

Theory of the d -density wave from a vertex model and its implications

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The thermal disordering of the d -density wave, proposed to be the origin of the pseudogap state of high-temperature superconductors, is suggested to be the same as that of the statistical mechanical model known as the six-vertex model. The low-temperature phase consists of a staggered order parameter of circulating currents, while the disordered high-temperature phase is a power-law phase with no order. A special feature of this transition is the complete lack of an observable specific heat anomaly at the transition. There is also a transition at a even higher temperature at which the magnitude of the order parameter collapses. These results are due to classical thermal fluctuations and are entirely unrelated to a quantum critical point in the ground state. The quantum-mechanical ground state can be explored by incorporating processes that causes transitions between the vertices, allowing us to discuss the quantum phase transition in the ground state as well as the effect of quantum criticality at a finite temperature as distinct from the power-law fluctuations in the classical regime. A generalization of the model on a triangular lattice that leads to a 20-vertex model may shed light on the Wigner glass picture of the metal-insulator transition in a two-dimensional electron gas. The power-law ordered high-temperature phase may be generic to a class of constrained systems, and its relation to recent advances in the quantum dimer models is noted.

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I. INTRODUCTION

A motivating factor for the present paper is the specific suggestion that a broken symmetry can explain the pseudogap phase of high-temperature superconductors.^{1,2} The corresponding order is a particle-hole condensate of “angular momentum” 2, termed the d -density wave (DDW).³ There is some telltale evidence of this unusual order parameter⁴ involving circulating orbital currents arranged in a staggered pattern, which is directly detected as a Bragg scattering signal in neutron measurements. The second motivating factor is to place this proposal in a wider context of many-body theory, where a strongly correlated electronic system can have unconventional broken symmetries in the ground state. In this respect, I shall briefly touch upon the topic of two-dimensional electron gas in Si-MOSFET devices.⁵

The building blocks of the low-energy theory corresponding to DDW are bond currents whose arrangements define the various order parameters that reflect particle-hole condensates.⁶ The idea is clearly similar to resonating valence bonds (RVB’s),⁷ where the building blocks are valence bonds that can be described in terms of particle-particle condensates. These may order in the ground state, or they may not, in which case one has a spin liquid.⁸ In either case, strong correlation effects are believed to play an important role. The parallel goes further: while a tractable RVB Hamiltonian is the quantum dimer model,⁹ a tractable model for bond currents will be seen to be a vertex model known in statistical mechanics and its suitable quantum generalization.

The actual statistical mechanics of the DDW transition is richer than the mean-field (Hartree-Fock) picture, in which the ordered pattern is frozen, until the magnitude of the circulating currents vanishes, which is the simplest possible description of the broken symmetry phase. While this is reasonable deep in the ordered state, it does not allow for

fluctuations of the order parameter.

A natural modification that I shall describe, involving the six-vertex model,^{10,11} leads to striking consequences for the pseudogap phase: (1) As the d -density wave disorders with increasing temperature, the system exits to a power-law phase that is not due to any underlying quantum critical points. (2) The pseudogap transition does not have any specific heat anomaly.¹² (3) Because of the power-law nature of the orbital current correlations of the disordered state, it is likely that the electron spectral function exhibits a cut spectrum, unlike a Fermi liquid. (4) As the temperature is raised, a second transition takes place at a higher temperature, where the local amplitude of the pseudogap vanishes. It is unusual, but true, that the two-dimensional classical statistical mechanics of the pseudogap state viewed as a DDW allows for a power-law phase with an infinite correlation length in the high-temperature regime, while the low-temperature phase is ordered with a finite correlation length.

The outline of the paper is as follows. In Sec. II, I introduce and discuss the six-vertex model for the pseudogap phase, and, in Sec. III, I discuss its quantum generalization. Section IV is a brief analysis of the relation with the quantum dimer model. In Sec. V, I introduce the 20-vertex model on a triangular lattice and discuss its relevance to the Wigner glass picture of the in metal-insulator transition. Finally, the conclusions are summarized in Sec. VI.

II. SIX-VERTEX MODEL FOR THE PSEUDOGAP PHASE

The ordered singlet DDW state consists of a staggered circulating pattern of currents flowing on the square planar CuO lattice, as shown in Fig. 1. The configuration is a result of juxtaposing two sets of vertices, shown in Fig. 2, centered on the Cu atoms in a current-conserving manner. There are only two possible choices, resulting in two distinct broken

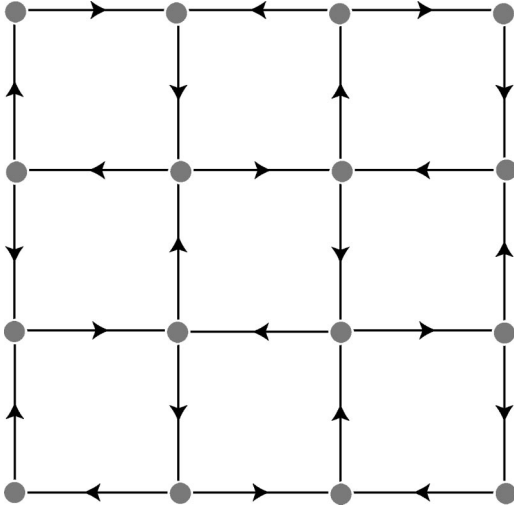


FIG. 1. A segment of the circulating current pattern in the DDW ordered state.

symmetry states, which are related by time reversal and a lattice translation.

More mathematically, the order parameter $\Phi_{\mathbf{Q}}$ is given by

$$\langle \psi^{\sigma\dagger}(\mathbf{k} + \mathbf{Q}, t) \psi_{\rho}(\mathbf{k}, t) \rangle = i \frac{\Phi_{\mathbf{Q}}}{2} (\cos k_x - \cos k_y) \delta_{\rho}^{\sigma}, \quad (1)$$

where σ and ρ are the spin indices, and $\mathbf{Q} = (\pi, \pi)$. The lattice constant has been set to unity, and the operator ψ_{ρ} is the electron destruction operator. The order parameter is called a density wave because it is a particle-hole condensate, even though what is actually modulated is current, not density. The order parameter $\Phi_{\mathbf{Q}}$ is a spin singlet; there is also a triplet version in which it is the spin current that is modulated.⁶ The order parameter is called a d wave because of the internal form factor of the particle and the hole, which is $(\cos k_x - \cos k_y)$. Of course, on a crystalline lattice, angular momentum is not a good quantum number. This is the closest we can come to the angular momentum 2 of a $d_{x^2-y^2}$ wave function in free space. Note that for particle-hole condensates there are no exchange requirements governing order parameter symmetries that enslave the orbital wave function to the spin wave function, in contrast to superconductivity where the condensate is of particle-particle type. Thus, one can have a DDW that is either a spin singlet, or a spin triplet.

In a mean field (Hartree-Fock) picture, the only way the state can disorder is by the thermal collapse of the magnitude

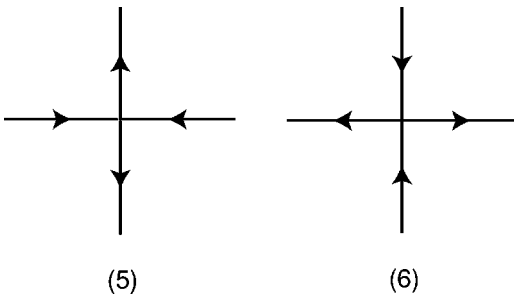


FIG. 2. Vertices 5 and 6.

of the order parameter, which is a highly restrictive mechanism because it does not allow for fluctuations. The system may disorder long before the magnitude of the order parameter collapses. To build fluctuation effects, we consider the basic building blocks that are bond currents between the nearest neighbor sites \mathbf{x} and $\mathbf{y} = \mathbf{x} \pm \hat{x}$, or $\mathbf{y} = \mathbf{x} \pm \hat{y}$, defined by

$$\mathbf{j}_{\mathbf{x},\mathbf{y}} = -i \epsilon \Phi_{\mathbf{Q}} \langle \psi_{\mathbf{x}}^{\dagger} \psi_{\mathbf{y}} - \text{H. c.} \rangle, \quad (2)$$

where $\epsilon = \pm 1$ determines the direction of the current flow. The status of the bond currents is identical to the local order parameter in the Ginzburg-Landau-Wilson formalism, in which the partition function is a sum over all complexions of the order parameter weighted by the effective coarse-grained action. To include quantum fluctuations, one must supplement the theory with a suitable dynamics, which I shall discuss in Sec. II A.

I shall assume that there is a regime of coupling constant and temperature such that the bond current order is well developed, and the fluctuations of the magnitudes of the bond currents can be neglected. If this is the case, there should be a second transition at which the magnitude of the bond current itself vanishes. This would imply a second pseudogap transition at a higher temperature. The existence of a specific-heat anomaly at this upper pseudogap transition is not entirely clear, as it will be seen to be a transition from a power-law ordered phase. However, in any case, the temperature may be too large to extract it from the large phonon or other backgrounds. I shall therefore ignore this transition altogether.

It is easy to convince oneself that low-lying thermal or quantum fluctuations can be expected to reverse a set of bond currents (without changing their magnitudes) provided no sources or sinks are generated, that is, $\nabla \cdot \mathbf{j} = 0$. Given that there are two incoming and two outgoing currents at a vertex of a square lattice, there are altogether $4!/2!2! = 6$ possible vertices. Thus there are four additional allowed vertices, beyond the two shown in Fig. 2. These are shown below in Fig. 3. In general, each of the six local arrangements can have a distinct energy ϵ_i , but if we impose the restriction that there are no net external currents, then

$$\epsilon_1 = \epsilon_2, \quad \epsilon_3 = \epsilon_4, \quad \epsilon_5 = \epsilon_6. \quad (3)$$

The model is then unchanged by reversing all the arrows. For tetragonal symmetry, there are only two independent energy scales ϵ_1 and ϵ_5 , because $\epsilon_1 = \epsilon_3$, but for generality we shall assume that they are distinct.

From a Hartree-Fock analysis,⁶ it can be argued that ϵ_1 and ϵ_5 are close in energy. This is because vertices 1–4 correspond to $(p_x + p_y)$ -density wave (PDW) states in the order parameter language, and the energetics of both the singlet p -density wave and the DDW are controlled by small pair-hopping matrix elements, smaller than the scale of the antiferromagnetic exchange constant J . A singlet p_x -density wave state has the ordering

$$\langle \psi^{\sigma\dagger}(\mathbf{k} + \mathbf{Q}', t) \psi_{\rho}(\mathbf{k}, t) \rangle = \Phi'_{\mathbf{Q}'} \sin k_x \delta_{\rho}^{\sigma}, \quad (4)$$

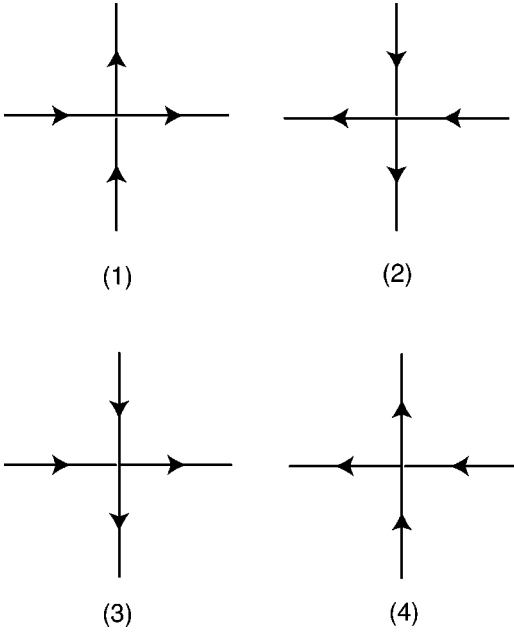


FIG. 3. Vertices 1, 2, 3, and 4.

where $\mathbf{Q}' = (0, \pi)$. Note that in this case $\Phi'_{\mathbf{Q}'}$ is real, but because of the form factor $\sin k_x$, a Fourier transform to the real space brings out the pattern of currents shown in vertices in Fig. 3, when it is superposed with the corresponding p_y -density wave. The order parameter $\Phi'_{\mathbf{Q}'}$ is the closest analog on a lattice of angular momentum 1, the p_x wave function in free space. Note, as before, that it is the current that is modulated, not the density.

I shall first consider thermal fluctuations and take into account quantum fluctuations in the following section. A basic assumption I shall make is that the energy of a given state is a simple sum of energies associated with the configuration at each vertex. This is a reasonable assumption because any long-ranged interaction between the vertices are unlikely, as the thermal smearing will cause any interactions mediated by the nodal quasiparticles to be exponentially decaying. Following the conventional notation, I shall define

$$a = \omega_1 = \omega_2, \quad b = \omega_3 = \omega_4, \quad c = \omega_5 = \omega_6, \quad (5)$$

where the Boltzmann factors are defined by $\omega_i = e^{-\varepsilon_i/T}$, where the Boltzmann constant k_B is set to unity. The partition function is

$$Z = \sum a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}, \quad (6)$$

where n_k is the number of vertices of type k . The sum is over all arrangements that fit together continuously without generating sources and sinks. The partition function is precisely the partition of the six-vertex model for which many exact results are known.^{10,11}

A. Phase diagram

The phase diagram is shown in Fig. 4. Regions I and II have orbital ferromagnetic order due to macroscopic currents

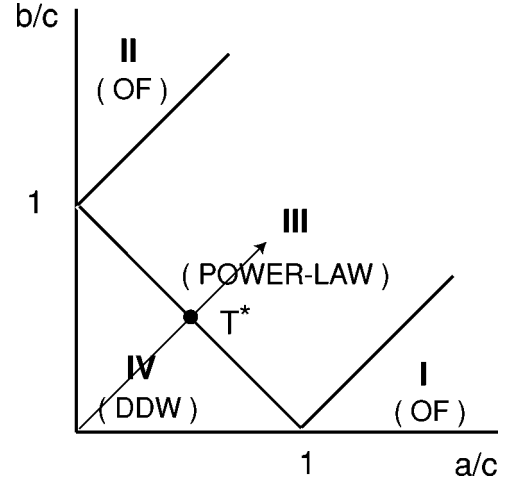


FIG. 4. The phase diagram. Regions I and II are orbital ferromagnets (OF). Region III is the power-law phase and region IV is the orbital antiferromagnet with DDW order. The arrow marks a path (tetragonal symmetry assumed) from the low-temperature phase to the high-temperature phase, with the pseudogap transition at $T_c = T^*$.

and correspond to $a > b + c$ and $b > a + c$, respectively. Region III is disordered, corresponding to $a, b, c < \frac{1}{2}(a + b + c)$, but the current-current correlation function exhibits a power law. It includes the infinite-temperature case $a = b = c = 1$. This entire high-temperature region is on the critical line of the eight-vertex model.^{10,11} Region IV is the DDW phase, which is an orbital antiferromagnet and corresponds to $c > a + b$. The phase boundary between regions III and IV is given by $b/c = -(a/c) + 1$, or $c = a + b$, which implicitly defines the transition temperature T_c .

Consider a given set values of a , b , and c . As the temperature increases from 0 to ∞ , this point follows a path always ending at (1,1). This path may or not cross a phase boundary. The path followed in the phase diagram when $a = b$ is shown as an arrow. On this path, at $T_c = T^*$, there is a phase transition from the ordered DDW phase to the disordered high temperature phase with a power-law correlation in the current-current correlation function. The temperature T^* is given by

$$T^* = \frac{\varepsilon_1 - \varepsilon_5}{\ln 2}, \quad (7)$$

determining, phenomenologically, the energy difference $\Delta\varepsilon = \varepsilon_1 - \varepsilon_5$.

In region IV, the correlation length ξ is^{10,13}

$$\xi = -\ln \left[2x^{1/2} \prod_{m=1}^{\infty} \left(\frac{1+x^{4m}}{1+x^{4m-2}} \right) \right], \quad (8)$$

where $x = e^{-\lambda}$ and $-\cosh \lambda = (a^2 + b^2 - c^2)/2ab$. As $T \rightarrow T_c^-$, $\lambda \propto (T_c - T)^{1/2}$, and it can be seen by applying Poisson summation formula that

$$\xi^{-1} = 8e^{-\pi^2/2\lambda}, \quad \lambda \rightarrow 0. \quad (9)$$

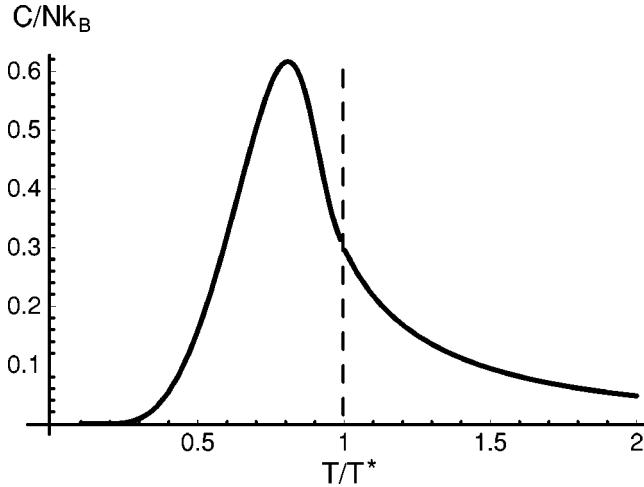


FIG. 5. The specific heat of the six-vertex model as a function of T/T^* for $a=b$. Note that there are no observable anomalies at the transition at T^* , and the peak of the specific, which is entirely due to the analytic part of the free energy, is significantly below the transition.

B. Free energy and specific heat

The exact free energy of the six-vertex model was obtained by Lieb,¹⁵ and it is quite remarkable. If we denote the free energy density in region III by $f_{\text{III}}(T > T_c)$, then its analytic continuation to $T < T_c$ is complex, equal to $f_{\text{IV}}(T < T_c) + if_{\text{sing}}$. The singular part of the free energy has only an essential singularity at $T = T_c$, which is given by

$$f_{\text{sing}} \propto \xi^{-2}. \quad (10)$$

In fact, all temperature derivatives of the free energy exist and are identical on both sides of the transition. An asymptotic expansion at T_c , which is the same above and below the transition, was given in Ref. 11.

The essential singularity implies that there are no observable specific-heat anomalies, as any derivative of the free energy vanishes at this infinite order transition. The situation is exactly the same as that of the Kosterlitz-Thouless phase transition of the two-dimensional XY model. The transition of the six-vertex model is inverted, however, as the low-temperature phase has a finite correlation length, while the high-temperature phase is a power-law phase with a continuously varying critical exponent. The analytic part of the free energy results in a broad heat capacity peak above the transition¹⁴ for the XY model, but below the transition in the six-vertex model.

The specific heat of the six-vertex model was also discussed in Ref. 11, but it is illuminating to plot it here. In Fig. 5, we show C/Nk_B , where N is the number of vertices, as a function of T/T^* , where we have set $a=b$. There is no observable anomaly at T^* despite the thermodynamic transition at this temperature to the ordered DDW phase. Nonetheless, since this is an order-disorder transition, the high-temperature entropy must disappear as $T \rightarrow 0$. At $T = \infty$, $a = b = c = 1$, and the entropy is that of square ice, which was calculated by Lieb¹⁶ to be $S = (3k_B/2) \ln \frac{4}{3}$. The system loses

this entropy by a specific-heat hump clustered around a definite temperature, which is significantly below T^* . The specific heat per vertex is also a universal function of T/T^* . That it vanishes at high temperatures—true for many statistical mechanical models such as the Ising model—simply reflects the bounded energy spectrum on the high-energy side and must not be taken seriously. The high energy spectrum is not modeled correctly by such low-energy effective models. As a result of the boundedness of the energy spectrum, the entire passage from complete order to complete disorder takes place over a limited range of temperature. However, we should probably take seriously the fact that the DDW transition results in a specific heat of order $\sim 0.5k_B$ per vertex considerably below the DDW transition, and may be mistaken as the pretransitional effect as the system enters the superconducting state from the pseudogap phase. The specific-heat peaks at the superconducting transition in the underdoped regime indeed appear to be anomalous and very broad. The low-temperature behavior of the specific heat is given by

$$\frac{C}{Nk_B} = 2 \left(\frac{T^* \ln 2}{T} \right)^2 e^{-(T^*/T) \ln^2}, \quad T \rightarrow 0. \quad (11)$$

C. Floating power-law phase

How does the coupling between the two-dimensional CuO planes change the behavior of the system? Since the low-temperature phase is ordered with a finite correlation length, a small coupling between the planes will in general have a small effect, except close to the critical region where the correlation length diverges and a crossover to three dimensions should take place. This can be checked explicitly from a mean field analysis. The coupling between the planes due to the orbital currents is indeed very small, as the magnetic field at the center of a plaquette generated by the circulating currents can be estimated to be of order 10 G from the magnitude of the pseudogap.² There is some experimental support to this fact,⁴ as the correlation length in the perpendicular direction does not extend much beyond a unit cell.

In contrast, the effect of three-dimensional (3D) coupling in the high-temperature power-law phase is much more interesting. In principle, the 2D power-law phase can be destroyed by an arbitrarily weak coupling between the planes. However, because of the miniscule coupling between the planes due to an estimated 10-G magnetic field of a plaquette,² the crossover to 3D order is likely to take place at very long length scales. Thus this is effectively a floating phase in which the system behaves as a stack of decoupled 2D power-law phases. The argument, in principle, is the same as that of Josephson coupled 2D XY systems in a direction perpendicular to the planes,¹⁷ although the details are different.

Consider the current-current coupling of a given layer n , with a neighboring layer $(n+1)$, in the continuum limit, which is

$$H_c[n] = g \int \frac{d^2 r}{s^2} \mathbf{j}_n(r) \cdot \mathbf{j}_{n+1}(r), \quad (12)$$

where s is a short distance cutoff of the order of the lattice spacing. It is sufficient to consider the coupling between the nearest-neighbor planes for our purposes. We wish to determine the relevancy of the coupling g under the renormalization-group transformation. Let

$$\langle \mathbf{j}_n(r) \cdot \mathbf{j}_{n+1}(r) \mathbf{j}_n(0) \cdot \mathbf{j}_{n+1}(0) \rangle \sim \frac{1}{r^{2x}}, \quad (13)$$

where the average is taken with respect to the fixed-point Hamiltonians of the uncoupled planes. Since every point in the power-law phase corresponds to a fixed point with a continuously varying critical exponent, we merely need to determine the critical exponent x of the current-current correlation function.

The exponent of the arrow-arrow correlation function of the six-vertex model (or the current-current correlation function in the present problem) in the power-law phase can be obtained from the exponent of the eight-vertex model along the critical line, which in turn can be obtained from the S_z - S_z correlation function of the 1D quantum XXZ Hamiltonian. The correspondence can also be seen by considering the Trotter-Suzuki decomposition of the 1D XXZ Hamiltonian, in which the arrow correlations along a row of the six-vertex model (or along a column) correspond to correlations along the diagonal of the space-imaginary time lattice.¹⁸ This subtlety is of no consequence, as we are interested in the exponent along a critical line at which the correlation length is infinite.

The S_z - S_z correlation has a staggered part and a uniform part. The exponent for the uniform part is 2, independent of the temperature, while the exponent of the staggered part is given by $1/\theta$, where^{13,19}

$$\theta = 1 - \frac{\mu}{\pi}. \quad (14)$$

Assuming a tetragonal symmetry ($a=b$),

$$\cos \mu = \frac{1}{2} e^{(2 \ln 2) \frac{T^*}{T}} - 1. \quad (15)$$

Therefore, the exponent $1/\theta$ varies monotonically from 1 at $T=T^*$ to 3 at $T=\infty$. The exponent of the uniform part follows from a local conservation law combined with the conformal invariance in two dimensions.²⁰ Clearly, the slowest decay will determine the 3D crossover. At T^* , it is determined by the staggered part since its exponent is less than the exponent of the uniform part. As the temperature increases, the exponent of the staggered part increase and crosses the uniform exponent. We therefore define the exponent x to be either the exponent of the uniform part, or that of the staggered part of the current-current correlation function, whichever is smaller. Under a renormalization group transformation, for which $s \rightarrow \lambda s$, where $\lambda > 1$ is a scale factor, $\mathbf{j}_n(r) \cdot \mathbf{j}_{n+1}(r) \rightarrow \lambda^x \mathbf{j}_n(r) \cdot \mathbf{j}_{n+1}(r)$ from Eq. (13). Thus, for

the free energy to remain invariant, it follows that $g \rightarrow \lambda^{2-x} g$; note that $d=2$. Therefore, at $T=T^*$, where $x=1$, the 3D coupling g is strongly relevant. As the temperature $T \rightarrow \infty$, the uniform exponent ($x=2$) takes over and g becomes marginal.

III. QUANTUM VERTEX MODEL

It is interesting to ask what a quantum generalization of this model could be. At the very least, we must allow the bond currents to flip as a result of quantum fluctuations. The most local move is to flip the currents around an elementary plaquette of the square lattice. But it is easy to see that such a process can create sources and sinks, if we do not impose any constraints. From a path integral description, the configuration at an intermediate (imaginary) time slice may contain a source or a sink even though the states at the initial and the final time slices at 0 and β do not.

I start by considering the eight-vertex model written in terms of Ising spin variables, and then incorporate quantum mechanics by introducing a transverse field on a dual lattice site. The advantage is that the description is in terms of the familiar Ising basis.²¹ One must then explicitly introduce projection operators to eliminate the unwanted sources and sinks.

The eight-vertex model (allowing only an even number of arrows into and out of each site) can be written as two coupled Ising models on interpenetrating square lattices with an additional four spin coupling, as is well known.^{22,23} The spins are situated on a dual lattice consisting of the sites at the center of the plaquettes of the original square lattice of bond currents. An arrow to the right (or upward) represents adjacent parallel spins; similarly, an arrow to the left (or downward) corresponds to adjacent antiparallel spins. The mapping is shown explicitly in Figs. 6 and 7. The additional vertices shown in Fig. 7 are assigned energies $\varepsilon_7 = \varepsilon_8$, and the number of such vertices must satisfy $n_7 = n_8 = n_d$. We shall denote these vertices as of type d . However, ultimately, we must impose an additional constraint on the Hamiltonian in which the vertex configurations shown in Fig. 7 are removed from the Hilbert space of states.

Quantum fluctuations are incorporated by a transverse field h . The resulting model, the quantum six-vertex model, is then defined by the partition function

$$Z_Q = \text{Tr} e^{-\beta P H P}, \quad (16)$$

where the Hamiltonian H is given by

$$H = -h \sum_{j,k} \sigma_{j,k}^x - \sum_{j,k} (J \sigma_{j,k}^z \sigma_{j+1,k+1}^z + J' \sigma_{j+1,k}^z \sigma_{j,k+1}^z) - J'' \sum_{j,k} \sigma_{j,k}^z \sigma_{j+1,k+1}^z \sigma_{j+1,k}^z \sigma_{j,k+1}^z. \quad (17)$$

The operators σ^x and σ^z are the standard Pauli matrices, and $P^2 = P$ is the projection operator that projects out the sources and sinks. It is easy to check that a single spin flip at a dual lattice site flips the bond currents on the surrounding plaquette, which causes transitions between the vertices on

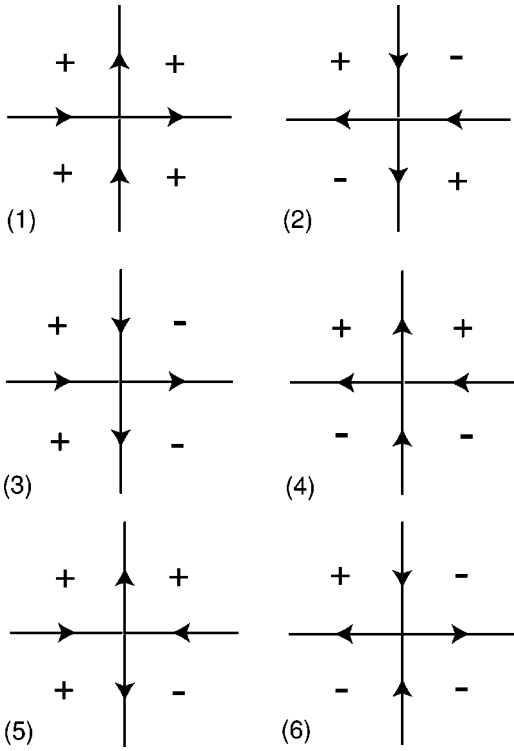


FIG. 6. The mapping of the first six vertices to equivalent spins on the dual lattice.

the four corners, as shown in Fig. 8. The previously defined vertex weights a , b , and c are given by

$$a = e^{(J+J'+J'')/T}, \tag{18}$$

$$b = e^{(-J-J'+J'')/T}, \tag{19}$$

$$c = e^{(-J+J'-J'')/T}. \tag{20}$$

The vertex weight d is

$$d = e^{(J-J'-J'')/T}. \tag{21}$$

The classical statistical mechanics that controls the finite-temperature phase transition is unchanged because there are no quantum-mechanical spin flips. Since the trace is a sum over states from which sources and sinks are projected out, the model is clearly the classical six-vertex model—the number of vertices of type d , $n_d=0$. We can arrive at the same conclusion by considering an imaginary-time path integral from a Trotter decomposition along the imaginary-time di-

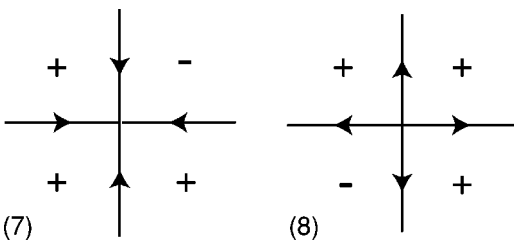


FIG. 7. The mapping of the vertices involving sinks and sources to spins on the dual lattice.

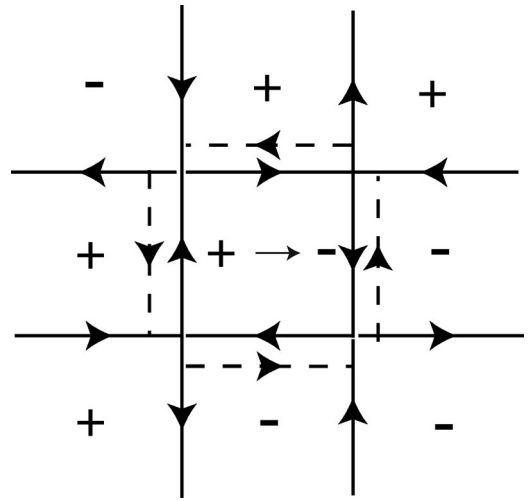


FIG. 8. The flip of the central spin from $+ \rightarrow -$ flips the surrounding bound currents around the plaquette. The flipped currents are denoted by dashed arrows.

rection τ between 0 and β , with a periodic boundary condition. The constraint can be enforced by including a delta functional $\delta[\nabla \cdot \mathbf{j}(\mathbf{x}, \tau)]$ at each space-time point. In the classical limit, we are left with a single time slice, because quantum spin flips due to the transverse field are punished. The partition function is now a sum over all spin configurations subject to the condition that there are no sources or sinks. Of course, at any finite temperature, the effective parameters of the model will be renormalized by quantum fluctuations at short distances, but that cannot change the universality class of the finite-temperature phase transition.

At $T=0$, the model is somewhat complex because of the projection operators. At $h=0$, the ground state is the ordered DDW state with no quantum fluctuations, if the vertices c have the lowest possible energy. As the transverse field h increases from zero, quantum fluctuations increase and nucleate PDW vertices. It is not clear what the ultimate fate of the system is. There are four distinct possibilities: (1) a quantum phase transition to a gapped quantum disordered state, which I find unlikely because of the constraints. If this is the case, the constraints have to be somehow irrelevant. (2) A transition to a PDW state, which I also find unlikely because there are no flippable plaquettes, and hence the state is disfavored from kinetic-energy considerations. (3) A transition to a power-law phase since a local deformation may not be able to heal sufficiently fast. (4) There is also a remote possibility that the system does not lose its DDW order. In any case, the model is sufficiently complex that at present I cannot provide a quantitative analysis. It is probably best studied by quantum Monte Carlo methods involving loop algorithms.²⁴

I shall assume that it is the third possibility discussed above that is realized. If this is correct, such a quantum critical point is unusual from the conventional perspective.²⁵ To illustrate this point, consider a phase diagram in which a system undergoes a continuous phase transition as a function of tuning parameters, temperature T , and a coupling h such as the transverse field. This is shown in Fig. 9, where the

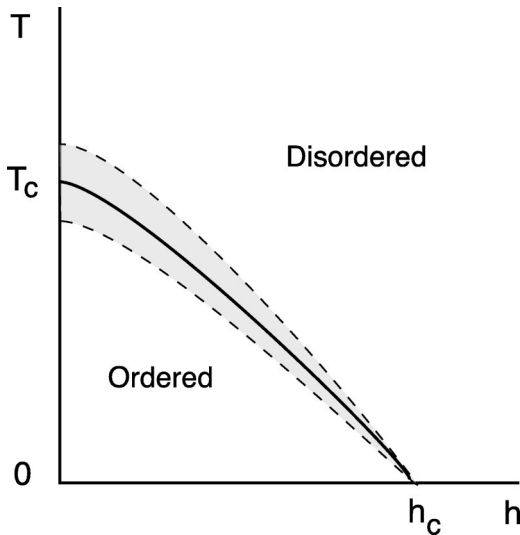


FIG. 9. Conventional phase diagram in which a finite temperature phase transition exists—the solid curve. The shaded region is the domain of classical critical fluctuations. The influence of the quantum critical point h_c , at $T=0$, has to lie outside this domain.

shaded region is the domain of classical critical fluctuations extending all the way to zero energy. Any influence of the quantum critical point at h_c has to be outside this domain. Thus we may argue that, on some scale, the disordered phase at a finite temperature could be influenced by the quantum critical point.

The phase diagram shown in Fig. 10, pertinent to our present problem, is strikingly different. Here the existence of a classical power-law phase of the six-vertex model has en-

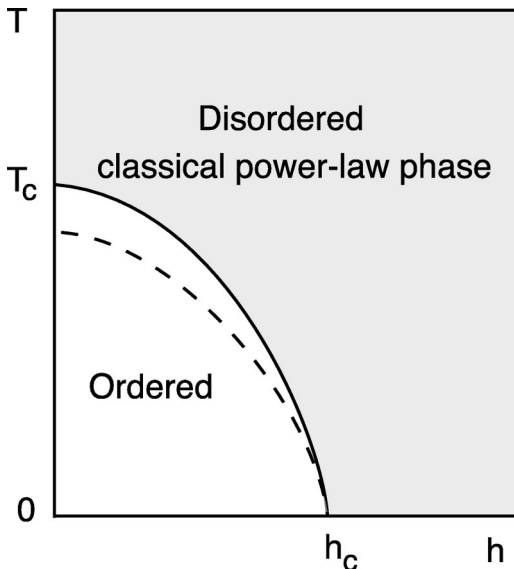


FIG. 10. The phase diagram in which the entire finite-temperature disordered phase is a classical power-law phase—the shaded region. The dashed curve is the crossover scale for the classical critical fluctuations within the ordered phase. The quantum critical regime is entirely eliminated, except precisely at $T=0$. The ordered regime is under the influence of the stable broken symmetry fixed point.

tirely eliminated the quantum fluctuations, which are, strictly speaking, cut off by the thermal length, while the classical power-law correlations have no such long-distance cutoff. Within the ordered phase, there is a classical critical region, but otherwise the ordered phase is under the influence of the broken symmetry fixed point. Of course, we are assuming that the magnitude of the order parameter can be assumed to be fixed. If not, there will be a further transition at a higher temperature where its magnitude collapses. The region above this transition can now be influenced by the quantum critical point. Nonetheless, there may be a sizable region of the phase diagram in which the classical power-law phase dominates.

IV. RELATION TO THE QUANTUM DIMER MODEL

As remarked upon in Sec. I, there are some similarities with the RVB ideas, particularly with the recent developments involving quantum dimer models,^{8,26} but the distinctions are equally important. At high temperatures, the quantum dimer model on a square lattice maps on to the classical dimer model with power-law correlations,²⁷ which is a $c = 1$ (conformal charge) theory, equivalent to two massless Majorana fermions. It is also the fully frustrated Ising model at $T=0$.

This critical phase arises from the Rokhsar-Kivelson quantum critical point.^{9,28} Instead, in the present case, the quantum vertex model maps on to the classical six-vertex model, which undergoes a phase transition with an essential singularity from an ordered region IV to the power-law region III. But this power-law phase does not stem from a quantum critical point, instead the entire region is situated on the critical line of the classical 8-vertex model.

A recent thrust in quantum dimer models focused on the existence of a spin liquid phase, which is made plausible by the fact that the classical dimer model on a triangular lattice is noncritical, with exponentially decaying dimer correlations.⁸ Thus the ground state of the triangular case has no gapless collective excitations, only deconfined, gapped spinons for a finite range of parameters. In contrast, we have focused on the existence of a power-law phase above the DDW phase and the nonexistence of the specific-heat anomaly at the transition. We have also argued that it is also possible to have a quantum phase transition in the ground state, which is induced by a sufficiently large transverse field mimicking the actual pseudogap transition in the ground state. There are also interesting resemblances of our work with the quantum and sliding ice pictures of Ref. 26.

V. VERTEX MODELS ON A TRIANGULAR LATTICE

Vertex models may be useful in other contexts. For example, consider the ideas that were put forward recently regarding the metal-insulator transition in two-dimensional electron systems in which the transition is identified as a quantum phase transition in a disordered Wigner crystal state.^{29,30} The chain of reasoning is as follows: extensive calculations of multiparticle exchange Hamiltonians in pure^{29–31} and disordered Wigner crystals³⁰ suggest that there

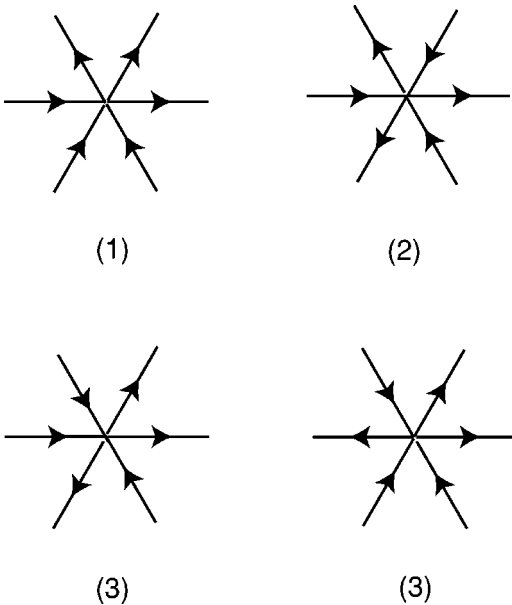


FIG. 11. Examples of three types of allowed vertices.

exist ground states with well-defined bond currents. These bond currents can be represented by vertex models on a triangular lattice. It is notable that recent studies revealed that a metal-insulator transition could occur in such models. A prototypical example is provided by a two-dimensional tight-binding Hamiltonian in a random magnetic field and on-site disorder.^{32,33} It is in this context that it is useful to consider a vertex model of bond currents on a triangular lattice and study its quantum dynamics. Although, even in the classical limit, such models are unsolved in general,^{10,34,35} some exact results, known for certain special cases, are sufficiently encouraging and similar to the six-vertex model to pursue further.

If we restrict ourselves to three incoming currents and three outgoing currents at a site of a triangular lattice, this generates a $6!/3!3! = 20$ vertex model. If the vertices which differ only by rotation and reflection are treated alike, they may be classified as follows: (i) six vertices in which the incoming arrows are adjacent, (ii) two vertices in which the incoming and the outgoing arrows alternate round the vertex, and (iii) 12 vertices containing two incoming arrows directly opposite each other. Examples are shown in Fig. 11.

Unlike the eight-vertex or six-vertex model on a square lattice, the solution to the 20-vertex model is not known in general. It is only solved for certain values of the vertex configuration probabilities. As with the six-vertex model, this model undergoes a phase transition to a power-law ordered phase, with an essential singularity in the free energy, from an ordered state with finite correlation length in which the arrows alternate along a row. Since the exact solution is not known in general, it would be extremely useful to explore this model numerically.²⁴

The quantum dynamics of this model can once again be implemented by going over to a Ising spin model on the dual lattice with a transverse field at a site. It would be interesting to explore this model, which could provide us with an interesting perspective on the old problem of a Wigner crystal,

which, in the conventional picture, is an ordered state with at most small zero-point oscillations of an electron about a lattice site; see, however, Refs. 29 and 30.

VI. CONCLUSIONS

I have argued that the natural generalization of the Hartree-Fock model of the DDW are certain vertex models involving bond currents. The finite-temperature transition from the ordered DDW phase to the disordered phase belongs to the universality class of the six-vertex model and is striking because of the absence of any observable specific-heat anomalies. I have also shown that the disordered phase is a power-law phase of the bond current correlations. These power-law correlations do not arise from any quantum critical point, but this entire phase is poised on the critical line of another hidden model, the eight-vertex model. The origin of the power-law correlations is the constraints that the configurations of the six-vertex model must satisfy. Thus a local deformation cannot heal within a finite correlation length. In this respect, there is a strong similarity with the RVB dimer models on a square lattice, where the constraints in the dimer model are once again responsible for the high-temperature power-law correlations.

It is important to note that the mere knowledge of the nature of the order parameter does not necessarily determine the universality class of the phase transition. For this, it is also necessary to know the nature of the possible excitations that can be thermally populated. Thus labeling the DDW transition as an Ising transition is an oversimplified view.

I was also able to construct a quantum generalization involving an eight-vertex model in a transverse field, but with projection operators that ultimately project out the sources and the sinks of the eight-vertex model. The model is sufficiently complex to present any in-depth discussion at this time, but it is not difficult to motivate a quantum phase transition as a function of the transverse field. Nonetheless, it is remarkable that the effect of the quantum critical point is completely swamped by the classical power-law correlations at any finite temperature. This is a strict departure from the conventional theory of quantum critical point.²⁵

I have also speculated on the possible application of a similar vertex model to the Wigner crystal, and its implications to the metal-insulator transition in the two-dimensional electron systems. Similarly, a zoo of density wave theories⁶ can be usefully formulated in terms of vertex models opening up a whole class of questions in the theory of quantum statistical mechanics. Some of these would correspond to modulations of the bond kinetic energy.^{6,36} In fact, it is quite possible that certain regions of the cuprate phase diagram are described by these models,³⁷ but the aim here was to consider only those models that are natural generalizations of the proposed DDW state.

The present analysis is incomplete in one important respect. I have been unable to make firm statements regarding the quasiparticle excitations and doping in the present model, although it is clear that power-law bond current correlations in the disordered phase probably leads to a non-Fermi-liquid behavior of the electronic excitations. Since we have consid-

ered models of the spin-singlet variety, there cannot be any spin-charge separation by definition. The non-Fermi-liquid behavior must then be akin to that of the spinless Luttinger model in one dimension. One should note that a rigorous description of electronic excitations are also unavailable for the RVB quantum dimer model.

In the future, it may also be interesting to allow for sources and sinks, as they may represent an effective means of incorporating decay of collective excitations involving the order parameter into low-lying nodal quasiparticles in the actual physical system.³⁸ From this perspective, the correct quantum model is actually an eight-vertex model in a trans-

verse field, as discussed above, but without any complications of the projection operators.

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- ¹S. Chakravarty and H. Y. Kee, Phys. Rev. B **61**, 14 821 (2000).
- ²S. Chakravarty, R. B. Laughlin, D. K. Morr, and C. Nayak, Phys. Rev. B **63**, 094503 (2001).
- ³H. J. Schulz, Phys. Rev. B **39**, 2940 (1989); I. Affleck and J. B. Marston, *ibid.* **37**, 3774 (1988); G. Kotliar *ibid.* **37**, 3664 (1988); D. A. Ivanov, P. A. Lee, and X.-G. Wen, Phys. Rev. Lett. **84**, 3958 (2000).
- ⁴H. A. Mook, P. Dai, and F. Dogan, Phys. Rev. B **64**, 012502 (2001); H. A. Mook, Pengcheng Dai, S. M. Hayden, A. Hiess, J. W. Lynn, S.-H. Lee, and F. Dogan, *ibid.* **66**, 144513 (2002). For a theoretical analysis of these experiments, see S. Chakravarty, H.-Y. Kee, and C. Nayak, Int. J. Mod. Phys. B **15**, 2901 (2001); in *Physical Phenomena at High Magnetic Fields IV*, edited by G. Boebinger, Z. Fisk, L. P. Gor'kov, A. Lacerda, and J. R. Schrieffer (World Scientific, Singapore, 2002), p. 242.
- ⁵E. Abrahams, S. V. Kravchenko, and M. P. Sarachik, Rev. Mod. Phys. **73**, 251 (2001).
- ⁶C. Nayak, Phys. Rev. B **62**, 4880 (2000).
- ⁷P. W. Anderson, Science **235**, 1196 (1987).
- ⁸R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
- ⁹D. S. Rokhsar and S. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
- ¹⁰R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, San Diego, 1982).
- ¹¹E. H. Lieb and F. Y. Wu, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic Press, New York, 1972), Vol. 1.
- ¹²For an alternative approach to the question of the specific heat anomaly, see H. -Y. Kee and Y. B. Kim, Phys. Rev. B **66**, 012505 (2002).
- ¹³J. D. Johnson, S. Krinsky, and B. M. McCoy, Phys. Rev. A **8**, 2526 (1973).
- ¹⁴J. E. Van Himbergen, and S. Chakravarty, Phys. Rev. B **23**, 359 (1981).
- ¹⁵E. H. Lieb, Phys. Rev. Lett. **18**, 1046 (1967).
- ¹⁶E. H. Lieb, Phys. Rev. **162**, 162 (1967).
- ¹⁷C. S. O'Hern, T. C. Lubensky, and J. Toner, Phys. Rev. Lett. **83**, 2745 (1999).
- ¹⁸M. Barma and B. S. Shastry, Phys. Rev. B **18**, 3351 (1978).
- ¹⁹A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).
- ²⁰See, for example, P. Ginsparg, in *Les Houches, Session XLIX, 1988, Champs, Cordes et Phénomènes Critiques/ Fields Strings and Critical Phenomena*, edited by E. Brézin and J. Zinn-Justin (Elsevier, Amsterdam, 1989) and references therein.
- ²¹An alternative quantum generalization would be to follow C. Nayak and K. Shtengel [Phys. Rev. B **64**, 064422 (2001)]. In this approach (also pointed out to me by C. L. Henley and S. L. Sondhi), one recognizes that the kinetic energy term is a plaquette operator that flips the currents around a flippable plaquette (as is also the case in the eight-vertex approach). But to this we can add a suitable set of potential energy terms that describe the energetics of the PDW and DDW states. Thus the quantum phase transition in the ground state will be between a DDW state, with many flippable plaquettes, to a PDW state, with no flippable plaquettes. In this alternative approach, the quantum phase transition would be very similar to the quantum dimer model exhibiting a transition between an ordered state with many flippable plaquettes to the ordered valence bond crystal state with no flippable plaquettes (Ref. 9). This is also a possible, but I feel unlikely, alternative in the eight-vertex approach taken in the main part of the text. Some care is necessary for setting up the parallel with the quantum dimer model, however: (a) the number of states in the dimer model is 4 not 6. (b) The time reversal symmetry is broken in the DDW state, but not in the dimer model. As of this moment, I have been unable to construct a precise Hamiltonian in this manner, which reduces exactly to the six-vertex model in the classical limit, but I have no doubts that it could be done.
- ²²L. P. Kadanoff and F. Wegner, Phys. Rev. B **4**, 3989 (1971).
- ²³F. Y. Wu, Phys. Rev. B **4**, 2312 (1971).
- ²⁴H. G. Evertz, cond-mat/9707221 (unpublished).
- ²⁵J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
- ²⁶R. Moessner, O. Tchernyshov, and S. L. Sondhi, cond-mat/0106286 (unpublished).
- ²⁷M. E. Fisher and J. Stephenson, Phys. Rev. **132**, 1411 (1963).
- ²⁸C. L. Henley, cond-mat/9607222 (unpublished); and (unpublished).
- ²⁹S. Chakravarty, S. Kivelson, C. Nayak, and K. Voelker, Philos. Mag. B **79**, 859 (1999).
- ³⁰K. Voelker and S. Chakravarty, Phys. Rev. B **64**, 235125 (2001).
- ³¹B. Bernu, L. Candido, and D. M. Ceperley, Phys. Rev. Lett. **86**, 870 (2001).
- ³²D. N. Sheng and Z. Y. Weng, Europhys. Lett. **50**, 776 (2000).
- ³³H. K. Nuygen, Phys. Rev. B **66**, 144201 (2002).
- ³⁴R. J. Baxter, J. Math. Phys. **10**, 1211 (1969).
- ³⁵S. B. Kelland, Aust. J. Phys. **27**, 813 (1974); J. Phys. A **7**, 1907 (1974).
- ³⁶S. Sachdev and N. Read, Int. J. Mod. Phys. B **5**, 219 (1991).
- ³⁷K. Park and S. Sachdev, Phys. Rev. B **64**, 184510 (2001), and references therein.
- ³⁸S. Kivelson (private communication).