-n)!} = 2^{2N-1}.$$

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