Uemura relation in phase-fluctuation-dominated superconductors

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Starting from the "negative-U" Hubbard Hamiltonian of a superconductor with nonretarded interaction on a lattice in three dimensions, we establish an effective theory in terms of the modulus and phase of the superconducting pairing field. This facilitates a scenario where pseudogap is associated with pairing without phase order at T^* while superfluidity corresponds to the onset of phase coherence at T_c. Building on this framework we calculate from the linear-response theory the penetration depth λ as a function of temperature and interaction strength U. By studying the crossover from the amplitude to the phase-fluctuation-dominated regime, we find a remarkable quasiuniversal behavior in form of the parametric Uemura plot for the superfluid density $n_s \sim 1/\lambda^2$ which we discuss in the context of experiments in underdoped and overdoped high-T_c cuprates.

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I. INTRODUCTION

After more than a decade of research effort in high-T_c oxide superconductors (HTSC), there is still no general consensus on the pairing mechanism and the essential physics involved. Traditionally, a superconducting phase in many-body system is represented by the existence of the so-called off-diagonal long-range order in the system.[1] According to the theory of Bardeen, Cooper, and Schrieffer[2] (BCS) in metallic superconductors, the pairing of charge carriers, the formation of the gap in the density of states, and the setting of the coherence between electron pairs occur at the same temperature marking the superconducting transition at critical temperature T_c. It turns out that in the BCS theory the phase of the complex order parameter is unimportant for determining T_c and other physical properties brought about by the transition. However, it has been realized that phase fluctuations are crucial in oxide superconductors since they are doped insulators with a very low superfluid density n_s.[3,4,6,7] This quantity is related to the magnetic-field penetration depth λ being the basic experimentally measurable parameter of superconductivity. In the framework of the clean-limit London model, $1/\lambda^2(T) \sim n_s(T)/m^*$ (where m^* is the effective mass). Remarkably, Uemura and co-workers found out that muon spin relaxation (μSR) measurements of λ in HTSC reveal intriguing correlation between T_c and $n_s(0)$: T_c increases linearly with $n_s(0)$ in the underdoped region, followed by a saturation with increasing superfluid density.[8] Surprisingly, μSR measurements in heavily overdoped Tl-2201 system show a *depression* of T_c in the overdoped region accompanied by the decrease of $n_s(0)$ even though the normal-state measurements clearly indicate an *increase* of carrier concentration with doping.[9] This is in contrast to the common expectation (supported by the BCS theory) that $n_s(0)$ should continue to increase with growing carrier concentration in the overdoped regime. The existing data imply a Uemura diagram, where T_c vs $n_s(0)$ describes "boomerang" (or reflex loop) shaped path from the underdoped to the heavily overdoped regime. Apparently, a number of experiments suggest that in HTSC the process of pair formation and their coherence may take place at different temperatures: pairing at the so-called pseudogap temperature T^* and phase coherence at T_c (see, e.g., review in Ref. 10). It is clear that a systematic theory of this phenomenon has to include the effect of phase fluctuations on the same footing as the size of the local superconducting gap.[11,12] Support for the paramount role of the phase fluctuations in HTSC comes also from the recent high-frequency transport experiments.[13]

To assess the relevance of phase-fluctuation picture in HTSC it is important to know how it is related to the basic phenomenology of superconducting cuprates. For example, it is an interesting question as to how the picture proposed by Uemura correlates with the phase-fluctuation scenario, when the behavior of the superfluid density is controlled by the strength of pairing interaction. Theoretically, the problem can be addressed without having to resort to a specific microscopic mechanism for electron pairing. In the present work to model both quantal and spatial phase fluctuations we adopt a "negative-U" Hubbard model on a three-dimensional (3D) simple cubic lattice where pairing can be conveniently described in terms of a single interaction parameter. We perform the separation of the superconducting order parameter into the modulus *and* phase variables (i.e., the Goldstone mode), which is essential to capture the intricate relations between the onset of pairing of electrons and their condensation. Furthermore, we approach the problem by analyzing the "phase-only" effective action resulting from the microscopic model within the functional integral framework. Subsequently, we study in the behavior of the superfluid density n_s, by calculating the penetration depth from the electromagnetic response of the system. In particular we examine the T_c vs $n_s(0) \sim 1/\lambda^2(0)$ dependence in a form of the parametric Uemura plot.

The outline of this paper is the following. In Sec. II we introduce a negative-U Hubbard model for strongly correlated superconductor. To explicate the low-energy physics of the relevant Goldstone modes we pass to the effective phase-only action and subsequently to the nonlinear σ model for the fluctuating superconducting order parameter. Furthermore, in Sec. III, by analyzing the electromagnetic response within the Matsubara "'imaginary-time" formalism we calculate the interaction and temperature dependence of the penetration depth. Finally, we discuss and compare our findings with the basic phenomenology of high-T_c cuprates.

II. THE MODEL AND EFFECTIVE PHASE-ONLY ACTION

Theoretically, the problem can be addressed without having to resort to a specific microscopic mechanism for electron pairing and can be studied on the negative-U Hubbard model. This is the simplest lattice model which can be studied where the pair size can be varied by changing the strength of the on-site attraction $|U|$ relative to the nearest-neighbor hopping. The Hamiltonian reads

$$H=\sum_{\langle ij\rangle\sigma}(t_{ij}e^{i\chi_{ij}}-\mu\delta_{ij})c^{\dagger}_{i\sigma}c_{j\sigma}-|U|\sum_{i}n_{i\uparrow}n_{i\downarrow}. \tag{1}$$

Here $c^{\dagger}_{i\sigma}(c_{i\sigma})$ annihilates an electron of spin σ at the ith site ($i=1,\ldots,N$) and $n_{i\sigma}=c^{\dagger}_{i\sigma}c_{i\sigma}$; $-|U|$ is the strength of (on-site) attractive interaction, while μ denotes the chemical potential. Here, t_{ij} is the hopping integral. For the latter the bare band dispersion was presumed to be $t(\mathbf{k})=-2t\epsilon(\mathbf{k})$, where $\epsilon(\mathbf{k})=\cos(ak_x)+\cos(ak_y)+\cos(ak_z)$ is the structure factor of the simple cubic lattice in three dimensions with lattice spacing a.

To study electromagnetic response, the magnetic-field effect is included through the Peierls phase factor:[14]

$$\chi_{ij}=\frac{2\pi}{\Phi_0}\int_{\mathbf{R}_j}^{\mathbf{R}_i}\mathbf{A}\cdot d\mathbf{l}, \tag{2}$$

where $\Phi_0=hc/e$ is the magnetic-flux quantum and $\mathbf{A}$ is the vector potential. Introducing the auxiliary (pair) bosonic fields $\Delta_i(\tau)$ depending on the imaginary–time τ ($0\leqslant\tau\leqslant\beta\equiv 1/k_BT$) and employing the Hubbard–Stratonovich transformation to decouple the quartic (in fermionic variables) Coulomb interaction term in Eq. (1) one arrives at the partition function[15]

$$Z=\int\prod_i \mathcal{D}\Delta_i^*\mathcal{D}\Delta_i\mathcal{D}\overline{\Psi}_i\mathcal{D}\Psi_i\exp\left[-\sum_i\int_0^{\beta}d\tau\frac{|\Delta_i(\tau)|^2}{U}-\sum_{\langle ij\rangle}\int_0^{\beta}d\tau\overline{\Psi}_i(\tau)(t_{ij}e^{i\chi_{ij}}\hat{\sigma}_3+\mathbf{G}_i^{-1}\delta_{ij})\Psi_j(\tau)\right]. \tag{3}$$

To compactify our notation we introduce Nambu spinor and its Hermitian adjoint:

$$\Psi_i=\begin{pmatrix}c_\uparrow\\ \overline{c_\downarrow}\end{pmatrix}_i,\qquad \overline{\Psi}_i=(\overline{c_\uparrow}c_\downarrow)_i, \tag{4}$$

and the Nambu Green's function $\mathbf{G}_i$ satisfies

$$\left[-\frac{\partial}{\partial\tau}\mathbf{1}+\mu\hat{\sigma}_3+\hat{\Delta}_j(\tau)\hat{\sigma}_1\right]\mathbf{G}_i(\tau-\tau')=\delta(\tau-\tau'). \tag{5}$$

Here,

$$\hat{\Delta}_i(\tau)=\Delta_i(\tau)\hat{\sigma}^{+}+\Delta_i^*(\tau)\hat{\sigma}^{-},$$
$$\hat{\sigma}^{\pm}=(\hat{\sigma}_1\pm i\hat{\sigma}_2)/2 \tag{6}$$

and σ_α ($\alpha=1,2,3$) are Pauli matrices.

III. EFFECTIVE PHASE-ONLY ACTION

A. Modulus-phase representation

Since superconductivity emerges with the spontaneous breaking of a local U(1) phase invariance, the most important degree of freedom is the phase of the pair field which is the relevant Goldstone variable. It is well known that the BCS limit is a rather exotic case of a fermionic system with infinitesimal coupling strength when the disappearance of superconductivity can be described as the pair-breaking transition. On the contrary, in the strong coupling bosonic regime the temperature scales for pair decomposition and the U(1) symmetry breakdown can be arbitrarily separated. As a result, the BCS limit hides the bosonic scenario of phase coherence, while the strong-coupling regime misses the fermionic aspect of gap formation. To combine both features we, therefore, set

$$\hat{\Delta}_j(\tau)=|\Delta_j(\tau)|e^{-i\phi_j(\tau)\hat{\sigma}_3}\hat{\sigma}_1 \tag{7}$$

and decouple the phase field $\phi_i(\tau)$ from the order-parameter amplitude by performing a singular gauge transformation to new fermionic variables:

$$f_{j\sigma}=c_{j\sigma}e^{-i\phi_i(\tau)/2}. \tag{8}$$

Physically, this amounts to replacing the original charged fermion $c_{j\sigma}(\tau)$ with spin carrying neutral fermion $f_{j\sigma}(\tau)$ and a spinless charged boson $e^{-i\phi_i(\tau)/2}$. By integrating out neutral fermionic fields, one obtains the partition function as a functional integral over the modulus and phase of the superconducting pair field:

$$Z=\int\prod_i|\Delta_i|\mathcal{D}|\Delta_i|\mathcal{D}\phi_i\exp\{\mathcal{S}_{\rm eff}[\Delta,\phi]\},$$
$$\mathcal{S}_{\rm eff}[\Delta,\phi]=\sum_i\int_0^{\beta}d\tau\frac{|\Delta_i(\tau)|^2}{U}-{\rm Tr}\,\ln[\hat{\mathcal{T}}_\phi(\chi)+\mathbf{G}_\phi^{-1}], \tag{9}$$

where

$$[\hat{\mathcal{T}}_\phi(\chi)]_{ij}=t_{ij}e^{-i[\phi_i(\tau)-\phi_j(\tau)-2\chi_{ij}]\hat{\sigma}_3/2}\hat{\sigma}_3,$$
$$\mathbf{G}_{\phi i}^{-1}=e^{-i\phi\hat{\sigma}_3/2}\mathbf{G}_i^{-1}(\tau,\tau')e^{i\phi\hat{\sigma}_3/2},$$
$$=-\frac{\partial}{\partial\tau}\mathbf{1}+\left[\mu+\frac{i}{2}\frac{\partial\phi_j(\tau)}{\partial\tau}\right]\hat{\sigma}_3+|\Delta_j(\tau)|\hat{\sigma}_1. \tag{10}$$

For large $|U|$ amplitude fluctuations can be regarded as highly massive excitations and all the important collective variables are then in the phase sector. The saddle point of the effective action with respect to the modulus of the pairing field $\delta\mathcal{S}_{\rm eff}[\Delta,\phi]/\delta|\Delta_i(\tau)|=0$ generates Hartree-Fock mean-field theory for the modulus of static and spatially uniform pairing amplitudes $\Delta_i(\tau)\equiv|\Delta|$. Now, focusing on the phase degrees of freedom, we expand the logarithm in the action (9) in powers of $\widehat{\mathcal{T}_\phi}(\mathbf{A})$ in a close analogy to the derivation of the phase action for Josephson-coupled superconductors.[16] The resulting lattice phase-only action then reads

$$\mathcal{S}_{\rm eff}[\phi]=\frac{E_C^{-1}(\Delta)}{16}\sum_i\int_0^\beta d\tau\left[\frac{\partial\phi_i(\tau)}{\partial\tau}\right]^2 -\sum_{\langle ij\rangle}\int_0^\beta d\tau E_J(\Delta)\cos[\phi_i(\tau)-\phi_j(\tau)-2\chi_{ij}]. \tag{11}$$

The Josephson-like coupling

$$E_J(\Delta)=2t^2\int_0^\beta d\tau G^{12}(\tau)G^{12}(-\tau) \tag{12}$$

then will contribute to the bare phase stiffness, while the kinetic-energy term

$$E_C^{-1}(\Delta)=-\int_0^\beta d\tau[G^{11}(-\tau)G^{11}(\tau)-G^{12}(-\tau)G^{12}(\tau)] \tag{13}$$

reflects the quantum fluctuations in the number density of charge carriers. These terms incorporate the fermionic degrees of freedom via the local electron Green's function

$$\mathbf{G}(\tau)=(1/\beta)\sum_{\nu_l}\mathbf{G}(\nu_l)e^{-i\nu_l\tau}, \tag{14}$$

where

$$G^{ab}(\nu_l)=\begin{pmatrix}\dfrac{-i\nu_l+\mu}{\nu_l^2+\mu^2+|\Delta|^2} & \dfrac{|\Delta|}{\nu_l^2+\mu^2+|\Delta|^2}\\ \dfrac{|\Delta|}{\nu_l^2+\mu^2+|\Delta|^2} & \dfrac{-i\nu_l-\mu}{\nu_l^2+\mu^2+|\Delta|^2}\end{pmatrix}^{ab}, \tag{15}$$

where $\nu_l=\pi(2l+1)/\beta$ $(l=0,\pm1,\pm2,\dots)$ are the (Fermi) Matsubara frequencies.

B. Mapping on a quantum Hamiltonian of the 3D *XY* model

By performing the standard Legendre transformation from the "phase velocities" $\partial\phi_j/\partial\tau$ to the conjugate "momenta" Q_j (that are interpreted as the number density fluctuations of charge carriers) we obtain the basic Hamiltonian for the system in a form of the *quantum* 3D *XY* model:

$$\mathcal{H}_{\rm XY}^{\rm qm}=E_C(\Delta)\sum_i\hat{Q}_i^2-\sum_{\langle ij\rangle}E_J(\Delta)\cos(\phi_i-\phi_j), \tag{16}$$

where $\hat{Q}_j=-2i(\partial/\partial\phi_j)$ is the charge (pair) number operator. A fundamental property of the quantum *XY* model is provided by the fact that $\hat{Q}_j$ and ϕ_j are canonically conjugate variables and therefore satisfy the following commutation relation $[\hat{Q}_m,\phi_j]=-i\delta_{mj}$—thus, the fluctuation of these are intimately related. The phase transition to the long–range phase-coherent state is governed by the competition between the bare phase stiffness $E_J(\Delta)$ and the effective Coulomb interaction $E_C(\Delta)$. When $E_C(\Delta)$ is comparable to $E_J(\Delta)$, charging effects give rise to profound quantum effects: zero-point fluctuations of the phase may destroy the long-range superconducting state even at $T=0$.

IV. NONLINEAR σ MODEL FORMULATION

The action (11) implies the phase-fluctuation algebra of the Euclidean group E_2 for the number density operator (angular momentum) $\hat{N}_j\equiv i\partial/\partial\phi_j$ and the charged boson (linear momentum) $\hat{P}_j\equiv e^{i\phi_j}$, with the square of linear momentum restricted to unity:[17] $\hat{P}_i\hat{P}_i^\dagger=\sin^2\phi_i+\cos^2\phi_i=1$. This constraint suggests that an effective continuum field theory in the form of a quantum-mechanical nonlinear σ model (QNLσM) would be appropriate to capture the dynamics in the phase sector beyond the mean-field level.[18] Using the Fadeev-Popov method with the Dirac δ functional (which facilitates both the change of integration variables and the imposition of the spherical constraint) we obtain

$$Z=\int\prod_i\mathcal{D}\psi_i\mathcal{D}\psi_i^*\,\delta\left(\sum_i|\psi_i|^2-N\right)\times\int\prod_i\mathcal{D}\phi_i e^{-\mathcal{S}_{\rm eff}[\phi]}\prod_i\delta[\mathrm{Re}\,\psi_i-\cos(\phi_i)]\times\delta[\mathrm{Im}\,\psi_i-\sin(\phi_i)]. \tag{17}$$

The convenient way to enforce the spherical constraint is to use the functional analog of the δ-function representation $\delta(x)=\int_{-\infty}^{+\infty}(d\zeta/2\pi)e^{i\zeta x}$, which introduces the Lagrange multiplier $\zeta(\tau)$.[19] The evaluation of Eq. (17) in terms of the order-parameter fields ψ_i yields the partition function of the corresponding QNLσM model:

$$Z_\sigma=\int\prod_j\mathcal{D}\psi_j\mathcal{D}\psi_j^*\int\left[\frac{\mathcal{D}\zeta}{2\pi i}\right]e^{-\mathcal{S}_\sigma[\psi,\zeta]}, \tag{18}$$

where

$$\mathcal{S}_\sigma[\psi,\zeta]=\frac{1}{\beta N}\sum_{\mathbf{k}}\sum_{\omega_l}\psi_{\mathbf{k}}^*(\omega_l)\mathcal{G}_{\mathbf{k}}^{-1}(\omega_l)\psi_{\mathbf{k}}(\omega_l). \tag{19}$$

The summation in Eq. (19) is performed over all wave-vector components $\mathbf{k}$ $(-\pi p/aN\leqslant k_\alpha\leqslant\pi p/aN$, where $\alpha=x,y,z$; p are integers, $0\leqslant p\leqslant N-1)$ and the (Bose) Matsubara frequencies $\omega_l=2\pi l/\beta$ $(l=0,\pm1,\pm2,\dots)$. Here, $\mathcal{G}_{\mathbf{k}}(\omega_l)$ is the Fourier transform of the order-parameter susceptibility,

$$\mathcal{G}_{ij}(\tau-\tau')=[E_J(\Delta)e^{2i\chi_{ij}}\delta_{i,j+d}+\zeta\delta_{ij}]\delta(\tau-\tau')+W_{ij}^{-1}(\tau-\tau'), \tag{20}$$

with the vector d running over nearest neighbors. Furthermore,

$$W_{ij}(\tau-\tau')=\langle e^{i[\phi_i(\tau)-\phi_j(\tau')]}\rangle \tag{21}$$

is the phase-phase correlation function, where

$$\langle \cdots \rangle = Z_0^{-1} \int \prod_i \mathcal{D}\phi_i(\tau) \cdots e^{-\mathcal{S}_C[\phi]}, \tag{22}$$

where $\mathcal{S}_C[\theta]$ refers to the first term of the action (11) and Z_0 is the statistical sum of the ''noninteracting'' system described by $\mathcal{S}_C[\theta]$. In the spherical model the critical boundary (in our case marking the onset of the superconducting phase-coherent state) is determined by the divergence of the order-parameter susceptibility $\mathcal{G}^{-1}_{\mathbf{k}=0}(\omega_l=0)=0$. This fixes the saddle-point value ζ_0:[20] with the onset of the phase-transition saddle-point value of the Lagrange multiplier ζ_0 ''sticks'' to that value at criticality ($\zeta=\zeta_0^{\mathrm{crit}}$) and stays constant in the whole low-temperature phase.

V. PENETRATION DEPTH

To proceed with the penetration depth calculation, we examine the response kernel as a function of the vector $\mathbf{q}$ and frequency ω_l:

$$\Lambda_{xx}(\mathbf{q},\omega_l) = \sum_i \int_0^\beta d\tau \left. \frac{e^{i\omega_l\tau + i\mathbf{q}\cdot\mathbf{r}_i}\delta^2 \ln Z}{\delta A_x(\mathbf{r}_i,\tau)\,\delta A_x(0,0)} \right|_{\mathbf{A}=0}. \tag{23}$$

Then, in the static and uniform limit the magnetic-field penetration depth becomes $\lambda^{-2}=4\pi\lim_{\mathbf{q}\to 0}\Lambda_{xx}(\mathbf{q},\omega_l=0)$. Explicitly,

$$\frac{1}{\lambda^2} = \frac{32\pi e^2}{\hbar^2 c^2 a}\psi_0^2 E_J(\Delta) \tag{24}$$

so that $n_s/m^* \equiv 8\psi_0^2 E_J(\Delta)/\hbar^2 a$. Here, $\psi_0=\langle e^{i\phi_j}\rangle$ is the order parameter describing phase-coherent state. The technical steps in obtaining ψ_0 and the modulus $|\Delta|$ in the form of the self-consistent equations are quite similar to those in Ref. 15—we will therefore be quite brief. The value of the order parameter ψ_0 then reads

$$\psi_0^2 = 1 - \sqrt{2\alpha(\Delta)} \int_{-\infty}^{+\infty} \frac{dy\,\rho(y)}{\sqrt{3-y}} \coth[\beta E_p(y,\Delta)] \tag{25}$$

with $E_p(y,\Delta)=E_J(\Delta)\sqrt{2(3-y)\alpha(\Delta)}$ and $\alpha(\Delta) \equiv E_C(\Delta)/E_J(\Delta)$. Further, $\rho(y)=\int_{-\pi}^{\pi}[d^3\mathbf{k}/(2\pi/a)^3]\delta[y-\epsilon(\mathbf{k})]$ is the density of states for 3D simple cubic lattice:

$$\rho(y) = \frac{1}{\pi^3}\int_{\max(-1,-2-y)}^{\min(1,2-y)} du\,\Theta(|y|/3-1) \times \frac{1}{\sqrt{1-u^2}}\mathbf{K}[\sqrt{1-(u+y)^2/4}], \tag{26}$$

where $\mathbf{K}(x)$ and $\Theta(x)$ are the elliptic integral of the first kind and the unit step function, respectively. Subsequently, the gap parameter $|\Delta|$ is determined self-consistently by

$$1 = |U| \int_{-\infty}^{+\infty} \frac{dy\,\rho(y)}{2E_g(y,\Delta)} \tanh\left[\frac{\beta}{2}E_g(y,\Delta)\right], \tag{27}$$

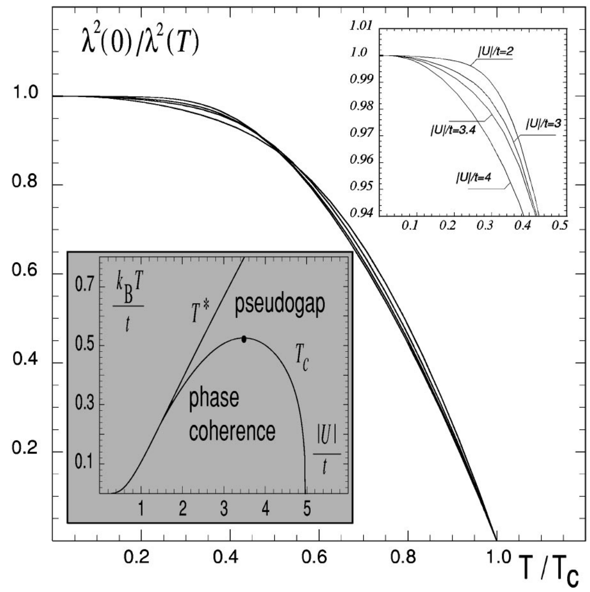

FIG. 1. Temperature vs the inverse squared penetration depth, [scaled by T_c and the zero-temperature value $\lambda^2(0)$, respectively] for several values of the interaction $|U|/t$ and $f=0.8$. Inset: a closeup of the low-temperature region.

where $E_g(y,\Delta)=\sqrt{[2yt(\Delta)+\mu]^2+|\Delta|^2}$ and $t(\Delta)$ is the renormalized (due to phase fluctuations) bandwidth parameter:[15]

$$t^2(\Delta) = \frac{2}{9}t^2\alpha(\Delta)\int_{-\infty}^{+\infty}\frac{dy\,\rho(y)y}{\sqrt{3-y}}\coth[\beta E_p(y,\Delta)]. \tag{28}$$

Finally, the band filling f is determined self-consistently by

$$f-1 = \int_{-\infty}^{+\infty}\frac{dy\,\rho(y)[2yt(\Delta)+\mu]}{E_g(y,\Delta)}\tanh\left[\frac{\beta}{2}E_g(y,\Delta)\right] \tag{29}$$

and $f=1$ corresponds to half-filling. From Eqs. (25)–(29) it follows that the modulus-phase representation (7) results in a theory which combines the (pseudo) gap equation (27) in a form resembling the standard BCS-like expression for Δ with the equation for the ordering of the phase degrees of freedom in the real space (25)—in close analogy to the local-pair superconductor scenario given by the hard-core bosonic limit of the negative-U Hubbard model.

The superfluid density (or inverse square penetration depth) normalized to its $T=0$ value calculated from Eq. (24) is plotted in Fig. 1 as a function of T/T_c for several representative values of $|U|/t$ ranging from the strong-coupling ($|U|>|U_{\mathrm{opt}}|$) to weak-coupling ($|U|<|U_{\mathrm{opt}}|$) regimes. Here, $|U_{\mathrm{opt}}|$ is the ''optimal'' interaction strength for which $T_c^{\max}=T_c(U_{\mathrm{opt}})$. Plots in Fig. 1 show a quasiuniversal behavior with respect to the interaction strength: $n_s(T)/n_s(0)$ depends only weakly on $|U|$ [more precisely, according to numerical analysis of Eq. (24), $1/\lambda^2(T)$ is weakly interaction dependent for roughly $|U|/t>2$]. This clearly indicates that T_c is the only energy scale involved in the temperature dependence of $\lambda(T)$ for interaction strength ranging from weak to strong

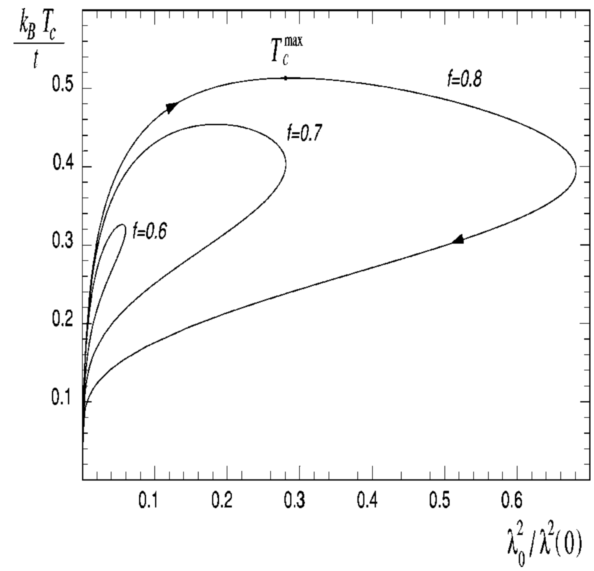

FIG. 2. Uemura-type plot of T_c vs zero-temperature superfluid density for several values of the filing parameter *f*. Variation of $|U|/t$ as the control parameter for the parametric plot; arrows indicate the crossover from the strong to weak coupling (here $\lambda_0 = 32\pi e^2 t/\hbar^2 c^2 a$). Inset: $T-|U|$ phase diagram showing T^* and T_c for $f=0.8$ (see Ref. 15).

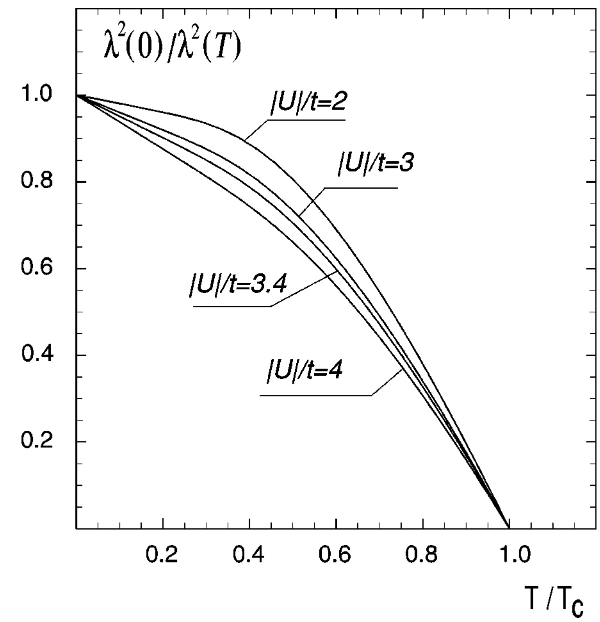

FIG. 3. Effect of classical phase fluctuations. Plot of $1/\lambda^2(T)$ as in Fig. 1 but calculated for the classical version of 3D XY Hamiltonian [see Eq. (30)].

coupling.[21] Indeed, the inspection of Eqs. (11) and (24) reveals that the bare phase stiffness $E_J(\Delta)$, which sets the energy scale of $k_B T_c$, also enters the expression for $1/\lambda^2(T)$ linearly. It is interesting to note, that in high-T_c cuprates similar behavior was found experimentally:[22] when the temperature scale is normalized by T_c, the overall temperature dependence of $\lambda^2(0)/\lambda^2(T)$ is remarkably independent of doping *x*, so the data collapse approximately onto one curve. Furthermore, we plot in Fig. 2 the critical temperature T_c vs the zero-temperature value of the superfluid density $n_s(0)$ in a form of the Uemura plot. It can be seen that $n_s(0)$ follows a reentrant loop, resembling the outline of a fly's wing, as the control parameter for the parametric plot—the interaction $|U|$ moves from the strong to weak regime. Experimentally, the relation between the critical temperature and the zero-temperature muon spin depolarization rate $\sigma(T=0)$ $[\sim 1/\lambda^2(T=0)]$ in the form resembling the plots in Fig. 2. was seen in a number of cuprates.[8,9]

VI. EFFECT OF CLASSICAL PHASE FLUCTUATIONS

It is interesting to consider the effect of *classical* phase fluctuations on the low-temperature properties of the penetration depth. It was suggested that classical phase fluctuations are capable to produce a linear temperature dependence of $\lambda^2(0)/\lambda^2(T)$ which may be relevant to explain the behavior of the penetration depth in high-T_c superconductors.[4] A suitable treatment of classical phase fluctuations is readily obtained by neglecting the quantum term with the Coulomb energy [for $E_C(\Delta)\to 0$] in the effective phase-only action [see Eq. (11)]. A "reverse engineering" of the action from the Lagrange to the Hamilton description results then in the classical XY-model Hamiltonian [see also Eq. (16)]

$$\mathcal{H}^{\rm cl}_{\rm phase} = -\sum_{\langle ij\rangle} E_J(\Delta)\cos(\phi_i-\phi_j), \tag{30}$$

with the bare phase stiffness given by $E_J(\Delta)$ as a single energy parameter. The transition temperature T_c into phase-ordered state which can be easily obtained from Eq. (25) in the limit $\alpha(\Delta)\to 0$:

$$\frac{E_J(\Delta)}{k_B T_c} = \int_{-\infty}^{+\infty} d\epsilon \frac{\rho(\epsilon)}{3-\epsilon} = K_c \approx 0.505\,462\,019\,7. \tag{31}$$

Furthermore, the order parameter (25) in this limit becomes

$$\psi_0^2 = 1 - K_c \frac{k_B T}{E_J(\Delta)}, \tag{32}$$

which implies a linear temperature behavior of the superfluid density in the low-temperature region [cf. Eq.(24) and Fig. 3] in agreement with the results of Refs. 4 and 5 and Monte Carlo simulations on the classical 3D XY model.[23] In particular, the slope of the superfluid density at $T=0$ according to Eqs. (24) and (32) becomes

$$\frac{d}{d(T/T_c)}\left[\frac{\lambda^2(0)}{\lambda^2(T)}\right]\Bigg|_{T=0} = -1. \tag{33}$$

For reference, estimates of the quantity $d/[d(T/T_c)][\lambda^2(0)/\lambda^2(T)]|_{T=0}$ for various cuprates and doping levels are in the range -0.94 to -0.51.[24] Interestingly, by comparing Figs. 1 and 3 we see that charging energy (quantum) effects alleviate the low-temperature linear-T dependence of $1/\lambda^2(T)$: the linear temperature behavior is replaced by the dominant exponential temperature dependence in the quantum case. However, as shown in Ref.4 the linear temperature behavior can be restored in the quantum case when the combined effect of charging interactions and dissipation is included. Whether or not the low-temperature features of $\lambda(T)$ due to classical phase fluctuations can solely account experimental data of the penetration depth in high-T_c cuprates without invoking, e.g., excitations near the nodes of the d-wave gap, is an interesting but presently unsettled issue.

Approaching T_c from below $\lambda^2(0)/\lambda^2(T)$ vanishes with a finite slope

$$\frac{d}{d(T/T_c)}\left[\frac{\lambda^2(0)}{\lambda^2(T)}\right]\Bigg|_{T=T_c}=-\frac{E_J(\Delta,T=T_c)}{E_J(\Delta,T=0)}. \qquad (34)$$

This is distinct from the Monte Carlo result of Ref.23 where $\lambda^2(0)/\lambda^2(T)\sim(T/T_c-1)^{0.673}$ implies an infinite slope at the critical temperature. Close to T_c, however, the effect of fluctuations is profound and the spherical model approach (being effectively at a one-loop level) is unable to reproduce corrections due to critical fluctuations in all details. Note that the universal behavior of the quantum model [see Eq. (11)] shown in Fig. 1 exhibits neither the low-temperature linearity nor (also due to the fluctuation corrections mentioned above) the genuine critical 3D XY behavior near T_c as the classical XY model of Ref.23.

VII. DISCUSSION AND FINAL REMARKS

Following the Uemura suggestion relating the Bose Einstein–BCS (BE-BCS) crossover scenario[25–27] to the under/overdoped phenomenology of high-T_c superconductors,[21,28] it is tempting to associate the small-$|U|$ regime with the overdoped and large-$|U|$ regime with the underdoped region of high-T_c superconductors. In this analogy the interaction parameter $|U|$ translates then to the doping parameter x for cuprates (with $|U_{\rm opt}|$ corresponding to the optimal doping). In doing so, one should bear in mind that, of course, the negative-U Hubbard system cannot serve as a realistic model of high-T_c cuprates. However, it can be employed as a useful tool to explore mutual effects of pairing and phase fluctuations. Strong suppression of the superfluid density seen in experiments in high-T_c cuprates disagrees, as we noted, from the predictions of the BCS theory. It is well known that the superfluid stiffness $n_s(0)$ in a system of fermions is finite at zero temperature for infinitesimal interaction and drops discontinuously to $n_s(0)=0$ at exactly zero interaction. Specifically, as a direct consequence of the the Galilean invariance of the continuum BCS model (with parabolic dispersion) one has $n_s(0)=n_f/4m$ where n_f is the fermion density. The fact that $n_s(0)$ approaches a finite value for $U\rightarrow 0$ can be comprehended by realizing that though Δ becomes small in this limit, the coherence length becomes very large resulting in a finite value for the stiffness. In other words, in the BCS limit the weak-coupling superfluid density is dominated by the concentration of the available charge carriers. It might be then surprising that Eq. (12) which follows from modulus-phase representation predicts the *vanishing* of the $T=0$ superfluid density in the $|U|\rightarrow 0$ limit where the BCS and modulus-phase model Eq. (11) formally should overlap. To account for this discrepancy we note that there is a fundamental difference between BCS continuum formulation in the momentum space and the discrete modulus-phase representation on a lattice in real space: the latter manifestly violates Galilean invariance for arbitrary interaction U.[29] Clearly, even in the $|U|\rightarrow 0$ limit the action (11) describes charges moving through activated (hopping) process in real space rather than moving as plane waves in momentum domain. As a result radically different behavior of $n_s(0)$ in the weak-coupling limit for both models emerges: the vanishing of $T=0$ superfluid stiffness in the modulus–phase model follows then from the loss of coherence between sites due to the diminishing of the amplitude pairing (or pair decay) in the $|U|\rightarrow 0$ limit. Therefore, if one consistently adopts the point of view that the changeover from underdoped to overdoped behavior in high-T_c cuprates is somehow related to the passage from strong to weak coupling (as the pseudogap phenomenon seems to suggest) then the "weak-coupling" overdoped regime cannot be described as the genuine-BCS limit (in momentum space) although other characteristics as, e.g., amplitude ratios $2\Delta/k_BT_c$ seem to suggest a typical BCS behavior.[30,31] This last observation is perhaps less striking when one realizes that virtually all approaches to high-T_c superconductivity underline the important role of inhomogeneity both in real and in momentum space.

In conclusion, in making the analogy between the interaction driven BE-BCS crossover with the phenomenology of superconducting cuprates, there are differences that should be recognized and points whose precise clarification presumably would require to go beyond an s-wave attractive Hubbard model. However, the similarities and analogies we found are very intriguing and we believe that the concept of pairing without phase coherence is a promising scenario while exploiting the properties of high-T_c superconductors.

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