Possible realization of an ideal quantum computer in Josephson junction array

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We introduce a class of Josephson arrays which have nontrivial topology and exhibit a novel state at low temperatures. This state is characterized by long-range order in a *two* Cooper pair condensate and by a discrete topological order parameter. These arrays have degenerate ground states with this degeneracy "protected" from the external perturbations (and noise) by the topological order parameter. We show that in ideal conditions the low order effect of the external perturbations on this degeneracy is exactly zero and that deviations from ideality lead to only exponentially small effects of perturbations. We argue that this system provides a physical implementation of an ideal quantum computer with a built-in error correction and show that even a small array exhibits interesting physical properties such as superconductivity with double charge, $4e$, and extremely long decoherence times.

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I. INTRODUCTION

Quantum computing[1,2] is, in principle, a very powerful technique for solving classic "hard" problems such as factorizing large numbers[3] or sorting large lists.[4] The remarkable discovery of quantum error correction algorithms[5] shows there is no problem of principle involved in building a functioning quantum computer. However, implementation still seems dauntingly difficult: the essential ingredient of a quantum computer is a quantum system with 2^K (with $K \sim 10^4 - 10^6$) quantum states which are degenerate (or nearly so) in the absence of external perturbations and are insensitive to the "random" fluctuations which exist in every real system, but which may be manipulated by controlled external fields with errors less than $10^{-4} - 10^{-6}$ (big system sizes, K, are needed to correct the errors; for smaller errors the size of the system, K, gets much smaller).[6] Insensitivity to random fluctuations means that any coupling to the external environment neither induces transitions among these 2^K states nor changes the phase of one state with respect to another. Mathematically, this means that one requires a system whose Hilbert space contains a 2^K-dimensional subspace (called "the protected subspace"[7]) within which any local operator $\hat{O}$ has (to a high accuracy) only state-independent diagonal matrix elements: $\langle n|\hat{O}|m\rangle = O_0 \delta_{mn} + o[\exp(-L)]$ where L is a parameter such as the system size that can be made as large as desired. It has been very difficult to design a system which meets these criteria. Many physical systems (for example, spin glasses[8]) exhibit exponentially many essentially degenerate states, not connected to each other by local operators. However, the requirement that all diagonal matrix elements be equal (up to vanishingly small terms) is highly nontrivial and puts such systems in a completely new class. Parenthetically we note that such systems were discussed in philosophical terms by I. Kant (who termed them noumenons; in his thinking the noumenal world is impenetrable but contains comprehensible information) in Ref. 9.

One very attractive possibility, proposed in an important paper by Kitaev[7] and developed further in Ref. 10 involves a protected subspace[11,12] created by a topological degeneracy of the ground state. Typically such degeneracy happens if the system has a conservation law such as the conservation of the *parity* of the number of "particles" along some long contour. Physically, it is clear that two states that differ only by the parity of some large number that cannot be obtained from any local measurement are very similar to each other. The model proposed in Ref. 7 has been shown to exhibit many properties of the ideal quantum computer; however, before now no robust and practical implementation was known. In a recent paper we and others proposed a Josephson-junction network which is an implementation of a similar model with protected degeneracy and which is possible (although difficult) to build in the laboratory.[13]

In the present paper we develop ideas of Ref. 13 proposing a new Josephson-junction network that has a number of practical advantages. (i) This network operates in a phase regime (i.e., when Josephson energy is larger than the charging energy), which reduces undesired effects of parasitic stray charges. (ii) All Josephson junctions in this array are similar which should simplify the fabrication process. (iii) This system has 2^K degenerate ground states "protected" to an even higher extent than in Ref. 13: matrix elements of local operators scale as ε^L, where $\varepsilon \leqslant 0.1$ is a measure of nonideality of the system's fabrication (e.g., the spread of critical currents of different Josephson junctions and geometrical areas of different elementary cells in the network). (iv) The new array does not require a fine tuning of its parameters into a narrow region. The relevant degrees of freedom of this new array are described by the model analogous to the one proposed in Ref. 7. Even when such system is small and contains only a few "protected" states its physical properties are remarkable: it is a superconductor with the elementary charge $4e$ and the decoherence time of the protected states can be made macroscopic, allowing "echo" experiments.

Below we first describe the physical array, and identify its relevant low-energy degrees of freedom and the mathemati-

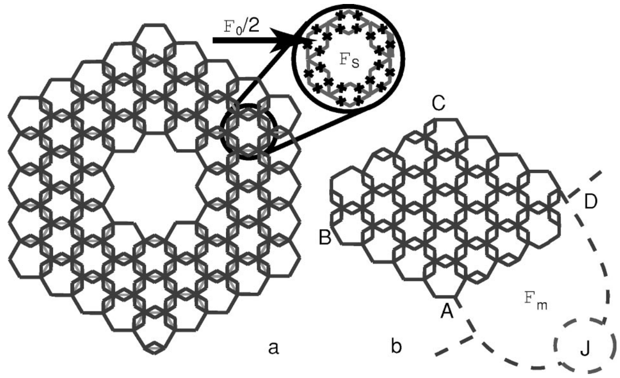

FIG. 1. Examples of the proposed Josephson-junction array. Thick lines show superconductive wires, and each wire contains one Josephson junction as shown in the detailed view of one hexagon. The width of each rhombi is such that the ratio of the area of the Star of David to the area of one rhombi is an odd integer. The array is put in magnetic field such that the flux through each elementary rhombus and through each Star of David (inscribed in each hexagon) is a half integer. Thin lines show the effective bonds formed by the elementary rhombi. The Josephson coupling provided by these bonds is π periodic. (a) Array with one opening; generally the effective number of qubits, K, is equal to the number of openings. The choice of boundary condition shown here makes the superconducting phase unique along the entire length of the outer (inner) boundary, and the state of the entire boundary is described by a single degree of freedom. The topological order parameter controls the phase difference between inner and outer boundaries. Each boundary includes one rhombus to allow experiments with flux penetration; magnetic flux through the opening is assumed to be $\Phi_0/2(\frac{1}{2}+m)$ with any integer m. (b) With this choice of boundary circuits the phase is unique only inside the sectors AB and CD of the boundary; the topological degree of freedom controls the difference between the phases of these boundaries. This allows a simpler setup of the experimental test for the signatures of the ground state described in the text, e.g., by a superconducting quantum interference device interference experiment sketched here that involves a measuring loop with flux Φ_m and a very weak junction J balancing the array.

cal model that describes their dynamics. We then show how the protected states appear in this model, derive the parameters of the model, and identify various corrections appearing in a real physical system and their effects. Finally, we discuss how one can manipulate these states in a putative quantum computer and the physical properties expected in small arrays of this type.

II. ARRAY

The basic building block of the lattice is a rhombus made of four Josephson junctions with each side of the rhombus containing one Josephson contact. These rhombi form a hexagonal lattice as shown in Fig. 1. We denote the centers of the hexagons by letters $a,b\ldots$ and the individual rhombi by $(ab),(cd)\ldots$, because each rhombus has a one-to-one correspondence with the link (ab) between the sites of the triangular lattice dual to the hexagonal lattice. The lattice is placed in a uniform magnetic field so that the flux through each rhombus is $\Phi_0/2$. The geometry is chosen in such a way that the flux Φ_s through each Star-of-David model is a half-integer multiple of Φ_0: $\Phi=(n_s+\frac{1}{2})\Phi_0$.[14] Finally, globally the lattice contains a number, K, of large openings [the size of the opening is much larger than the lattice constant; a lattice with $K=1$ is shown in Fig. 1(a)]. The dimension of the protected space will be shown to be equal to 2^K. The system is characterized by the Josephson energy $E_J=(\hbar/2e)I_c$ of each contact and by the capacitance matrix of the islands (vertices of the lattice). We shall assume (as is usually the case) that the capacitance matrix is dominated by the capacitances of individual junctions; we write the charging energy as $E_C=e^2/2C$. The "phase regime" of the network mentioned above implies that $E_J>E_C$. The whole system is described by the Lagrangian

$$\mathcal{L}=\sum_{(ij)}\frac{1}{16E_C}(\dot{\phi}_i-\dot{\phi}_j)^2+E_J\cos(\phi_i-\phi_j-a_{ij}), \qquad (1)$$

where ϕ_i are the phases of individual islands and a_{ij} are chosen to produce the correct magnetic fluxes. The Lagrangian (1) contains only gauge-invariant phase differences, $\phi_{ij}=\phi_i-\phi_j-a_{ij}$, so it will be convenient sometimes to treat them as independent variables satisfying the constraint $\Sigma_\Gamma\phi_{ij}=2\pi\Phi_\Gamma/\Phi_0+2\pi n$ where the sum is taken over closed loop Γ and n is arbitrary integer.

As will become clear below, it is crucial that the degrees of freedom at the boundary have dynamics identical to those in the bulk. To ensure this one needs to add additional superconducting wires and Josephson junctions at the boundary. There are a few ways to do this; two examples are shown in Fig. 1(a) and Fig. 1(b): type-I boundary [entire length of boundaries in Fig. 1(a), parts AB, CD] and type-II boundary (BC, AD). For both types of boundaries one needs to include in each boundary loop the flux which is equal to $Z_b^*\pi/2$ where Z_b is the coordination number of the dual triangular lattice site. For instance, for the four coordinated boundary sites one needs to enclose the integer flux in these contours. In the type-I boundary the entire boundary corresponds to one degree of freedom (phase at some point) while the type-II boundary includes many rhombi so it contains many degrees of freedom.

Note that each (inner and outer) boundary shown in Fig. 1(a) contains one rhombus. We included it to allow flux to enter and exit through the boundary when it is energetically favorable.

III. GROUND STATE, EXCITATIONS, AND TOPOLOGICAL ORDER

In order to identify the relevant degrees of freedom in this highly frustrated system we consider first an individual rhombus. As a function of the gauge-invariant phase difference between the far ends of the rhombus the potential energy is

$$U(\phi_{ij})=-2E_J[|\cos(\phi_{ij}/2)|+|\sin(\phi_{ij}/2)|]. \qquad (2)$$

This energy has two equivalent minima, at $\phi_{ij}=\pm\pi/2$ which can be used to construct an elementary unprotected qubit, see

Ref. 15. In each of these states the phase changes by $\pm\pi/4$ in each junction clockwise around the rhombus. We denote these states as $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. In the limit of large Josephson energy the space of low-energy states of the full lattice is described by these binary degrees of freedom, and the set of operators acting on these states is given by Pauli matrices $\sigma_{ab}^{x,y,z}$. We now combine these rhombi into hexagons forming the lattice shown in Fig. 1. This gives another condition: the sum of phase differences around the hexagon should equal the flux, Φ_s, through each Star of David inscribed in this hexagon. The choice $\phi_{ij}=\pi/2$ is consistent with flux Φ_s that is equal to a half-integer number of flux quanta. This state minimizes the potential energy (2) of the system. This is, however, not the only choice. Although flipping the phase of one dimer changes the phase flux around the Star by π and thus is prohibited, flipping two, four, and six rhombi is allowed; generally the low-energy configurations of $U(\phi)$ satisfy the constraint

$$\hat{P}_a=\prod_b \sigma_{ab}^z=1, \tag{3}$$

where the product runs over all neighbors, b, of site a. The number of (classical) states satisfying the constraint (3) is still huge: the corresponding configurational entropy is extensive (proportional to the number of sites). We now consider the charging energy of the contacts, which results in the quantum dynamics of the system. We show that it reduces this degeneracy to a much smaller number 2^K. The dynamics of the individual rhombus is described by a simple Hamiltonian $H=\tilde{t}\sigma_x$ but the dynamics of a rhombus embedded in the array is different because individual flips are not compatible with the constraint (3). The simplest dynamics compatible with Eq. (3) contains flips of three rhombi belonging to the elementary triangle, (a,b,c), $\hat{Q}_{(abc)}=\sigma_{ab}^x\sigma_{bc}^x\sigma_{ca}^x$ and therefore the simplest quantum Hamiltonian operating on the subspace defined by Eq. (3) is

$$H=-r\sum_{(abc)} Q_{(abc)}. \tag{4}$$

We discuss the derivation of the coefficient r in this Hamiltonian and the correction terms and their effects below but first we solve the simplified model (3) and (4), and show that its ground state is "protected" in the sense described above and that excitations are separated by the gap.[16]

Clearly, it is very important that the constraint is imposed on all sites, including boundaries. Evidently, some boundary hexagons are only partially complete but the constraint should still be imposed on the corresponding sites of the corresponding triangular lattice. This is ensured by additional superconducting wires that close the boundary hexagons in Fig. 1.

We note that constraint operators commute not only with the full Hamiltonian but also with individual $\hat{Q}_{(abc)}$: $[\hat{P}_a,\hat{Q}_{(abc)}]=0$. The Hamiltonian (4) without constraint has an obvious ground state, $|0\rangle$, in which $\sigma_{ab}^x=1$ for all rhombi. This ground state, however, violates the constraint. This can be fixed noting that since operators $\hat{P}_a$ commute with the Hamiltonian, any state obtained from $|0\rangle$ by acting on it by $\hat{P}_a$ is also a ground state. We can now construct a true ground state satisfying the constraint by

$$|G\rangle=\prod_a \frac{1+\hat{P}_a}{\sqrt{2}}|0\rangle. \tag{5}$$

Here $(1+\hat{P}_a)/\sqrt{2}$ is a projector onto the subspace satisfying the constraint at site a and preserving the normalization.

Obviously, the Hamiltonian (4) commutes with any product of $\hat{P}_a$ which is equal to the product of σ_{ab}^z operators around a set of closed loops. These integrals of motion are fixed by the constraint. However, for a topologically nontrivial system there appear a number of other integrals of motion. For a system with K openings a product of σ_{ab}^z operators along the contour, γ, that begins at one opening and ends at another (or at the outer boundary, see Fig. 2)

$$\hat{T}_q=\prod_{(\gamma_q)} \sigma_{ab}^z \tag{6}$$

commutes with the Hamiltonian and is not fixed by the constraint. Physically these operators count the parity of "up" rhombi along such a contour. The presence of these operators results in the degeneracy of the ground state. Note that multiplying such an operator by an appropriate $\hat{P}_a$ gives a similar operator defined on the shifted contour so all topologically equivalent contours give one new integral of motion. Further, multiplying two operators defined along the contours beginning at the same (e.g., outer) boundary and ending in different openings, A, B, is equivalent to the operator defined on the contour leading from A to B, so the independent operators can be defined, e.g., by the set of contours that begin at one opening and end at the outer boundary. The state $|G\rangle$ constructed above is not an eigenstate of these operators but this can be fixed by defining

$$|G_f\rangle=\prod_q \frac{1+c_q\hat{T}_q}{\sqrt{2}}|G\rangle, \tag{7}$$

where $c_q=\pm1$ is the eigenvalue of the $\hat{T}_q$ operator defined on contour γ_q. Equation (7) is the final expression for the ground-state eigenfunctions.

Construction of the excitations is similar to the construction of the ground state. First, one notices that since all operators $\hat{Q}_{abc}$ commute with each other and with the constraints, any state of the system can be characterized by the eigenvalues ($Q_{abc}=\pm1$) of $\hat{Q}_{abc}$. The lowest excited state corresponds to only one Q_{abc}, being -1. Notice that a simple flip of one rhombus [by operator $\sigma_{(ab)}^z$] somewhere in the system changes the sign of *two* Q_{abc} eigenvalues corresponding to two triangles to which it belongs. To change only one Q_{abc} one needs to consider a continuous string of these flip operators starting from the boundary $|(abc)\rangle=v_{(abc)}|0\rangle$ with $v_{(abc)}=\Pi_{\gamma'}\sigma_{(cd)}^z$ where the product is over all rhombi (cd) that belong to the path, γ', that begins at the

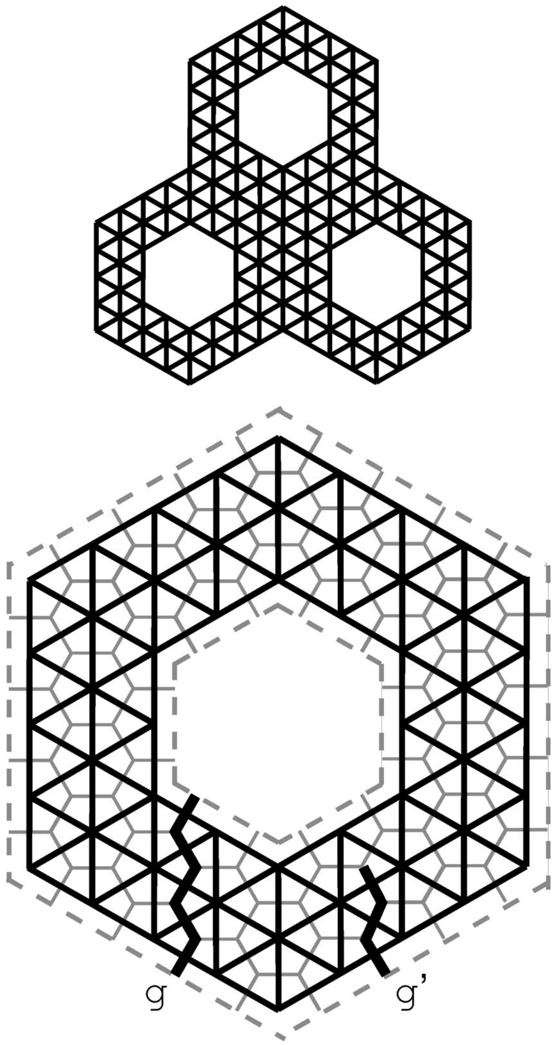

FIG. 2. Lower pane: Location of the discrete degrees of freedom responsible for the dynamics of the Josephson-junction array shown in Fig. 1. The spin degrees of freedom describing the state of the elementary rhombi are located on the bonds of the triangular lattice (shown by thick lines) while the constraints are defined on the sites of this lattice. The dashed line indicates the boundary condition imposed by a physical circuitry shown in Fig. 1(a). Contours γ and γ' are used in the construction of topological order parameter and excitations. Upper pane: The lattice with $K=3$ openings; the ground state of the Josephson-junction array on this lattice 2^k is eightfold degenerate.

boundary and ends at (abc) (see Fig. 2 which shows one such path). This operator changes the sign of only one Q_{abc}, the one that corresponds to the ''last'' triangle. This construction does not satisfy the constraint, so we have to apply the same ''fix'' as for the ground-state construction above,

$$|v_{(abc)}\rangle=\prod_q \frac{1+c_q\hat{T}_q}{\sqrt{2}}\prod_a \frac{1+\hat{P}_a}{\sqrt{2}} v_{(abc)}|0\rangle, \tag{8}$$

to get the final expression for the lowest energy excitations. The energy of each excitation is $2r$. Note that a single flip excitation at a rhombus (ab) can be viewed as a combination of two elementary excitations located at the centers of the triangles to which rhombus (ab) belongs and has twice their energy. Generally, all excited states of the model (4) can be characterized as a number of elementary excitations (8), so they give an exact quasiparticle basis. Note that creating a quasiparticle at one boundary and moving it to another is equivalent to the $\hat{T}_q$ operator, so this process acts as τ_q^z in the space of the 2^K degenerate ground states. As will be shown below, in the physical system of Josephson junctions, these excitations carry charge $2e$ so that the τ_q^z process is equivalent to the charge $2e$ transfer from one boundary to another.

Consider now the matrix elements $O_{\alpha\beta}=\langle G_\alpha|\hat{O}|G_\beta\rangle$ of a local operator, $\hat{O}$, between two ground states, e.g., of an operator that is composed of a small number of σ_{ab}. To evaluate this matrix element we first project a general operator onto the space that satisfies the constraint: $\hat{O}\rightarrow\mathcal{P}\hat{O}\mathcal{P}$ where $\mathcal{P}=\Pi_a(1+\hat{P}_a)/2$. The new (projected) operator is also local, in that it has the same matrix elements between ground states but it commutes with all $\hat{P}_a$. Since it is local it can be represented as a product of σ_z and $\hat{Q}$ operators which implies that it also commutes with all $\hat{T}_q$. Thus, its matrix elements between different states are exactly zero. Further, using the fact that it commutes with $\hat{P}_a$ and $\hat{T}_q$ we write the difference between its diagonal elements evaluated between the states that differ by a parity over contour q as

$$O_+ - O_- = \left\langle 0\left|\prod_i \frac{1+\hat{P}_i}{\sqrt{2}}\hat{T}_q\hat{O}\right|0\right\rangle. \tag{9}$$

This equation can be viewed as a sum of products of σ_z operators. Clearly to get a nonzero contribution each σ^z should enter an even number of times. Each $\hat{P}$ contains a closed loop of six σ^z operators, so any product of these terms is also a collection of a set of closed loops of σ^z. In contrast to it, operator $\hat{T}_q$ contains a product of σ^z operators along the loop γ, so the product of them contains a string of σ_z operators along the contour that is topologically equivalent to γ. Thus, one gets a nonzero $O_+ - O_-$ only for the operators $\hat{O}$ which contain a string of σ^z operators along the loop that is topologically equivalent to γ which is impossible for a local operator. Thus, we conclude that for this model all nondiagonal matrix elements of a local operator are *exactly* zero while all diagonal are *exactly* equal.

IV. EFFECT OF PHYSICAL PERTURBATIONS

We now come back to the original physical system described by the Lagrangian (1) and derive the parameters of the model (4) and discuss the most important corrections to it and their effect. We begin with the derivation. In the limit of small charging energy the flip of three rhombi occurs by a virtual process in which the phase, ϕ_i, at one (six coordinated) island, *i*, changes by π. In the quasiclassical limit the phase differences on the individual junctions are $\phi_{ind}=\pm\pi/4$; the leading quantum process changes the phase on one junction by $3\pi/2$ and on others by $-\pi/2$ changing the phase across the rhombus $\phi\rightarrow\phi+\pi$. The phase differences,

ϕ, satisfy the constraint that the sum of them over the closed loops remains $2\pi(n+\Phi_s/\Phi_0)$. The simplest such a process preserves the symmetry of the lattice, and changes simultaneously the phase differences on the three rhombi containing island i keeping all other phases constant. The action for such process is three times the action of elementary transitions of individual rhombi, S_0:

$$r \approx E_J^{3/4} E_C^{1/4} \exp(-3S_0), \quad S_0 = 1.61\sqrt{E_J/E_C}. \qquad (10)$$

In the alternative process the phase differences between i and other islands change in turn, via a high-energy intermediate state in which one phase difference has changed while others remained close to their original values. The estimate for this action shows that it is larger than $3S_0$, so Eq. (10) gives the dominating contribution. There are in fact many processes that contribute to this transition: the phase of island i can change by $\pm\pi$ and in addition in each rombus one can choose arbitrarily the junction in which the phase changes by $\pm 3\pi/2$; the amplitude of all these processes should be added. This does not change the result qualitatively unless these amplitudes exactly cancel each other, which happens only if the charge of the island is exactly half integer [because phase and charge are conjugate the amplitude difference of the processes that are different by 2π is $\exp(2\pi i q)$]. We assume that in a generic case this cancellation does not occur. External electrical fields (created by, e.g., stray charges) might induce noninteger charges on each island which would lead to a randomness in the phase and amplitude of r. The phase of r can be eliminated by a proper gauge transformation $|\uparrow\rangle_{ab} \rightarrow e^{i\alpha_{ab}}|\uparrow\rangle_{ab}$ and has no effect at all. The amplitude variations result in a position-dependent quasiparticle energy.

We now consider the corrections to the model (4). One important source of corrections is the difference of the actual magnetic flux through each rhombus from the ideal value $\Phi_0/2$. If this difference is small it leads to the bias of "up" versus "down" states, and their energy difference becomes $2\epsilon = 2\pi\sqrt{2}(\delta\Phi_d/\Phi_0)E_J$. Similarly, the difference of the actual flux through the Star of David star and the difference in the Josephson energies of individual contacts leads to the interaction between "up" states:

$$\delta H_1 = \sum_{(ab)} V_{ab}\sigma^z_{ab} + \sum_{(ab),(cd)} V_{(ab),(cd)}\sigma^z_{ab}\sigma^z_{cd}, \qquad (11)$$

where $V_{ab}=\epsilon$ for uniform field deviating slightly from the ideal value and $V_{(ab),(cd)} \neq 0$ for rhombi belonging to the same hexagon. Consider now the effect of perturbations described by δH_1, Eq. (11). These terms commute with the constraint but do not commute with the main term, H, so the ground state is no longer $|G_\pm\rangle$. In other words, these terms create excitations (8) and give them kinetic energy. In the leading order of the perturbation theory the ground state becomes $|G_\pm\rangle + \frac{\epsilon}{4r}\Sigma_{(ab)}\sigma^z_{(ab)}|G_{i\pm}\rangle$. Qualitatively, it corresponds to the appearance of virtual pairs of quasiparticles in the ground state. The density of these quasiparticles is ϵ/r. As long as these quasiparticles do not form a topologically nontrivial string all previous conclusions remain valid. However, there is a nonzero amplitude to form such a string—it is now exponential in the system size. With exponential accuracy this amplitude is $(\epsilon/2r)^L$ which leads to an energy splitting of the two ground-state levels and the matrix elements of typical local operators of the same order $E_+ - E_- \sim O_+ - O_- \sim (\epsilon/2r)^L$.

The physical meaning of the $v_{(abc)}$ excitations becomes more clear if one considers the effect of the addition of one σ^z operator to the end of the string defining the quasiparticle: it results in the charge transfer of $2e$ across this last rhombus. To prove this, note that the wave function of a superconductor corresponding to the state which is a symmetric combination of $|\uparrow\rangle$ and $|\downarrow\rangle$ is periodic with period π and thus corresponds to a charge which is a multiple of $4e$ while the antisymmetric state corresponds to charge $(2n+1)2e$. The action of one σ_z induces the transition between these states and thus transfers the charge $2e$. Thus, these excitations carry charge $2e$. Note that continuous degrees of freedom are characterized by the long range order in $\cos(2\phi)$ and thus correspond to the condensation of pairs of Cooper pairs. In other words, this system superconducts with an elementary charge $4e$ and has a gap $2r$ to the excitations carrying charge $2e$. Similar pairing of Cooper pairs was shown to occur in a chain of rhombi in a recent paper;[19] formation of a classical superconductive state with an effective charge $6e$ in a frustrated *Kagomé* wire network was predicted in Ref. 20.

The model (4) completely ignores the processes that violate the constraint at each hexagon. Such processes might violate the conservation of the topological invariants $\hat{T}_q$ and thus are important for a long-time dynamics of the ground-state manifold. In order to consider these processes we need to go back to the full description involving the continuous superconducting phases ϕ_i. Since potential energy (2) is periodic in π it is convenient to separate the degrees of freedom into continuous (defined as modulus π) and discrete parts. Continuous parts have a long-range order: $\langle\cos(2\phi_0 - 2\phi_r)\rangle \sim 1$. The elementary excitations of the continuous degrees of freedom are harmonic oscillations and vortices. The harmonic oscillations interact with discrete degrees of freedom only through the local currents that they generate, further the potential (2) is very close to the quadratic, so we conclude that they are practically decoupled from the rest of the system. In contrast to this vortices have an important effect. By construction, the elementary vortex carries flux π in this problem. Consider the structure of these vortices in greater detail. The superconducting phase should change by 0 or 2π when one moves around a closed loop. In a half vortex this is achieved if the gradual change by π is compensated (or augmented) by a discrete change by π on a string of rhombi which costs no energy. Thus, from the viewpoint of discrete degrees of freedom the position of the vortex is the hexagon where constraint (3) is violated. The energy of the vortex is found from the usual arguments

$$E_v(R) = \frac{\pi E_J}{4\sqrt{6}}[\ln(R) + c], \quad c \approx 1.2; \qquad (12)$$

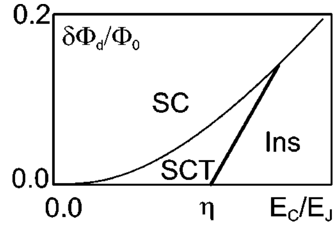

FIG. 3. Schematic of the phase diagram for half integer Φ_s at low temperatures: $\delta\Phi_d$ is the deviation of the magnetic flux through each rhombus from its ideal value. SC stands for usual superconducting phase, and SCT for the phase with $\cos(2\phi)$ long-range order of the continuous degrees of freedom and discrete topological order parameter discussed extensively in the bulk of the paper. The SCT phase and SC phase are separated by a two-dimensional quantum Ising phase transition.

it is logarithmic in the vortex size, R. The process that changes the topological invariant, $\hat{T}_q$, is the one in which one-half vortex completes a circle around an opening. The amplitude of such a process is exponentially small: $[\tilde{t}/E_v(D)]^{\Lambda}$ where $\tilde{t}$ is the amplitude to flip of one rhombus and Λ is the length of the shortest path around the opening. In the quasiclassical limit the amplitude $\tilde{t}$ can be estimated analogously to Eq. (10): $\tilde{t}\sim\sqrt{E_J E_Q}\exp(-S_0)$. The half vortices would appear in a realistic system if the flux through each hexagon is systematically different from the ideal half-integer value. The presence of free vortices destroys topological invariants, so a realistic system should either be not too large (so that deviations of the total flux do not induce free vortices) or these vortices should be localized in prepared traps (e.g., Stars of David with fluxes slightly larger or smaller than Φ_s). If the absence of half vortices the model is equivalent to the Kitaev model[7] placed on triangular lattice in the limit of the infinite energy of the excitation violating the constraints.

Quantitatively, the expression for the parameters of the model (4) become exact only if $E_J \gg E_C$. One expects, however, that the qualitative conclusions remain the same and the formulas derived above provide good estimates of the scales even for $E_J \sim E_C$, provided that charging energy is not so large as to result in a phase transition to a different phase. One expects this transition to occur at $E_c^* = \eta E_J$ with $\eta \sim 1$ whose exact value can be reliably determined only from numerical simulations. Practically, since the perturbations induced by flux deviations from Φ_0 are proportional to $(\delta\Phi/\Phi_0)E_J/r$ and r becomes exponentially small at small E_C, the optimal choice of the parameters for the physical system is $E_C \approx E_C^*$. We show the schematics of the phase diagram in Fig. 3.[21] The "topological" phase is stable in a significant part of the phase diagram; further since the vortex excitations have logarithmic energy, we expect that this phase survives at finite temperatures as well. In the thermodynamic limit, at $T\neq 0$, one gets a finite density of $2e$ carrying excitations $[n_v \sim \exp(-2r/T)]$ but the vortices remain absent as long as temperature is below a BKT-like depairing transition for half vortices.

V. QUANTUM MANIPULATIONS

We now discuss the manipulation of the protected states formed in this system. First, we note that the topological invariant $\hat{T}_q$ has a simple physical meaning—it is the total phase difference (modulus 2π) between the inner and outer boundaries. This means that measuring this phase difference measures the state of the qubit. Also, introducing a weak coupling between these boundaries by a very weak Josephson circuit (characterized by a small energy ϵ_J) would change the phase of these states in a controllable manner, e.g., in a unitary transformation $U^z = \exp(i\epsilon_J t \tau_q^z)$. The transformation coupling two qubits can be obtained if one introduces a weak Josephson circuit that connects two different inner boundaries (corresponding to different qubits). Analogously, the virtual process involving half vortex motion around the opening gives the tunneling amplitude, ϵ_t, between topological sectors, e.g., unitary transformation $U^x = \exp(it\epsilon_t \tau_q^x)$. This tunneling can be controlled by magnetic field if the system is prepared with some number of vortices that are pinned in the idle state in a special plaquette where the flux is an integer. The slow (adiabatic) change of this flux towards a normal (half-integer) value would release the vortex and result in the transitions between topological sectors with $\epsilon_t \sim \tilde{t}/D^2$.

These operations are analogous to usual operations on a qubit and are prone to the usual source of errors. This system, however, allows another type of operation that is naturally discrete. As we show above the transmission of the elementary quasiparticle across the system changes its state by τ_q^z. This implies that a discrete process of one pair transfer across the system is equivalent to the τ_q^z transformation. Similarly, a controlled process in which a vortex is moved around a hole results in a discrete τ_q^x transformation. Moreover, this system allow one to make discrete transformations such as $\sqrt{\tau^{x,z}}$. Consider, for instance, a process in which by changing the total magnetic flux through the system one-half vortex is placed in a center of the system shown in Fig. 1(b) and then released. It can escape through the left or through the right boundary; in one case the state does not change, in another it changes by τ_x. The amplitudes add, resulting in the operation $(1+i\tau^x)/\sqrt{2}$. Analogously, using the electrostatic gate(s) to pump one charge $2e$ from one boundary to the island in the center of the system and then releasing it results in a $(1+i\tau^z)/\sqrt{2}$ transformation. These types of processes allow a straightforward generalization for the array with many holes: there an extra half vortex or charge should be placed at equal distances from the inner and outer boundaries.

VI. PHYSICAL PROPERTIES OF SMALL ARRAYS

Even without these applications for quantum computation the physical properties of this array are remarkable: it exhibits a long-range order in the square of the usual superconducting order parameter: $\langle\cos[2(\phi_0-\phi_r)]\rangle\sim 1$ without the usual order: $\langle\cos(\phi_0-\phi_r)\rangle=0$; the charge transferred through the system is quantized in units of $4e$. This can be tested in an interference experiment sketched in Fig. 1(b), as a function of external flux, Φ_m; the supercurrent through the loop should be periodic with half the usual period. This simpler array can be also used for a kind of ''spin-echo'' experiment: applying two consecutive operations $(1+i\tau_x)/\sqrt{2}$ described above should give again a unique classical state, while applying only one of them should result in a quantum superposition of two states with equal weight.

The echo experiment can be used to measure the decoherence time in this system. Generally one distinguishes between processes that flip the classical states and the ones that change their relative phases. In NMR literature the former are referred to as transverse relaxation and the latter as longitudinal. The transverse relaxation occurs when a vortex is created and then moved around the opening by an external noise. Assuming a thermal noise, we estimate the rate of this process $\tau_{\perp}^{-1} \sim \tilde{t} \exp[-E_V(L)/T]$. Similarly, the transfer of a quasiparticle from the outer to the inner boundary changes the relative phase of the two states, leading to longitudinal relaxation. This rate is proportional to the density of quasiparticles, $\tau_{\parallel}^{-1} = \mathcal{R} \exp(-2r/T)$. The coefficient $\mathcal{R}$ depends on the details of the physical system. In an ideal system with some nonzero uniform value of ϵ [defined above, Eq. (11)] quasiparticles are delocalized and $\mathcal{R} \sim \epsilon/L^2$. Random deviations of fluxes Φ_r from a half-integer value produce randomness in ϵ, in which case one expects Anderson localization of quasiparticles due to *off-diagonal* disorder, with localization length of the order of lattice spacing, thus $\mathcal{R} \sim \bar{\epsilon} \exp(-cL)$ with $c \sim 1$, and $\bar{\epsilon}$ is the typical value of ϵ. Stray charges induce randomness in the values of r, i.e., they add some *diagonal* disorder. When the random part of r, δr, becomes larger than $\bar{\epsilon}$ the localization becomes stronger: $\mathcal{R} \sim \bar{\epsilon}(\bar{\epsilon}/\overline{\delta r})^L$ where $\overline{\delta r}$ is the typical value of δr. Upon a further increase of stray charge field there appear rare sites where r_i is much smaller than an average value. Such a site acts as additional openings in the system. If the density of these sites is significant, the effective length that controls the decoherence becomes the distance between these sites. For typical $E_V(L) \approx E_J \approx 2$ K the transverse relaxation time reaches seconds for $T \sim 0.1$ K while realistic $\epsilon/r \sim 0.1$ implies that due to a quasiparticle localization in a random case the longitudinal relaxation reaches the same scale for systems of size $L \sim 10$; note that temperature T has to be only somewhat lower than the excitation gap, $2r$, in order to make the longitudinal rate low.

Most properties of the array are only weakly sensitive to the effect of stray charges: as discussed above, they result in a position-dependent quasiparticle potential energy which has very little effect because these quasiparticles had no kinetic energy and were localized anyway. A direct effect of stray charges on the topologically protected subspace can be also physically described as a effect of the electrostatic potential on the states with even and odd charges at the inner boundary; since the absolute value of the charge fluctuates strongly this effect is exponentially weak.

Finally, we remark that the properties of the excitations and topological order parameter exhibited by this system are in many respects similar to the properties of the ring exchange and frustrated magnets models discussed recently in Refs. 11, 12, and 22–32.

VII. CONCLUSION

We have shown that a Josephson-junction array of a special type (shown in Fig. 1) has a degenerate ground state described by a topological order parameter. The manifold of these states is protected in the sense that local perturbations have an exponentially weak effect on their relative phases and transition amplitudes. The main building block of the array is the rhombus which has two (almost) degenerate states. In the array discussed here these rhombi are assembled into hexagons but we expect that lattices in which these rhombi form other structures would have similar properties. However, the dynamics of these arrays is described by quartic- (or higher-) order spin-exchange terms which have a larger barrier in a quasiclassical regime, implying that their parameter r is much smaller than for the array considered here. This makes them more difficult to build in the interesting regime.

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