Phonon decoherence of quantum entanglement: Robust and fragile states

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We study the robustness and fragility of entanglement of open quantum systems in some exactly solvable models in which the decoherence is caused by a pure dephasing process. In particular, for the toy models presented in this paper, we identify two different time scales, one is responsible for local dephasing, while the other is for entanglement decay. For a class of fragile entangled states defined in this paper, we find that the entanglement of two qubits, as measured by concurrence, decays faster asymptotically than the quantum dephasing of an individual qubit.

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I. INTRODUCTION

Central to quantum information and quantum computation is the concept of entanglement of qubits.[1] In ideal situations, entangled quantum states would not decohere during processing and transmission of quantum information. However, real quantum systems will inevitably be influenced by surrounding environments. The interaction between the environment and a qubit system of interest can lead to decoherence. This is manifest in the loss of unitary evolution.[2,3] The decoherence process varies for different quantum states. On the one hand, the pointer basis is formed by those states that are unaffected by the environmental variables, so in this sense they constitute a set of robust states.[2] If such robust states are also entangled, the entanglement is expected to be stable.[4–8]

Our intuition strongly suggests that a specified entanglement, as a nonlocal property of a composed quantum system, should be very fragile under the influence of the environment. This fragility is a main obstacle for the realization of practical quantum computers. Among the various proposals to combat the decoherence in quantum computing and quantum information processing are ion traps and nuclear magnetic resonance (NMR).[9–13]

The main purpose of the present paper is to focus on the issue of fragility. We show in a specific case that the environment affects the type of coherence called entanglement quantitatively more severely than it affects the coherence associated with off-diagonal matrix elements of a single qubit. We show that this demonstration can be carried out without an approximation for a nontrivial open system described by a reasonably general Hamiltonian. Here we write H_{tot} as the sum of Hamiltonians for the system itself, the environment, and the coupling between them (we use $\hbar = 1$),

$$H_{\text{tot}} = H + \sum_{\lambda} (g_\lambda L a_\lambda^\dagger + g_\lambda^* L^\dagger a_\lambda) + \sum_{\lambda} \omega_\lambda a_\lambda^\dagger a_\lambda , \quad (1)$$

where H is the Hamiltonian of the system of interest. The coupling operator L is a system operator coupled to the environment, and the g_λ are coupling constants.

We cannot discuss all issues in full generality. Specifically, we present a toy model in which the environment is represented as a pure dephasing process. Our final conclusions, based on calculations of concurrence,[14] will apply to a two-qubit pair, and it is fair to say that entanglement of two qubits is a well-discussed topic.[15] However, we think that the way in which entanglement itself decays (or does not decay) when a system is exposed to a nonlocal noisy channel is worth a quantitative examination.

The organization of the paper is as follows: In Sec. II, we present an exactly solvable model and the solution of the exact non-Markovian master equation. In Secs. III and IV, we first present a two-qubit model and then provide some detailed studies of robust and fragile entangled states that are initially pure, as they relax to mixed states. We compare the entanglement decay time with the local dephasing time in Sec. V. We conclude in Sec. VI.

II. AN EXACTLY SOLVABLE MODEL

Equation (1) describes a standard model for open quantum systems in the system-plus-reservoir framework in which a system is coupled linearly to a bath of harmonic oscillators, the excitations of which we can interpret as phonons or photons, for example. In any event, the bath has distributed eigenfrequencies ω_λ and creation and annihilation operators $a_\lambda^\dagger, a_\lambda$ satisfying $[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda,\lambda'}$. We assume that the system and the environment are initially uncorrelated: $\rho(0) = \rho_s(0) \otimes \rho_{\text{bath}}(0)$, where $\rho_{\text{bath}}(0)$ represents the thermal state of the heat bath at temperature T.

In the following, we will consider a specialized model such that $H = H^\dagger$ and $L = L^\dagger$ and the two self-adjoint operators satisfy $[L,H] = i\kappa I$, where κ is a constant, and I is the identity operator acting on the Hilbert space of the system. A particular interesting case is when $\kappa = 0$. That is, H and L commute with each other. Let us note that, except for these conditions, for the time being, we do not assign operators H and L any concrete forms.

The exact non-Markovian master equation for the model presented above takes the following form:[16]

$$\dot{\rho}_t = -i[H,\rho_t] + F(t)[L\rho_t, L] + F^*(t)[L, \rho_t L] + G(t)[\rho_t, L] + G^*(t)[L,\rho_t], \quad (2)$$

where

$$F(t)=\int_0^t \alpha(t,s)ds \quad \text{and} \quad G(t)=\kappa\int_0^t \alpha(t,s)(t-s)ds, \tag{3}$$

where $\alpha(t,s)=\eta(t,s)+i\nu(t,s)$ is the bath correlation function at temperature T,

$$\eta(t,s)=\sum_\lambda |g_\lambda|^2 \coth\left(\frac{\omega_\lambda}{2k_BT}\right)\cos[\omega_\lambda(t-s)], \tag{4}$$

$$\nu(t,s)=-\sum_\lambda |g_\lambda|^2 \sin[\omega_\lambda(t-s)]. \tag{5}$$

Note first that this master equation is in a *time-local* form. So finite memory and thus non-Markovian effects are encoded in the time-dependent coefficients $F(t)$ and $G(t)$.[17,18] In particular, the $G(t)$ term is a pure non-Markovian term which does not appear in the Markov approximation, when $\alpha(t-s)\to\delta(t-s)$. This term gives a non-Markovian phase shift.

Let us now turn to the special case: $\kappa=0$, i.e., the system's Hamiltonian H and the coupling operator L commute with each other. Suppose the states $|n\rangle$ are simultaneous eigenkets of H and L,

$$H|n\rangle=E_n|n\rangle \quad \text{and} \quad L|n\rangle=l_n|n\rangle. \tag{6}$$

We can explicitly solve the corresponding non-Markovian master equation (2), and the solution is surprisingly simple,

$$\rho_{nm}(t)=\langle n|\rho_t|m\rangle=e^{-i(E_n-E_m)t-i(l_n^2-l_m^2)Y(t)} \times e^{-(l_n-l_m)^2X(t)}\rho_{nm}(0) \tag{7}$$

with the symbols $X(t)$ and $Y(t)$ defined as

$$X(t)=\int_0^t F_R(s)ds, \quad Y(t)=\int_0^t F_I(s)ds,$$

where the functions $F_R(t)$ and $F_I(t)$ are the real and imaginary parts of $F(t)$, respectively. The first exponential factor in (7) represents a phase shift and the second one introduces decay, i.e., the decoherence effect.[19] In what follows we always assume that $F(t)$ has an asymptotically positive real part ensuring that the decoherence is an irreversible process.

We can see from the solution (7) that eigenvectors of L are robust states. Precisely, for any initial pure state of the system $|\psi(0)\rangle=|n\rangle$, we have $\rho(0)=|\psi(0)\rangle\langle\psi(0)|=\rho(t)=|\psi(t)\rangle\langle\psi(t)|$.

III. TWO-QUBIT SYSTEM

So far, we have not made any concrete assumptions about the structure of Hamiltonian H and the coupling operator L. In order to discuss entangled states, we have to specify the Hamiltonian. This section is devoted to discuss a simple yet interesting example.

The system we consider consists of two coupled qubits A and B, where the Hamiltonian for two qubits is taken to be nonlinear and nonlocal,

$$H=\omega_A\sigma_z^A+\omega_B\sigma^B+J\sigma_z^A\otimes\sigma_z^B, \tag{8}$$

and the coupling operator is given by

$$L=\sigma_z^A+\sigma_z^B. \tag{9}$$

The coupling operator L commutes with the Hamiltonian H and this guarantees that the energy is conserved at any time. Thus the decoherence is a pure dephasing process in which the loss of quantum phase of the system into the environment is the source of decoherence. The dephasing is a key issue in practical implementations of quantum computers.[11,12]

As shown in the last section, the eigenvectors $|n\rangle$ of L are robust states. The solution of (2) for the two-qubit system is given by

$$\rho_t=\begin{bmatrix} \rho_{11}(0) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) \\ \rho_{21}(t) & \rho_{22}(0) & \rho_{23}(0) & \rho_{24}(t) \\ \rho_{31}(t) & \rho_{32}(0) & \rho_{33}(0) & \rho_{34}(t) \\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(0) \end{bmatrix}, \tag{10}$$

where we have employed the "standard" eigenbasis,

$$|1\rangle_{AB}=|++\rangle, \quad |2\rangle_{AB}=|+-\rangle,$$

$$|3\rangle_{AB}=|-+\rangle, \quad |4\rangle_{AB}=|--\rangle,$$

and where the matrix elements are

$$\rho_{12}(t)=e^{-i(E_1-E_2)t-4iY(t)}e^{-4X(t)}\rho_{12}(0) \tag{11}$$

$$\rho_{13}(t)=e^{-i(E_1-E_3)t-4iY(t)}e^{-4X(t)}\rho_{13}(0), \tag{12}$$

$$\rho_{14}(t)=e^{-i(E_1-E_4)t}e^{-16X(t)}\rho_{14}(0), \tag{13}$$

$$\rho_{24}(t)=e^{-i(E_2-E_4)t+4iY(t)}e^{-4X(t)}\rho_{24}(0), \tag{14}$$

$$\rho_{34}(t)=e^{-i(E_3-E_4)t+4iY(t)}e^{-4X(t)}\rho_{34}(0). \tag{15}$$

Here the eigenvalues of H and L are given by

$$E_1=\omega_1+\omega_2+J, \quad E_2=\omega_1-\omega_2-J,$$

$$E_3=-\omega_1+\omega_2-J, \quad E_4=-\omega_1-\omega_2+J,$$

and

$$l_1=2, \quad l_2=l_3=0, \quad l_4=-2,$$

respectively.

IV. DEGREE OF ENTANGLEMENT: ROBUST VS FRAGILE

In order to quantify the degree of entanglement, we will adopt the *concurrence* C defined by Wootters.[14] The concurrence varies from $C=0$ for an unentangled state to $C=1$ for a maximally entangled state. For qubits, the concurrence may be calculated explicitly from the density matrix ρ for qubits A and B,

$$C(\rho)=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \tag{16}$$

where the quantities λ_i are the square roots of the eigenvalues in decreasing order of the matrix

$$\varrho=\rho(\sigma_y^A\otimes\sigma_y^B)\rho^*(\sigma_y^A\otimes\sigma_y^B), \tag{17}$$

where ρ^* denotes the complex conjugation of ρ in the standard basis.

The most general pure states in the case of the two-qubit model can be written as

$$|\Psi\rangle_{AB}=a_1|1\rangle_{AB}+a_2|2\rangle_{AB}+a_3|3\rangle_{AB}+a_4|4\rangle_{AB}, \tag{18}$$

where $\Sigma_{i=1}^4|a_i|^2=1$. The concurrence of the pure state (18) is simply given by[14]

$$C(|\Psi\rangle_{AB})=2|a_2a_3-a_1a_4|. \tag{19}$$

Thus, the pure state (18) is entangled if and only if

$$a_1a_4\neq a_2a_3. \tag{20}$$

By a robust entangled state we mean one whose entanglement will not decay to zero in temporal evolution (10). In what follows, we will prove that the following two special cases with $a_4=0$ and $a_1=0$:

$$|\psi_1\rangle_{AB}=a_1|1\rangle_{AB}+a_2|2\rangle_{AB}+a_3|3\rangle_{AB}, \tag{21}$$

$$|\psi_2\rangle_{AB}=a_2|2\rangle_{AB}+a_3|3\rangle_{AB}+a_4|4\rangle_{AB}, \tag{22}$$

are robust entangled states if $a_2a_3\neq 0$. In doing so, let us compute the concurrence of the density matrix with the initial state (21). The density matrix for qubits A and B at time t is given by

$$\rho_t=\begin{bmatrix} |a_1|^2 & \rho_{12}(t) & \rho_{13}(t) & 0\\ \rho_{21}(t) & |a_2|^2 & a_2a_3^* & 0\\ \rho_{31}(t) & a_2^*a_3 & |a_3|^2 & 0\\ 0&0&0&0\end{bmatrix}. \tag{23}$$

It is easy to check that the concurrence,

$$C(\rho_t)=C(\rho_0)=2|a_2a_3|. \tag{24}$$

That is, state $|\psi_1\rangle_{AB}$ has robust entanglement (which is more than simply being a robust state). Similarly, we can show, for the entangled pure initial state (22), that the degree of entanglement is completely preserved.

The above result is slightly surprising since the pure states (21) and (22) are the superpositions of the eigenvectors of L with different eigenvalues. One would expect that decoherence would eventually degrade the degree of entanglement. In fact, if we use

$$P(t)=\mathrm{Tr}\,\rho_t^2 \tag{25}$$

to quantify the loss of purity of a quantum state, for the initial pure state (18) we have

$$P(t)=\sum_{i,j=1}^{4}|a_i|^2|a_j|^2\exp[-(l_i-l_j)^2X(t)]. \tag{26}$$

As a measure of purity of a quantum state, it is easy to see that $0\leqslant P\leqslant 1$ where $P=1$ if and only if ρ represents a pure state. From (24) and (26) we can easily see that in temporal evolution the purity of the state (21) deteriorates, but the degree of entanglement remains constant.

In what follows, we will investigate another class of entangled pure states whose entanglement tends to vanish under the influence of the environment. Specifically, we will demonstrate, for the entangled bipartite pure states (18) with $a_3=0$ and $a_2=0$,

$$|\phi_1\rangle_{AB}=a_1|1\rangle_{AB}+a_2|2\rangle_{AB}+a_4|4\rangle_{AB}, \tag{27}$$

$$|\phi_2\rangle_{AB}=a_1|1\rangle_{AB}+a_3|3\rangle_{AB}+a_4|4\rangle_{AB}, \tag{28}$$

that the entanglement will vanish after an *entanglement decay time* denoted by τ_e. We refer those states as fragile entangled states.

Now we are in the position to discuss the entanglement decay of the fragile states by explicitly computing the concurrence. The density matrix with the initial entangled state (27) at t is given by

$$\rho_t=\begin{bmatrix} |a_1|^2 & \rho_{12}(t) & 0 & \rho_{14}(t)\\ \rho_{21}(t) & |a_2|^2 & 0 & \rho_{24}(t)\\ 0&0&0&0\\ \rho_{41}(t) & \rho_{42}(t) & 0 & |a_4|^2\end{bmatrix}, \tag{29}$$

and the concurrence can easily be obtained

$$C(\rho_t)=2|\rho_{14}(t)|=2|a_1a_4|e^{-16\int_0^t ds F_R(s)}. \tag{30}$$

The time scale for the entanglement of a fragile state decaying to zero is determined by the function $F_R(t)$. For large times, the function $F_R(t)\to\Gamma=\int_0^\infty\eta(t)dt$. Then in this long time limit, the entanglement decay time can be identified as

$$\tau_e^{-1}\equiv 16\Gamma. \tag{31}$$

V. ENTANGLEMENT DECOHERENCE VS LOCAL DEPHASING

The dephasing rate of an individual qubit can directly be estimated from the density matrix for qubit A, which can be obtained from the density matrix (10) by further tracing out the variables of qubit B, and vice versa; that is, $\rho^A=\mathrm{Tr}_B\,\rho$, $\rho^B=\mathrm{Tr}_A\,\rho$. The reduced density matrix for qubit A is thus obtained from (10),

$$\rho_t^A=\begin{bmatrix}\rho_{11}+\rho_{22} & \rho_{13}+\rho_{24}\\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44}\end{bmatrix}. \tag{32}$$

Thus, the dephasing rate denoted by τ_ϕ for qubit A is determined by the off-diagonal elements of ρ_t^A,

$$|\rho^A_{12}|=|\rho_{13}+\rho_{24}|\sim e^{-4\int_0^t ds F_R(s)}. \tag{33}$$

Similar analysis applies to qubit B as well. Clearly, the dephasing time depends on the behavior of the function $F_R(t)$. Similar to the entanglement decay time, by ignoring the details of the heat bath we can immediately identify the dephasing time τ_ϕ for the large times as

$$\tau_\phi^{-1}\equiv 4\Gamma. \tag{34}$$

We have thus used the explicit solution of master equation (10) to evaluate the time development of both the degree of entanglement and the dephasing rate. As seen from the expressions (30) and (33), the entanglement for qubits A and B and local quantum coherence for an individual qubit, say qubit A, decay at different rates. We have shown, for a large class of fragile entangled states, that the entanglement decay time is shorter than the local dephasing time on which the quantum coherence of each local qubit is destroyed. Moreover, it can easily be shown, in the all cases with the entangled initial states (18), that the entanglement decay time is not longer than the dephasing time, i.e.,

$$\tau_e\leqslant\tau_\phi. \tag{35}$$

VI. CONCLUSION

Robust and fragile states have been discussed in some exactly solvable models. Particularly, for the two-qubit toy model, we have investigated the dynamics of the robust and fragile entangled states in terms of a measure of entanglement called concurrence. We have emphasized and identified that there exist two different decoherence time scales—entanglement decay time and local dephasing time. For the coupling considered here, we find that the entanglement decay occurs faster than the local dephasing for a large class of fragile entangled states. If the size of the active qubits were to greatly increase, the entanglement decoherence time would be expected to become exceedingly small, reflecting the classical limit.[20] Our example supports one's intuition about the nonlocal nature of the entanglement.

Since the amount of entanglement contained in an entangled quantum state is dependent on the choice of a specific measure of entanglement, the entanglement decoherence rate is also dependent on such a choice. We also emphasize that classification of both robust and fragile entangled states will depend on the concrete form of the interaction between qubits and the environment. In addition, we do not expect that our toy model exhibits all interesting aspects of the decay processes of the quantum entanglement. We do believe, however, that the fast decay rate of quantum entanglement is a generic feature in a variety of physical processes where decoherence is important.

Finally, it may be worth noting that our two-qubit model allows two competing processes: creation and annihilation of entanglement.[21,22] Thus, the maximal entanglement generation under the dephasing process is an interesting problem that will be addressed in future publications.

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