Bose-Einstein condensation and superfluidity of a dilute Bose gas in a random potential

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There is a growing interest in the relation between Bose-Einstein condensation (BEC) and the superfluidity. A Bose system confined in random media such as porous glass is suitable for studying this relation because BEC and superfluidity can be suppressed and controlled in such a disordered environment. However, it is not clear how this relation is affected by disorder and there are few theoretical studies that can be quantitatively tested by experiment. In this work we develop a dilute Bose gas model with a random potential that takes into account the pore-size dependence of porous glass. Then we compare our model with the measured low-temperature specific heat, condensate density, and superfluid density of ^{4}He in Vycor glass. This comparison uses no free parameters. We predict phenomena at low temperatures: First, the random potential causes a T-linear specific heat instead of the T^3 dependence that is usually caused by phonons. Second, the BEC can remain even when the superfluidity disappears at low densities. Third, the system makes a reentrant transition at low densities; that is, the superfluid phase changes to the normal phase again as the temperature is reduced. This reentrant transition is more likely to be observed when the strength of the random potential is increased.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) and the superfluidity of liquid ^{4}He in random environments including Aerogel and Vycor glass are active problems in quantum fluid research. In particular, finding out how spatial confinement affects the Bose fluid has stimulated both experimental and theoretical studies.

Below the λ temperature of 2.17 K, liquid ^{4}He enters a superfluid state and behaves as though it has no viscosity. Superfluidity is a macroscopic quantum phenomenon as well as superconductivity, and understanding both has been one of the major goals of quantum statistical physics. The various observations of superfluidity was successfully explained by the phenomenological two fluid model,[1] which is based on the idea that the system consists of an inviscid superfluid and a viscous normal fluid. On the other hand, the lambda transition had been thought to be caused by BEC, which was confirmed by neutron scattering experiments.[2] With a BEC, a macroscopic number of particles occupies a single particle ground state and is described by a macroscopic wave function. The inviscid superflow can be described by this wave function.[3] However, the relation between BEC and superfluidity is not completely understood. Although superfluidity and BEC are closely related to each other, one is not necessary or sufficient for the other. For example, in a two-dimensional Bose system, Kosterlitz and Thouless proved that superfluidity can exist even without BEC,[4] and the superfluidity was actually observed in ^{4}He films.[5] The Bose system in a random environment might be another good example for studying the relation between BEC and superfluidity.[6,7] This system has received considerable attention because localization effects allow condensed particles to belong to the normal fluid rather than to the superfluid, and BEC can separate from the superfluid. The phase diagram of this system has been discussed, showing a specific nonsuperfluid phase. Thus studies of this system can reveal a relation between BEC and superfluidity.

Porous glass such as Vycor is often used as a random media in experimental studies. Vycor glass is porous from 30% to 70%, containing wormholelike pores, the characteristic diameters of which vary from 30 to 100 Å. By adjusting the pore size and the adsorbed ^{4}He coverage, we can change the density of ^{4}He and the superfluid transition. By using torsional oscillators, Reppy and co-workers[8,9] showed interesting features as various pore sizes of Vycor glass or coverages were observed, particularly the behavior of the superfluid critical temperature and the temperature dependence of the superfluid density. The superfluid component has a two-dimensional behavior when the pore size of Vycor glass is large, and becomes three dimensional as the pore size is reduced. The superfluid density in such porous glass is smaller than that of bulk ^{4}He and its critical temperature decreases with the coverage. Below a certain coverage, the superfluid density can no longer exist, even near 0 K. These results show that superfluidity is broken by the random environment. It is also important to find out how disorder affects the BEC. BEC and its elementary excitation in liquid ^{4}He can be observed by neutron scattering. Bulk ^{4}He has typical excitations such as those of phonons, maxons, and rotons.[1] There are collective excitations on BEC, all excitations except for phonons being absent above the critical temperature. Dimeo *et al.*[10] and Plantevin *et al.*[11] used neutron scattering and a torsional oscillator to measure the elementary excitations and the superfluid transition, respectively, of ^{4}He in porous glass. Surprisingly, the dispersion curve in porous glass was the same as that in the bulk, which means that the disorder does not affect the elementary excitations. Furthermore, these elementary excitations were observed even above the superfluid critical temperature. Hence the BEC might persist in a disordered system even above the superfluid critical temperature. These results are not yet understood completely; thus the exact relationship between BEC and superfluidity remains puzzling.[12]

This problem is also interesting from the following theoretical standpoint. In a Bose system confined in a random media, the long-range-order correlation due to BEC can compete with the disorder, so that the BEC critical temperature can be reduced. Huang and Meng proposed a model for a three-dimensional dilute Bose gas in a random potential[13] that assumed a small coverage of ^{4}He in Vycor glass . Because it is difficult to formulate the random potential for the porous glass, they used a delta-functional impurity potential, and that analyzed their model using the Bogoliubov transformation and taking an ensemble average. They found that both BEC and superfluidity are depressed by the random potential, and that the superfluidity disappears below a critical density, even at 0 K, which is qualitatively consistent with the observations by Reppy and co-workers. They also predicted a reentrant transition at low densities; that is, the superfluid phase enters a normal phase again with decreasing temperatures. However, the random potential of their model does not include the pore size, and thus it is difficult to quantitatively compare to experimental results for a range of pore sizes. Another model is the Bose Hubbard model with the random potential. By considering the transfer energy, the on-site repulsion, and the random potential, Fisher *et al.*[14] found that the Bose glass phase can exist with the superfluid phase and the Mott insulating phase. The Bose glass phase is similar to the Anderson insulating phase[15] in metal. In the Bose glass phase, the condensed particles are localized and thus do not contribute to superfluidity. Thus the Bose glass phase could influence the collective excitations even above the superfluid critical temperature. However, the theoretical excitation energy[16] for the Bose glass phase disagrees with measurements from neutron scattering experiments,[11] so it is not yet clear whether the Bose glass phase has actually been detected. Finally, it should be noted that Huang and Meng's model cannot describe the Bose glass phase because the ensemble average makes the system uniform.

Few theoretical studies of this random system can be quantitatively compared to the experiment. Thus, in this work, we improve Huang and Meng's model[13] by adding the size dependence of the random potential instead of using their the delta-functional potentials. The strength of the random potential can be estimated by comparing calculated and experimental critical coverages below which the superfluid density disappears, even at 0 K. As a result, our model has no free parameters and can be used for quantitative comparisons to experimental data. This enables us to determine whether or not our picture of the three-dimensional dilute Bose gas in a random potential is applicable to a real system. Our formulation cannot address this question at high temperatures due to the high number of thermally excited quasiparticles. As far as the condensate density is almost independent of temperature at low temperatures, however, our formulation works well, leading to the following results. (1) The specific heat agrees quantitatively with experimental data at low temperatures. (2) Because of the random potential, the specific heat is not proportional to T^3, as occurs for phonons, but to T. Furthermore, by obtaining the condensate density and the superfluid density, we found the following. (3) When the total density is sufficiently low, BEC can persist even when the superfluid density disappears below that critical coverage. (4) The random potential causes a reentrant transition of the superfluid phase. Finally, we show why decreasing the open pore density of the Vycor glass should allow the reentrant phase to be detected experimentally

A brief summary of our paper is as follows. In Sec. II, we describe our model of the dilute Bose gas in a random potential, and derive the partition function. Section III tests our model by quantitatively comparing calculated to experimental specific heats. In Sec. IV, the BEC density and the superfluid density are obtained and their characteristics are discussed. Section V contains a discussion and conclusions.

II. MODEL

Superfluid ^{4}He adsorbed in Vycor glass can be modeled by a three-dimensional dilute Bose gas in a random external potential.[13] The grand canonical Hamiltonian is

$$\hat{H}-\mu\hat{N}\equiv\hat{K}=\int dx^3\hat{\Psi}^\dagger(\boldsymbol{x})\left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+U(\boldsymbol{x})-\mu\right]\hat{\Psi}(\boldsymbol{x})+\frac{v_0}{2}\int dx^3\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x}), \qquad (1)$$

where $\hat{\Psi}(\boldsymbol{x})$ is the field operator for Bose particles of mass m, $\hat{N}=\int dx^3\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})$ is the number operator, μ is the chemical potential, and $U(\boldsymbol{x})$ is the external random potential that represents the effect of Vycor glass. The first term of the Hamiltonian is the kinetic energy and the external potential, whereas the second term refers to the hard-sphere interaction between particles, with $v_0=4\pi a\hbar^2/m$ being the coupling constant with the s-wave scattering length a. This repulsive interaction prevents all particles from being localized at the minimum of $U(\boldsymbol{x})$. This has similarities to the Fermi system with disorder;[15,17,18] for example, fermions cannot be localized in a single orbital in space due to the Pauli exclusion principle. Therefore, the fermion system is stable even if it is free from the repulsive interaction. On the other hand, to prevent the system from collapsing into the minimum of $U(\boldsymbol{x})$, the Bose system should include a repulsive interaction. This makes the problem more complicated than that of the Fermi system.

Proceeding in a standard fashion, we introduce the free particle annihilation and creation operators $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^\dagger_{\boldsymbol{k}}$. We assume that the level with $\boldsymbol{k}=\boldsymbol{0}$ is macroscopically occupied with an occupation number N_0, so $\hat{a}_0$ and $\hat{a}^\dagger_0$ are replaced by a c number $\sqrt{N_0}$. By making a Fourier transformation and neglecting all off-diagonal terms $U_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}''}$ and $v_0\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}'}\hat{a}_{\boldsymbol{k}''}\hat{a}_{\boldsymbol{k}'''}$, we obtain

$$\hat{K}_{\mathrm{eff}}=V\left(-\mu n_0+\frac{1}{2}v_0n_0^2+\frac{n}{V}U_0\right)+\sum_{\boldsymbol{k}\neq0}\left[\frac{\hbar^2k^2}{2m}-\mu+v_0(n+n_0)\right]\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}+\sqrt{\frac{n_0}{V}}\sum_{\boldsymbol{k}\neq0}(U_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}+U_{-\boldsymbol{k}}\hat{a}_{\boldsymbol{k}})+\frac{1}{2}v_0n_0\sum_{\boldsymbol{k}\neq0}(\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}+\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}), \qquad (2)$$

where V is the volume of the system, $n_0 = N_0/V$ is the number density of condensate, and $U_{\boldsymbol{k}}$ is the Fourier transformation of $U(\boldsymbol{x})$. By neglecting the off-diagonal terms, we are neglecting the interactions between the excited particle and the random potential and that between pairs of excited particles; these become important as the temperature rises and the condensate density decreases. Hence this approximation is poor when many particles are thermally excited. Nevertheless, this approximation is useful at low temperatures where the condensate density is almost independent of the temperature. All results here are obtained for these low temperatures.

This Hamiltonian can be diagonalized by the Bogoliubov transformation

$$\hat{a}_{\boldsymbol{k}} = \frac{\hat{c}_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}} \hat{c}^{\dagger}_{-\boldsymbol{k}}}{\sqrt{1-\gamma_{\boldsymbol{k}}^2}} + g_{\boldsymbol{k}}. \tag{3}$$

Then the coefficients $\gamma_{\boldsymbol{k}}$ and $g_{\boldsymbol{k}}$ and the quasiparticle spectrum $\omega_{\boldsymbol{k}}$ are given by

$$\gamma_{\boldsymbol{k}} = -\xi - 1 + \sqrt{\xi(\xi+2)}, \tag{4a}$$

$$g_{\boldsymbol{k}} = -\sqrt{\frac{n_0}{V}} \frac{U_{\boldsymbol{k}}}{(\xi+2) v_0 n_0}, \tag{4b}$$

$$\omega_{\boldsymbol{k}} = v_0 n_0 \sqrt{\xi(\xi+2)}, \tag{4c}$$

$$\xi = \frac{\hbar^2 k^2}{2 m v_0 n_0} + \Delta, \quad \Delta = \frac{v_0 n_0 - \mu}{v_0 n_0}. \tag{4d}$$

Next, we take an ensemble average to quench the random potential. The random potential simulates Vycor glass with a characteristic pore size r_p as follows. The quenched potential $U_{\boldsymbol{k}}$ may decay above the characteristic wave number $k_p = 2\pi/r_p$. Thus we assume the averaged potential

$$\frac{1}{V}\langle U_{\boldsymbol{k}} U_{-\boldsymbol{k}}\rangle_{\text{av}} = R_0 \exp\left[-\frac{k^2}{2k_p^2}\right], \tag{5}$$

where av denotes the ensemble-average. R_0, with dimension $(\text{energy})^2(\text{length})^3$, is the characteristic strength of the random potential. Equation (5) makes our model completely different from Huang and Meng's, and we will show that the results are also different. The coherence length of the BEC is thought to be from hundreds to thousands of Å, whereas the spatial scale of disorder is the pore size in the glass, which is dozens of Å. Hence the macroscopic wave function of BEC is not sensitive to disorder in and between pores, but instead depends on the disorder averaged over the coherence length. Hence the ensemble-averaged system can become nearly uniform. For a uniform Bose system, it has been proven that the elementary excitation spectrum becomes the gapless Goldstone mode.[19] Thus we set $\Delta = 0$ in Eq. (4d).

The resultant diagonalized and ensemble-averaged Hamiltonian is

$$\hat{K}_{\text{eff}} = V\left(-\mu n_0 + \frac{n}{V} U_0 + \epsilon_1 + \epsilon_R\right) + \sum_{\boldsymbol{k}\neq 0} \hbar \omega_k \hat{c}^{\dagger}_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}, \tag{6a}$$

$$\omega_k = \frac{\hbar}{2m} k \sqrt{k^2 + 16\pi a n_0}, \tag{6b}$$

$$\epsilon_1 = \frac{2\pi a n_0^2 \hbar^2}{m}\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{n_0 a^3}\right], \tag{6c}$$

$$\epsilon_R = \frac{m\sqrt{a} n_0^{3/2} R_0}{4\sqrt{\pi}\hbar^2}\left[-e^{2\alpha}(5+4\alpha)\left\{1 - \text{erf}(\sqrt{2\alpha}\right\} + \sqrt{\frac{2}{\pi\alpha}}(1+\alpha)\right], \tag{6d}$$

$$\alpha = \frac{4\pi a n_0}{k_p^2}, \tag{6e}$$

where ϵ_1 is the hard sphere interaction energy at 0 K, similarly, ϵ_R is that for the random potential. The quasiparticle spectrum ω_k is the same as that in the hard sphere Bose gas model[7] and is independent of the random potential. This independence is confirmed by neutron scattering experiments,[11] which justifies the above assumption of $\Delta = 0$; conversely, if $\Delta \neq 0$, the spectrum would depend on the random potential.

This Hamiltonian enables us to obtain the grand partition function $Q = \text{Tr}\{\exp(-\beta\hat{K})\}$ and various physical quantities. The condensate density is defined by the relation

$$n_0 = n - \frac{1}{V}\sum_{\boldsymbol{k}\neq 0} \langle \hat{a}^{\dagger}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}\rangle, \tag{7}$$

where n is the particle number density. The second term represents the noncondensate particle number as

$$\frac{1}{V}\sum_{\boldsymbol{k}\neq 0} \langle \hat{a}^{\dagger}_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}\rangle = n_1 + n_R, \tag{8a}$$

$$n_1 = \frac{8}{3\sqrt{\pi}}(n_0 a)^{3/2} + \frac{4}{\sqrt{\pi}\lambda^3}\int_0^{\infty} dt \frac{t(t^2+\theta/2)}{\sqrt{t^2+\theta}\{e^{t\sqrt{t^2+\theta}} - 1\}}, \tag{8b}$$

$$n_R = \frac{m^2 R_0}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n_0}{a}}\left[e^{2\alpha}(1+4\alpha)\{1-\text{erf}(\sqrt{2\alpha})\} - 2\sqrt{\frac{2\alpha}{\pi}}\right], \tag{8c}$$

$$\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}}, \quad \theta = \frac{8\pi a \hbar^2 \beta n_0}{m}, \quad t^2 = \frac{\hbar^2\beta}{2m}k^2. \tag{8d}$$

Here n_1 is the noncondensate density excited by the hard sphere interaction, n_R is the density due to the scattering of condensate particles with the random potential, and λ is the thermal de Broglie wave length. When a vanishes, n_R becomes infinite. This means that the system would collapse if there were no repulsive interactions between particles.

Because superfluidity is described by the two fluid model, the particle density n consists of the normal fluid density n_n and the superfluid density n_s. The superfluid density n_s can be calculated by linear response theory.[20] Because of its viscosity, only the normal fluid responds to a small, applied velocity field. Thus the normal fluid density can be defined by the response of the momentum density $j_i(\boldsymbol{x},t)$ to the external velocity field $v_i(\boldsymbol{x},t)$. Linear response theory gives the relations

$$j_i(\boldsymbol{x},t)=\chi_{ij}(\boldsymbol{x},t)v_j(\boldsymbol{x},t), \tag{9a}$$

$$\chi_{ij}(\boldsymbol{x},t)=\langle[j_i(\boldsymbol{x},t),j_j(0,0)]\rangle, \tag{9b}$$

$$j_i(\boldsymbol{x},t)=\frac{\hbar}{2i}\left\{\hat{\Psi}^\dagger(\boldsymbol{x},t)\frac{\partial\hat{\Psi}(\boldsymbol{x},t)}{\partial x_i}-\frac{\partial\hat{\Psi}^\dagger(\boldsymbol{x},t)}{\partial x_i}\hat{\Psi}(\boldsymbol{x},t)\right\}, \tag{9c}$$

$$\hat{\Psi}(\boldsymbol{x},t)=e^{\mathrm{i}(\hat{H}-\mu\hat{N})t/\hbar}\hat{\Psi}(\boldsymbol{x})e^{-i(\hat{H}-\mu\hat{N})t/\hbar}, \tag{9d}$$

where $\hat{\Psi}(\boldsymbol{x},t)$ is the Heisenberg field operator. The static susceptibility $\chi_{ij}(\boldsymbol{k})$ is defined as

$$\chi_{ij}(\boldsymbol{x},t)=\int\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}e^{-\mathrm{i}\omega t}e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\chi_{ij}(\boldsymbol{k},\omega), \tag{10a}$$

$$\chi_{ij}(\boldsymbol{k})=\lim_{\omega\to 0}\chi_{ij}(\boldsymbol{k},\omega). \tag{10b}$$

Because of the rotational invariance, the static susceptibility $\chi_{ij}(\boldsymbol{k})$ can be written

$$\chi_{ij}(\boldsymbol{k})=\frac{k_ik_j}{k^2}A(\boldsymbol{k})+\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)B(\boldsymbol{k}), \tag{11}$$

where $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$ are the longitudinal and transverse parts, respectively. The transverse susceptibility $B(0)$ is the normal fluid mass density. The superfluid number density n_s is $n-B(0)/m$. The susceptibility $B(0)$ can be calculated by the Bogoliubov transformation in Eq. (3). After some tedious calculations, the resultant superfluid density is given by

$$n_s=n-n_{n1}-n_{nR}, \tag{12a}$$

$$n_{n1}=\frac{8}{3\sqrt{\pi}\lambda^3}\int_0^\infty dt\frac{t^4e^{-t\sqrt{t^2+\theta}}}{(1-e^{-t\sqrt{t^2+\theta}})^2}, \tag{12b}$$

$$n_{nR}=\frac{4}{3}n_R, \tag{12c}$$

where n_{n1} is the normal fluid density due to the elementary excitations, and n_{nR} is that due to scattering with the random potential. The density n_{n1} can be also obtained using Khalatnikov's method that is based on Galilean invariance.[21] The relation $n_{nR}=4/3n_R=n_R+1/3n_R$ shows that the random potential causes the larger normal fluid density than the noncondensate density; some condensate particles are captured by the random potential to participate in the normal fluid. This makes it possible to destroy superfluidity even at 0 K when n_{nR} becomes comparable to n. This formulation can be used to obtain various physical quantities including the condensate density, the superfluid density, and the specific heat.

III. COMPARISON WITH EXPERIMENTS

In this section, we compare the calculated specific heat and the superfluid density with experimental results. Quantitative agreement is shown to be good at low temperatures, which supports our assumption of a dilute Bose gas in the random potential. Furthermore, we show that the random potential leads to as-yet unobserved behavior of the specific heat.

To make a quantitative comparison, we give the following numerical values to the parameters: $m\simeq 6.6\times 10^{-27}$ kg and $a\simeq 5\times 10^{-10}$ m are the mass and the s-wave scattering length of a ^{4}He atom. Other parameters are from the experiments of Reppy.[8] The volume V of open pores in the Vycor glass (about 40% of the total volume of the Vycor glass) is about 1 cm^3. The particle density n of ^{4}He inside the Vycor glass is estimated as follows. In Vycor glass, the atoms are adsorbed and fully cover the surfaces of the open pores due to the van der Waals attraction. The pore area is about 10^8 m^2/m^3. The rest of the atoms, which do not participate in the first-layer solid, can behave as a dilute gas inside the pores. The particle density n of the dilute gas is obtained by subtracting the adsorbed amount from the total amount. This density is estimated to be from 0.001% to 70% of the density of bulk liquid ^{4}He $n_{bulk}\sim 2.1\times 10^{28}$/m^3. Because the first layer of ^{4}He adsorbed on the surfaces cannot move and behaves as a solid, we assume the pore size 30 Å of Vycor glass is effectively reduced by $2a$. Thus r_p is estimated to be 20 Å. The last parameter R_0, which is the strength of the random potential, can be fixed by comparing to experiment. Shown in Fig. 1(a) are the data of zero temperature superfluid signals taken in an experiment that used a torsional oscillator (Fig. 12 of Ref. 8). Because the superfluid component does not contribute to the moment of inertia, the resonant frequency, and the period of oscillation differ from those without superfluid. The period difference ΔP is approximately proportional to the superfluid component. Here the superfluid density is nearly proportional to ΔP and disappears at a coverage of 17.5 mg. Figure 1(b) shows the superfluid density at 0 K from Eq. (12). As in the experiment, the superfluid density becomes zero at a certain coverage that depends on R_0. Thus the value of R_0 can be fixed using the comparison with Fig. 1(a); i.e., $R_0=5\times 10^{-75}$ J^2 m^3. Here we define $R_w\equiv\sqrt{R_0 n}$, which is the single particle energy converted from R_0. In the Vycor glass, R_w/k_B is about $0.001\sim 1$ K. Just above the critical coverage, the superfluid density increases linearly for both the experiment and the calculation; however, their slopes cannot be compared because the amplitude of n_s is unknown in the experiment.

Because all parameters are now fixed, we will quantitatively compare calculations to experiments. The specific heat can be obtained from temperature differentiation of the free energy,

$$\Omega=\Omega_1+\Omega_R, \tag{13a}$$

$$\Omega_1=V(-\mu n_0+\epsilon_1)+\frac{4V}{\sqrt{\pi}\beta\lambda^3}\int_0^\infty dt\{t^2\log[1-e^{-t\sqrt{t^2+\theta}}]\}, \tag{13b}$$

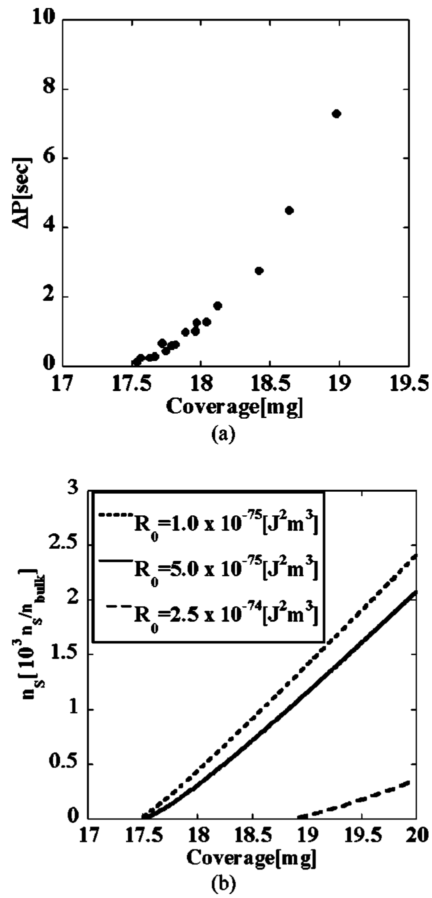

FIG. 1. Superfluid signals of experiments (a) and calculations (b) near 0 K. ΔP is the resonant period difference in a torsional balance experiment, which is approximately proportional to the superfluid density (Ref. 8).

$$\Omega_R = V\left(\epsilon_R + \frac{n}{V}U_0\right). \tag{13c}$$

where Ω_1 is the free energy of the elementary excitation and the hard sphere interaction, and Ω_R is the free energy from the random potential. Figure 2 compares our results to data of low temperature specific heat taken from Fig. 1 of Ref. 8 In Fig. 2(a), which shows the data for high density, the density n is fixed from the experimental coverage, whereas we fix the density from the superfluid critical temperature in Fig. 2(b) at low density. This is because we have no information about the data on the coverage. Figure 2(b) also shows the superfluid density. The theoretical results agree quantitatively with experiment without using free parameters. Above 1.0 K in Fig. 2(a), the calculated condensate density begins to decrease rapidly; here our criterion of constant condensate density fails, which likely causes the discrepancy with experiment. However, Fig. 2(b) shows that the calculated specific heat agrees with experiment up to temperatures near the superfluid critical temperature; in this temperature region, the calculated condensate density hardly decreases. This means that the system is more dilute than that of Fig. 2(a) and thus is affected by the random potential rather than the

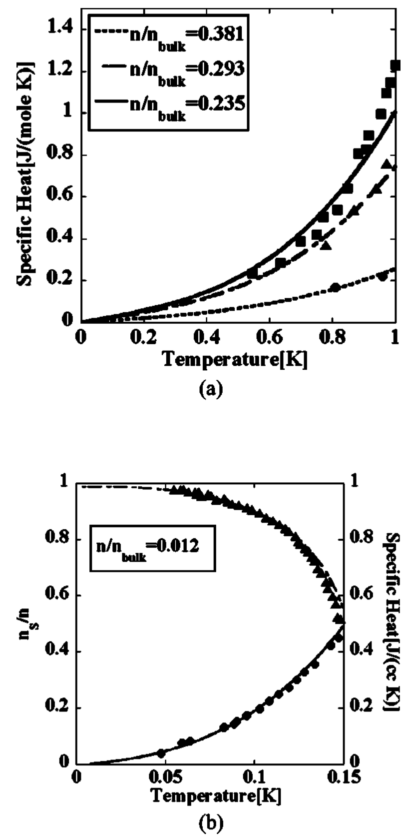

FIG. 2. The specific heat data from in experiments (plot) and calculations (line). In (a), experimental data are given by Fig. 1 of Ref. 8. The circles, triangles, and squares, respectively, correspond to full pores $(\sigma=1)$, $\sigma=0.780$ and 0.636. Here σ is the ratio of the coverage to the full pore coverage. In (b), calculated and experimental superfluid densities are compared $(T_c=0.163\ \mathrm{K})$.

elementary excitations. We discuss this in the next section. Nevertheless, these comparisons show that our model is accurate at low temperatures.

Our model predicts an effect at low temperature that is due to the random potential but has not yet been observed. This is shown in Fig. 3, which is the log-log plot of Fig. 2(a). When the system is free from the random potential, the specific heat should increase to T^3 because of the contribution from the phonons. However, with a random potential, the

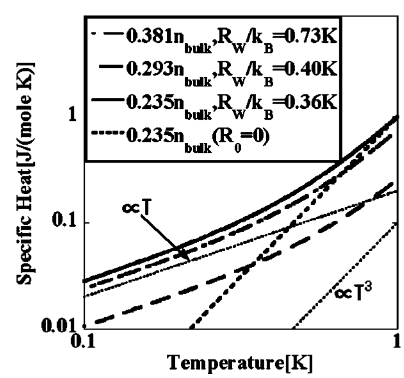

FIG. 3. The log-log plot of Fig. 2(a). Data of the specific heat at $R_0=0(n/n_{bulk}=0.35)$. Two lines for $\propto T$ and $\propto T^3$ are added.

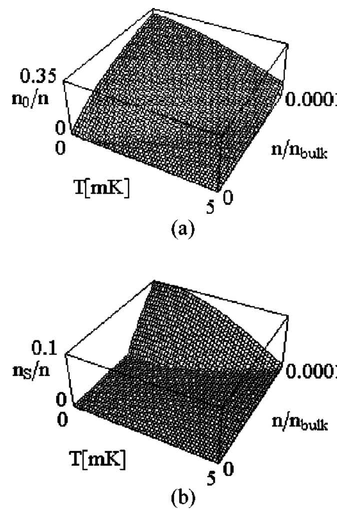

FIG. 4. Temperature and density dependence of n_0 (a) and n_s (b) at $R_0 = 5.0\times 10^{-75}$ J^2 m^3.

dependence is linear in T at low temperatures. This means that at low temperatures, the contribution from the random potential is larger than that from the elementary excitations (phonons). The free energy Ω_R in Eq. (13) depends on the temperature only through the condensate density n_0. Ω_R is the energy from the scattering of the condensate particles with the random potential, and the resultant specific heat is given by the energy that the condensate particles need to slip out of the random potential. An experimental observation of this T-linear dependence might clearly identify the influence from the random potential.

IV. CONDENSATE DENSITY AND SUPERFLUID DENSITY

This section describes some characteristic behavior of the condensate density n_0 and the superfluid density n_s derived from our model. Figure 4 shows the dependence of n_0 and n_s on temperature and density. Both n_0 and n_s decrease with decreasing density, even at 0 K. This means that the effect of the random potential on n_0 and n_s becomes larger as the density is reduced. Figure 4 clearly shows the difference between n_s and n_0. Below the critical density, the superfluid density disappears, although the condensate persists. This situation indicates that the condensate particles cannot move as a superfluid because they are trapped by the random potential. We expect that this theoretical result will be confirmed by measurements of condensate density.

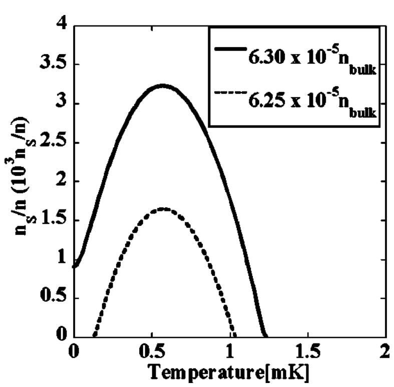

FIG. 5. Temperature dependence of n_s at low temperature and low density. In this regime, the superfluid density n_s goes to zero with a decrease of temperature (reentrant transition).

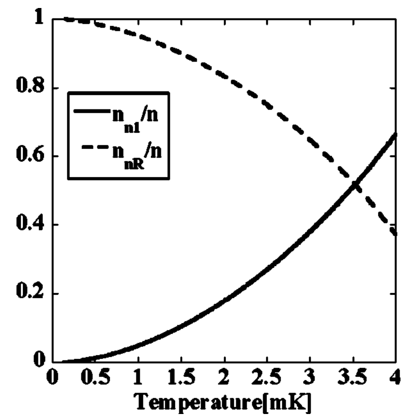

FIG. 6. Temperature dependence of n_{n1} and n_{nR} near the reentrant distribution at $n = 6.25\times 10^{-5}\ n_{bulk}$ (dashed line in Fig. 5).

Figure 5 shows the temperature dependence of the superfluid density just before superfluidity disappears. This figure shows the reentrant transition at which the superfluid density n_s goes to zero with a decrease of temperature. In this temperature region, the condensate density is almost constant; hence our formulation should work well in accordance with

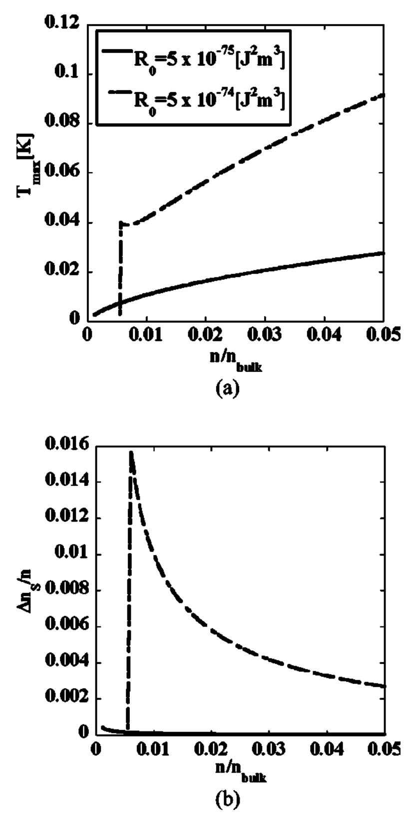

FIG. 7. Density dependence of T_{max} and Δn_s for two values of R_0. T_{max} is the temperature that maximizes the superfluid density n_s, and $\Delta n_s = n_s(T = T_{max}) - n_s(T=0)$.

the criterion described in Sec. II. This reentrant transition is understood as follows. The condensate depletion n_R of Eq. (8c) comes from the scattering of the condensate particles with the random potential. This decreases with n_0 as the temperature rises, so that $n_{nR}=4/3n_R$ of Eq. (12c) also decreases. As shown in Fig. 6, the magnitude of this decrease in n_{nR} exceeds the increase in n_{n1} of Eq. (12b), which is the normal fluid density due to the elementary excitations, in the very low temperature region $T\leqslant 0.5$ mK. In other words, condensate particles that are trapped by the random potential at lower temperatures can escape at higher temperatures and thus participate in the superfluidity. This reentrant transition has not been observed experimentally, probably because it should only occur at very low densities and low temperatures. However, large values of R_0 can make the reentrant transition observable as follows. We define the temperature T_{max} as that which maximizes the superfluid density n_s, and define $\Delta n_s \equiv n_s(T=T_{max})-n_s(T=0)$. Figure 7 shows the density dependence of T_{max} and Δn_s. Both variables increase with R_0. Therefore, the reentrant transition is more likely to be measured at larger R_0. The parameter R_0 is the strength of the random potential over the entire space, and one way to increase R_0 is to decrease the open pore density of the Vycor glass.

V. CONCLUSIONS

The present paper describes the dilute Bose gas system in a random potential. The outcomes of our studies are as follows. By including the pore size dependence of Vycor glass in the random potential, our model could closely match the experimental conditions of liquid ^{4}He in Vycor glass. We fixed the strength of the random potential by equating the theoretical and experimental critical coverages below which the superfluid density at 0 K vanishes. No other parameters could be adjusted, and thus we could quantitatively compare theory to experiment for other physical quantities.

First, we showed that the calculated specific heat for Vycor glass quantitatively agrees with measurements. This indicates that liquid ^{4}He in Vycor glass behaves as a dilute Bose gas in a random potential. For low temperatures, the calculated specific heat was linear in T because of the random potential. Second, the BEC was shown to persist even when superfluidity disappears below the critical density. Finally, we showed that a reentrant transition of the superfluid phase is more likely to be observed experimentally by increasing the strength of the random potential.

Because we neglected interactions between pairs of excited particles and between excited particles and the random potential, this model does not apply to systems at high temperatures. To overcome this limitation, we are improving the model to include these interactions, and will report on this more general model in the near future.

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