# Algebra of observables in the Calogero model and in the Chern-Simons matrix model

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The algebra of observables of an $N$-body Calogero model is represented on the $S_N$-symmetric subspace of the positive-definite Fock space. We discuss some general properties of the algebra and construct four different realizations of the dynamical symmetry algebra of the Calogero model. Using the fact that the minimal algebra of observables is common to the Calogero model and the finite Chern-Simons (CS) matrix model, we extend our analysis to the CS matrix model. We point out the algebraic similarities and distinctions of these models.

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## I. INTRODUCTION

The Calogero model[1] is a completely integrable model[2] that describes a system of $N$ interacting particles in one dimension. Even though the interaction is highly nontrivial, one can construct the needed constants of motion, find a spectrum and expressions for the wave functions. Surprisingly, the model and its various generalizations have been found relevant to a host of problems in physics (and mathematics). It was long realized that for three special values of the interaction parameter $\nu=1/2,1,2$ the model was closely related to the random matrix theory[3] of the three Wigner-Dyson ensembles: orthogonal, unitary and symplectic, respectively. The particles subject to the Calogero dynamics obey fractional statistics,[4] and this motivated investigations of the connection between the Chern-Simons based anyonic physics in the fractional quantum Hall effect and the Calogero model.[5] The collective-field theory approach proved useful in constructing solitonic solutions,[6] and in establishing relations to the $d=1$ string theory.[7] Recently, the interest in the Calogero model has been renewed. It was proposed[8] that the supersymmetric extension of the model could provide a microscopic description of the extremal Reissner-Nordström black hole. The supersymmetric extensions of the Calogero model themselves were analyzed in detail.[9] Many more applications of the model have been found, whih have intensified the research of the model intrinsic properties.

Investigations of the algebraic properties of the Calogero model in terms of the $S_N$-extended Heisenberg algebra[10] defined a basic algebraic setup for further research. Floreanini *et al.*[11] showed that the dynamical symmetry algebra of the two-body Calogero model was a polynomial generalization of the SU(2) algebra. The three-body problem was also treated,[12] and the dynamical algebra of the polynomial type and the action of its generators on the orthonormal basis were obtained. It was shown that in the two-body case the polynomial SU(2) algebra could be linearized, but an attempt to generalize this result to the $N$-body case led to $(N-1)$ linear SU(2) subalgebras that operated only on subsets of the degenerate eigenspace.[13] The general construction of the dynamical symmetry algebra was given in Ref. 14. Also, the bosonic realization of a nonlinear symmetry algebra describing the structure of degenerate energy levels of the Calogero model was obtained.[15] In this paper we give a more detailed analysis of the algebra of observables of the Calogero model, discuss the spin representation of the algebra, and present a realization of the dynamical symmetry algebra.

The construction of the algebra of symmetric one-particle operators for the Calogero model[16] resulted in an infinite-dimensional Lie algebra, independent of the particle number and the constant of interaction. This algebra was interpreted as the algebra of observables for a system of identical particles on the line. The problem with a fixed number of particles $N$, which we discuss here, can be viewed as an irreducible representation of the aforementioned algebra. In Ref. 17 it was claimed that this infinite-dimensional Lie algebra was common to the matrix model. We argue that although this is true, the actual models are different, as can be seen from the the algebraic structure obtained for a fixed number of particles $N$.

The renewed interest in the matrix model came from the connection with the non-commutative field theory. Namely, following the Susskind conjecture[18] that the noncommutative Chern-Simons (CS) theory provides an effective description of a quantum Hall effect, Polychronakos proposed[19] a finite matrix version of the model. He claimed that this matrix model was in fact equivalent to the Calogero model. We analyzed the algebra of observables acting on the physical Fock space of that model,[20] and observed that the minimal algebra of observables was identical with that of the Calogero model, although the complete algebraic stuctures are different. We identified the states in the $l$th tower of the CS matrix model Fock space with the states in the physical Fock space of the Calogero model with the interaction parameter $\nu=l+1$. Also, we described quasiparticle and quasihole states in the both models in terms of Schur functions. Using a coherent-state representation, the wave functions for the Chern-Simons matrix model proposed by Polychronakos were constructed and compared with the Laughlin ones.[21] The same authors also studied the spectrum of the model and identified the orthogonal set of states.[22]

Taking into consideration all relations between the Calogero model, the matrix model and the QH physics, we feel that a more detail algebraic analysis is in order. In this paper we analyze the complete algebra of observables of the CS matrix model and confirm by explicit construction that the models in question have different algebraic structures.

The paper is organized as follows. In Sec. II we review basic steps in the construction of the $S_N$-extended Heisenberg algebra of the Calogero model represented on the

$S_N$-symmetric subspace of the positive-definite Fock space. In Sec. III we construct the minimal algebra of observables, $\mathcal{A}_N$ and discuss its general properties. We establish a mapping to the Heisenberg algebra and this gives us a natural, orthogonal basis for the model. In the following section we consider the larger algebra $\mathcal{B}_N$ and show that its spin representation gives an irreducible representation of the algebra constructed in Ref. 16. In Sec. V we present four different realizations of the dynamical symmetry algebra describing the structure of the degenerate eigenspace in detail. The analysis of the algebra of obervables is extended to the finite CS matrix model in Sec. VI, and in the last section we compare the algebraic structure of the two models. Finally, in the appendixes we confirm our findings by explicit calculations.

## II. THE CALOGERO MODEL ON THE SYMMETRIC FOCK SPACE

The Hamiltonian of the (rational) Calogero model describes $N$ identical particles (bosons) interacting through an inverse square interaction subject to a common confining harmonic force:

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{m\omega^2}{2}\sum_{i=1}^{N}x_i^2+\frac{\nu(\nu-1)\hbar^2}{2m}\sum_{i\neq j}^{N}\frac{1}{(x_i-x_j)^2}. \tag{1}$$

In the following we set $\hbar$, the mass of particles $m$ and the frequency of harmonic oscillators $\omega$ equal to 1. The dimensionless constant $\nu$ is the coupling constant and $N$ is the number of particles. The ground-state wave function is, up to normalization,

$$\psi_0(x_1,\ldots,x_N)=\theta(x_1,\ldots,x_N)\exp\left(-\frac{1}{2}\sum_{i=1}^{N}x_i^2\right), \tag{2}$$

where

$$\theta(x_1,\ldots,x_N)=\prod_{i<j}^{N}|x_i-x_j|^{\nu}, \tag{3}$$

with the ground-state energy $E_0=N[1+(N-1)\nu]/2$.

The Dunkl operators[23]

$$D_i=\partial_i+\nu\sum_{j,j\neq i}^{N}\frac{1}{x_i-x_j}(1-K_{ij})$$

are the basic building blocks of the Calogero model creation and annihilation operators.[10] The elementary generators $K_{ij}$ of the symmetry group $S_N$ exchange the labels $i$ and $j$:

$$K_{ij}x_j=x_iK_{ij},\quad K_{ij}=K_{ji},\quad (K_{ij})^2=1,$$

$$K_{ij}K_{jl}=K_{jl}K_{il}=K_{il}K_{ij}\quad \text{for}\quad i\neq j,\quad i\neq l,\quad j\neq l,$$

and we choose $K_{ij}|0\rangle=|0\rangle$. Now we can introduce the creation and annihilation operators:

$$a_i^\dagger=\frac{1}{\sqrt{2}}(-D_i+x_i),\ a_i=\frac{1}{\sqrt{2}}(D_i+x_i), \tag{4}$$

where the operator $a_i$ annihilates the vacuum. Using the well-known properties of the Dunkl operators

$$[D_i,D_j]=0,\quad K_{ij}D_j=D_iK_{ij},$$

$$[D_i,x_j]=\delta_{ij}\left(1+\nu\sum_{k=1}^{N}K_{ik}\right)-\nu K_{ij},$$

one can easily check that the commutators of the creation and annihilation operators (4) are

$$[a_i,a_j]=[a_i^\dagger,a_j^\dagger]=0,$$

$$[a_i,a_j^\dagger]=\left(1+\nu\sum_{k=1}^{N}K_{ik}\right)\delta_{ij}-\nu K_{ij}. \tag{5}$$

After performing a similarity transformation on the Hamiltonian (1), we obtain the reduced Hamiltonian

$$H'=\theta^{-1}H\theta=\frac{1}{2}\sum_{i=1}^{N}\{a_i,a_i^\dagger\}=\sum_{i=1}^{N}a_i^\dagger a_i+E_0, \tag{6}$$

acting on the space of symmetric functions. We restrict the Fock space $\{a_1^{\dagger n_1}\cdots a_N^{\dagger n_N}|0\rangle\}$ to the $S_N$-symmetric subspace $F_{\mathrm{symm}}$, where $\mathcal{N}=\sum_{i=1}^{N}a_i^\dagger a_i$ acts as the total number operator. In the following we demand that all states should have positive norm, i.e., $\nu>-1/N$.[24] Next, we introduce the collective $S_N$-symmetric operators

$$B_n=\sum_{i=1}^{N}a_i^n,\quad n=0,1,\ldots,N, \tag{7}$$

where $B_0$ is the constant $N$ multiplied by the identity operator, and $B_1$ represents the center-of-mass operator. The complete $F_{\mathrm{symm}}$ can be described as $\{B_1^{\dagger n_1}B_2^{\dagger n_2}\cdots B_N^{\dagger n_N}|0\rangle\}$. We wish to construct the operators $X_k^\dagger$ such that $[B_1,X_k^\dagger]=0$ for every $k$ greater than 1, in order to separate the center-of-mass coordinate. The general solution of this equation is described by any symmetric monomial polynomial $m_\lambda(\overline{a}_1,\ldots,\overline{a}_N)=\sum\overline{a}_1^{\lambda_1}\overline{a}_2^{\lambda_2}\cdots\overline{a}_N^{\lambda_N}$, where $\overline{a}_i=a_i-B_1/N$, and the sum goes over all distinct permutations of $(\lambda_1,\lambda_2,\ldots,\lambda_N)$. The multiset $(\lambda_1,\lambda_2,\ldots,\lambda_N)$ denotes any partition of $\mathcal{N}$ such that $\sum_{i=1}^{N}\lambda_i=\mathcal{N}$ and $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_N\geqslant 0$. The "shifted" operators $\overline{a}_i$ and $\overline{a}_i^\dagger$ satisfy the following commutation relations:

$$[\overline{a}_i,\overline{a}_j]=[\overline{a}_i^\dagger,\overline{a}_j^\dagger]=0,$$

$$[\overline{a}_i,\overline{a}_j^\dagger]=\left(1+\nu\sum_{k=1}^{N}K_{ik}\right)\delta_{ij}-\nu K_{ij}-\frac{1}{N}. \tag{8}$$

The simplest choice of the $(N-1)$ operators commuting with the center-of-mass operator is

$$A_n=\sum_{i=1}^{N}\bar{a}_i^n=\sum_{i=1}^{N}\left(a_i-\frac{B_1}{N}\right)^n,\quad n=2,\ldots,N. \tag{9}$$

Another possible choice is $\bar{h}_n=\Sigma_{\text{dist.}}\bar{a}_{i_1}\cdots\bar{a}_{i_n}$ describing one-quasihole states,[25] spanning $\{\bar{h}_2^\dagger\cdots\bar{h}_N^\dagger\}|0\rangle$. The symmetric Fock space $F_{\text{symm}}$ is now $\{B_1^{\dagger n_1}A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle\}$ and after removing the center-of-mass operator it is $\{A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}\}|0\rangle$. We have reduced the problem to the $(N-1)$ Jacobi-type operators. The norm of the state $\langle 0|A_nA_n^\dagger|0\rangle$ can be calculated recursively. We present results up to $n=4$:

$$\langle 0|A_2A_2^\dagger|0\rangle=2(N-1)(1+N\nu),$$

$$\langle 0|A_3A_3^\dagger|0\rangle=3\frac{(N-1)(N-2)}{N}(1+N\nu)(2+N\nu),$$

$$\begin{aligned}\langle 0|A_4A_4^\dagger|0\rangle=4\frac{(N-1)}{N^2}(1+N\nu)[6(N^2-3N+3)\\ +N\nu(5N^2-18N+18)+N^2\nu^2(N-2)\\ \times(N-3)].\end{aligned} \tag{10}$$

We see from Eq. (10) that for the positive-definite Fock space $\nu$ is larger than $-1/N$. Two different states $A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle$ and $A_2^{\dagger n_2'}\cdots A_N^{\dagger n_N'}|0\rangle$ with the same energy $(\Sigma i n_i=\Sigma i n_i')$ are not orthogonal. The total number operator on $F_{\text{symm}}$ splits into

$$\mathcal{N}^\dagger=\mathcal{N}=\mathcal{N}_1+\bar{\mathcal{N}},$$

$$\mathcal{N}_1^\dagger=\mathcal{N}_1=\frac{1}{N}B_1^\dagger B_1,$$

$$\bar{\mathcal{N}}^\dagger=\bar{\mathcal{N}}\equiv\sum_{k=2}^{N}k\mathcal{N}_k. \tag{11}$$

Note that $\mathcal{N}_k$ are the number operators of $A_k^\dagger$ but not of $A_k$. Namely, $[\mathcal{N}_k,A_l^\dagger]=\delta_{kl}A_k^\dagger$, and $\mathcal{N}_k(\cdots A_k^{\dagger n_k}\cdots|0\rangle)=n_k(\cdots A_k^{\dagger n_k}\cdots|0\rangle)$ for every $k$ larger than 1, but $\mathcal{N}_k^\dagger\neq\mathcal{N}_k$. If $\mathcal{N}_k$ were Hermitian, then the eigenstates $A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle$ would be orthogonal, and vice versa.

## III. THE $\mathcal{A}_N$ ALGEBRA AND BOSONIC REALIZATION

Next, we discuss the $\mathcal{A}_N$ algebra of the collective $S_N$-invariant operators $A_k$ defined in Eq. (9) and acting on $F_{\text{symm}}$. It is convenient to add two additional operators, $A_0=N\cdot\mathbb{1}$ and $A_1=0$. We easily see that

$$[A_i,A_j]=[A_i^\dagger,A_j^\dagger]=0,$$

$$[A_i,[A_j,X^\dagger]]=[A_j,[A_i,X^\dagger]],\quad\forall\ i,j,\ \text{any X}^\dagger. \tag{12}$$

The second relation in Eq. (12) is a consequence of the Jacobi identity. The commutator

$$\begin{aligned}[A_m,A_n^\dagger]=mn\left(\sum_{i=1}^{N}\bar{a}_i^{\dagger(n-1)}\bar{a}_i^{(m-1)}-\frac{1}{N}A_{n-1}^\dagger A_{m-1}\right)\\ +\sum_{k=2}^{\min(m,n)}c_k(m,n)\left(\prod\bar{a}^\dagger\right)^{n-k}\left(\prod\bar{a}\right)^{m-k}\end{aligned} \tag{13}$$

is an $S_N$ symmetric and normally ordered operator. We have separated the $c_1$ coefficient because it determines the structure of the algebra of observables. The coefficients $c_k(m,n)$ depend also on the precise index structure (as can be seen in the $k=1$ case), and the symbolical expression $(\Pi\mathcal{O})^k$ denotes a product of operators $\mathcal{O}_i$ of the total order $k$ in $\bar{a}_i(\bar{a}_i^\dagger)$. Hence, the structure of the $\mathcal{A}_N(\nu)$ algebra is of the following type:

$$[A_{i_1},[A_{i_2},\ldots,[A_{i_j},A_j^\dagger]]\cdots]=\sum\left(\prod A\right)^{I-j}, \tag{14}$$

where $I=\Sigma_{\alpha=1}^{j}i_\alpha\geqslant 2j$, and similarly for the hermitian conjugate case. Generally, $j$ successive commutators of $A_{i_1},\ldots,A_{i_j}$ with $A_j^\dagger$, form a homogeneous polynomial $\Sigma(\Pi A)^{I-j}$ in $a_i$ of order $I-j$ with coefficients independent of $\nu$. Therefore, we stress that the algebraic relations (14) are common to all sets of operators $\{A_k,A_k^\dagger\}$, with $k=2,3,\ldots,N$, satisfying

$$[A_i,A_j]=[A_i^\dagger,A_j^\dagger]=0,\ [\mathcal{N}_i,A_j^\dagger]=\delta_{ij}A_j^\dagger. \tag{15}$$

Two different sets of operators satisfying the same algebra (14) differ only in the generalized vacuum conditions, see below. So, we denote the common algebra of the operators $\{A_k,A_k^\dagger\}$ by $\mathcal{A}_N$, and its representation for a given $\nu>-1/N$ by $\mathcal{A}_N(\nu)$.

The term $A_{I-j}$ on the right-hand side (rhs) of Eq. (14) appears with the coefficient $(\Pi_\alpha i_\alpha)j!$. There are $\binom{2N-1}{N}-N$ linearly independent relations (14). Specially, we find

$$\underbrace{[A_2,[A_2,\ldots[A_2}_{n},A_n^\dagger]\ldots]=2^n n!A_n, \tag{16}$$

and more generally

$$\begin{aligned}\underbrace{[A_2,[A_2,\ldots[A_2}_{m},\underbrace{[A_2^\dagger,[A_2^\dagger,\ldots[A_2^\dagger}_{m},A_n]\ldots]\\ =(-)^m2^{2m}\frac{m!n!}{(n-m)!}A_n.\end{aligned} \tag{17}$$

Relations (16) and (17) can be proved using induction and Jacobi identities. Note that any $A_n$, $n>N$, can be algebraically expressed in terms of $A_m$, $m\leqslant N$, see Appendix A.

Hence, the $\mathcal{A}_N$ algebra expressed in terms of $2(N-1)$ algebraically independent operators, is closed, finite, and of the polynomial type.

For general $N$, the $\mathcal{A}_N(\nu)$ algebra of the collective $S_N$-symmetric operators (9) completely determines the action of $A_k$ on any state:

$$A_k A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle=\sum\left(\prod A^{\dagger}\right)^{\mathcal{N}-k}|0\rangle,\quad k\leqslant\mathcal{N}. \quad (18)$$

In order to calculate the precise form of the rhs of Eq. (18), we apply the hermitian conjugate relation Eq. (14) on the lhs of Eq. (18) shifting the operator $A_k$ to the right, at least by one place. We repeat this iteratively as long as the number of $A^{\dagger}$'s on the right from $A_k$ is larger or equal to the index $k$. For $k>\Sigma n_i$, we calculate the finite set of relations, so-called generalized vacuum conditions, directly from Eqs. (9) and (8). We show that the *minimal* set of generalized vacuum conditions needed to completely define the representation of the algebra (14) on the Fock space is

$$A_2A_2^{\dagger}|0\rangle=2(N-1)(1+\nu N)|0\rangle,$$

$$A_3A_3^{\dagger}|0\rangle=3(N-1)(N-2)(1+\nu N)(2+\nu N)/N|0\rangle,$$

$$A_3A_3^{\dagger 2}|0\rangle=3(N-2)(2+\nu N)\times[2(N-1)(1+\nu N)+18]/N|0\rangle. \quad (19)$$

Namely, the operators $A_2$, $A_3$, and Hermitian conjugates play a distinguished role in the algebra, as all other operators $A_n$, $A_n^{\dagger}$ for $n\geqslant 4$ can be expressed as successive commutators (14), using only $A_2$, $A_3$ ,and their Hermitian conjugates. Therefore, one can derive all other generalized vacuum conditions using Eqs. (14) and (19).

Note that the action of $A_iA_j^{\dagger}$ (and $\mathcal{N}_i$) on the symmetric Fock subspace can be written as an infinite, normally ordered expansion

$$A_iA_j^{\dagger}=\sum_{k=0}^{\infty}\left(\prod A^{\dagger}\right)^{k+j}\left(\prod A\right)^{k+i}\quad\forall\ i,j. \quad (20)$$

Applied to a monomial state of the finite order $\mathcal{N}$ in $F_{\text{symm}}$, only the finite number of terms in Eq. (20) will contribute. For the $N=3$ case, we provide explicit calculations in Appendix B, demonstrating the main features of the $\mathcal{A}_N(\nu)$ algebra.

There is a useful (and orthogonal) basis for the problem at hand. We define the operators $\widetilde{A}_i$, $i=2,\ldots,N$, satisfying the vacuum condition $\widetilde{A}_i|0\rangle=0$ and

$$\widetilde{A}_iA_2^{\dagger n_2}\cdots A_i^{\dagger n_i}\cdots A_N^{\dagger n_N}|0\rangle=n_iA_2^{\dagger n_2}\cdots A_i^{\dagger(n_i-1)}\cdots A_N^{\dagger n_N}|0\rangle, \quad (21)$$

for all $n_2,\ldots,n_N\in\mathbb{N}_0$. As a consequence of $[A_i^{\dagger},A_j^{\dagger}]=0$, we immediately find

$$[\widetilde{A}_i,\widetilde{A}_j]=0,\quad[\widetilde{A}_i,A_j^{\dagger}]=\delta_{ij},\quad i,j=2,\ldots,N. \quad (22)$$

We define a dual Fock space as a set of states $\langle 0|\widetilde{A}_2^{n_2}\cdots\widetilde{A}_N^{n_N}$, $n_i\in\mathbb{N}_0$, such that

$$\langle 0|\widetilde{A}_2^{n_2'}\cdots\widetilde{A}_N^{n_N'}A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle=\prod_i n_i!\,\delta_{n_in_i'}. \quad (23)$$

Although the operators $\widetilde{A}_i$ and $A_i^{\dagger}$ are not Hermitian conjugates, the second equation in (22) induces a new scalar product with respect to which the states $A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle$ are orthogonal to $\langle 0|\widetilde{A}_2^{n_2}\cdots\widetilde{A}_N^{n_N}$, according to Eq. (21). Generally, the operators $\widetilde{A}_i$ can be written in the form

$$\widetilde{A}_i=A_i+\sum_{k=2}^{\infty}\left(\prod A^{\dagger}\right)^{k}\left(\prod A\right)^{k+i}. \quad (24)$$

We give an example for the $N=3$ case in Appendix B. Then we define the number operators $\mathcal{N}_i$ and the transition number operators $\mathcal{N}_{i,j}$ as

$$\mathcal{N}_{i,j}=A_i^{\dagger}\widetilde{A}_j,\quad\mathcal{N}_i=\mathcal{N}_{i,i}=A_i^{\dagger}\widetilde{A}_i,\quad\mathcal{N}_1=\frac{1}{N}B_1^{\dagger}B_1,$$

$$[\mathcal{N}_{i,j},A_k^{\dagger}]=\delta_{jk}A_i^{\dagger},\quad[\mathcal{N}_{i,j},\widetilde{A}_k]=-\delta_{ik}\widetilde{A}_j,$$

$$[\mathcal{N}_{i,j},\mathcal{N}_{k,l}]=\delta_{jk}\mathcal{N}_{i,l}-\delta_{il}\mathcal{N}_{k,j}. \quad (25)$$

One can define a new Hermitian conjugation operation $(*)$ in the following way:

$$\mathcal{N}_{i,j}^{*}=\mathcal{N}_{j,i},\quad(A_k^{\dagger})^{*}=\widetilde{A}_k,\quad(\widetilde{A}_k)^{*}=A_k^{\dagger},\quad B_1^{*}=B_1^{\dagger},$$

$$i,j,k=2,\ldots,N.$$

This realization of the algebra of observables provides an orthogonal basis in the dual Fock space and leads to a simple realization of the dynamical symmetry algebra.

We point out that if the set of operators $\{A_k,A_k^{\dagger}\}$ satisfies relations (15), then there exists the mapping $A_i=f(b_i,b_i^{\dagger})$ from the ordinary Bose oscillators $\{b_i,b_i^{\dagger}\}$ to $\{A_k,A_k^{\dagger}\}$. If there are no null-norm vectors in the $\{A\}$ Fock space, there exists the inverse mapping $b_i=f^{-1}(A_k,A_k^{\dagger})$. It has been found in Ref. 25 that the full Fock space $\{a_1^{\dagger n_1},\ldots,a_N^{\dagger n_N}|0\rangle\}$ for $\nu>-1/N$ does not contain any additional null-norm states when compared with the $\{b\}$ Fock space. As $F_{\text{symm}}$ is a subspace of the full Fock space, it also does not contain any additional null-norm states, so $F_{\text{symm}}$ is isomorphic to the $\{b\}$ Fock space, and the mapping $f$ is invertible for $\nu>-1/N$. The point $\nu=-1/N$ is a critical point of the algebra and a description of the system near this point is given in Ref. 26.

In our case of interest, namely, in the $A_N(\nu)$ algebra, we have started with positive-norm states, i.e., $\nu>-1/N$, so there exists a real mapping $f$ and its inverse $f^{-1}$. In general, one can write the infinite series as

$$A_n=\sum\left(\prod b_i^{\dagger n_i}\right)^{k}\left(\prod b_j^{n_j}\right)^{k+n},$$

$$\sum in_i=k,\quad\sum jn_j=k+n,$$

$$b_n=\sum \left(\prod A^\dagger\right)^k\left(\prod A\right)^{k+n}, \tag{26}$$

and then calculate the coefficients. There is freedom in mapping Eq. (26), which appears in the subspaces spanned by the monomials $(\Pi A^\dagger)^{\mathcal{N}}$ [or $(\Pi b^\dagger)^{\mathcal{N}}$] of the same order $\mathcal{N}$. A simple and natural choice of fixing these boundary conditions is

$$A_2^{\dagger n_2}|0\rangle=\sqrt{\frac{\langle 0|A_2^{n_2}A_2^{\dagger n_2}|0\rangle}{n_2!}}b_2^{\dagger n_2}|0\rangle,$$

$$A_3^{\dagger n_3}|0\rangle\sim\sum\left(\prod b_2^\dagger b_3^\dagger\right)^{n_3}|0\rangle,$$

$$\vdots$$

$$A_N^{\dagger n_N}|0\rangle\sim\sum\left(\prod b_2^\dagger\cdots b_N^\dagger\right)^{n_N}|0\rangle, \tag{27}$$

and generally,

$$A_2^{\dagger n_2}A_3^{\dagger n_3}\cdots A_N^{\dagger n_N}|0\rangle \sim\sum b_2^{\dagger n_2}\sum\left(\prod b_2^\dagger b_3^\dagger\right)^{n_3}\cdots\left(\prod b_2^\dagger\cdots b_N^\dagger\right)^{n_N}|0\rangle. \tag{28}$$

Only after fixing this freedom, one can determine the coefficients in Eq. (26) in a unique way. For the $N=3$ case, we present results for the first few coefficients in Eq. (26), up to $k+n\leqslant 5$, for the operators $A_2$ and $A_3$ in Appendix B. The states in the $\{A\}$ Fock space are not orthogonal. However, the monomial states $\Pi b_i^{\dagger n_i}/\sqrt{n_i!}|0\rangle$ in the $\{b\}$ Fock space are orthogonal, so when we express $b_i=f^{-1}(A_k,A_k^\dagger)$, we obtain natural orthogonal states in the $\{A\}$ Fock space, labeled by $(n_2,\ldots,n_N)$, i.e., by free oscillator quantum numbers. Degenerate, orthogonal energy eigenstates of level $\mathcal{N}$ are then defined by $\mathcal{N}=\Sigma i n_i$.

## IV. THE $\mathcal{B}_N(\nu)$ ALGEBRA AND SPIN REPRESENTATION

One can construct the larger closed $\mathcal{B}_N(\nu)$ algebra containing $\mathcal{A}_N(\nu)$ as a subalgebra. This larger algebra appears naturally when one calculates the commutators between the operators $A_i$ and $A_j^\dagger$, defined in Eq. (9). We discussed this algebra in Ref. 14; here we repeat the main results for completeness and present an alternative construction of the algebra.

One can define $S_N$-symmetric operators $\overline{B}_{n,m}$:

$$\overline{B}_{n,m}=\sum_{i=1}^N \overline{a}_i^{\dagger n}\overline{a}_i^m=\overline{B}_{m,n}^\dagger,\quad n,m\in\mathbb{N}_0. \tag{29}$$

The operators $\overline{B}_{n,m}$ can be represented in the symmetric Fock space:

$$\overline{B}_{n,m}A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle\equiv\overline{B}_{n,m}\left(\prod A^\dagger\right)^{\mathcal{N}}|0\rangle =\sum\left(\prod A^\dagger\right)^{\mathcal{N}+n-m}|0\rangle.$$

There are $1/2(N+4)(N-1)$ algebraically independent $\overline{B}_{n,m}$ operators contained in the algebra $\mathcal{B}_N(\nu)$, namely, $2(N-1)$ operators $\overline{B}_{n,0}=A_n^\dagger$ for $n=2,3,\ldots,N$ and their Hermitian conjugates, and $N(N-1)/2$ operators $\overline{B}_{n,m}$ for $n,m\geqslant 1, n+m\leqslant N$. One can express the operators $\overline{B}_{n,m}$ for $n+m>N$ in terms of the algebraically independent operators $\overline{B}_{n,m}$ with $n+m\leqslant N$, see Appendix A. This is a consequence of Cayley-Hamilton theorem. Generally,

$$\overline{B}_{1,1}=\overline{\mathcal{N}},\ [B_1,\overline{B}_{n,m}]=0,\ [\overline{B}_{1,1},\overline{B}_{n,m}]=(n-m)\overline{B}_{n,m},$$

$$[\overline{a}_i,\overline{B}_{n,m}]=n\left[\overline{a}_i^{\dagger(n-1)}\overline{a}_i^m-\frac{1}{N}\overline{B}_{n-1,m}\right]. \tag{30}$$

In the case of $N$ free harmonic oscillators with $\nu=0$, we find the general $\mathcal{B}_N(0)$-algebra relation

$$\begin{aligned}[\overline{B}_{m',m},\overline{B}_{n,n'}]=&\sum_{k=1}^{\min(n,m)}\beta_k(m,n)\left(\frac{1}{N}\right)^k\Bigg\{[(N-1)^k+1]\\&\times\overline{B}_{m'+n-k,m+n'-k}-\overline{B}_{m',m-k}\overline{B}_{n-k,n'}\\&+\sum_{s=1}^{\min(n-k,m-k)}\beta_s(m-k,n-k)\\&\times\overline{B}_{m'+n-k-s,m+n'-k-s}\left(1-\frac{1}{N}\right)^s\Bigg\}\\&-\{m'\leftrightarrow n,n'\leftrightarrow m\},\end{aligned} \tag{31}$$

where

$$\beta_k(m,n)=\frac{m!n!}{k!(m-k)!(n-k)!}.$$

For $\nu\neq 0$, the structure of the $\mathcal{B}_N(\nu)$ algebra becomes more complicated. New polynomial terms of the form $(\Pi_\alpha\overline{B}_{n_\alpha,m_\alpha})$ with $\Sigma_\alpha n_\alpha\leqslant n+m'-1$, $\Sigma_\alpha m_\alpha\leqslant n'+m-1$ appear on the rhs of the commutation relation (31). The corresponding coefficients are polynomial in $\nu$, vanishing when $\nu$ goes to zero. The coefficients of the leading terms [$k=1$ in Eq. (31)] do not depend on $\nu$, i.e., they are the same for any $\nu$. For example, for arbitrary $N$ and $\nu$, we find

$$\begin{aligned}[A_2,A_n^\dagger]=&2n\overline{B}_{n-1,1}+n\left(\frac{N-1}{N}\right)A_{n-2}^\dagger(n-1+\nu N)\\&+n\nu\sum_{i=1}^{n-2}(A_{n-2-i}^\dagger A_i^\dagger-A_{n-2}^\dagger),\end{aligned} \tag{32}$$

It is known that for $N\to\infty$ and $\nu=0$ the corresponding algebra is $W_{1+\infty}$, so it would be interesting to see what kind of deformation the $\mathcal{B}_\infty(\nu)$ algebra (for nontrivial $\nu$) represents. Some investigation in this direction has already been done, see Ref. 27.

Alternatively, the $\mathcal{B}_N(\nu)$ algebra can be constructed by grouping the generators into su(1,1)-spin multiplets. Note that the operators

$$J_+=\tfrac{1}{2}A_2^\dagger,\ \ J_-=\tfrac{1}{2}A_2,\ \ J_0=\tfrac{1}{8}[A_2,A_2^\dagger] \tag{33}$$

generate the sl(2) algebra. The complete set of generators spanning the $\mathcal{B}_N(\nu)$ algebra is given by $(N-1)$ nondegenerate spin multiplets with $s=1,3/2,2,\ldots,N/2$. The unique generator with spin $s$ and projection $s_z$ is defined as

$$J_{s,s_z}=\frac{1}{2^{(s-s_z)}}\sqrt{\frac{(s+s_z)!}{(s-s_z)!}}\underbrace{[A_2,\cdots,[A_2}_{(s-s_z)},A_{2s}^\dagger]\cdots]. \tag{34}$$

Its Hermitian conjugate is simply $J_{s,s_z}^\dagger=J_{s,-s_z}$. One can show that by proving relation (17)

$$\underbrace{[A_2,[A_2,\ldots[A_2}_{s-s_z},\underbrace{[A_2^\dagger,[A_2^\dagger,\ldots[A_2^\dagger}_{s-s_z},A_{2s}]\ldots]$$
$$=(-)^{(s-s_z)}2^{2(s-s_z)}\frac{(s-s_z)!(2s)!}{(s+s_z)!}A_{2s},$$

by induction and using Jacobi identities. The action of elements of the sl(2) algebra on the generator $J_{s,s_z}$ is defined in the following way:

$$[J_-,J_{s,s_z}]=\sqrt{(s+s_z)(s-s_z+1)}J_{s,s_z-1},$$
$$[J_+,J_{s,s_z}]=-\sqrt{(s-s_z)(s+s_z+1)}J_{s,s_z+1},$$
$$[J_0,J_{s,s_z}]=s_zJ_{s,s_z}. \tag{35}$$

Let us write down the first few spin multiplets for any $N$ in terms of $\overline{B}_{i,j}$ operators:
$s=1$, $N\geqslant 2$,

$$J_{1,1}=\sqrt{2}A_2^\dagger=2\sqrt{2}J_+=J_{1,-1}^\dagger,$$
$$J_{1,0}=2\overline{B}_{1,1}+(N-1)(1+N\nu)=4J_0, \tag{36}$$

$s=\frac{3}{2}$, $N\geqslant 3$,

$$J_{3/2,3/2}=\sqrt{6}A_3^\dagger=J_{3/2,-3/2}^\dagger,\ J_{3/2,1/2}=3\sqrt{2}\overline{B}_{2,1}=J_{3/2,-1/2}^\dagger, \tag{37}$$

$s=2$, $N\geqslant 4$,

$$J_{2,2}=2\sqrt{6}A_4^\dagger=J_{2,-2}^\dagger,$$
$$J_{2,1}=2\sqrt{6}\left\{2\overline{B}_{3,1}+\left[3\left(\frac{N-1}{N}\right)+\nu(2N-3)\right]A_2^\dagger\right\}=J_{2,-1}^\dagger,$$
$$J_{2,0}=\sqrt{2}\left\{6\overline{B}_{2,2}+\overline{B}_{1,1}\left[12\left(\frac{N-1}{N}\right)+\nu(7N-12)\right]+(N-1)\times(1+N\nu)\left[3\left(\frac{N-1}{N}\right)+\nu(2N-3)\right]\right\}. \tag{38}$$

The algebra of the operators $J_{s,s_z}$ is finite, closed and of general form,

$$[J_{s_1,s_{1z}},J_{s_2,s_{2z}}]=\sum\left(\prod J\right)_{S,s_{1z}+s_{2z}}, \tag{39}$$

where $|s_{1z}+s_{2z}|\leqslant S\leqslant s_1+s_2-1$. Using the definition (35) and relation (36), one can easily obtain commutation relations involving operators of the $s=1$ spin multiplet,

$$[J_{1,0},J_{s,s_z}]=4s_zJ_{s,s_z},$$
$$[J_{1,\pm1},J_{s,s_z}]=\mp\sqrt{8(s\mp s_z)(s\pm s_z+1)}J_{s,s_z\pm1}. \tag{40}$$

For arbitrary $N$, the algebra of $s=3/2$ operators is

$$[J_{3/2,1/2},J_{3/2,3/2}]=\frac{9}{\sqrt{2}}J_{2,2}-\frac{9\sqrt{3}}{N}J_{1,1}^2,$$
$$[J_{3/2,-1/2},J_{3/2,3/2}]=\frac{9}{\sqrt{2}}J_{2,1}-\frac{9\sqrt{2}}{N}J_{1,1}J_{1,0}+9\sqrt{6}J_{1,1}\left[-\frac{2}{N}+\nu(2-N)\right],$$
$$[J_{3/2,-3/2},J_{3/2,3/2}]=\frac{9}{\sqrt{2}}J_{2,0}-\frac{27}{N}J_{1,1}J_{1,-1}-9\left(\frac{6}{N}+\frac{N}{2}\nu\right)J_{1,0}+a,$$
$$[J_{3/2,-1/2},J_{3/2,1/2}]=\frac{9}{\sqrt{2}}J_{2,0}+\frac{9}{N}J_{1,1}J_{1,-1}-\frac{18}{N}J_{1,0}^2+18J_{1,0}\left[\frac{1}{N}+\left(\frac{3}{4}N+4\right)\right]+b,$$

where the constants $a$ and $b$ are

$$a=+9(N-1)(1+N\nu)\left[\frac{N+1}{N}+\left(\frac{N}{2}-1\right)\nu\right],$$
$$b=-18(N-1)(1+N\nu)\times\left[-\frac{1}{2}+\frac{3}{2N}+N+\left(N^2-\frac{11}{4}N-\frac{1}{2}\right)\nu\right].$$

For $N=3$, the $\mathcal{B}_3(\nu)$ algebra (including $s=1$ and $s=3/2$ spin-mutiplets) is in full agreement with Ref. 12. The exact correspondence between $Y_s$ and $J$ defined in Ref. 12 and our operators $J_{s,s_z}$ is

$$Y_1=-\sqrt{2}J_{1,1},\ \ Y_{3/2}=2J_{3/2,\frac{3}{2}},\ \ Y_{1/2}=-2\sqrt{3}J_{3/2,1/2},$$

$$J=\tfrac{1}{4}(J_{1,0}-2),\quad J_y^2=\tfrac{1}{16}(J_{1,0}-2)^2-\tfrac{1}{8}J_{1,1}J_{1,-1}. \tag{41}$$

This representation of the algebra of observables can be viewed as a generalization of the polynomial algebras for $N=2$ (Ref. 11) and $N=3$ (Refs. 12 and 28) to the general $N$.

The $\mathcal{B}_N(\nu)$ represents an irreducible representation of the infinite-dimensional Lie algebra $\mathcal{G}$ of all possible strings of consecutive commutators $[B_{i_n},[B_{i_{n-1}},\ldots[B_{i_2},B_{i_1}]\ldots]$ introduced in Ref. 16. The $\mathcal{G}$ algebra depends neither on the particle number $N$ nor on the constant of interaction $\nu$. The elements of the algebra $\mathcal{G}$ fall into spin multiplets. There is a unique spin multiplet with a maximal spin $s_{\max}=(m+n)/2$, no spin multiplet with spin $s=s_{\max}-1$, and many spin multiplets with $s<s_{\max}-1$. On the other hand, we have started with a fixed number $N$ of Calogero particles (oscillators) and an arbitrary $\nu>-1/N$. Then we have defined a finite closed algebra $\mathcal{B}_N(\nu)$ of operators $\bar{B}_{m,n}$, with $m+n\leqslant N$. We have shown (see Appendix A) that all operators $\bar{B}_{m,n}$, with $m+n\geqslant N$ can be expressed in terms of $\bar{B}_{m,n}$ with $m+n\leqslant N$. These operators have a unique su(1,1) decomposition into unique spin multiplets with spin $s=1,3/2,\ldots,N/2$. Degenerate spin multiplets in the approach of Isakov and Leinaas[16] are just composites of lower-spin multiplets in our picture.

## V. DYNAMICAL SYMMETRY OF THE CALOGERO MODEL

The dynamical symmetry algebra $\mathcal{C}_N(\nu)$ of the Calogero model is defined as maximal algebra commuting with the Hamiltonian (6), on the restricted Fock space $F_{\text{symm}}$. The generators of the $\mathcal{C}_N(\nu)$ algebra act among the degenerate states with a fixed energy $E=\mathcal{N}+E_0$, $\mathcal{N}$ a non-negative integer. Starting from any of the degenerate states with energy $E$, all other states can be reached by applying the generators of the algebra. Degeneracy appears for $\mathcal{N}\geqslant 2$. The vacuum $|0\rangle$ and the first excited state $B_1^\dagger|0\rangle$ are nondegenerate. For $\mathcal{N}=2$, the degenerate states are $B_1^{\dagger 2}|0\rangle$, $A_2^\dagger|0\rangle$; for $\mathcal{N}=3$, the degenerate states are $B_1^{\dagger 3}|0\rangle$, $B_1^\dagger A_2^\dagger|0\rangle$ and $A_3^\dagger|0\rangle$, etc. The number of degenerate states of level $\mathcal{N}$ is given by partitions $\mathcal{N}_1,\ldots,\mathcal{N}_k$ of $\mathcal{N}$ such that $\mathcal{N}=\Sigma_k k\mathcal{N}_k$. The generators of the algebra $\mathcal{C}_N(\nu)$ can be chosen in different ways, and in the following we present four different sets of generators.

*Set 1.* Let us choose $1/2(N+4)(N-1)$ algebraically independent generators $X_{i,j}$ $(i+j\leqslant N)$ in the following way:

$$X_{i,j}=\bar{B}_{i,j}\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)},\quad X_{j,i}=X_{i,j}^\dagger=\bar{B}_{j,i}\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^{(i-j)},\quad i\geqslant j. \tag{42}$$

For example,

$$X_{i,0}=A_i^\dagger\left(\frac{B_1}{\sqrt{N}}\right)^i,\quad X_{0,i}=A_i\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^i,$$

$$[X_{i,0},X_{j,0}]=[X_{0,i},X_{0,j}]=0,\quad X_{i,i}^\dagger=X_{i,i}=\bar{B}_{i,i}. \tag{43}$$

The generators $X_{i,i}$ are Hermitian but they do not commute because $\bar{B}_{i,i}$'s do not commute even for $\nu=0$.[14] On the other hand, the number operators $\mathcal{N}_k$ (11) commute but are not Hermitian because the states $A_2^{\dagger n_2}\cdots A_N^{\dagger n_N}|0\rangle$ are not mutually orthogonal. Generally,

$$[\mathcal{N}_1,X_{i,j}]=-(i-j)X_{i,j},\quad [\bar{\mathcal{N}},X_{i,j}]=(i-j)X_{i,j},$$
$$[H,X_{i,j}]=[\mathcal{N},X_{i,j}]=0,\quad \text{for all } i,j. \tag{44}$$

The general structure of the commutation relations for $i\geqslant j$, $k\geqslant l$ is

$$[X_{i,j},X_{k,l}]=[\bar{B}_{i,j},\bar{B}_{k,l}]\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)+(k-l)}$$
$$=\sum\left[\prod_\alpha X_{n_\alpha,m_\alpha}\right], \tag{45}$$

and for $i>j$, $k<l$,

$$[X_{i,j},X_{k,l}]=\sum\left[\prod_\alpha X_{n_\alpha,m_\alpha}g_{n_\alpha,m_\alpha}(\mathcal{N}_1)\right]$$
$$+X_{i,j}X_{k,l}f_{ijkl}(\mathcal{N}_1), \tag{46}$$

with the restriction $0\leqslant\Sigma m_\alpha\leqslant j+l-1$, $0\leqslant\Sigma n_\alpha\leqslant i+k-1$, and similarly for Hermitian conjugate relations. The functions $f$ and $g$ are generally rational functions of $\mathcal{N}_1$, with the finite action on all states. One can show that for $i\geqslant j$,

$$\left(\frac{B_1}{\sqrt{N}}\right)^i\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^j=\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)}(\mathcal{N}_1+1)\cdots(\mathcal{N}_1+j)$$
$$=(\mathcal{N}_1+1+i-j)\cdots(\mathcal{N}_1+i)\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)}$$

and

$$\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^j\left(\frac{B_1}{\sqrt{N}}\right)^i=\mathcal{N}_1(\mathcal{N}_1-1)\cdots(\mathcal{N}_1-j+1)\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)}$$
$$=\left(\frac{B_1}{\sqrt{N}}\right)^{(i-j)}(\mathcal{N}_1-i+j)\cdots(\mathcal{N}_1-i+1),$$

and similarly for $i<j$. Now it is easy to see that

$$\left[\left(\frac{B_1}{\sqrt{N}}\right)^i,\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^j\right]=\sum_{k=1}^{\min(i,j)}\beta_k(i,j)\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^{(j-k)}\left(\frac{B_1}{\sqrt{N}}\right)^{(i-k)}. \tag{47}$$

The $\mathcal{C}_N(\nu)$ algebra is intrinsically polynomial. For $N=2$, the $\mathcal{C}_2(\nu)$-Calogero algebra is the SU(2)-polynomial (cubic) algebra,[11] i.e., $[X_{2,0},X_{0,2}]=P_3(\mathcal{N}_1,\bar{\mathcal{N}})$. In this case, the $\mathcal{C}_2(\nu)$ algebra can be linearized to the ordinary SU(2) algebra owing to the fact that there are two independent, uncoupled oscillators $B_1$ and $A_2$, which can be mapped to two ordinary Bose oscillators.[29] The SU(2) generators are

$$\mathcal{J}_+=\frac{1}{4\sqrt{(\mathcal{N}_1-1)(\bar{\mathcal{N}}+1+2\nu)}}B_1^{\dagger 2}A_2,$$

$$\mathcal{J}_-=A_2^\dagger B_1^2\frac{1}{4\sqrt{(\mathcal{N}_1-1)(\bar{\mathcal{N}}+1+2\nu)}}=(\mathcal{J}_+)^\dagger,$$

$$\mathcal{J}_0=\frac{1}{16}\left(\frac{1}{(\mathcal{N}_1-1)}B_1^{\dagger 2}B_1^2-\frac{4}{(\bar{\mathcal{N}}-1+2\nu)}A_2^\dagger A_2\right)$$
$$=\frac{1}{4}(\mathcal{N}_1-\bar{\mathcal{N}}), \tag{48}$$

satisfying $[\mathcal{J}_+,\mathcal{J}_-]=2\mathcal{J}_0$, $[\mathcal{J}_0,\mathcal{J}_\pm]=\pm\mathcal{J}_\pm$. The generators $\mathcal{J}_+$ and $\mathcal{J}_-$ are hermitian conjugates to each other and in this respect differ from the construction done in Ref. 13. For $N=3$, the $\mathcal{C}_3(\nu)$ algebra in Eqs. (45) and (46) is the same as in Ref. 12. One can easily find the exact correspondence using Eq. (41).

*Set 2*. We can construct a new set of generators of the dynamical $\mathcal{C}_N(\nu)$ algebra in terms of the operators $J_{s,s_z}$. For general $N$, we define

$$\tilde{X}_{s,s_z}=J_{s,s_z}\left(\frac{B_1}{\sqrt{N}}\right)^{2s_z}\quad \text{for}\quad s_z\geqslant 0,$$

$$\tilde{X}_{s,s_z}=J_{s,s_z}\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^{-2s_z}\quad \text{for}\quad s_z<0. \tag{49}$$

The operators $\tilde{X}_{s,s_z}$ satisfy a similar algebraic relation as the generators of the preceding realization, Eqs. (45) and (46):

$$[\tilde{X}_{s,s_z},\tilde{X}_{s',s_z'}]=[J_{s,s_z},J_{s',s_z'}]\left(\frac{B_1}{\sqrt{N}}\right)^{2(s_z+s_z')}$$
$$=\sum\left(\prod\tilde{X}\right)_{S,s_z+s_z'},\quad s_z,s_z'\geqslant 0, \tag{50}$$

where $|s_z+s_z'|\leqslant S\leqslant s+s'-1$, and for $s_z\geqslant 0$, $s_z'<0$:

$$[\tilde{X}_{s,s_z},\tilde{X}_{s',s_z'}]=\sum\left[\prod_\alpha \tilde{X}_{s^\alpha,s_z^\alpha}\tilde{g}_{s^\alpha,s_z^\alpha}(\mathcal{N}_1)\right]$$
$$+\tilde{X}_{s,s_z}\tilde{X}_{s',s_z'}\tilde{f}_{s,s_z,s',s_z'}(\mathcal{N}_1), \tag{51}$$

with the restriction $\Sigma s_z^\alpha=s_z+s_z'$, $\Sigma s^\alpha\leqslant s+s'-1$, and similarly for Hermitian conjugate relations. The functions $\tilde{f}$ and $\tilde{g}$ are generally rational functions of $\mathcal{N}_1$, with the finite action on all states. For general $N$, we present several typical commutators that demonstrate the general structure given by Eqs. (50) and (51),

$$[\tilde{X}_{1,0},\tilde{X}_{s,s_z}]=4s_z\tilde{X}_{s,s_z},$$

$$[\tilde{X}_{1,1},\tilde{X}_{s,s_z}]=-\sqrt{8(s-s_z)(s+s_z+1)}\tilde{X}_{s,s_z+1},\quad s_z\geqslant 0,$$

$$[\tilde{X}_{1,-1},\tilde{X}_{s,s_z}]=+\sqrt{8(s+s_z)(s-s_z+1)}\tilde{X}_{s,s_z-1}(\mathcal{N}_1-2s_z$$
$$+2)(\mathcal{N}_1-2s_z+1)-\frac{2s_z}{\mathcal{N}_1-2s_z}\tilde{X}_{1,-1}\tilde{X}_{s,s_z}$$
$$\times\left(2+\frac{2s_z-1}{\mathcal{N}_1-2s_z-1}\right),\quad s_z>0. \tag{52}$$

This construction can be viewed as a generalization of the polynomial algebras for $N=2$ (Ref. 11) and $N=3$ (Ref. 12) to the general $N$, using the $\mathcal{B}_N(\nu)$ algebra.

*Set 3*. Here we introduce a new set of generators of the dynamical $\mathcal{C}_N(\nu)$ algebra which obey very simple relations and possess interesting properties. The dynamical algebra can be defined through the set of $N^2$ generators in terms of the transition number operators (25)

$$Y_{i,j}=\mathcal{N}_{i,j}\left(\frac{B_1}{\sqrt{N}}\right)^{i-j}\quad\text{for}\quad i\geqslant j$$

$$Y_{i,j}=\mathcal{N}_{i,j}\left(\frac{B_1^\dagger}{\sqrt{N}}\right)^{j-i}\quad\text{for}\quad i\leqslant j,$$

$$Y_{i,i}=\mathcal{N}_i, \tag{53}$$

where $1\leqslant i,j\leqslant N$. The diagonal generators $Y_{i,i}$ mutually commute and are Hermitian with respect to * (not $^\dagger$), because the basis is now orthogonal, Eq. (21), in the dual Fock space (23). Note that

$$Y_{i,1}Y_{1,j}=\begin{cases}Y_{i,j}(\mathcal{N}_1+1)\cdots(\mathcal{N}_1+j), & i\geqslant j\\ (\mathcal{N}_1+1)\cdots(\mathcal{N}_1+j)Y_{i,j}, & i\leqslant j.\end{cases} \tag{54}$$

Hence, $Y_{i,j}$ generators can be written in terms of $2N-1$ $Y_{1,i}$ generators and their Hermitian conjugates. We find

$$[\mathcal{N}_1,Y_{i,j}]=-(i-j)Y_{i,j},\quad [\bar{\mathcal{N}},Y_{i,j}]=(i-j)Y_{i,j},$$

$$[H,Y_{i,j}]=[\mathcal{N},Y_{i,j}]=0,\quad [Y_{i,1},Y_{j,1}]=[Y_{1,i},Y_{1,j}]=0\quad \forall i,j,$$

$$[Y_{i,1},Y_{1,j}]=[\mathcal{N}_{i,j}-\delta_{ij}\mathcal{N}_1]B_1^{i-1}B_1^{\dagger(j-1)}$$
$$+\mathcal{N}_{i,1}\mathcal{N}_{1,j}[B_1^{i-1},B_1^{\dagger(j-1)}]$$
$$=-\delta_{ij}\mathcal{N}_1 g_{ii}(\mathcal{N}_1)+Y_{i,1}Y_{1,j}f_{ij}(\mathcal{N}_1),\quad i>j, \tag{55}$$

with $f$ and $g$ being some rational functions of $\mathcal{N}_1$. The dynamical symmetry algebra $\mathcal{C}_N(\nu)$ is expressed in a much simpler way through the above $\{Y_{i,j}\}$ generators than through the $\{X_{i,j}\}$ or $\{\tilde{X}_{s,s_s}\}$ generators, but in all three cases is still of the polynomial type.

The $\mathcal{C}_N(\nu)$ algebras are equivalent (isomorphic) for all $\nu$ parameters larger than $-1/N$. Also, they are equivalent (isomorphic) to the dynamical symmetry algebra $\mathcal{C}_N$ of the Hamiltonian $H_2=H'-E_0=\Sigma_{k=1}^N k b_k^\dagger b_k$, where $b_k$'s satisfy the standard bosonic commutation relation, with $b_1=B_1/\sqrt{N}$. The generators of the corresponding symmetry algebra are

$$Z_{i,j}=\begin{cases} b_i^\dagger b_j (b_1)^{i-j}, & i>j \\ b_i^\dagger b_j (b_1^\dagger)^{j-i}, & i<j \\ b_i^\dagger b_i, & i=j \end{cases} \tag{56}$$

and $[H_2,Z_{i,j}]=0$. They act among degenerate states with fixed energy $E=\mathcal{N}$, $\mathcal{N}\in\mathbb{N}_0$, of the Hamiltonian $H_2$. The commutation relations for $Z_{i,j}$'s are the same as those for $Y_{i,j}$'s, see Eqs. (55). This motivated us to look for a bosonic realization of the dynamical symmetry algebra.

*Set 4*. Finally, we present the bosonic realization of the $\mathcal{C}_N(\nu)$ algebra in terms of $\{\mathcal{J}_i^\pm,\mathcal{J}_i^0\}$, $i=2,\ldots,N$ generators, which represent a generalization of Eq. (48):

$$\mathcal{J}_i^+=\frac{1}{\sqrt{i(\mathcal{N}_1-1)\cdots(\mathcal{N}_1-i+1)}}b_1^{\dagger i}b_i,$$

$$\mathcal{J}_i^-=b_i^\dagger b_1^i\frac{1}{\sqrt{i(\mathcal{N}_1-1)\cdots(\mathcal{N}_1-i+1)}}=(\mathcal{J}_i^+)^\dagger,$$

$$\mathcal{J}_i^0=\frac{1}{2}\left(\frac{\mathcal{N}_1}{i}-b_i^\dagger b_i\right). \tag{57}$$

One can express the generators $\{\mathcal{J}_i^\pm,\mathcal{J}_i^0\}$ in terms of $\{A_k,A_k^\dagger\}$, using the expression for mapping, Eq. (26), for any $\nu>-1/N$. The coefficients in the expansion depend on the parameter $\nu$. The generators (57) satisfy the following new algebra for every $i,j=2,\ldots,N$:

$$[\mathcal{J}_i^0,\mathcal{J}_j^\pm]=\pm\frac{1}{2}\left(\frac{j}{i}+\delta_{ij}\right)\mathcal{J}_j^\pm,$$

$$\mathcal{J}_i^+\mathcal{J}_j^- -\left(\frac{\sqrt{\mathcal{N}_1(\mathcal{N}_1-i+j)}}{\mathcal{N}_1+j}\right)\mathcal{J}_j^-\mathcal{J}_i^+=\delta_{ij}\left(\frac{\mathcal{N}_1}{i}\right),$$

$$\mathcal{J}_i^+\mathcal{J}_j^+-\sqrt{\frac{\mathcal{N}_1-i}{\mathcal{N}_1-j}}\mathcal{J}_j^+\mathcal{J}_i^+=0,\ \ [\mathcal{J}_i^0,\mathcal{J}_j^0]=0. \tag{58}$$

Note that the generators $\mathcal{J}_i^+$ and $\mathcal{J}_i^-$ are Hermitian conjugate to each other, and that relations (58) do not depend on $\nu$. The $\mathcal{C}_N$ algebra contains $(N-1)$ ordinary SU(2) subalgebras. Different SU(2) subalgebras are connected in a nonlinear way, namely, relations (58) have nonlinear (algebraic in $\mathcal{N}_1$) deformations. We demonstrate these features by the simple $N=3$ example:

$$\mathrm{SU}(2):[\mathcal{J}_2^0,\mathcal{J}_2^\pm]=\pm\mathcal{J}_2^\pm,\ \ [\mathcal{J}_2^+,\mathcal{J}_2^-]=2\mathcal{J}_2^0,$$

$$\mathrm{SU}(2):[\mathcal{J}_3^0,\mathcal{J}_3^\pm]=\pm\mathcal{J}_3^\pm,\ \ [\mathcal{J}_3^+,\mathcal{J}_3^-]=2\mathcal{J}_3^0,$$

$$[\mathcal{J}_2^0,\mathcal{J}_3^\pm]=\pm\tfrac{3}{4}\mathcal{J}_3^\pm,\ \ [\mathcal{J}_3^0,\mathcal{J}_2^\pm]=\pm\tfrac{1}{3}\mathcal{J}_2^\pm,$$

$$\mathcal{J}_2^+\mathcal{J}_3^- -\frac{\sqrt{\mathcal{N}_1(\mathcal{N}_1+1)}}{\mathcal{N}_1+3}\mathcal{J}_3^-\mathcal{J}_2^+=0,$$

$$\mathcal{J}_2^+\mathcal{J}_3^+-\sqrt{\frac{\mathcal{N}_1-2}{\mathcal{N}_1-3}}\mathcal{J}_3^+\mathcal{J}_2^+=0.$$

The states with fixed energy $E=\mathcal{N}$ are given by $\Pi_{i=1}^N b_i^{\dagger n_i}/\sqrt{n_i!}|0\rangle$ with $\Sigma_{i=1}^N i n_i=\mathcal{N}$. These states are orthonormal and build an irreducible representation (irrep) of the symmetry algebra $\mathcal{C}_N$.

The dimension of the irrep on the states with fixed energy $\mathcal{N}$ of the $\mathcal{C}_N$ algebra can be obtained recursively in $N$:

$$D(\mathcal{N},N)=\sum_{i=0}^{[\mathcal{N}/N]}D(\mathcal{N}-Ni,N-1).$$

For example, for $N=1$, $D(\mathcal{N},1)=1$, i.e., for a single oscillator, there is no degeneracy; for $N=2$, $D(\mathcal{N},2)=[\mathcal{N}/2]+1$, $\mathcal{N}\geqslant 0$; for $N=3$,

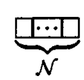
$$D(\mathcal{N},3)=\sum_{i=0}^{[\mathcal{N}/3]}\left[\frac{\mathcal{N}-3i}{2}\right]+\left[\frac{\mathcal{N}}{3}\right]+1.$$

More specifically, for $\mathcal{N}=6n+i$, for some integer $n$ and $i=0,\ldots,5$:

$$D(6n+i,3)=(3n+i)(n+1)\quad\text{for } i=1,\ldots,5,$$

$$D(6n,3)=3n(n+1)+1.$$

Note that the dynamical symmetry algebra of the Hamiltonian $H_1=\Sigma_{k=1}^N b_k^\dagger b_k$ is SU($N$). The corresponding generators are $b_i^\dagger b_j$, and $[H_1,b_i^\dagger b_j]=0$ for all $i$ and $j$. The degenerate states are $\Pi_{i=1}^N b_i^{\dagger n_i}/\sqrt{n_i!}|0\rangle$ with $\Sigma_i n_i=\mathcal{N}$. They form a totally symmetric irrep of SU($N$) described by the Young diagram with $\mathcal{N}$-boxes



(Ref. 28). The states of the same Fock space $\{\Pi_i b_i^{\dagger n_i}/\sqrt{n_i!}|0\rangle\}$ are differently arranged. With respect to $H_1$, degenerate states are organized into the SU($N$) symmetric irrep's. However, in respect to $H_2$, degenerate states build the $\mathcal{N}$-irrep of the $\mathcal{C}_N$ algebra. In this sense, one can (formally) say that the nonlinear algebra $\mathcal{C}_N$ of the Hamiltonian $H_2$ is related to the boson realization of the SU($N$) algebra describing the dynamical symmetry of the Hamiltonian $H_1$.

## VI. THE CHERN-SIMONS MATRIX MODEL

### A. Introduction—The physical Fock space

The Chern-Simons (CS) matrix model was obtained from the noncommutative $U(1)$ Chern-Simons theory, and it was conjectured that it described the quantum Hall fluid.[18] The regularized version of the model, introduced in Ref. 19, is related to the Calogero model.[19,20,22] In Ref. 20 it was shown that the minimal algebra $\mathcal{A}_N$ of the observables was identical in the CS matrix theory and in the Calogero model. Using the results obtained in sections herebefore, we discuss the algebraic structure of the finite CS matrix theory in detail.

Let us start from the action proposed in Ref. 19:

$$S=\int dt\frac{B}{2}\mathrm{Tr}\{\varepsilon_{ab}(\dot{X}_a+i[A_0,X_a])X_b+2\theta A_0-\omega X_a^2\}$$
$$+\Psi^\dagger(i\dot{\Psi}-A_0\Psi). \tag{59}$$

Here, $A_0$ and $X_a$, $a=1,2$, are $N\times N$ Hermitian matrices and $\Psi$ is a complex $N$-vector. The eigenvalues of the matrices $X_a$ represent the coordinates of electrons and $A_0$ is a gauge field. We choose the gauge $A_0=0$ and impose the equation of motion for $A_0$ as a constraint:

$$-iB[X_1,X_2]+\Psi\Psi^\dagger=B\theta. \tag{60}$$

The trace part of Eq. (2) gives $\Psi^\dagger\Psi=NB\theta$. Notice that the commutators have so far been classical matrix commutators. After quantization, the matrix elements of $X_a$ and the components of $\Psi$ become operators satisfying the following commutation relations:

$$[\Psi_i,\Psi_j^\dagger]=\delta_{ij},$$

$$[(X_1)_{ij},(X_2)_{kl}]=\frac{i}{B}\delta_{il}\delta_{jk}. \tag{61}$$

It is convenient to introduce the operator $A=\sqrt{B/2}(X_1+iX_2)$ and its Hermitian conjugate $A^\dagger$ obeying the following commutation relations:

$$[(A)_{ij},(A^\dagger)_{kl}]=\delta_{il}\delta_{jk},$$

$$[(A)_{ij},(A)_{kl}]=[(A^\dagger)_{ij},(A^\dagger)_{kl}]=0. \tag{62}$$

Then, one can write the Hamiltonian of the model at hand as

$$H=\omega\left(\frac{N^2}{2}+\mathrm{Tr}(A^\dagger A)\right)=\omega\left(\frac{N^2}{2}+\mathcal{N}_A\right), \tag{63}$$

$\mathcal{N}_A$ being the total number operator associated with $A$'s. Upon quantization, the constraints (60) become the generators of unitary transformations of both $X_a$ and $\Psi$. The trace part of the constraint (60) demands that (the lhs being the number operator for $\Psi$'s) $B\theta\equiv l$ be quantized to an integer. The traceless part of the constraint (60) demands that the wave function be invariant under SU($N$) transformations, under which $A$ transforms in the adjoint and $\Psi$ in the fundamental representation. Note that as $A$ transforms in the reducible representation $(N^2-1)+1$, with the singlet $B_1=\mathrm{Tr}\ A$, one can introduce a pure adjoint representation as $(\overline{A})_{ij}=(A)_{ij}-\delta_{ij}B_1/N$. This slightly modifies the commutator (62), and completely decouples $B_1$ from the Fock space. Physically, this corresponds to the separation of the center-of-mass coordinate as it has been done for the Calogero model. However, for the sake of simplicity, this will not be done in this section. So, one has to remember that when we compare the results for the Calogero model with those for the CS matrix model, it is always up to the separation of the center-of mass.

Energy eigenstates will be SU($N$) singlets, and explicit expressions for the wave functions were written in Ref. 30:

$$|\Phi\rangle=\prod_{i=1}^{N}(\mathrm{Tr}A^{\dagger i})^{c_i}C^{\dagger l}|0\rangle, \tag{64}$$

where $C^\dagger\equiv\varepsilon^{i_1\cdots i_N}\Psi_{i_1}^\dagger(\Psi^\dagger A^\dagger)_{i_2}\cdots(\Psi^\dagger A^{\dagger N-1})_{i_N}$, and $(A)_{ij}|0\rangle=\Psi_i|0\rangle=0$.

The system contains $N^2+N$ oscillators coupled by $N^2-1$ constraint equations in the traceless part of Eq. (60). Effectively, we can describe the system wih $N+1$ independent oscillators. Therefore, the physical Fock space that consists of all SU($N$)-invariant states can be spanned by $N+1$ algebraically independent operators:

$$B_n^\dagger\equiv\mathrm{Tr}A^{\dagger n}\quad\text{with}\quad n=1,2,\ldots,N, \tag{65}$$

and $C^\dagger$. Again, the operators $B_k^\dagger$ for $k>N$ can be expressed as a homogeneous polynomial of total order $k$ in $\{B_1^\dagger,\ldots,B_N^\dagger\}$, with constant coefficients which are common to all operators $A^\dagger$, see Appendix C. We use the same letter to denote observables in both models. In the CS matrix model, $B_n=\mathrm{Tr}A^n$ and in the Calogero model, $B_n=\Sigma_i a_i^n$, but from the context it is clear what $B_n$ represent. Since

$$\mathrm{Tr}A^kC^{\dagger l}|0\rangle\equiv B_kC^{\dagger l}|0\rangle=0\quad\forall\quad k,\forall\quad l, \tag{66}$$

the state $C^{\dagger l}|0\rangle\equiv|0,l\rangle$ can be interpreted as a ground state—vacuum with respect to all operators $B_k$. Note that the vacuum is not normalized to 1, i.e., $\langle 0,l|0,l\rangle\neq 1$. The whole physical Fock space can be decomposed into towers (modules) built on the ground states with different $l$:

$$F_{\mathrm{phys}}^{\mathrm{CS}}=\sum_{l=0}^{\infty}F_{\mathrm{phys}}^{\mathrm{CS}}(l)=\sum_{l=0}^{\infty}\left\{\prod B_k^{\dagger n_k}|0,l\rangle\right\}.$$

In Ref. 20 it was shown that the states in the $l$th tower of the Chern-Simons matrix model Fock space were identical to the states of the Calogero model with the interaction parameter $\nu=l+1$.

### B. The $\mathcal{A}_N$ algebra, bosonic realization, and dynamical symmetry

Using $[A_{ij},B_n^\dagger]=n(A^{\dagger n-1})_{ij}$, we find a general expression for the commutators between observables:

$$[B_m,B_n^\dagger]=n\sum_{r=0}^{m-1}\mathrm{Tr}(A^rA^{\dagger n-1}A^{m-r-1})$$
$$=m\sum_{s=0}^{n-1}\mathrm{Tr}(A^{\dagger s}A^{m-1}A^{\dagger n-s-1}). \tag{67}$$

One can normally order the rhs of Eq. (67) using the recurrent relation

$$\mathrm{Tr}(A^rA^{\dagger n-1}A^{m-1})=\mathrm{Tr}(A^{r-1}A^{\dagger n-1}A^m)$$
$$+\sum_{s=0}^{n-2}\mathrm{Tr}(A^{r-1}A^{\dagger s})\mathrm{Tr}(A^{\dagger n-s-2}A^{m-1}). \tag{68}$$

With the formal mapping $\mathrm{Tr}(A^r A^{\dagger s} A^k) \to \Sigma_i a_i^r a_i^{\dagger s} a_i^k$, relation (67) goes to the relation valid for the observables in the Calogero model. Also, the recurrent relation (68) has its counterpart in the Calogero model with $\nu=1$, with the same formal mapping. The minimal algebra $\mathcal{A}_N$ including only observables of the type $B_n$ and $B_n^\dagger$, defined by the following relations (including the corresponding Hermitian conjugate relations):

$$[B_{i_1},[B_{i_2},[\ldots,[B_{i_n},B_n^\dagger]\ldots]]=n!\prod_{\alpha=1}^{n} i_\alpha B_{I-n}, \quad (69)$$

where $I=\Sigma_{\alpha=1}^n i_\alpha$ and $i_1,\ldots,i_n$, $n=1,2,\ldots,N$. The identical successive commutators relations (69) hold for the observables acting on the $S_N$-symmetric Fock space of the Calogero model, see Eq. (14), the only difference being that in the Calogero model we have separated the center-of-mass motion. The *minimal* set of generalized vacuum conditions needed to completely define the representation of the algebra (69) on the Fock space is

$$B_2B_2^\dagger|0,l\rangle=2N(N+lN-l)|0,l\rangle,$$

$$B_3B_3^\dagger|0,l\rangle=3N[N^2+1+l(N-1)(2N-1)+l^2(N-1)(N-2)]|0,l\rangle=y|0,l\rangle,$$

$$B_3B_3^{\dagger 2}|0,l\rangle=54\{(l+1)B_1^\dagger B_2^\dagger+[N+(N-2)l+y/27]B_3^\dagger\}|0,l\rangle. \quad (70)$$

The generalized vacuum conditions for the CS matrix model (70) are the same as those for the Calogero model with the interaction parameter $\nu=l+1$.[20]

Using the results obtained in the Calogero model, we can construct a mapping from free Bose oscillators $\{b_i,b_i^\dagger\}$ to $\{B_i,B_i^\dagger\}$. However, the coefficients in the expansion

$$B_n^\dagger=\sum_k\left(\prod b^\dagger\right)^{n+k}\left(\prod b\right)^k$$

depend on $l$, i.e., they are not the same for all towers in the physical Fock space. On the other hand, we can construct a mapping that includes the $C$ operator and its bosonic conterpart $c_0$:

$$B_n^\dagger=\sum_{k,l}\left(\prod b^\dagger\right)^{n+k}c_o^{\dagger l}c_0^l\left(\prod b\right)^k,$$

$$C^\dagger=\sum_{k,l}\left(\prod b^\dagger\right)^{k}c_o^{\dagger l+1}c_0^l\left(\prod b\right)^k,$$

where

$$[b_i,b_j^\dagger]=\delta_{ij},\;\;[c_0,c_0^\dagger]=1,\;\;[b_i,c_0^\dagger]=0.$$

The inverse mapping is given similarly as

$$b_n^\dagger=\sum_{k,l}\left(\prod B^\dagger\right)^{n+k}C^{\dagger l}C^l\left(\prod B\right)^k,$$

$$c_0^\dagger=\sum_{k,l}\left(\prod B^\dagger\right)^{k}C^{\dagger l+1}C^l\left(\prod B\right)^k.$$

As in the case of the Calogero model, these mappings are not unique, but there exist a simple and natural choice of the boundary condition, see Eqs. (27) and (28). The mapping to bosons provides a natural orthogonal basis.

In order to describe the dynamical symmetry of the CS matrix model, we can use the same relations as those obtained for the Calogero model. The generators of the symmetry algebra obtained from the Calogero model act within a fixed tower of states in $F_{\mathrm{phys}}^{\mathrm{CS}}$. However, one can introduce an additional generator acting between different towers in order to describe the larger dynamical symmetry connecting all degenerate states in the CS matrix model.

### C. The $\mathcal{B}_N^{\mathrm{CS}}$ algebra

The problem of finding the underlying algebra of observables in the CS matrix model is more complicated than in the Calogero model. An additional problem is that in the CS matrix model there are more invariants of a given order. For example, let us consider the set of six observables of the fourth order $\{\mathrm{Tr}(A^{\dagger 2}A^2),\mathrm{Tr}(AA^{\dagger 2}A),\mathrm{Tr}(A^\dagger A^2A^\dagger),\mathrm{Tr}(A^2A^{\dagger 2}),\mathrm{Tr}(A^\dagger AA^\dagger A),\mathrm{Tr}(AA^\dagger AA^\dagger)\}$. After performing the normal ordering (see Appendix C), we find that there are two independent, normally ordered invariants $\{\mathrm{Tr}(A^{\dagger 2}A^2),A_{ij}^\dagger A_{kl}^\dagger A_{jk}A_{li}\}$ for $N\geq 4$. (For $N=3$ there is only one independent invariant of that order.) In the Calogero model, for $N\geq 4$, there is only one invariant of the fourth order $\Sigma a_i^{\dagger 2}a_i^2$. Hence, we cannot describe the $\mathcal{B}_N^{\mathrm{CS}}$ algebra in terms of the generators $B_{m,n}=\mathrm{Tr}(A^{\dagger m}A^n)$ in a way analogous to the procedure in the Calogero model.

There are two approaches to the construction of algebra of observables in the CS matrix model. Generally, these two approaches result in different algebra, although in the Calogero model they produce the same algebra. The first one is to write all possible invariants in $A^\dagger$ and $A$, and then to reduce them to the normally ordered ones. The main point is that this set of normally ordered invariants coincides with the set of all invariants in two $N\times N$ matrices $X$ and $Y$ of which the matrix elements are $c$ numbers. The problem thus reduces to finding the minimal number of algebraically independent invariants of two $N\times N$ matrices. This was solved explicitly for $N=2,3$ by Sibirsky[31] and for general $N$ by Donkin.[32] One starts with the finite set of the algebraically independent invariants, for example, for $N=3$ with $\{B_1,B_2,B_3,B_1^\dagger,B_2^\dagger,B_3^\dagger,\mathrm{Tr}(A^\dagger A),\mathrm{Tr}(A^{\dagger 2}A),\mathrm{Tr}(A^\dagger A^2),\mathrm{Tr}(A^{\dagger 2}A^2),{:}\mathrm{Tr}(A^{\dagger 2}A^2A^\dagger A){:}\}$, and calculates all possible commutators between them. The results are invariant operators that can be expressed generally in terms of polynomials in algebraically independent invariants. Note that the basic set of independent invariants is not unique and can be chosen in different ways. However, each invariant can be uniquely expressed in terms of a given basic set of algebraically independent invariants. Generally, one can express the commutator between any two observables as a polynomial in terms of basic generators. This approach is under investigation and will be reported separately.

The second approach is a generalization of our construction of the $\mathcal{A}_N$ algebra. Namely, we start with a minimal number of generators $\{B_1,B_2,\ldots,B_1^\dagger,B_2^\dagger,\ldots\}$ describing the complete physical Fock space, and try to close the algebra under the Lie bracket (commutator) with a minimal number of additional generators. For finite $N$, the algebra is finite and generally contains less generators compared with the first approach. Since the result of the Lie commutator of two generators is a polynomial in the basic generators, we call the algebra of this type finite polynomial Lie algebra.

For $N=2$, the $\mathcal{B}_2$ algebras for the Calogero model and the CS matrix model are equivalent, but for $N\geqslant 3$ they differ. We demonstrate this by the $N=3$ example. We start the construction of the $\mathcal{B}_3^{\rm CS}$ algebra with $\{B_1,B_2,B_3,B_1^\dagger,B_2,{}^\dagger B_3^\dagger\}$ and four additional generators $\mathcal{O}_{1,1}$, $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}$, $\mathcal{O}_{2,2}$ defined as

$$[B_i,B_j^\dagger]=ij\mathcal{O}_{i-1,j-1},\quad i,j=2,3.$$

The operators $\{B_2,B_2^\dagger,\mathcal{O}_{1,1}\}$ build the su(1,1) algebra. This algebra is a subalgebra of every polynomial Lie algebra $\mathcal{B}_N^{\rm CS}$. The operator $\mathcal{O}_{1,1}$ (the total number operator) satisfies the following relation:

$$[\mathcal{O}_{1,1},\mathcal{O}_{i,j}]=(i-j)\mathcal{O}_{i,j}.$$

The $\mathcal{B}_3^{\rm CS}$ algebra is defined by two types of commutation relations:

$$[B_i,\mathcal{O}_{j,k}]=ij\mathcal{O}_{j-1,k+i-1},\quad [B_i^\dagger,\mathcal{O}_{j,k}]=ik\mathcal{O}_{i+j-1,k-1} \tag{71}$$

and

$$[\mathcal{O}_{i+1,j+1},\mathcal{O}_{k+1,l+1}]=(jk-il)\mathcal{O}_{i+k+1,j+l+1},\quad i,j,l,k=0,1. \tag{72}$$

The latter commutator (72) follows from the former (71) and the Jacobi identities.

It is important to note that the invariant of the sixth order $\mathrm{Tr}(A^{\dagger 2}A^2A^\dagger A)$ does not participate in the above algebra. This means that although the above algebra $\mathcal{B}_3^{\rm CS}$ is the minimal algebra, closed under commutation, it is not complete in the sense that we do not know the commutator between any two observables. The second important point is that the above algebra $\mathcal{B}_3^{\rm CS}$ applies equally well to the three-body Calogero model. The only difference is that the invariant $\mathcal{O}_{2,2}$ in the Calogero model can be expressed in terms of the lower invariants, see Appendix A. However, when we reduce the set of generators for $\mathcal{B}_3^{\rm Cal}$ to nine, using the identity for $\mathcal{O}_{2,2}$, we obtain a different algebra with different commutation relations. The above construction can be generalized for any finite $N$. The generators are of the form $B_n$, $B_m^\dagger$, $[B_n,B_m^\dagger]$, $[B_k,[B_n,B_m^\dagger]]$, etc. The number of generators is finite, with an upper limit of $2N^2$ generators. Of course, the lower limit of the number of generators is the number of generators of $\mathcal{B}_N^{\rm Cal}$ in the Calogero model—$1/2(N+4)(N-1)$. Using Jacobi identities, the commutator of any two generators can be expressed as a combination of successive commutators. Some of them are generators of the algebra, and the others can be expressed as polynomials of generators. Here we give some simple, general results:

$$\frac{[B_{i_{n-1}},\ldots,[B_i,B_n^\dagger]\cdots]}{\left(\prod_{\alpha=1}^{n-1} i_\alpha\right)n!}=\frac{[B_{I'-n+2},B_2^\dagger]}{2(I'-n+2)},\ I'=\sum_{\alpha=1}^{n-1} i_\alpha,$$

$$\mathrm{Tr}(A^{\dagger r}AA^{\dagger s}+A^{\dagger s}AA^{\dagger r})=\mathrm{Tr}(A^{\dagger r+s}A+AA^{\dagger r+s}). \tag{73}$$

The authors of Ref. 17 considered the infinite set of all successive commutators of the type $[g_{i_1},[g_{i_2},\ldots,[g_{i_n},g_{i_{n+1}}]\cdots]$, where $g_i$ is either $B_i$ or $B_i^\dagger$, $i$ natural number. This set was closed under commutation, which followed from the Jacobi identities, and all generators were "mirror" symmetric. However, they did not know how the algebra $\mathcal{G}$ looked like explicitly, and gave a step-by-step construction. As their main results, they presented a Table with numbers of linearly independent invariants of a given type $(n,m)$, but they completely omitted the discussion of nonlinear identities between invariants that are crucial for the finite number of degrees of freedom. The conclusion of Ref. 17 was that the algebras $\mathcal{G}$ for the Calogero model and for the matrix model were the same. Although this is true, the actual models are different, as can be seen from the algebraic structure obtained for a fixed number of particles $N$.

## VII. CONCLUSION

The minimal algebra $\mathcal{A}_N$ of invariant operators $A_i$ defined by the successive commutation relation and the generalized vacuum conditions completely define the action of the operators $A_i$ on the states in the physical Fock space. The general structure of the $\mathcal{A}_N$ algebra can be viewed as a generalization of triple operator algebras[33] to the $(N+1)$-tuple operator algebra. For $N=2$, the $\mathcal{A}_2(\nu)$ algebra is just $[A_2,[A_2,A_2^\dagger]]=8A_2$ for a single oscillator $A_2=(a_1-a_2)^2/2$ describing the relative motion in the two-body problem.

We have constructed an orthogonal basis $\tilde{A}_i$ in the dual Fock space. The operators $\tilde{A}_i$ and $A_j^\dagger$ are conjugate to each other with respect to the new scalar product that they induce on the dual Fock space. This construction differs from the construction proposed in Ref. 13 where conjugated operators with different indices do not commute. Furthermore, we have shown that there exists a mapping from ordinary Bose oscillators to operators $A_k,A_k^\dagger$, and its inverse, and in this way we have obtained a natural orthogonal basis for the symmetric Fock space.

Since this algebraic structure is identical in the Calogero model with that in the CS matrix model, all above-mentioned results aplly equally to both models.

It has been shown[20,22] that the states in the $l$th tower of the CS matrix model Fock space are equivalent to the states in the physical Fock space of the Calogero model with the interaction parameter $\nu=l+1$. Therefore, in order to describe the dynamical symmetry of the CS matrix model, we can use the same relations as those obtained for the Calogero model.

The authors of Ref. 17 claim that the infinite algebra $\mathcal{G}$, analyzed in their paper, is common to both, the Calogero model and the CS matrix model. In fact, this algebra is common to all systems of infinitely many identical particles. However, the analysis of the finite CS matrix model proposed in Ref. 19 requires finite algebras of observables. We have shown that owing to the trace identities between observables for finite $N$, the effective, minimal, finite $\mathcal{B}_N$ algebras of observables for the Calogero model and the CS matrix model are quite different. The $\mathcal{B}_N^{\mathrm{CS}}$ algebra contains more algebraically independent observables than $\mathcal{B}_N^{\mathrm{Cal}}$. Also, in the CS matrix model one has to be careful when constructing the $\mathcal{B}_N^{\mathrm{CS}}$ algebra. There are two different methods of constructing the algebra and they lead to different results, as we have demonstrated by the $N=3$ example. Moreover, for the Calogero model, $\mathcal{B}_N^{\mathrm{Cal}}$ can be connected to $W$ algebras, whereas for the CS matrix model it is unclear whether $\mathcal{B}_N^{\mathrm{CS}}$ represents a generalization of the $W$ algebra.

## ACKNOWLEDGMENTS

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## APPENDIX A

It is known[34,35] that the sum of powers $B_n=\Sigma_{i=1}^N a_i^n$ for $n>N$ can be expressed in terms of $B_k$, $1\leqslant k\leqslant N$ in the form $B_n=\Sigma(\Pi B)^n$. In sections herebefore, we have claimed that there exist similar relations for the elements of the algebras $A_N(\nu)$ and $B_N(\nu)$ of the form

$$A_n=\sum\left(\prod A\right)^n,\quad n>N, \tag{A1}$$

and more generally,

$$\bar{B}_{n,m}=\sum\left(\prod \bar{B}_{n_\alpha,m_\alpha}\right),\quad n+m>N, \tag{A2}$$

where $\Sigma n_\alpha\leqslant n$, $\Sigma m_\alpha\leqslant m$. Here we give a method for calculating the coefficients in the identities (A1) and (A2).

For fixed $N$, let us denote

$$A_n=\sum_{i=1}^{N-1}\bar{a}_i^n+(-)^n\left(\sum_{i=1}^{N-1}\bar{a}_i\right)^n, \tag{A3}$$

where $\bar{a}_1,\ldots,\bar{a}_{N-1}$ are independent operators. Since these operators mutually commute, we treat the above identity as an identity in the ring of polynomials in real variables $a_i\in\mathbb{R}$. In order to calculate the coefficients in Eq. (A1), we construct a set of linear equations inserting some points $(a_1,a_2,\ldots,a_{N-1})$ in Eq. (A1). For example, the point

$$\underbrace{(1,k,0,\ldots,0)}_{N-1}$$

gives $A_n=1+k^n+(-)^n(1+k)^n$ for $A_n$ in Eq. (A3). For $N=2$, one easily finds

$$A_{2k}=\frac{1}{2^{k-1}}A_2^k,\quad A_{2k+1}=0,\quad k>1.$$

We also list some results for $N=3$:

$$A_4=\tfrac{1}{2}A_2^2,\quad A_5=\tfrac{5}{6}A_2A_3,\quad A_6=\tfrac{1}{4}A_2^3+\tfrac{1}{3}A_3^2,$$

and for $N=4$

$$A_5=\tfrac{5}{6}A_2A_3,$$

$$A_6=\tfrac{3}{4}A_2A_4-\tfrac{1}{8}A_2^3+\tfrac{1}{3}A_3^2,$$

$$A_7=\tfrac{7}{12}A_3A_4+\tfrac{7}{24}A_2^2A_3.$$

To calculate the coefficients in Eq. (A2), we proceed in two steps. First, we find the coefficients in the following relation:

$$\bar{B}_{n,m}=\sum :\prod \bar{B}_{n_\alpha,m_\alpha}:, \tag{A4}$$

where $\Sigma n_\alpha=n$, $\Sigma m_\alpha=m$, and $:\mathcal{O}:$ denotes normal ordering of the operator $\mathcal{O}$, $\bar{a}_i^\dagger$ on left and $\bar{a}_j$ on the right. Since by definition $:a_ia_j^\dagger:=:a_j^\dagger a_i:$, we consider the identity (A4) as an identity in the ring of polynomials in two sets of commuting, real variables. Then we construct a set of linear equations inserting some points in relations (A2) and (A4). For example, for points $\{\bar{a}_1^\dagger,\ldots,\bar{a}_{N-1}^\dagger\}=\{1,k,0,\ldots,0\}$ and $\{\bar{a}_1,\ldots,\bar{a}_{N-1}\}=\{l,0,\ldots,0,1\}$ the corresponding invariants are

$$\bar{B}_{n,m}=l+(-)^{m+n}(1+k)^n(1+l)^m,\quad m>0\quad\forall\ n,$$

$$A_n^\dagger=1+k^n+(-)^n(1+k)^n\quad\forall\ n,$$

$$A_n=1+l^n+(-)^n(1+k)^n\quad\forall\ n.$$

For the $N=2$ case, we find

$$\bar{B}_{n,m}=0\quad\text{for}\quad n+m=\text{odd},$$

$$\bar{B}_{2k,2l}=\frac{1}{2^{k+l-1}}A_2^{\dagger k}A_2^l,\quad \bar{B}_{2k+1,2l+1}=\frac{1}{2^{k+l-1}}A_2^{\dagger k}\bar{B}_{1,1}A_2^l.$$

We also have a set of relations for $N=3$:

$$\bar{B}_{3,1}=\tfrac{1}{3}A_2^\dagger\bar{B}_{1,1},$$

$$\bar{B}_{2,2}=\tfrac{1}{3}:\bar{B}_{1,1}^2:+\tfrac{1}{6}A_2^\dagger A_2,$$

$$\bar{B}_{4,1}=\tfrac{1}{3}A_3^\dagger\bar{B}_{1,1}+\tfrac{1}{2}A_2^\dagger\bar{B}_{2,1},$$

$$\bar{B}_{3,2}=\tfrac{7}{12}A_3^\dagger A_2+\tfrac{3}{4}A_2^\dagger\bar{B}_{1,2}-\tfrac{1}{2}:\bar{B}_{2,1}\bar{B}_{1,1}:.$$

In the four-particle case, the expressions for $\bar{B}_{4,1}$ and $\bar{B}_{3,2}$ are the same as in the three-particle case, and we present results for $B_{5,1}$, $\bar{B}_{4,2}$ and $\bar{B}_{3,3}$:

$$\overline{B}_{5,1}=\tfrac{1}{3}A_3^\dagger\overline{B}_{2,1}+\tfrac{1}{2}A_2^\dagger B_{3,1},$$

$$\overline{B}_{4,2}=\tfrac{1}{4}A_4^\dagger A_2-\tfrac{1}{8}A_2^{\dagger 2}A_2+\tfrac{1}{2}A_2^\dagger\overline{B}_{2,2}+\tfrac{1}{3}A_3^\dagger\overline{B}_{1,2},$$

$$\overline{B}_{3,3}=-\tfrac{1}{24}A_3^\dagger A_3+\tfrac{3}{4}:\overline{B}_{2,2}\overline{B}_{1,1}:-\tfrac{1}{8}:\overline{B}_{1,1}^3:+\tfrac{3}{8}:\overline{B}_{2,1}\overline{B}_{1,2}:.$$

Note that identities for $\overline{B}_{n,m}$ transform into identities for $A_{n+m}$ after identification $\overline{a_i^\dagger}=\overline{a}_i$.

In the second step we express the normally ordered products in terms of the elements of the $\overline{B}_N(\nu)$ algebra:

$$:\prod \overline{B}_{n_\alpha,m_\alpha}:=\sum\left(\prod \overline{B}_{n'_\alpha,m'_\alpha}\right),$$

$$\sum n'_\alpha\leqslant n_\alpha,\ \sum m'_\alpha\leqslant m_\alpha, \tag{A5}$$

using the commutation relations (8). We present few examples:

$$:\overline{B}_{1,1}^2:=\overline{B}_{1,1}(\overline{B}_{1,1}-1+N\nu),$$

$$:\overline{B}_{2,1}\overline{B}_{1,1}:=\overline{B}_{2,1}(\overline{B}_{1,1}-1+N\nu),$$

$$:\overline{B}_{2,1}\overline{B}_{1,2}:=\overline{B}_{2,1}\overline{B}_{1,2}-(1-N\nu)\left(\overline{B}_{2,2}-\frac{1}{N}A_2^\dagger A_2\right),$$

$$:\overline{B}_{2,2}\overline{B}_{1,1}:=\overline{B}_{2,2}(\overline{B}_{1,1}-2+N\nu)-\nu A_2^\dagger A_2,$$

$$:\overline{B}_{1,1}^3:=\overline{B}_{1,1}(\overline{B}_{1,1}-1+N\nu)(\overline{B}_{1,1}-2+2N\nu-2\nu)$$
$$-\nu(A_2^\dagger A_2+N\overline{B}_{2,2}).$$

Finally, we have

$$\overline{B}_{2,2}=\tfrac{1}{3}\overline{B}_{1,1}^2-\tfrac{1}{3}\overline{B}_{1,1}(1-N\nu)+\tfrac{1}{6}A_2^\dagger A_2,\quad N\leqslant 3,$$

$$\overline{B}_{3,2}=\tfrac{7}{12}A_3^\dagger A_2+\tfrac{3}{4}A_2^\dagger\overline{B}_{1,2}-\tfrac{1}{2}\overline{B}_{2,1}\overline{B}_{1,1}+\tfrac{1}{2}\overline{B}_{2,1}(1-N\nu),\quad N\leqslant 4,$$

$$\overline{B}_{3,3}=-\tfrac{1}{24}A_3^\dagger A_3-\tfrac{1}{8}\overline{B}_{1,1}(\overline{B}_{1,1}-1+N\nu)(\overline{B}_{1,1}-2+2N\nu-2\nu)$$
$$+\tfrac{3}{4}\overline{B}_{2,2}\overline{B}_{1,1}+\tfrac{3}{8}\overline{B}_{2,1}\overline{B}_{1,2}-\tfrac{5}{4}\overline{B}_{2,2}(\tfrac{3}{2}-N\nu)$$
$$+A_2^\dagger A_2\left(\frac{3}{8N}-\nu\right),\quad N\leqslant 5.$$

## APPENDIX B

In this appendix we perform some explicit calculations for the $\mathcal{A}_3(\nu)$ algebra. The minimal set of relations that define the $\mathcal{A}_3$ algebra of operators $\{A_2,A_3,A_2^\dagger,A_3^\dagger,\overline{\mathcal{N}}\}$ is

$$[A_i[A_2,A_2^\dagger]]=4iA_i,$$

$$[A_3,[A_3,A_2^\dagger]]=3A_2^2,$$

$$[A_i,[A_2,[A_3,A_3^\dagger]]]=6iA_iA_2,\quad i=2,3,$$

$$[A_2,[A_2,[A_2,A_3^\dagger]]]=48A_3,$$

$$[A_3,[A_3,[A_3,A_3^\dagger]]]=54A_3^2-\tfrac{9}{2}A_2^3. \tag{B1}$$

For a given representation with fixed parameter $\nu$, we also need generalized vacuum conditions:

$$A_2|0\rangle=A_3|0\rangle=A_2A_3^\dagger|0\rangle=A_3A_2^\dagger|0\rangle=A_3A_2^{\dagger 2}|0\rangle=0,$$

$$A_2A_2^\dagger|0\rangle=4(1+3\nu)|0\rangle,\quad A_3A_3^\dagger|0\rangle=2(1+3\nu)(2+3\nu)|0\rangle,$$

$$A_3A_2^\dagger A_3^\dagger|0\rangle=2(2+3\nu)(4+3\nu)A_2^\dagger|0\rangle,$$

$$A_3A_3^{\dagger 2}|0\rangle=2(2+3\nu)(11+6\nu)A_3^\dagger|0\rangle. \tag{B2}$$

The algebra (B1), with the vacuum conditions (B2), has a unique representation on $F_{\text{symm}}$. Using Eqs. (B1) and (B2) one finds the action of the operators $A_2$ and $A_3$ on any state in the Fock space:

$$A_2|n_2,n_3\rangle=3\binom{n_3}{2}|n_2+2,n_3-2\rangle+4n_2(3n_3+n_2+3\nu)$$
$$\times|n_2-1,n_3\rangle,$$

$$A_3|n_2,n_3\rangle=2\Bigg[n_3(2+3\nu)(1+3\nu+3n_2)+9\binom{n_3}{2}$$
$$\times(2+3\nu+n_2)+27\binom{n_3}{3}$$
$$+6n_3\binom{n_2}{2}\Bigg]|n_2,n_3-1\rangle+48\binom{n_2}{3}|n_2-3,n_3+1\rangle$$
$$-\frac{81}{2}\binom{n_3}{3}|n_2+3,n_3-3\rangle. \tag{B3}$$

The ket $|n_2,n_3\rangle$ denotes the state $A_2^{\dagger n_2}A_3^{\dagger n_3}|0\rangle$.

Here we demonstrate how to construct an orthogonal basis in the dual Fock space defined by (23). We write the general expression (24), up to $k+n\leqslant 5$:

$$\tilde{A}_2=g_2A_2+g_{22}A_2^\dagger A_2^2+g_{23}A_3^\dagger A_2A_3+\cdots$$

$$\tilde{A}_3=g_3A_3+g_{32}A_2^\dagger A_2A_3+\cdots, \tag{B4}$$

with unknown coefficients $g$. To calculate five unknown coefficients in Eq. (B4), we need the following relations:

$$A_2A_2^\dagger|0\rangle=4(1+3\nu)|0\rangle,$$

$$A_3A_3^\dagger|0\rangle=2(1+3\nu)(2+3\nu)|0\rangle,$$

$$A_2A_2^{\dagger 2}|0\rangle=8(2+3\nu)A_2^\dagger|0\rangle,$$

$$A_2A_2^\dagger A_3^\dagger|0\rangle=4(4+3\nu)A_3^\dagger|0\rangle,$$

$$A_3A_2^\dagger A_3^\dagger|0\rangle=2(2+3\nu)(4+3\nu)A_2^\dagger|0\rangle. \tag{B5}$$

Next, we apply Eqs. (B4) to the states in the Fock space, and using Eq. (B5) we obtain

$$\tilde{A}_2=\frac{1}{4(1+3\nu)}A_2-\frac{1}{16(1+3\nu)^2(2+3\nu)}A_2^\dagger A_2^2 -\frac{3}{8(1+3\nu)^2(2+3\nu)(4+3\nu)}A_3^\dagger A_3 A_2+\cdots,$$

$$\tilde{A}_3=\frac{1}{2(1+3\nu)(2+3\nu)}A_3 -\frac{3}{8(1+3\nu)^2(2+3\nu)(4+3\nu)}A_2^\dagger A_2 A_3+\cdots.$$

Next, we calculate the coefficients in the expressions (26), up to $k+n\leqslant 5$, thus finding the mapping from the Bose oscillators $\{b_i,b_i^\dagger\}$ to the operators $\{A_i,A_i^\dagger\}$, and vice versa. We write the general expression

$$A_2=f_2b_2+f_{22}b_2^\dagger b_2^2+f_{23}b_3^\dagger b_2 b_3+\cdots$$
$$A_3=f_3b_3+f_{32}b_2^\dagger b_2 b_3+\cdots, \tag{B6}$$

with unknown coefficients $f$. Inserting the expansion (B6) into Eqs. (B5) and solving for $f$'s, we obtain

$$A_2=2\sqrt{1+3\nu}b_2+2(\sqrt{2+3\nu}-\sqrt{1+3\nu})b_2^\dagger b_2^2 +2(\sqrt{4+3\nu}-\sqrt{1+3\nu})b_3^\dagger b_2 b_3+\cdots,$$

$$A_3=\sqrt{2(1+3\nu)(2+3\nu)}b_3+[\sqrt{2(4+3\nu)(2+3\nu)} -\sqrt{2(1+3\nu)(2+3\nu)}]b_2^\dagger b_2 b_3+\cdots, \tag{B7}$$

and similarly for Hermitian conjugates.

The norms in the $\{A_2^{\dagger n_2}A_3^{\dagger n_3}|0\rangle\}$ Fock space are positive if $\nu>-1/3$ [see Eq. (10)], and we look for inverse mapping. The general expressions are

$$b_2=f_2'A_2+f_{22}'A_2^\dagger A_2^2+f_{23}'A_3^\dagger A_2 A_3+\cdots$$
$$b_3=f_3'A_3+f_{32}'A_2^\dagger A_2 A_3+\cdots. \tag{B8}$$

Inserting these relations into

$$b_ib_j^\dagger|0\rangle=\delta_{ij}|0\rangle,\quad b_ib_i^\dagger b_j^\dagger|0\rangle=(1+\delta_{ij})b_j^\dagger|0\rangle,$$

and solving for unknown coefficients, we find

$$b_2=\frac{A_2}{2\sqrt{1+3\nu}}+\frac{A_2^\dagger A_2^2}{8(1+3\nu)^{3/2}}\left(\sqrt{\frac{1+3\nu}{2+3\nu}}-1\right) +\frac{A_3^\dagger A_2 A_3}{4(1+3\nu)^{3/2}(2+3\nu)}\left(\sqrt{\frac{1+3\nu}{4+3\nu}}-1\right)+\cdots,$$

$$b_3=\frac{A_3}{\sqrt{2(1+3\nu)(2+3\nu)}} +\frac{A_2^\dagger A_2 A_3}{4\sqrt{2(1+3\nu)^3(2+3\nu)}}\left(\sqrt{\frac{1+3\nu}{4+3\nu}}-1\right)+\cdots. \tag{B9}$$

After determining the mapping, we are in a position to construct the orthogonal states. The first few of these states are as follows:

$$b_2^\dagger|0\rangle=\frac{1}{2\sqrt{1+3\nu}}A_2^\dagger|0\rangle,\quad b_3^\dagger|0\rangle=\frac{1}{\sqrt{2(1+3\nu)(2+3\nu)}}A_3^\dagger|0\rangle,\quad b_2^{\dagger 2}|0\rangle=\frac{1}{4\sqrt{(1+3\nu)(2+3\nu)}}A_2^{\dagger 2}|0\rangle,$$

$$b_2^\dagger b_3^\dagger|0\rangle=\frac{1}{2\sqrt{2(1+3\nu)(2+3\nu)(4+3\nu)}}A_2^\dagger A_3^\dagger|0\rangle,\quad b_2^{\dagger 3}|0\rangle=\frac{1}{8\sqrt{3(1+3\nu)(2+3\nu)(1+\nu)}}A_2^{\dagger 3}|0\rangle,$$

$$b_3^{\dagger 2}|0\rangle=\alpha(-12(1+\nu)A_3^{\dagger 2}|0\rangle+A_2^{\dagger 3}|0\rangle),$$

$$\alpha^{-1}=12\sqrt{2(1+\nu)(1+3\nu)(2+3\nu)[(1+\nu)(2+3\nu)(11+6\nu)-2]}. \tag{B10}$$

## APPENDIX C

We propose a method for constructing algebraic relations between observables in the Chern-Simons matrix model. The starting point is the Cayley-Hamilton theorem. For a $N\times N$ matrix $A$, it expresses $A^N$ as a linear function of lower powers of $A$, with coefficients which are symmetric functions of eigenvalues of $A$:

$$A^N=e_1A^{N-1}-e_2A^{N-2}+\cdots(-)^{N-1}e_N\cdot\mathbb{1}, \tag{C1}$$

where $e_r$ is the $r$th elementary symmetric function[34] of eigenvalues of $A$. The trace of Eq. (C1) can be written in the following form:

$$p_N=\sum_{i=1}^{N}(-)^{i-1}e_ip_{N-i}, \tag{C2}$$

where $p_r$ is the $r$th power sum[34] of eigenvalues of $A$. Using Eqs. (C1) and (C2) we can generate algebraic relations between observables for fixed $N$. We will demonstrate the method for the $N=2$ and $N=3$ cases.

For $N=2$, the Cayley-Hamilton theorem gives

$$A^2=(\mathrm{Tr}\,A)A-\det A\cdot\mathbb{1}=B_1A-\det A\cdot\mathbb{1}, \tag{C3}$$

and the trace of Eq. (C3) gives

$$B_2=B_1^2-2\det A. \tag{C4}$$

First, we generate some algebraic relations expressing the observable $B_n$, defined in Eq. (65), for $n\geqslant 3$, using only $B_1$ and $B_2$. One multiplies Eq. (C3) by $A$, takes a trace and expresses $\det A$ using Eq. (C4), to obtain

$$B_3=\tfrac{3}{2}B_2B_1-\tfrac{1}{2}B_1^3. \tag{C5}$$

The expression for $B_4$ is obtained along the same lines, but we also have to use the relation (C5),

$$B_4=\tfrac{1}{2}B_2^2+B_2B_1^2-\tfrac{1}{2}B_1^4. \tag{C6}$$

Recursively, we can generete all algebraic relations expressing $B_n$, $n\geqslant 3$, using only $B_1$ and $B_2$. Of course, when we put $B_1=0$, we obtain results valid for $A_j$ operators in the Calogero model (see Appendix A).

For the $N=3$, case we present a few relations obtained along the same lines. The Cayley-Hamilton theorem gives

$$A^3=(\mathrm{Tr}A)A^2+\tfrac{1}{2}[\mathrm{Tr}A^2-(\mathrm{Tr}A)^2]A+\det A\cdot\mathbb{1}, \tag{C7}$$

and from the trace of Eq. (C5) we obtain

$$\det A=\tfrac{1}{3}B_3-\tfrac{1}{2}B_1B_2+\tfrac{1}{6}B_1^3. \tag{C8}$$

Using Eqs. (C7) and (C8) one easily obtains

$$B_4=\tfrac{1}{2}B_2^2-B_2B_1^2+\tfrac{4}{3}B_3B_1+\tfrac{1}{6}B_1^4,$$

$$B_5=\tfrac{5}{6}B_2B_3+\tfrac{5}{6}B_3B_1^2-\tfrac{5}{6}B_2B_1^3+\tfrac{1}{6}B_1^5,$$

$$B_6=\tfrac{1}{4}B_2^3+\tfrac{1}{3}B_3^2+B_3B_2B_1+\tfrac{1}{3}B_3B_1^3-\tfrac{3}{4}B_2^2B_1^2-\tfrac{1}{4}B_2B_1^4+\tfrac{1}{12}B_1^6. \tag{C9}$$

Next, we turn to the elements of the algebra $\mathcal{B}_N^{\mathrm{CS}}$. These observables are all of the type $\mathrm{Tr}(A^{\dagger n_1}A^{m_1}\cdots A^{\dagger n_k}A^{m_k})$, where $m_i,n_i\in\{1,2,\ldots,(N-1)\}$, and $k<N$. They are not all algebraically indepedent, and in the following we give a method for constructing the relations between them.

We are dealing with matrices whose matrix elements are operators, so we have to take care of ordering. It is important to observe that the relations between normally ordered invariants are identical to relations between invariants in two $N\times N$ matrices whose matrix elements are $c$ numbers. Therefore, our first step is to reduce observables to normally ordered ones. For example,

$$\mathrm{Tr}(AA^{\dagger n})=\mathrm{Tr}(A^{\dagger n}A)+\sum_{s=0}^{n-1}\mathrm{Tr}A^{\dagger s}\mathrm{Tr}A^{\dagger n-s-1},$$

$$\mathrm{Tr}(AA^{\dagger}AA^{\dagger})={:}\mathrm{Tr}(AA^{\dagger}AA^{\dagger}){:}+3N\mathrm{Tr}(A^{\dagger}A)+N^3,$$

$$\mathrm{Tr}(A^{\dagger}AA^{\dagger}A)={:}\mathrm{Tr}(A^{\dagger}AA^{\dagger}A){:}+\mathrm{Tr}(A^{\dagger}A),$$

$$\mathrm{Tr}(A^2A^{\dagger 2})=\mathrm{Tr}(A^{\dagger 2}A^2)+3N\mathrm{Tr}(A^{\dagger}A)+2\mathrm{Tr}A^{\dagger}\mathrm{Tr}A+N(N^2+1),$$

$$\mathrm{Tr}(AA^{\dagger 2}A)=\mathrm{Tr}(A^{\dagger 2}A^2)+\mathrm{Tr}A^{\dagger}\mathrm{Tr}A+N\mathrm{Tr}(A^{\dagger}A),$$

$$\mathrm{Tr}(A^{\dagger}A^2A^{\dagger})=\mathrm{Tr}(A^{\dagger 2}A^2)+\mathrm{Tr}A^{\dagger}\mathrm{Tr}A+N\mathrm{Tr}(A^{\dagger}A),$$

$$\mathrm{Tr}(A^{\dagger 2}AA^{\dagger})=\mathrm{Tr}(A^{\dagger 3}A)+N\mathrm{Tr}A^{\dagger 2},$$

$$\mathrm{Tr}(A^{\dagger}AA^{\dagger 2})=\mathrm{Tr}(A^{\dagger 3}A)+N\mathrm{Tr}A^{\dagger 2}+(\mathrm{Tr}A^{\dagger})^2. \tag{C10}$$

Now, we can construct the relations connecting normally orderd observables starting from the expressions for $B_k\equiv\mathrm{Tr}(A^k)$. For example, in the $N=3$ case, we use the first relation in Eq. (C9) to write

$${:}\mathrm{Tr}(A^{\dagger}+A)^4{:}={:}\tfrac{1}{2}{:}[\mathrm{Tr}(A^{\dagger}+A)^2]^2{:}-{:}\mathrm{Tr}(A^{\dagger}+A)^2{:}{:}[\mathrm{Tr}(A^{\dagger}+A)]^2{:}+\tfrac{4}{3}{:}\mathrm{Tr}(A^{\dagger}+A)^3\mathrm{Tr}(A^{\dagger}+A){:}+\tfrac{1}{6}{:}[\mathrm{Tr}(A^{\dagger}+A)]{:}^4. \tag{C11}$$

This relation gives us an identity expressing observables of order (3,1) as functions of observables of lower order. An observable of order $(m,n)$ is any observable of the type $\mathrm{Tr}(AA^{\dagger}\cdots A\cdots)$ with $m$ $A^{\dagger}$ and $n$ $A$ matrices in the trace. We take ${:}\mathrm{Tr}(A^{\dagger 3}A+A^2A^{\dagger}A+AA^{\dagger}A^2+AA^{\dagger 3}){:}$ on the lhs of Eq. (C11) and we pick up all terms of the same order in $A^{\dagger}$ and $A$ on the rhs Thus, we obtain the following result:

$$\mathrm{Tr}(A^{\dagger 3}A)=\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}(A^{\dagger}A)-\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}A^{\dagger}\mathrm{Tr}A-\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2\mathrm{Tr}(A^{\dagger}A)+\tfrac{1}{6}(\mathrm{Tr}A^{\dagger})^3\mathrm{Tr}A+\mathrm{Tr}A^{\dagger}\mathrm{Tr}(A^{\dagger 2}A)+\tfrac{1}{3}\mathrm{Tr}A^{\dagger 3}\mathrm{Tr}A. \tag{C12}$$

Using Eq. (C12) and relations given in Eq. (C10) we can express all observables of order (3,1) as functions of observables of lower order. From relation (C11) we also obtain a relation for observables of order (2,2)—we simply pick the terms of order (2,2) on both sides of Eq. (C11):

$$2\mathrm{Tr}(A^{\dagger 2}A^2)+{:}\mathrm{Tr}(A^{\dagger}AA^{\dagger}A)={:}{:}[\mathrm{Tr}(A^{\dagger}A)]^2{:}+\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}A^2-2\mathrm{Tr}A^{\dagger}\mathrm{Tr}(A^{\dagger}A)\mathrm{Tr}A-\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}(\mathrm{Tr}A)^2-\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2\mathrm{Tr}A^2+\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2(\mathrm{Tr}A)^2+2\mathrm{Tr}(A^{\dagger 2}A)\mathrm{Tr}A+2\mathrm{Tr}A^{\dagger}\mathrm{Tr}(A^{\dagger}A^2).$$

To obtain all relations among the invariants, one uses the Cayley-Hamilton theorem for matrix $C=A^{\dagger}+\lambda A$. For $k>N$ we can write $\mathrm{Tr}C^k$ as a polynomial in $\mathrm{Tr}C,\ldots,\mathrm{Tr}C^N$, and project terms with $\lambda^n$, $n=1,2,\ldots,(k-1)$. We start with $k=N+1$ and construct algebraicaly independent invariants, step by step, by going to higher $k$.

For completeness, we give some results for $N=2$ case:

$$\mathrm{Tr}(A^{\dagger 2}A)=\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}A+\mathrm{Tr}A^{\dagger}\mathrm{Tr}(A^{\dagger}A)-\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2\mathrm{Tr}A,$$

$$\mathrm{Tr}(A^{\dagger 2}A^2)=\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}A^2+\mathrm{Tr}A^{\dagger}\mathrm{Tr}(A^{\dagger}A)\mathrm{Tr}A-\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2(\mathrm{Tr}A)^2,$$

$$:\mathrm{Tr}(A^{\dagger}AA^{\dagger}A):=-\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}\mathrm{Tr}A^2+:(\mathrm{Tr}(A^{\dagger}A))^2:+\tfrac{1}{2}\mathrm{Tr}A^{\dagger 2}(\mathrm{Tr}A)^2+\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2\mathrm{Tr}A^2-\tfrac{1}{2}(\mathrm{Tr}A^{\dagger})^2(\mathrm{Tr}A)^2.$$

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