

Critical behavior of the two-dimensional N -component Landau-Ginzburg Hamiltonian with cubic anisotropy

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We study the two-dimensional N -component Landau-Ginzburg Hamiltonian with cubic anisotropy. We compute and analyze the fixed-dimension perturbative expansion of the renormalization-group functions to four loops. The relations of these models with N -color Ashkin-Teller models, discrete cubic models, the planar model with fourth-order anisotropy, and the structural phase transition in adsorbed monolayers are discussed. Our results for $N=2$ (XY model with cubic anisotropy) are compatible with the existence of a line of fixed points joining the Ising and the $O(2)$ fixed points. Along this line the exponent η has the constant value $1/4$, while the exponent ν runs in a continuous and monotonic way from 1 to ∞ [from Ising to $O(2)$]. In the four-loop approximation, for $N \geq 3$ we find a cubic fixed point in the region $u, v \geq 0$.

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I. INTRODUCTION

In the framework of renormalization-group (RG) approach to critical phenomena, the critical behavior at many continuous phase transitions can be investigated by considering an effective Landau-Ginzburg Hamiltonian having an N -component order parameter ϕ_i as fundamental field and containing up to fourth-order powers of the field components. The fourth-degree polynomial form of the potential depends essentially on the symmetry of the system. In fact according to the universality hypothesis, the critical properties of these systems can be described in terms of quantities that do not depend on the microscopic details of the system, but only on global properties such as the dimensionality and the symmetry of the order parameter and the range of the interactions.

The critical properties of many magnetic materials are computed using the $O(N)$ -invariant Landau-Ginzburg Hamiltonian. Uniaxial ferromagnets should be described by the Ising universality class ($N=1$), while magnets with easy-plane anisotropy should belong to the XY universality class. Ferromagnets are often described in terms of the $O(3)$ Hamiltonian. However, this is correct if the nonrotationally invariant interactions that have only the reduced symmetry of the lattice are irrelevant in the renormalization-group sense. In two dimensions the effect of anisotropy is very important: systems possessing continuous symmetry do not exhibit conventional long-range order at finite temperature, while models with discrete symmetry do undergo phase transitions into conventionally ordered phase.

For studying the effect of cubic anisotropies one usually consider the ϕ^4 theory:^{1,2}

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_{i=1}^N [(\partial_\mu \phi_i)^2 + r \phi_i^2] + \frac{1}{4!} \sum_{i,j=1}^N (u_0 + v_0 \delta_{ij}) \phi_i^2 \phi_j^2 \right\}, \quad (1)$$

in which the added cubic term breaks explicitly the $O(N)$ invariance, leaving a residual discrete cubic symmetry given by the reflections and permutations of the field components.³ This term favors the spin orientations towards the faces or the corners of an N -dimensional hypercube for $v_0 < 0$ or $v_0 > 0$, respectively.

The Hamiltonian (1) has received much attention also because it describes the two-dimensional order-disorder transition in some adsorbed monolayers. In fact, in the original classification⁴ on square and rectangular lattices, all these transitions belong to the universality class of the Ising model, three- or four-state Potts model, and XY model with cubic anisotropy. The Heisenberg ($N=3$) model with cubic anisotropy describes instead such transitions in non-Bravais lattices,⁵ as the honeycomb and kagomé ones (e.g., in the adsorption of diatomic molecules on graphite). The full classification of continuous phase transitions of magnetic symmetry in two dimensions⁶ reveals the interest of model (1) also for other kind of transitions.

An important application of the XY model with cubic anisotropy is in the oxygen ordering in $\text{YBaCu}_3\text{O}_{6+x}$,^{7,8} since it is one of the most studied high-temperature superconductors. Some other applications of Landau-Ginzburg Hamiltonian (1) are the buckling instabilities of a confined colloid crystal layer,⁹ some discrete models with competing nearest- and next-nearest-neighbor interactions,¹⁰ and, very recently,¹¹ it is argued that the phase diagram of a lattice-gas model for studying the micellar binary solution of water and amphiphile is partially described by the XY model with cubic anisotropy.

We also mention that, in the limit $N \rightarrow 0$, the cubic model (1) describes the Ising model with site-diluted disorder,¹²⁻¹⁴ but we do not consider it here, since in the framework of fixed dimension $d=2$ it was already analyzed in Ref. 15.

In this paper we present a field-theoretic study based on an expansion performed directly in two dimensions, as proposed for the $O(N)$ models by Parisi.¹⁶

The paper is organized as follows. In Sec. II we give an overview of all known results that we believe necessary for a

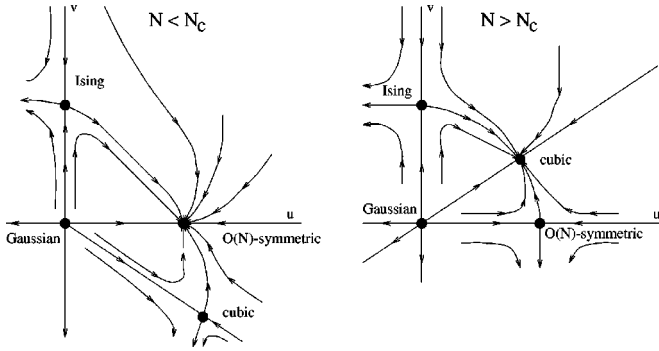


FIG. 1. Renormalization-group flow in the coupling plane (u, v) for $N < N_c$ and $N > N_c$.

full understanding of the two-dimensional cubic model (1). In Sec. III we derive the perturbative series for the renormalization-group functions at four loops and discuss the singularities of the Borel transform. The results of the analysis are presented in Sec. IV. The reader who is not interested in the details of the calculations can skip Secs. III and IV and read directly Sec. V where we summarize all our results and point out some questions which we think deserve further study.

II. OVERVIEW OF KNOWN RESULTS

A. Three-dimensional results

We shortly review the most interesting features appearing in the study of the Hamiltonian (1) in the framework of an ϵ expansion^{17,1} and at fixed dimension $d=3$,¹⁸ since they are necessary for a good understanding of the two-dimensional case.

The model (1) has four fixed points: the trivial Gaussian one, the Ising one in which the N components of the field decouple, and the $O(N)$ -symmetric and cubic fixed points. The Gaussian fixed point is always unstable and so is the Ising fixed point.¹⁹ Indeed, in the latter case, it is natural to interpret Eq. (1) as the Hamiltonian of N Ising-like systems coupled by the $O(N)$ -symmetric term. But this interaction is the sum of the products of the energy operators of the different Ising systems. Therefore, at the Ising fixed point, the crossover exponent associated with the $O(N)$ -symmetric quartic term should be given by the specific-heat critical exponent α_I of the Ising model, independently of N . Since α_I is positive for all $d > 2$, the Ising fixed point is unstable. Obviously in two dimensions this argument fails since $\alpha_I = 0$.

While the Gaussian and Ising fixed points are unstable for any number of components N , the stability properties of the $O(N)$ -symmetric and of the cubic fixed points depend on N . For sufficiently small values of N , $N < N_c$, the $O(N)$ -symmetric fixed point is stable and the cubic one is unstable. For $N > N_c$, the opposite is true: the renormalization-group flow is driven towards the cubic fixed point, which now describes the generic critical behavior of the system. The $O(N)$ -symmetric point corresponds to a tricritical transition. Figure 1 sketches the flow diagram in the two cases $N < N_c$ and $N > N_c$. At $N = N_c$, the two fixed points should coincide, and logarithmic corrections to the

$O(N)$ -symmetric critical exponents are expected. Outside the attraction domain of the fixed points, the flow goes away towards more negative values of u and/or v and finally reaches the region where the quartic interaction no longer satisfies the stability condition. These trajectories should be related to first-order phase transitions.²⁰ Some recent and very accurate calculations^{18,21–29} suggest that in three-dimensions $N_c < 3$.

If $N > N_c$, the cubic anisotropy is relevant and therefore the critical behavior of the system is not described by the Heisenberg isotropic Hamiltonian. If the cubic interaction favors the alignment of the spins along the diagonals of the cube—i.e., for a positive coupling v_0 —the critical behavior is controlled by the cubic fixed point and the cubic symmetry is retained even at the critical point. On the other hand, if the system tends to magnetize along the cubic axes—this corresponds to a negative coupling v_0 —then the system undergoes a first-order phase transition.^{1,30–32}

In the limit $N \rightarrow \infty$, keeping Nu and v fixed, one can derive exact expressions for the exponents at the cubic fixed point for all dimensions. Indeed, in this limit the model can be reinterpreted as a constrained Ising model,³³ leading to a Fisher renormalization of the Ising critical exponents.³⁴ One has^{35,33,1}

$$\eta = \eta_I + O\left(\frac{1}{N}\right), \quad \nu = \frac{\nu_I}{1 - \alpha_I} + O\left(\frac{1}{N}\right), \quad (2)$$

where η_I , ν_I , and α_I are the critical exponents of the Ising model.

In all dimensions, a simple argument based on the symmetry of the two-component cubic model³⁶ shows that the cubic fixed point for $N=2$ has the same stability properties of the Ising one. Indeed, for $N=2$, a $\pi/4$ internal rotation, i.e.,

$$(\phi_1, \phi_2) \rightarrow \frac{1}{\sqrt{2}}(\phi_1 + \phi_2, \phi_1 - \phi_2), \quad (3)$$

maps the cubic Hamiltonian (1) onto a new one of the same form but with new couplings (u'_0, v'_0) given by

$$u'_0 = u_0 + \frac{3}{2}v_0, \quad v'_0 = -v_0. \quad (4)$$

This symmetry maps the Ising fixed point onto the cubic one. So for all $d > 2$, since the Ising point is unstable, the cubic point is unstable too, and the stable point is the isotropic one. In two dimensions, this is no longer true. Indeed, one expects the cubic interaction to be truly marginal for $N=2$ (Refs. 37 and 38) and relevant for $N > 2$, (Ref. 39) so $N_c = 2$ in two dimensions.

B. Two-dimensional case

As is already clear from the previous subsection, in two dimensions several new and interesting features appear. First of all we have no general argument to understand the stability properties of the Ising fixed point; in fact the specific heat of the two-dimensional Ising model has a logarithmic diver-

gence (i.e., $\alpha_I=0$). The vanishing of this crossover exponent—and so the presence of a marginal operator—can bring us to several different scenarios. For example we may have that the Ising fixed point is marginally stable or unstable because of higher-order corrections to RG equations or that there exists a line of fixed points (see Ref. 40 for a detailed review about the effects of marginal operators).

When the cubic anisotropy becomes marginal (at $N=N_c=2$),³⁹ the cubic fixed point is in the region with $\nu<0$ [cf. Eq. (4)], and so it cannot coincide with the $O(N)$ one, as in the case of 3 and $4-\epsilon$ dimensions.

There are several studies on systems related to Landau-Ginzburg Hamiltonian (1). In Ref. 37 the model

$$\mathcal{H} = \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} J \mathbf{S}(\mathbf{r}) \mathbf{S}(\mathbf{r}') + \sum_{\mathbf{r}} \mathbf{h}_p \cos \mathbf{p} \theta(\mathbf{r}) \quad (5)$$

was considered, where the first sum is only over nearest neighbors and $\theta(\mathbf{r})$ is the angle that the two-component spin $\mathbf{S}(\mathbf{r})$ forms with some arbitrary axis. This system clearly belongs to the same universality class of the XY model with cubic anisotropy if $p=4$ (the “field” h_4 maps onto v_0).

The phase diagram found in Ref. 37 consists of three distinct lines of fixed points showing continuously varying exponents. The line with $h_4=0$ is the standard Kosterlitz-Thouless (KT) one.⁴¹ There are two lines with $h_4 \neq 0$, starting from the KT transition (the end point of the KT line) and continuing to infinitely—positive or negative—large values of h_4 . They map onto each other with changing the sign of h_4 . The latter are lines of second-order phase transitions with conventional power-law singularities characterized by $\eta=1/4$. The exponent ν diverges at the confluence of these critical lines as

$$\nu \sim \frac{1}{|h_4|} \quad (6)$$

when $h_4 \rightarrow 0$. For this reason it is often said that the XY model with cubic anisotropy has a nonuniversal behavior, although there is the so-called weak universality of Suzuki⁴² (i.e., η is constant).

In a successive work Kadanoff⁴³ identifies the two fixed-point lines for $h_4 \neq 0$ with the dual line of the Ashkin-Teller (AT) and the eight-vertex Baxter model (8V) (see the Baxter book⁴⁴ for a review about these models). One remark is necessary at this point: the AT and the 8V models are equivalent on the dual line⁴⁴ and they show a continuously varying exponent of the correlation length:

$$\nu = \frac{2-y}{3-2y}, \quad (7)$$

where y is a parameter that appears in the Hamiltonians defined in the range $[0, 4/3]$ for the AT (Ref. 45) and $[0, 2]$ for the 8V (Ref. 44). The $O(2)$ multicritical point identified by Kadanoff is the F-model limit of the 8V that is characterized by $y=3/2$, allowed for the 8V but forbidden for the AT.

Another class of models related to the Landau-Ginzburg Hamiltonian (1) is the discrete-cubic N -component model.^{30,46–50} It is a short-range interacted system with

N -component spins \mathbf{S}_i pointing to the faces of an N -dimensional hypercube (face-centered-cubic model). The Hamiltonian may be written as

$$\mathcal{H} = \sum_{\langle i, j \rangle} (C \delta_{a_i, a_j} s_i s_j + P \delta_{a_i, a_j}), \quad (8)$$

where a_i is a Potts-like variable that determines which component of \mathbf{S}_i is nonzero and s_i an Ising variable that determines the sign of that component. For $P=0$ the Hamiltonian (8) reduces to the $2N$ -state Potts model for $C=0$ to two decoupled N -state Potts models, and for $N=2$ to the AT model. The continuous cubic model (1) reduces to the discrete one (8) in the limit of strong anisotropy ($|v| \gg |u|$). In Ref. 48 it is shown that the iteration of RG transformations enforces the continuous model, with $v_0 < 0$, to have spins pointing only to the faces of the hypercube, and so, also for finite anisotropy, the two models are equivalent at criticality. The model (8) exhibits four competing possible types of critical behavior, related to the Ising model, the N and $2N$ -state Potts models, and to a “cubic” fixed point. In Refs. 49 and 50 it was found that the critical behavior of the discrete face-centered-cubic model is $O(N)$ like for $N < N_c=2$, AT-like for $N=2$, and characterized by a first-order phase transition for $N > 2$. This result is not surprising since the model (8) is related to (1) in the region with $v_0 < 0$ where we expect a first-order phase transition for $N > N_c=2$ and $O(N)$ behavior for $N < 2$.

In the region with $v_0 > 0$ the continuous model (1) is related to the corner-cubic model in which N -component spins \mathbf{S}_i point to the corners of an N -dimensional hypercube. This model is equivalent to the face-centered one for $N=2$ [this is the symmetry (4) for the Hamiltonian (1) and the changing of the sign of h_4 for (5)]. For other values of N the critical behavior of the face-centered- and the corner-cubic model is in principle very different.

This difference is clarified by the study of N -color Ashkin-Teller model, first introduced by Grest and Widom,⁵¹ which is equivalent to the corner-cubic model. The Hamiltonian of this model reads

$$\mathcal{H} = - \sum_{\langle i, j \rangle} \left[J \sum_{a=1}^N s_i^a s_j^a + J_4 \left(\sum_{a=1}^N s_i^a s_j^a \right)^2 \right], \quad (9)$$

where $s_i^a = \pm 1$ for $a=1, \dots, N$ are N Ising variables and the sum is only over nearest-neighbors [the sign of J_4 is the opposite of u_0 in Eq. (1)]. In Ref. 51 it is concluded that the order J_4^2 in the RG equations makes the decoupled Ising fixed point stable for perturbations with $J_4 < 0$ ($u_0 > 0$) and unstable for $J_4 > 0$ ($u_0 < 0$). In the latter case the system flows away towards more negative values of u and finally the transition is first order. This has been confirmed⁵¹ by a Monte Carlo simulation for $N=3$. This argument was proved exactly in the $N \rightarrow \infty$ limit.^{52,53} After, using a mapping of the Hamiltonian (9) onto the $O(N)$ Gross-Neveu model⁵⁴ it was argued^{54–57} that the critical behavior of the Ising fixed point is affected by logarithmic corrections.

Finally the three couplings ($N, 2$) model of Domany and Riedel⁴⁸ [equivalent for $N=3$ to the $Z(6)$ model⁵⁸] reduces

to the N -component discrete cubic model, for particular values of the parameters entering in the Hamiltonian. The model (3,2) was studied in Refs. 48 and 58 with the Migdal-Kadanoff renormalization-group (MKRG) approach. Unfortunately the MKRG approach is expected to give neither precise values of the exponents nor the correct nature of the transition, but only a good description of the phase diagram.

C. Order-disorder transition in adsorbed monolayers

We have already mentioned that the Hamiltonian (1) describes some order-disorder transitions in adsorbed monolayers. The great interest in these models is justified since they provide a unique possibility to study experimentally a rich variety of two-dimensional systems.

Several experimental works have confirmed that these transitions belong to the universality class of the Ising,^{59–61} the three-^{62,63} and four-state Potts models,^{64,63} and the XY model with cubic anisotropy,^{65–68} according to their classification.^{4–6} Monte Carlo simulations and other numerical works confirm this scenario.^{69–71}

According to the standard classification, the critical behavior of the order-disorder transition of diatomic molecules on a honeycomb lattice in the $p(2 \times 1)$ structure is described by the $N=3$ cubic model.^{5,48} The experimental investigation⁷² of oxygen on Ru(001) shows that this transition has critical exponents in agreement with the three-state Potts model within about 10%. Several numerical simulations (after the first ones favoring a first-order phase transition^{73,74}) confirm this critical behavior.^{75,76} So it may be possible that a fixed point of the Heisenberg model with cubic anisotropy has the critical behavior of the three-state Potts model.

We want still to note the possibility for systems having exponents close to the four-state Potts model, to belong to the universality class of the XY model with cubic anisotropy and residing near the four-state Potts transition characterized by $y=0$.

III. FIXED-DIMENSION PERTURBATIVE EXPANSION IN TWO DIMENSIONS

A. Renormalization of the theory

The fixed-dimension field-theoretical approach¹⁶ represents an effective procedure in the study of the critical properties of systems belonging to the $O(N)$ universality class (see, e.g., Ref. 77). The idea is to extend this procedure to models where there are two ϕ^4 couplings with different symmetry.^{22,18} One performs an expansion in powers of appropriately defined zero-momentum quartic couplings and renormalizes the theory by a set of zero-momentum conditions for the (one-particle irreducible) two-point and four-point correlation functions:

$$\Gamma_{ab}^{(2)}(p) = \delta_{ab} Z_\phi^{-1} [m^2 + p^2 + O(p^4)], \quad (10)$$

$$\Gamma_{abcd}^{(4)}(0) = Z_\phi^{-2} m^2 \left[\frac{u}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) + v \delta_{ab} \delta_{ac} \delta_{ad} \right]. \quad (11)$$

They relate the second-moment mass m and the zero-momentum quartic couplings u and v to the corresponding Hamiltonian parameters r , u_0 , and v_0 :

$$u_0 = m^2 u Z_u Z_\phi^{-2}, \quad v_0 = m^2 v Z_v Z_\phi^{-2}. \quad (12)$$

In addition, one introduces the function Z_t defined by the relation

$$\Gamma_{ab}^{(1,2)}(0) = \delta_{ab} Z_t^{-1}, \quad (13)$$

where $\Gamma^{(1,2)}$ is the (one-particle-irreducible) two-point function with an insertion of $\frac{1}{2} \phi^2$.

From the perturbative expansion of the correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ and the above relations, one derives the functions $Z_\phi(u, v)$, $Z_u(u, v)$, $Z_v(u, v)$, and $Z_t(u, v)$ as a double expansion in u and v .

The fixed points of the theory are given by the common zeros of the β functions

$$\beta_u(u, v) = m \frac{\partial u}{\partial m} \Big|_{u_0, v_0},$$

$$\beta_v(u, v) = m \frac{\partial v}{\partial m} \Big|_{u_0, v_0}. \quad (14)$$

The stability properties of the fixed points are controlled by the eigenvalues ω_i of the matrix

$$\Omega = \begin{pmatrix} \frac{\partial \beta_u(u, v)}{\partial u} & \frac{\partial \beta_u(u, v)}{\partial v} \\ \frac{\partial \beta_v(u, v)}{\partial u} & \frac{\partial \beta_v(u, v)}{\partial v} \end{pmatrix}, \quad (15)$$

computed at the given fixed point: a fixed point is stable if both eigenvalues are positive. The eigenvalues ω_i are related to the leading scaling corrections, which vanish as $\xi^{-\omega_i} \sim |t|^{\Delta_i}$ where $\Delta_i = \nu \omega_i$.

One also introduces the functions

$$\eta_\phi(u, v) = \frac{\partial \ln Z_\phi}{\partial \ln m} \Big|_{u_0, v_0} = \beta_u \frac{\partial \ln Z_\phi}{\partial u} + \beta_v \frac{\partial \ln Z_\phi}{\partial v}, \quad (16)$$

$$\eta_t(u, v) = \frac{\partial \ln Z_t}{\partial \ln m} \Big|_{u_0, v_0} = \beta_u \frac{\partial \ln Z_t}{\partial u} + \beta_v \frac{\partial \ln Z_t}{\partial v}. \quad (17)$$

Finally, the critical exponents are obtained from

$$\eta = \eta_\phi(u^*, v^*), \quad (18)$$

$$\nu = [2 - \eta_\phi(u^*, v^*) + \eta_t(u^*, v^*)]^{-1}, \quad (19)$$

$$\gamma = \nu(2 - \eta), \quad (20)$$

where (u^*, v^*) is the position of the stable fixed point.

B. Four-loop series

In this section we present the perturbative expansion of the RG functions (14), (16), and (17) up to four loops. The diagrams contributing to the two-point and four-point functions are reported in Ref. 78. We do not calculate the inte-

TABLE I. The coefficients $b_{ij}^{(u)}$; cf. Eq. (22).

i, j	$(N+8)^i b_{ij}^{(u)}$
2,0	$-(10.3350N + 47.675)$
1,1	-8.39029
0,2	-0.21608
3,0	$524.377 + 149.152N + 5.00028N^2$
2,1	$144.813 + 7.27755N$
1,2	$10.0109 + 0.0583278N$
0,3	0.231566
4,0	$-(7591.108 + 2611.15N + 179.697N^2 + 0.0888427N^3)$
3,1	$-(2872.09 + 291.255N - 0.126814N^2)$
2,2	$-(330.599 + 5.97086N)$
1,3	$-(16.0559 + 0.0578955N)$
0,4	-0.311695

grals associated with each diagram, but we use the numerical results compiled in Ref. 78. Summing all contributions with the right symmetry and group factors (see Ref. 18) we obtain all RG functions.

The results are written in terms of the rescaled couplings

$$u \equiv \frac{8\pi}{3} R_N \bar{u}, \quad v \equiv \frac{8\pi}{3} \bar{v}, \quad (21)$$

where $R_N = 9/(8+N)$. We adopt this rescaling to have finite fixed-point values in the limit $N \rightarrow \infty$.

The resulting series are

$$\bar{\beta}_u = -\bar{u} + \bar{u}^2 + \frac{2}{3} \bar{u} \bar{v} + \bar{u} \sum_{i+j \geq 2} b_{ij}^{(u)} \bar{u}^i \bar{v}^j, \quad (22)$$

$$\bar{\beta}_v = -\bar{v} + \bar{v}^2 + \frac{12}{8+N} \bar{u} \bar{v} + \bar{v} \sum_{i+j \geq 2} b_{ij}^{(v)} \bar{u}^i \bar{v}^j, \quad (23)$$

$$\eta_\phi = \sum_{i+j \geq 2} e_{ij}^{(\phi)} \bar{u}^i \bar{v}^j, \quad (24)$$

$$\eta_t = -\frac{2(2+N)}{(8+N)} \bar{u} - \frac{2}{3} \bar{v} + \sum_{i+j \geq 2} e_{ij}^{(t)} \bar{u}^i \bar{v}^j, \quad (25)$$

where

$$\bar{\beta}_u = \frac{3}{16\pi} R_{2N}^{-1} \beta_u, \quad \bar{\beta}_v = \frac{3}{16\pi} R_{2N}^{-1} \beta_v. \quad (26)$$

The coefficients $b_{ij}^{(u)}$, $b_{ij}^{(v)}$, $e_{ij}^{(\phi)}$, and $e_{ij}^{(t)}$ are reported in Tables I, II, III, and IV. Note that due to the rescaling (26), the matrix elements of Ω are 2 times the derivative of $\bar{\beta}$ with respect to \bar{u} and \bar{v} .

We have verified the exactness of our series by the following relations.

(i) $\bar{\beta}_u(\bar{u}, 0)$, $\eta_\phi(\bar{u}, 0)$, and $\eta_t(\bar{u}, 0)$ reproduce the corresponding functions of the $O(N)$ -symmetric model.^{79,80}

TABLE II. The coefficients $b_{ij}^{(v)}$; cf. Eq. (23).

i, j	$(N+8)^i b_{ij}^{(v)}$
2,0	$-(92.6834 + 5.83417N)$
1,1	-17.392
0,2	-0.716174
3,0	$1228.63 + 118.504N - 1.83156N^2$
2,1	$358.882 + 2.84758N$
1,2	31.4235
0,3	0.930766
4,0	$-(20723.1 + 2692.0N + 25.4854N^2 + 0.824655N^3)$
3,1	$-(8273.28 + 233.78N - 0.574757N^2)$
2,2	$-(1134.8 + 1.91402N)$
1,3	-68.4022
0,4	-1.58239

(ii) $\bar{\beta}_v(0, \bar{v})$, $\eta_\phi(0, \bar{v})$, and $\eta_t(0, \bar{v})$ reproduce the corresponding functions of the Ising-like ($N=1$) ϕ^4 theory.

(iii) The following relations hold for $N=1$:

$$\bar{\beta}_u(u, x-u) + \bar{\beta}_v(u, x-u) = \bar{\beta}_v(0, x), \quad (27)$$

$$\eta_\phi(u, x-u) = \eta_\phi(0, x),$$

$$\eta_t(u, x-u) = \eta_t(0, x).$$

(iv) For $N=2$, using the symmetry (3) and (4), and taking into account the rescalings (21), one can easily obtain the identities

$$\bar{\beta}_u\left(\bar{u} + \frac{5}{3}\bar{v}, -\bar{v}\right) + \frac{5}{3}\bar{\beta}_v\left(\bar{u} + \frac{5}{3}\bar{v}, -\bar{v}\right) = \bar{\beta}_u(\bar{u}, \bar{v}),$$

$$\bar{\beta}_v\left(\bar{u} + \frac{5}{3}\bar{v}, -\bar{v}\right) = -\bar{\beta}_v(\bar{u}, \bar{v}),$$

TABLE III. The coefficients $e_{ij}^{(\phi)}$; cf. Eq. (24).

i, j	$(N+8)^i e_{ij}^{(\phi)}$
2,0	$0.917086(N+2)$
1,1	0.611391
0,2	0.033966
3,0	$-0.054609(N+2)(N+8)$
2,1	$-0.054609(N+8)$
1,2	-0.054609
0,3	-0.002023
4,0	$(N+2)(26.7676 + 4.24178N - 0.092684N^2)$
3,1	$27.6901 + 5.6557N - 0.123579N^2$
2,2	$5.404243 + 0.132800N$
1,3	0.410151
0,4	0.011393

TABLE IV. The coefficients $e_{ij}^{(l)}$; cf. Eq. (25).

i, j	$(N+8)^i e_{ij}^{(l)}$
2,0	6.751 258 (N+2)
1,1	4.500 839
0,2	0.250 047
3,0	$-(96.7105 + 65.1686N + 8.406\ 68N^2)$
2,1	$-(48.3553 + 8.406\ 68N)$
1,2	$-(6.190\ 23 + 0.116656N)$
0,3	$-0.233\ 588$
4,0	$1135.05 + 844.5N + 139.656N^2 + 0.583\ 377N^3$
3,1	$756.697 + 184.652N + 0.777\ 836N^2$
2,2	$149.467\ 584 + 7.553\ 459N$
1,3	$11.515\ 397 + 0.115\ 791N$
0,4	$0.323\ 089$

$$\eta_\phi\left(\bar{u} + \frac{5}{3}\bar{v}, -\bar{v}\right) = \eta_\phi(\bar{u}, \bar{v}),$$

$$\eta_t\left(\bar{u} + \frac{5}{3}\bar{v}, -\bar{v}\right) = \eta_t(\bar{u}, \bar{v}). \quad (28)$$

These relations are exactly satisfied by our four-loop series. Note that, since the Ising fixed point is $(0, g_I^*)$ and g_I^* is known with very high precision from Ref. 81,

$$g_I^* = 1.7543637(25), \quad (29)$$

the above symmetry gives us the location of the cubic fixed point $(\frac{5}{3}g_I^*, -g_I^*)$.

(v) In the large- N limit the critical exponents of the cubic fixed point are related to those of the Ising model: $\eta = \eta_I$ and $\nu = \nu_I$. One can easily see that, for $N \rightarrow \infty$, $\eta_\phi(u, v) = \eta_I(v)$, where $\eta_I(v)$ is the perturbative series that determines the exponent η of the Ising model. Therefore, the first relation is trivially true. On the other hand, the second relation $\nu = \nu_I$ is not identically satisfied by the series and is verified only at the critical point.¹⁸

(vi) For $N=0$ the series reproduce the results of Ref. 15.

C. Resummations of the series

The field-theoretic perturbative expansion generates asymptotic series that must be resummed to extract the physical information about the critical behavior of the real systems.

Exploiting the property that these series are Borel summable for ϕ^4 theories in two and three dimensions,⁸² one can resum these perturbative expressions considering the Borel transform combined with a method for its analytical extension. In the case of the $O(N)$, symmetric model with only a coupling g , all perturbative series are of the form $F(g) = \sum f_k g^k$. Exploiting the knowledge of the large order behavior of the coefficients f_k (Ref. 77),

$$f_k \sim k!(-a)^k k^b [1 + O(k^{-1})] \quad \text{with} \quad a = -1/g_b \quad (30)$$

(a large-order behavior related to the singularity g_b of the Borel transform closest to the origin), one can perform the mapping⁸³

$$y(g) = \frac{\sqrt{1-g/g_b} - 1}{\sqrt{1-g/g_b} + 1} \quad (31)$$

to extend the Borel transform of $F(g)$ to all positive values of g . The singularity g_b depends only on the considered model and can be obtained from a steepest-descent calculation in which the relevant saddle point is a finite-energy solution (instanton) of the classical field equations with negative coupling.^{84,85} Instead the coefficient b depends on which Green's function is considered.

Note that the function $F(g)$ can be Borel summable only if there are no singularities of the Borel transform on the positive real axis.

This resummation procedure has worked successfully for the $O(N)$ -symmetric theory, for which accurate estimates for the critical exponents and other physical quantities have been obtained.^{83,86,87}

For this reason we want to extend the resummation procedure cited above to multicoupling models, as has been done for the three-dimensional cubic model^{18,80} and for the frustrated system with noncollinear order in two dimensions.⁸⁸ Considering a double expansion in \bar{u} and \bar{v} at fixed $z = \bar{v}/\bar{u}$ and studying the large-order behavior (following the same procedure used in Refs. 18 and 88) of the new expansion in powers of \bar{u} to calculate the singularity of the Borel transform closest to the origin \bar{u}_b we have

$$\frac{1}{\bar{u}_b} = -a(R_N + z) \quad \text{for} \quad 0 < z,$$

$$\frac{1}{\bar{u}_b} = -a\left(R_N + \frac{1}{N}z\right) \quad \text{for} \quad 0 > z > -\frac{2NR_N}{N+1}, \quad (32)$$

where $a = 0.238\ 659\ 217 \dots$

Note that the series in powers of \bar{u} keeping z fixed is not Borel summable for $\bar{u} > 0$ and $z < -R_N$. This fact will not be a real limitation for us, since we will only consider values of z such that $\bar{u}_b < 0$. It should be noted that these results do not apply to the case $N=0$. Indeed, in this case, additional singularities in the Borel transform are expected.^{89,90}

The exponent b in Eq. (30) is related to the number of symmetries broken by the classical solution.⁸⁵ It depends on the quantity considered. In the cubic model, for $v \neq 0$, we have $b=2$ for the function η_ϕ , and $b=3$ for the β functions and η_t . For $v=0$, we recover the results of the $O(N)$ -symmetric model—that is, $b=(3+N)/2$ for η_ϕ and $b=(5+N)/2$ for the β function and η_t .⁸³

Good estimates of the critical exponent could be obtained also using a Padé Borel analysis of the series, as shown in the case of the two-dimensional $O(N)$ models^{79,80} and the random Ising model.¹⁵

An important issue in the fixed-dimension approach to critical phenomena (and in general of all the field-theoretical

methods) concerns the analytic properties of the β functions. As shown in Ref. 91 for the $O(N)$ model, the presence of confluent singularities in the zero of the perturbative β function causes a slow convergence of the resummation of the perturbative series to the correct fixed point value. The $O(N)$ two-dimensional field-theory estimates of physical quantities^{83,80} are less accurate than the three-dimensional ones, due to the stronger nonanalyticities at the fixed point.⁹¹⁻⁹⁴ In Ref. 91 it is shown that the nonanalytic terms may cause large imprecisions in the estimate of the exponent related to the leading correction to the scaling ω ; instead the result for the fixed point value is a rather good approximation of the correct one (if one compare the field-theoretical results for the four-point renormalized coupling in the g expansion⁸⁰ and in the ϵ expansion^{94,95} with the estimates of other nonperturbative methods,^{81,96,97} one finds that the systematic error is always less than 10%). The nonanalyticities also have a strong influence in the resummation of the critical exponents of the $O(N)$ models (compare the five-loop results of Ref. 80 with the exact ones⁹⁸). We think that this scenario holds also for the cubic models.

IV. FOUR-LOOP EXPANSION ANALYSIS

A. Analysis method

In order to study the critical properties of the continuous anisotropic cubic model (1), we use two different resummation procedures: the Padé-Borel method and the conformal mapping of the Borel-transformed series.

Explicitly, let us consider an l -loop series in \bar{u} and \bar{v} of the form

$$R(\bar{u}, \bar{v}) = \sum_{k=0}^l \sum_{h=0}^{l-k} R_{hk} \bar{u}^h \bar{v}^k = \sum_{k=0}^l R_k(z) \bar{u}^k, \quad (33)$$

where R is one of the RG functions and z is the ratio \bar{v}/\bar{u} that we will consider always fixed to the same value. In this manner we have an asymptotic series of only one variable \bar{u} , depending on the additional parameter z , which has to run from 0 to ∞ in order to reproduce all the quadrants with $\bar{u}, \bar{v} \geq 0$. In order to use a variable defined in a finite range we set

$$z = \frac{\bar{v}}{\bar{u}} = \tan \frac{\pi}{2} x = \tan \frac{\pi}{2} \frac{k}{N_{max}}, \quad (34)$$

where $x \in [0, 1]$ and k is an integer running from 0 to N_{max} (we will fix $N_{max} = 25$).

In the Padé-Borel method we consider the Padé approximants $[L/M]$ of the Borel-Leroy transform of $R(\bar{u}, z)$. Explicitly the approximants of R are

$$E(R)(L, M, b; \bar{u}, z) = \int_0^\infty dt t^b e^{-t} \frac{N_L(\bar{u}t, z)}{D_M(\bar{u}t, z)}, \quad (35)$$

where N_L and D_M are two polynomials of degree L and M , respectively, with $L + M \leq l$, which are determined by the

condition that the expansion of $E(R)(L, M, b; \bar{u}, z)$ in powers of \bar{u} give $R(\bar{u}, z)$ to order $L + M$.

In this manner we have several approximants of the function $R(\bar{u}, \bar{v})$ with varying the three parameters b, L , and M . As usual in this case⁸⁰ the best estimates of the resummed function are given by the diagonal and near-diagonal approximants (i.e., for our four-loop series we have three reasonable choices $[2, 2]$, $[3, 1]$, and $[2, 1]$). Then we search for the value of b (called b_{opt}), integer or half-integer, minimizing the differences between the considered approximants; a reasonable estimate will be the mean value of $E(b_{opt}, L, M)$ on all values of L and M considered. Then we could take the error bar proportional to the deviations from the mean value of all the approximants with $b_{opt} - 1 \leq b \leq b_{opt} + 1$ (always considering integer and half-integer values of b).

In the analysis using the conformal-mapping method we essentially follow the procedure used in Ref. 18. We exploit the knowledge of the value of the singularity of the Borel transform closest to the origin (a value given in the previous section), and we generate a set of approximants to our asymptotic series, varying the two parameters α and b appearing in

$$E(R)_p(\alpha, b; \bar{u}, z) = \sum_{k=0}^l B_k(\alpha, b; z) \times \int_0^\infty dt t^b e^{-t} \frac{y(\bar{u}t; z)^k}{[1 - y(\bar{u}t; z)]^\alpha}, \quad (36)$$

where

$$y(x; z) = \frac{\sqrt{1 - x/\bar{u}_b(z)} - 1}{\sqrt{1 - x/\bar{u}_b(z)} + 1}. \quad (37)$$

The coefficients B_k are determined by the condition that the expansion of $E(R)_p(\alpha, b; \bar{u}, z)$ in powers of \bar{u} and \bar{v} give $R(\bar{u}, \bar{v})$ to order l .

The procedure to choose the range of the parameters α and b used to find good estimates and reasonable error bars of the various quantities is the same of Ref. 18.

B. Stability properties of the $O(N)$ and the Ising fixed points

First of all, we analyze the stability properties of the $O(N)$ -symmetric fixed point. Since

$$\frac{\partial \beta_v^-}{\partial \bar{u}}(\bar{u}, 0) = 0, \quad (38)$$

the eigenvalues are simply

$$\omega_1 = 2 \frac{\partial \bar{\beta}_u^-}{\partial \bar{u}}(\bar{u}^*, 0), \quad \omega_2 = 2 \frac{\partial \bar{\beta}_v^-}{\partial \bar{v}}(\bar{u}^*, 0), \quad (39)$$

where \bar{u}^* is the fixed-point value of the $O(N)$ vector model of which accurate estimates are available from the five-loop analysis of Ref. 80. The exponent ω_1 is the usual exponent of the $O(N)$ -symmetric theory⁷⁷ that gives the first correction

TABLE V. Half of the exponent ω_2 at the $O(N)$ fixed point. CM is the value obtained using conformal mapping technique and PB the one using the Padé-Borel method.

N	$u_{O(N)}^*$	CM	PB
2	1.80(3)	0.03(3)	0.06(4)
3	1.75(2)	-0.08(3)	-0.07(3)
4	1.70(2)	-0.18(4)	-0.17(5)
8	1.52(1)	-0.45(5)	-0.44(6)

to the scaling. For all $N > 2$ it is known that ω_1 assumes the constant value 2 (see Ref. 91 and references therein). Instead ω_2 is the eigenvalue that determines the stability of the fixed point respect to an anisotropic cubic perturbation.

In Table V we report the results for ω_2 for several values of N . It is quite evident that the $O(N)$ fixed point is unstable for $N \geq 3$ (we note that for small N these values are very close to zero). In the limit $N \rightarrow \infty$ it holds $\omega_2 = -2$. For $N = 2$ our result is compatible with the presence of a marginal operator, i.e., $\omega_2 = 0$.

Then we focus our attention on the stability properties of the Ising fixed point. Also in this case the eigenvalues are simply

$$\omega_1 = 2 \frac{\partial \bar{\beta}_v^-}{\partial \bar{v}}(0, \bar{v}^*), \quad \omega_2 = 2 \frac{\partial \bar{\beta}_u^-}{\partial \bar{u}}(0, \bar{v}^*), \quad (40)$$

where \bar{v}^* is the fixed-point value of the Ising model (29). The exponent ω_1 gives the correction to the scaling of the Ising model and it should be equal to $7/4$,⁹¹ while ω_2 is the eigenvalue determining the stability of this fixed point. We find that the series $\omega_2(\bar{v})$ is independent of N (we already know that its fixed-point value must be equal to α_I for all N):

$$\frac{\omega_2^I(\bar{v})}{2} = -1 + \frac{2}{3}\bar{v} - 0.2161\bar{v}^2 + 0.23157\bar{v}^3 - 0.31169\bar{v}^4. \quad (41)$$

The fixed-point value of this exponent is $\omega_2^I/2 = -0.10(5)$, using the conformal-mapping method, and $-0.08(5)$, using the Padé-Borel analysis. These values are quite close to $-\alpha_I = 0$; we attribute this small discrepancy to the presence of nonanalyticities at the zero of the $\bar{\beta}_v^-$. To support this thesis we note that the value of $\omega_2/2$ is approximately the same of the resummation of the exponent $-\alpha_I = -0.081$ found in Ref. 80.

C. Evaluation of nontrivial fixed points

Since the Ising fixed point has always a marginal operator and the $O(N)$ is unstable for $N \geq 3$, we search for the presence of nontrivial fixed points in the region $\bar{u}, \bar{v} > 0$.

Let us consider first the case $N = 2$. The curves of the zeros of β_u^- and β_v^- are parallel and very close. Named $(\bar{u}_1^z, \bar{v}_1^z)$ the location of the zero of β_u^- at fixed k [cf. Eq. (34)] and $(\bar{u}_2^z, \bar{v}_2^z)$ the same for β_v^- , we report in Fig. 2 the quan-

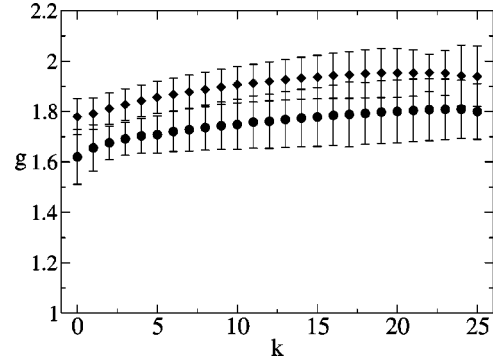


FIG. 2. Sum of the coordinates of the zeros of the β functions for $N = 2$ vs k using the Padé-Borel method. The diamonds correspond to β_u^- and the circle to β_v^- . The results with the conformal-mapping method are substantially equivalent.

ties $g_i = \bar{u}_i^z + \bar{v}_i^z$, with their error bars, as a function of k (note that, at fixed k , \bar{u}_i^z and \bar{v}_i^z are not independent). This figure clearly shows that the two lines of zeros are indistinguishable within their error bars. So our result is consistent with the presence of the line of fixed points conjectured by Kadanoff,⁴³ joining the decoupled Ising and the $O(2)$ fixed points. In the Kadanoff identification this line of fixed points is the dual line of the 8V model in the region that goes from the F-model limit [$O(2)$ multicritical point with $y = 3/2$] to the decoupled Ising point ($y = 1$). This line of fixed points is identifiable with the one of Ref. 37 with $h_4 < 0$.

For the symmetry (4) the fixed-point line continues for negative \bar{v} and reaches the “cubic” fixed point (which in this case is a standard Ising) located at $(5/3g_I^*, -g_I^*)$. This second line is the one of Ref. 37 with $h_4 > 0$, which is isomorphic to the other one with $h_4 < 0$. We do not know if the line continues in the region with negative \bar{u} , which may be a region of second-order phase transitions, having the critical exponent of the 8V model with $y < 1$ (only for $v > -u$, from the stability condition). We tried to perform the resummation also for these values of \bar{u}, \bar{v} but we obtained strongly oscillating results.

For $N > 2$ the two curves of zeros seem to intersect. For example, the results of the conformal-mapping analysis for $N = 3$ are shown in Fig. 3 (the error bars are not shown to make the figure more readable). All the results about the cubic fixed points are summarized in Table VI. We note that, within the error bar, there is full agreement between the two methods.⁹⁹

In the limit $N \rightarrow \infty$ the series $\bar{\beta}_u^-(\bar{u}, \bar{v})$ and $\bar{\beta}_v^-(\bar{u}, \bar{v})$ simplify to

$$\bar{\beta}_v^-(\bar{u}, \bar{v}) \rightarrow \bar{\beta}_{\text{Ising}}^-(\bar{v}), \quad (42)$$

$$\bar{\beta}_u^-(\bar{u}, \bar{v}) \rightarrow \bar{u} [P_1(\bar{v}) - \bar{u} P_2(\bar{v})], \quad (43)$$

with

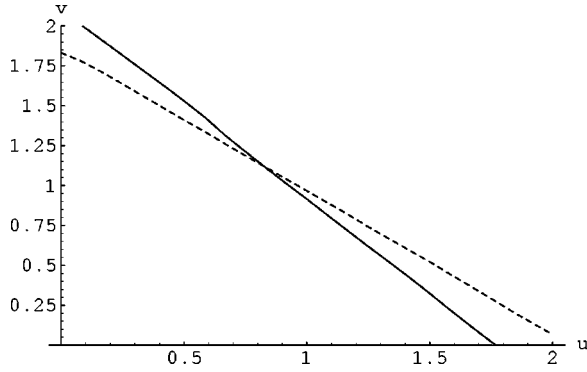


FIG. 3. Zeros of the $\bar{\beta}$ functions for $N=3$ in the (\bar{u}, \bar{v}) plane using the conformal-mapping method. The straight line correspond to the zeros of $\bar{\beta}_{\bar{u}}(\bar{u}, \bar{v})$ and the dashed one to the zeros of $\bar{\beta}_{\bar{v}}(\bar{u}, \bar{v})$.

$$P_1(\bar{v}) = -1 + \frac{2}{3}\bar{v} - 0.21608\bar{v}^2 + 0.23157\bar{v}^3 - 0.31168\bar{v}^4, \quad (44)$$

$$P_2(\bar{v}) = 1 + 0.058328\bar{v}^2 - 0.057896\bar{v}^3.$$

The zero of $\bar{\beta}_{\bar{v}}(\bar{u}, \bar{v})$ is constant with varying \bar{u} at the value $\bar{v}^* = \bar{g}_I^*$ Eq. (29), and so the \bar{u} coordinate of the cubic fixed point is simply

$$u_{N \rightarrow \infty}^* = \frac{P_1(\bar{g}_I^*)}{P_2(\bar{g}_I^*)} = 0.09(4). \quad (45)$$

We note that the small value of u^* for $N \rightarrow \infty$ does not exclude that the Ising and the cubic fixed points coalesce in this limit.

We want to stress that this cubic fixed point is obtained in the four-loop approximation and so it could also be an artifact of the relatively low number of terms considered. Knowledge of higher-loop terms is needed in order to clarify this point. Nevertheless, we analyze its stability and critical exponents in the next paragraphs.

D. Stability properties of the cubic fixed point

For $N=2$ we check that one eigenvalue of the stability matrix Ω vanishes on the whole line of fixed points. We evaluate the Ω matrix on the straight line joining the Ising fixed point [at $(0, g_I^*)$] and the $O(2)$ [at $(1.80(3), 0)$ from Ref. 80] since the real curve of the common zeros of the β

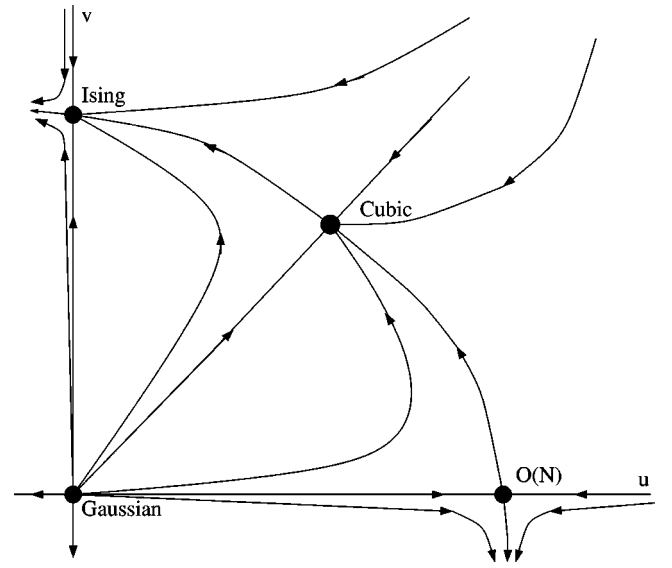


FIG. 4. Renormalization-group flow in presence of two “marginal” fixed points.

functions is well approximated by this. Within the precision of our calculation we confirm that one marginal operator exists on this line.

The results for the smallest eigenvalue of the Ω matrix at the cubic fixed point for $N \geq 3$ are summarized in Table VI. This is positive for all considered values of N , but it is very close to zero. Since the Ising fixed point is stable against perturbations with positive u , the cubic fixed point should be unstable for “left” perturbations. A possible scenario compatible with this result is that both cubic and Ising fixed points have a marginal operator, so the resulting RG flow might be the one sketched in Fig. 4.¹⁰⁰ For $u, v < 0$ we expect a first-order phase transition, and for $0 < v/u \leq v^*/u^*$ the transition is second order and its critical behavior might be characterized by the cubic fixed point, while for $v/u \geq v^*/u^*$ the transition is governed by the Ising fixed point. Obviously if the cubic fixed point is an artifact of the four-loop approximation, this has to coalesce with the Ising one.

In order to have more precise estimates we also consider the pseudo- ϵ expansion using substantially the same procedure of Ref. 27. By using this trick we have a smallest error bar since we avoid the uncertainty of the fixed point (for a detailed discussion see Ref. 83). The results are equivalent to the ones presented above (see Table VI).

E. Critical exponents

The direct evaluation of the critical exponents for the two-dimensional $O(N)$ model leads to erroneous values because

TABLE VI. Critical properties of cubic fixed point $N \geq 3$.

N	(\bar{u}^*, \bar{v}^*)	(\bar{u}^*, \bar{v}^*)	$\omega_1/2$		η		$\eta - \eta_t$		
	CM	PB	CM	PB	pseudo- ϵ	CM	PB	CM	PB
8	(0.24(8), 1.72(10))	(0.15(9), 1.72(12))		0.07(6)	0.08(4)	0.26(2)	0.25(2)	1.20(15)	1.15(15)
4	(0.54(10), 1.43(8))	(0.38(11), 1.51(9))	0.11(6)	0.06(6)	0.06(4)	0.23(4)	0.20(6)	1.22(15)	1.18(20)
3	(0.83(12), 1.12(9))	(0.71(12), 1.17(9))	0.09(6)	0.04(5)	0.05(5)	0.17(8)	0.15(9)	1.39(20)	1.23(20)

of the strong effect of nonanalytic terms. In fact from the analysis of the four- and five-loop series of the $N=1$ model is found $\eta \sim 0.131$ (Ref. 80) and $\eta = 0.13(7)$ (Ref. 83) instead of the exactly known $\eta = 1/4$. The value of ν for the Ising universality class is quite good ($\nu \sim 0.96$ (Refs. 80 and 83)). Our unpublished analysis of the $O(N)$ series (already cited in Ref. 88) shows that for higher values of N the effect of nonanalyticities is very dangerous. In fact, studying four-loop series, we find $\eta = 0.11(6)$ both for the XY and Heisenberg models instead of $1/4$ and 0 . In the same way for $\eta - \eta_t$ we find $1.18(5)$ and $1.36(4)$, for $N=2,3$, instead of 2 . This latter systematic error brings to a finite value of ν . The above standard analysis applied to the XY model with cubic anisotropy gives $\eta \sim 0.11$ along the fixed-point line and values ranging from 1.18 to 0.97 for $\eta - \eta_t$.

In order to reduce the effect of nonanalytic terms in the estimates of the critical exponents of the cubic model we adopt a new strategy. We use a constrained analysis on the two variables series η and $\eta - \eta_t$, fixing the values assumed at the fixed point on the axes $\bar{u} = 0$ and $\bar{v} = 0$. Explicitly, let us consider a generic l -loop series in \bar{u} and \bar{v} without a constant term,

$$R(\bar{u}, \bar{v}) = \sum_{k=0}^l \sum_{h=0}^{l-k} R_{hk} \bar{u}^h \bar{v}^k = \sum_{k=1}^l R_k(z) \bar{u}^k, \quad (46)$$

of which we know the values $R(\bar{u}^*, 0) = a$ and $R(0, \bar{v}^*) = b$. We can rewrite the previous function in the form

$$R(\bar{u}, \bar{v}) = R(\bar{u}, 0) + R(0, \bar{v}) + \bar{u}\bar{v}A_R(\bar{u}, \bar{v}), \quad (47)$$

where $\bar{u}\bar{v}A_R(\bar{u}, \bar{v})$ is the difference between the original function $R(\bar{u}, \bar{v})$ and the value that it assumes on the axes. We resum the three functions $R(\bar{u}, 0)$, $R(0, \bar{v})$, and $A_R(\bar{u}, \bar{v})$, appearing in Eq. (47), in an independent way. In the case of $R(\bar{u}, 0)$ and $R(0, \bar{v})$ we adopt the standard method to resum one variable function with some constraints, as done for the ϵ expansion in Refs. 86, 94, and 18. A full description of this method may be found in Ref. 94.

First of all we consider the peculiar case $N=2$. We evaluate the critical exponents on the straight line joining the Ising and the $O(2)$ fixed points (as in the case of the Ω matrix). The results for the exponents η and $\eta - \eta_t$ are presented in Figs. 5 and 6, respectively. Within the precision of our calculation we find an exponent η assuming the constant value $1/4$ along the line as predicted in Ref. 37 (the uncertainty is about 0.05 for $x \sim 0.5$ and decreases near the borders). The exponent $\eta - \eta_t$ interpolates in a monotonic and continuous way from 1 at the Ising fixed point to 2 at the $O(2)$ fixed point and so it is compatible with the Kadanoff's conjecture identifying this line with the dual line of the $8V$ model with the continuous varying exponent given by Eq. (7).

The parameter y in Eq. (7) belongs to the range $[1, 3/2]$, respectively, from Ising to $O(2)$; instead x ranges between 0 and 1 from $O(2)$ to Ising. So we try to identify

$$y = \frac{3-x}{2}. \quad (48)$$

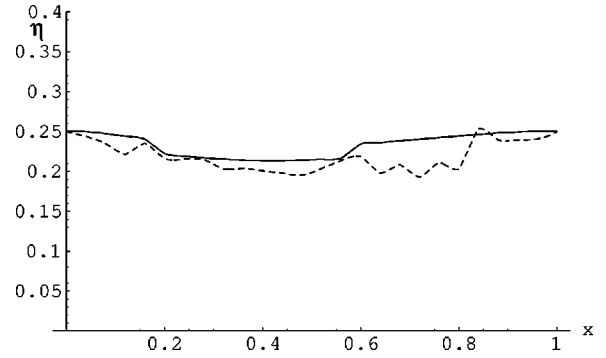


FIG. 5. Values of η with varying the parameter x , Eq. (34). The straight line represents CM results and the dashed the PB ones.

Using this assumption the exponent $\eta - \eta_t$ could be written in terms of x as

$$\eta - \eta_t = \frac{2}{1+x}. \quad (49)$$

This curve is shown in Fig. 6 together with the numerical values obtained from the resummation of the perturbative series: all curves are very close. This result strongly supports both Kadanoff's conjecture about the nature of the line of fixed points and the relation (48) between the parameter y of the $8V$ model and x of the XY model with cubic anisotropy.

The exponents at the cubic fixed point obtained using this constrained analysis for some $N \geq 3$ are reported in Table VI. We find an η exponent varying very slowly with N . Our estimates are also compatible with an η constant and equal to $1/4$ for all N . The estimates of $\eta - \eta_t$ are instead affected by a big error bar, mainly due to the uncertainty of the location of the fixed point. We could reduce the last error using the pseudo- ϵ expansion, but in this way we will find again the systematic error of nonanalyticities that we cannot control.

From Table VI it is clear that when N increases the exponents get closer to the ones of the Ising model—that is, the limit for $N \rightarrow \infty$; cf. Eq. (2). In this limit the series η_ϕ reproduces order by order the one of the Ising model. For the difference $\eta - \eta_t$ the equality of the exponent is expected only at the fixed point. In fact it holds that

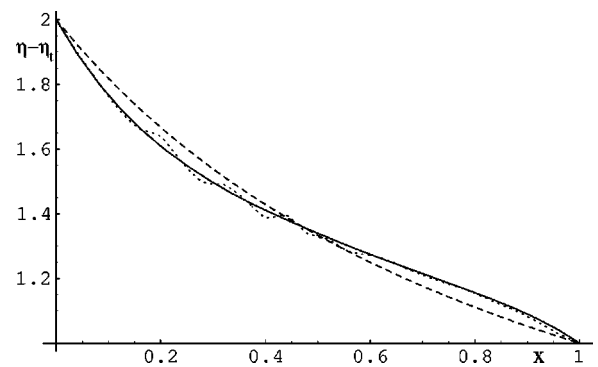


FIG. 6. Values of $\eta - \eta_t$ (straight line CM and pointed line PB) with varying the parameter x , Eq. (34). The dashed line is our conjecture (49).

$$\lim_{N \rightarrow \infty} \eta_i(\bar{u}, \bar{v}) = -2\bar{u} + \eta_i^I(\bar{v}) - \bar{u}\bar{v}^2 A(\bar{v}), \quad (50)$$

where $\eta_i^I(\bar{v})$ is the series of the Ising model, and $A(\bar{v})$ a function that at four loops reads $A(\bar{v}) = 0.1166 - 0.1157\bar{v}$. At the cubic fixed point we obtain

$$\eta_i - \eta_i^I = 0.13(10), \quad (51)$$

which is compatible with zero.

For $N=3$ the result for $\eta - \eta_i$ via scaling law leads to $\nu = 1.3(3)$, so there are no fixed points (stable or unstable) with the three-state Potts exponent $\nu = 5/6$. In Sec. II C we have stressed that experimental investigations⁷² and Monte Carlo simulations^{75,76} in the adsorption of diatomic molecules on a honeycomb lattice in the $p(2 \times 1)$ structure display the critical behavior of three-state Potts, although, according to the standard classification,⁵ this system should be described by the $N=3$ cubic model. It may be possible that the system undergoes a weak first-order phase transition in the face-cubic region with $v < 0$, with effective exponents close to the ones of the three-state Potts model, which appears in the discrete face-centered model. This point needs further studies for a full understanding of the phenomenon.

V. CONCLUSIONS

In the present paper we have studied the critical behavior of N -component spin models with cubic anisotropy by applying the field-theoretic renormalization-group technique directly in two dimensions.

We have first focused our attention on the XY model ($N=2$) with cubic anisotropy. We found that this model has a line of fixed points joining the decoupled Ising and the $O(2)$. Along this line the critical exponent η assumes the constant value $1/4$. The exponent ν runs from 1, at Ising, to ∞ , at $O(2)$, according to the standard expression of the 8V model (7). This argument is based on a mapping of the y parameter of the 8V model onto the x parameter of the Hamiltonian (1):

$$x = 3 - 2y = \frac{2}{\pi} \arctan \frac{\bar{v}^*}{u^*}. \quad (52)$$

The predicted expression for ν [Eq. (49)] is in very good agreement with the resummation of perturbative series (see

Fig. 6). In this manner we relate the measured exponents to the strength of the anisotropy (the parameter x).¹⁰¹ The XY model with cubic anisotropy for $u < 0$ could display the critical behavior of the Askin-Teller model with $y < 1$, i.e., the region on the dual line from the decoupled Ising fixed point ($y=1$) to the four-state Potts ($y=0$). We tried to check if the line of fixed points continues for negative values of u , but the results of the resummation for these values of the renormalized couplings are strongly oscillating.

For all other values of N we found one fixed point in the region with $\bar{u}, \bar{v} > 0$, which seems marginally stable and unstable for perturbations with $v/u < v^*/u^*$ and $v/u > v^*/u^*$ respectively. A possible RG flow diagram is sketched in Fig. 4. The estimates of this fixed point and of the critical exponents for several values of N are reported in Table VI. For the physically relevant case of $N=3$ at the cubic fixed point we found $\eta = 0.17(8)$ and $\nu = 1.3(3)$. This value of ν is different from the three-state Potts model. This fact deserves further studies about the order-disorder transitions of some adsorbed monolayers that should belong to the universality class of the Heisenberg cubic model, but that experimentally display the exponents of the three-state Potts model. One possible scenario is that the system undergoes a very weak first-order phase transition with effective exponents close to the ones of three-state Potts model.

Finally we mention that in Ref. 56, using an argument based on conformal field theory, it is argued that the cubic fixed point and the Ising one merge for arbitrary $N > 2$, and so the critical behavior will be Ising like in the whole $u_0, v_0 > 0$ plane. Our analysis seems to contradict this picture. However, we cannot exclude that the existence of a marginal cubic fixed point is just an artifact of the relatively short series that we have analyzed. It would be important to get further confirmation, either by extending the perturbative series or by alternative analyses, such as Monte Carlo simulations.

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- ¹⁰⁰The presence of a marginal operator both at Ising and at cubic fixed points means that we should expect slowly decaying cross-over effects that could produce systematic errors in Monte Carlo simulations and in the experimental investigations.
- ¹⁰¹In order to relate the physical anisotropy to the experimentally measured exponents, it remains to understand the relation between the ratio of the bare parameter u_0 and v_0 (which are the physical couplings) and their fixed-point values. This could be obtained by a Monte Carlo simulation of the Hamiltonian (1).
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