Berry phase and spin quantum Hall effect in the vortex state of superfluid ^{3}He in two dimensions

Jun Goryo* and Mahito Kohmoto

Institute for Solid State Physics, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8581, Japan

(Received 13 June 2002; published 4 November 2002)

We show that the spin quantum Hall effect in the vortex state of two-dimensional rotating superfluid ^{3}He can be described as an adiabatic spin transport of Bloch quasiparticles. We show that the spin Hall conductivity is written by the Berry phase as well as the Chern number. The results have similarity to the adiabatic pumping of Bloch electrons and the spontaneous polarization in crystalline dielectrics.

DOI: 10.1103/PhysRevB.66.174503 PACS number(s): 67.57.−z, 03.65.Vf, 73.43.−f, 47.32.−y

I. INTRODUCTION

The Berry phase (the geometrical phase) arises in quantum-mechanical systems with an adiabatic change on a closed loop in a parameter space.[1] In spite of the fact that it is a phase of the wave function, it could be related to physical effects and, in some cases, has a connection with topological numbers. The quantum Hall effect in Bloch electron systems is described as an adiabatic charge transport whose process is closed in a parameter space. A Berry phase is generated and the quantized Hall conductivity is written by the Berry phase as well as the Chern number.[2–7] Recently, it has been pointed out that quasiparticles in the vortex lattice of $d_{x^2-y^2}$-wave superconductors are in the Bloch states.[8] The spin quantum Hall effect occurs and its conductivity is written by a Chern number.[9] Then, one can expect that the spin Hall conductivity in the vortex state is written by a Berry phase, when the effect can be described as an adiabatic spin transport of a closed process.

In this paper, we discuss Bloch quasiparticles in the vortex state of p-wave superfluid ^{3}He in two dimensions. We consider a magnetic field with a weak and homogeneous gradient. Such a field cannot be introduced in superconductors due to the Meissner effect. The magnetic field couples to spin through the Zeeman term and does not to orbital currents because of the neutrality of the superfluid. The spin Hall current flows as an adiabatic spin transport. Its conductivity is written by a Chern number and quantized when an excitation gap exists. We show that the conductivity is closely related to the Berry phase. We also point out that the results have some similarity to the adiabatic pumping in Bloch electrons[10] and the spontaneous polarization in the crystalline dielectrics.[11] We set $\hbar=c=\mu_{\rm B}=1$, where $\mu_{\rm B}$ is the Bohr magneton.

II. SUPERFLUID HELIUM 3 IN TWO DIMENSIONS

Let $\psi_\alpha(\mathbf{x})$ stands for the Fermion field with spin $\alpha=\uparrow,\downarrow$. The mean-field Hamiltonian for Fermionic superfluid (or superconductors) in D-dimensional space is written with the gap matrix $\hat\Delta_{\alpha\beta}(\mathbf{x},\mathbf{y})$ as

$$H_{\rm MF}=\int d^Dx\,\psi_\alpha^\dagger(\mathbf{x})\epsilon(\hat{\mathbf{p}})\psi_\alpha(\mathbf{x}) +\frac{1}{2}\int d^Dx\,d^Dy[\hat\Delta_{\alpha\beta}(\mathbf{x},\mathbf{y})\psi_\alpha^\dagger(\mathbf{x})\psi_\beta^\dagger(\mathbf{y})+\text{H.c.}], \tag{1}$$

$$\epsilon(\hat{\mathbf{p}})=\frac{\hat{\mathbf{p}}^2-p_{\rm F}^2}{2m},$$

where $\hat{\mathbf{p}}=-i\boldsymbol{\nabla}$ and the repeated Greek indices are summed up. We may consider the Fourier transform of the gap matrix in terms of the relative coordinate of $\mathbf{x}$ and $\mathbf{y}$, i.e.,

$$\hat\Delta_{\alpha\beta}(\mathbf{x},\mathbf{y})=\int\frac{d^Dp}{(2\pi)^D}e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}}\hat\Delta_{\alpha\beta}(\mathbf{r},\mathbf{p}), \tag{2}$$

where $\mathbf{r}=(\mathbf{x}+\mathbf{y})/2$.

Superfluid ^{3}He is in the spin triplet p-wave states.[12] In general, the gap function for spin triplet pairing is $\hat\Delta(\mathbf{r},\mathbf{p})=i\sigma_y\boldsymbol{\sigma}\cdot\mathbf{d}(\mathbf{r},\mathbf{p})$, where $\mathbf{d}(\mathbf{r},-\mathbf{p})=-\mathbf{d}(\mathbf{r},\mathbf{p})$ is a three-dimensional (3D) vector in the spin space. In p-wave states, the magnitude of the relative angular momentum of the Cooper pair $|\mathbf{l}|=1$ and $\mathbf{d}$ vectors have a linear dependence on $\mathbf{p}$. It is well known that the three phases are observed in the superfluid ^{3}He (A, B, and A_1 phases).[12] Those phases are represented by the different $\mathbf{d}$ vectors, respectively.

We consider a two dimensions system. To realize 2D, we introduce a strong confine potential along the z axis to avoid quasiparticle excitations along the z axis. The boundary effect introduced by the confine potential locks the relative angular momentum of all the Cooper pairs in the same direction along the z axis.[12] Then, we take $l_z=1$ (l_z: the z component of the angular momentum) in the whole region. The direction of the $\mathbf{d}$ vector becomes parallel to the angular momentum (i.e., $\mathbf{d}//\mathbf{e}_z$) because of the existence of the magnetic dipole interaction which couples spin and orbit.[12] Then, the $\mathbf{d}$ vector in our situation is

$$\mathbf{d}(\mathbf{r},\mathbf{p})=\mathbf{e}_z\phi(\mathbf{r})(p_x+ip_y). \tag{3}$$

This state corresponds to the A phase.[12] Since the direction of the $\mathbf{d}$ vector and the relative angular momentum are frozen, we may neglect the textures and coreless vortices.[12] The Hamiltonian for ^{3}He-A is

$$H_{\rm MF}=\int d^2x\,\psi_\alpha^\dagger(\mathbf{x})\epsilon(\hat{\mathbf{p}})\psi_\alpha(\mathbf{x}) +\int d^2x\,d^2y[\Delta_{\rm A}(\mathbf{x},\mathbf{y})\psi_\uparrow^\dagger(\mathbf{x})\psi_\downarrow^\dagger(\mathbf{y})+\text{H.c.}], \tag{4}$$

$$\Delta_{\rm A}(\mathbf{x},\mathbf{y})=\frac{1}{2}\text{Tr}[\sigma_x\hat\Delta(\mathbf{x},\mathbf{y})].$$

III. VORTEX STATE IN HELIUM 3 *A* PHASE WITH A ROTATION

It is well known that rotating superfluid is direct analogy of type-II superconductors, and actually the vortex states in superfluid ^{3}He are detected by the experiments.[13] Then, we consider superfluid in a container that rotates around the z axis with an angular velocity Ω. Hereafter, we use the rotating frame which is fixed on the container. In the rotating frame, H_{MF} is transformed as

$$H_{\mathrm{MF}} \to H = H_{\mathrm{MF}} - \mathbf{\Omega} \cdot \mathbf{L},$$
$$= \int d^2x \psi^\dagger_\alpha(\mathbf{x}) \left(\epsilon(\hat{\mathbf{p}} - m\mathbf{R}) - \frac{m\mathbf{R}^2}{2} \right) \psi_\alpha(\mathbf{x})$$
$$+ (\text{pairing terms}), \tag{5}$$

$$\mathbf{R} = \mathbf{\Omega} \times \mathbf{x},$$

where $\mathbf{L}$ is the total angular momentum of Fermions.[14,15] The kinetic energy for the quasiparticle is transformed as $\epsilon(\mathbf{p}) \to \epsilon(\mathbf{p} - m\mathbf{R}) - m\mathbf{R}^2/2$. We consider $\Omega \sim 1$ rad/s and $|\mathbf{r}| \leqslant r_0 \sim 1$ mm (r_0: the radius of the container),[13] and we can neglect the $-m\mathbf{R}^2/2$ term. Otherwise, we can cancel out this term by introducing a parabolic trap.[16] *To avoid this term is essential to introduce the translational invariance, as we will discuss below.*

Then, the one to one correspondence can be seen between our system and *charged* superfluid in a magnetic field with an infinite London penetration depth, i.e., the strongly type-II superconductors. The vector field $\mathbf{R}$ corresponds to a "vector potential $\mathbf{A}$" and the Fermion mass m corresponds to "the electric charge e." And "the magnetic field" is $2\mathbf{\Omega} = \nabla \times \mathbf{R}$.

Let us consider a vortex state, i.e., $\Omega > \Omega_{c1}$ and set up a square vortex lattice. We would like to note that our discussion is applicable to other types of lattices. In the vortex state, the gap function has the singular phase $\varphi(\mathbf{x})$ which satisfies

$$\Delta_A(\mathbf{x},\mathbf{y}) = \tilde{\Delta}_{\mathrm{A}}(\mathbf{x},\mathbf{y}) e^{-(i/2)[\varphi(\mathbf{x}) + \varphi(\mathbf{y})]}, \tag{6}$$

$$\nabla \times \nabla \varphi(\mathbf{x}) = 2\pi \mathbf{e}_z \sum_i \delta^2(\mathbf{x} - \mathbf{r}_i),$$

where $\tilde{\Delta}_{\mathrm{A}}(\mathbf{x},\mathbf{y})$ is the gauge invariant part of the gap function, $\mathbf{r}_i = (\mathbf{e}_x l_i + \mathbf{e}_y n_i)a$ with integers l_i and n_i is the ith lattice point, and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vector of the Cartesian coordinate in the rotating frame. When $\Omega \sim 1$ rad/s the vortex lattice constant $a = \sqrt{\pi/m\Omega} \sim 10^{-2}$ cm. From Eq. (5), the Hamiltonian density operator can be written in the Nambu representation as

$$\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}) = \begin{pmatrix} \epsilon(\hat{\mathbf{p}} - m\mathbf{R})\delta(\mathbf{x}-\mathbf{y}) & \tilde{\Delta}_{\mathrm{A}}(\mathbf{x},\mathbf{y}) e^{-(i/2)[\varphi(\mathbf{x})+\varphi(\mathbf{y})]} \\ -\tilde{\Delta}^*_{\mathrm{A}}(\mathbf{x},\mathbf{y}) e^{(i/2)[\varphi(\mathbf{x})+\varphi(\mathbf{y})]} & -\epsilon(\hat{\mathbf{p}} + m\mathbf{R})\delta(\mathbf{x}-\mathbf{y}) \end{pmatrix}. \tag{7}$$

The Bogoliubov–de Gennes (BdG) equation[17] is

$$\int d^2y \mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}) \Phi_E(\mathbf{y}) = E \Phi_E(\mathbf{x}), \tag{8}$$

$$\Phi_E(\mathbf{x}) = [U_E(\mathbf{x}), -V^*_E(\mathbf{x})]^{\mathrm{T}},$$

where

$$\psi_\uparrow(\mathbf{x}) = \sum_E [U_E(\mathbf{x})\gamma_{E\uparrow} + V_E(\mathbf{x})\gamma^\dagger_{E\downarrow}],$$

$$\psi^\dagger_\downarrow(\mathbf{x}) = \sum_E [-V^*_E(\mathbf{x})\gamma_{E\uparrow} + U^*_E(\mathbf{x})\gamma^\dagger_{E\downarrow}],$$

and $\gamma^\dagger_{E\alpha}$ and $\gamma_{E\alpha}$ are the creation and annihilation operator of the Bogoliubov quasiparticles, respectively.

Let us discuss the periodicity of the system.[18] The multivalued phase field $\varphi(\mathbf{x})$ which satisfies Eq. (6) has an ambiguity for deformations which does not change the topology of its configuration, i.e., the ambiguity remains in terms of the gauge degrees of freedom. So, we may take a constraint

$$\begin{cases} \varphi(\mathbf{x} + \mathbf{e}_x a) = \varphi(\mathbf{x}) - a\mathbf{e}_x \cdot m\mathbf{R}, \\ \varphi(\mathbf{x} + \mathbf{e}_y a) = \varphi(\mathbf{x}) - a\mathbf{e}_y \cdot m\mathbf{R}. \end{cases} \tag{9}$$

Obviously, it is consistent with Eq. (6). Then, let us define a translation operator

$$T_{\delta\mathbf{r}} = \exp[i\delta\mathbf{r} \cdot (\hat{\mathbf{p}} + m\mathbf{R}\tau_3)], \tag{10}$$

which is the direct analogy of the magnetic translation operator. The symbol τ_3 denotes the third Pauli matrix in the Nambu (particle-hole) space. The coordinates are translated by $T_{\delta\mathbf{r}}$ as $\mathbf{x} \to \mathbf{x} + \delta\mathbf{r}$ and $\mathbf{y} \to \mathbf{y} + \delta\mathbf{r}$. Then, one can see easily that the operator $T_{\mathbf{e}_x a}$ and $T_{\mathbf{e}_y a}$ commute with $\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})$, but does not commute with each other.

We define a unit cell, in which a "unit flux" $2\pi/m$ penetrates. It is a direct analogy of the magnetic unit cell, where the magnetic unit flux $2\pi/e$ penetrates. A vortex has a "half unit flux" π/m and two vortices are contained in a unit cell. Assume that there are even numbers of vortices, and one may choose the unit cell as Fig. 1. Consider the translations in terms of the cell, $T_{\mathbf{e}'_x d} = T_{\mathbf{e}_x a + \mathbf{e}_y a}$ and $T_{\mathbf{e}'_y d} = T_{\mathbf{e}_x a - \mathbf{e}_y a}$, where $d = \sqrt{2}a$. One can see easily that the operators satisfy

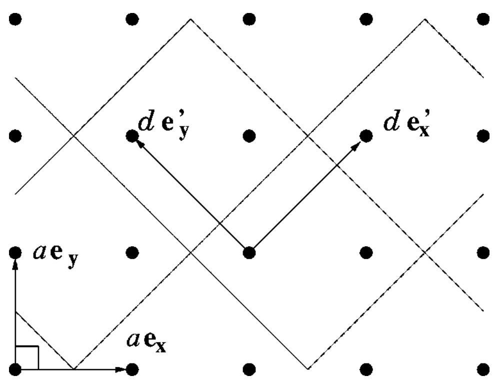

FIG. 1. The unit cells surrounded by dotted lines. Black dots denote the vortices.

$$[\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y}),T_{\delta\mathbf{r}}]=[T_{\mathbf{e}'_x d},T_{\mathbf{e}'_y d}]=0. \tag{11}$$

Therefore *the eigenstates of* $\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})$ *are in the Bloch state*, i.e.,

$$\Phi_{\mathbf{k}}(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}u_{\mathbf{k}}(\mathbf{x}), \tag{12}$$

where $\mathbf{k}$ is in the Brillouin zone (BZ), $-\pi/d\leqslant(k_x,k_y)\leqslant\pi/d$. Here, we omit the band index. Define

$$\mathcal{H}_{\mathbf{k}}(\mathbf{x},\mathbf{y})\equiv e^{-i\mathbf{k}\cdot\mathbf{x}}\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}. \tag{13}$$

From Eq. (7),

$$\mathcal{H}_{\mathbf{k}}(\mathbf{x},\mathbf{y})=\mathcal{H}(\hat{\mathbf{p}}+\mathbf{k},\mathbf{x},\mathbf{y})e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}. \tag{14}$$

Then, from Eqs. (8) and (12), one can see that the function $u_{\mathbf{k}}(\mathbf{x})$ satisfies

$$\int d^2y\mathcal{H}_{\mathbf{k}}(\mathbf{x},\mathbf{y})u_{\mathbf{k}}(\mathbf{y})=E_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{x}), \tag{15}$$

and its translation in terms of the unit cell satisfies a generalized Bloch condition[3]

$$u_{\mathbf{k}}(\mathbf{x}+\mathbf{e}'_x d)=\exp[id\mathbf{e}'_x\cdot m\mathbf{R}\tau_3]u_{\mathbf{k}}(\mathbf{x}),$$

$$u_{\mathbf{k}}(\mathbf{x}+\mathbf{e}'_y d)=\exp[id\mathbf{e}'_y\cdot m\mathbf{R}\tau_3]u_{\mathbf{k}}(\mathbf{x}). \tag{16}$$

The spectrum for lattice quasiparticles in the square vortex array of the p_x+ip_y-wave superconductors has been investigated by using the singular gauge transformation.[8,19] A zero energy state and gaps around it are found. In our situation, the continuum system is considered. There might be some problems to apply the singular gauge transformation approach to the continuum systems. It was pointed out that the quasiparticle spectrum depends on the choice of the singular gauge transformations.[21] Here, we assume the existence of an excitation gap.

IV. THE ADIABATIC PROCESS AND THE BERRY PHASE

Let us introduce a magnetic field, which is directed to the z axis and has a homogeneous gradient in the rotating frame, which is written $B_z(\mathbf{x})=\mathbf{x}\cdot\nabla B_z$ and $\boldsymbol{\nabla}B_z$ is a constant vector. The field will be a driving force of the spin transport. For a moment, we consider in the Lagrange formalism. In superfluid, the magnetic field couples to spin through the Zeeman term and does not couple to orbital currents. Then, the Lagrangian is written in the form

$$\mathcal{L}=\int d^2x\Psi^\dagger(\mathbf{x})\{i\partial/\partial_0-(\mathbf{x}\cdot\boldsymbol{\nabla}B_z/2)\}\Psi(\mathbf{x})$$

$$-\int d^2xd^2y\Psi^\dagger(\mathbf{x})\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})\Psi(\mathbf{y}),$$

$$\Psi(\mathbf{x})=(\psi_\uparrow(\mathbf{x}),\psi_\downarrow^\dagger(\mathbf{x}))^{\mathrm{T}}. \tag{17}$$

We consider a phase transformation of Eq. (17),

$$\Psi(\mathbf{x})\rightarrow\exp[-it\mathbf{x}\cdot\boldsymbol{\nabla}B_z/2]\Psi(\mathbf{x}). \tag{18}$$

Then, the term $-(\mathbf{x}\cdot\boldsymbol{\nabla}B_z/2)$ is absorbed and the Hamiltonian density operator is transformed as

$$\mathcal{H}(\hat{\mathbf{p}},\mathbf{x},\mathbf{y})\rightarrow\mathcal{H}(\hat{\mathbf{p}}-\mathbf{f}(t),\mathbf{x},\mathbf{y}), \tag{19}$$

where

$$\mathbf{f}(t)=t\boldsymbol{\nabla}B_z/2. \tag{20}$$

By using the analogy of U(1) (electromagnetic) gauge theory, we may regard $\mathbf{f}(t)$ as a vector potential that couples to the spin current, since it is introduced by the local spin rotation Eq. (18). We assume that $\mathbf{f}(t)$ changes adiabatically, i.e., $|\boldsymbol{\nabla}B_z|\ll1$. For simplicity, we write

$$\mathcal{H}(\hat{\mathbf{p}}-\mathbf{f}(t),\mathbf{x},\mathbf{y})\equiv\mathcal{H}(t,\mathbf{x},\mathbf{y}). \tag{21}$$

Then, we solve a time-dependent equation of motion with the adiabatic parameter $\mathbf{f}(t)$,

$$i\frac{\partial}{\partial t}\Psi(t,\mathbf{x})=\int d^2y\mathcal{H}(t,\mathbf{x},\mathbf{y})\Psi(t,\mathbf{y}). \tag{22}$$

We use the adiabatic approximation and an eigenvalue equation at fixed t is

$$\int d^2y\mathcal{H}(t,\mathbf{x},\mathbf{y})\Phi_{E(t)}(t,\mathbf{y})=E(t)\Phi_{E(t)}(t,\mathbf{x}). \tag{23}$$

Obviously, it is equivalent to the BdG equation (8) at $t=0$. The Hamiltonian $\mathcal{H}(t,\mathbf{x},\mathbf{y})$ has a spatial periodicity as well as $\mathcal{H}(t=0,\mathbf{x},\mathbf{y})$ because $\boldsymbol{\nabla}B_z$ is homogeneous. Then, eigensolutions are written in the Bloch form, i.e., $\Phi_{\mathbf{k}}(t,\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}u_{\mathbf{k}}(t,\mathbf{x})$ [see Eq. (12)]. The function $u_{\mathbf{k}}(t,\mathbf{x})$ obeys the equation

$$\int d^2y\mathcal{H}_{\mathbf{k}}(t,\mathbf{x},\mathbf{y})u_{\mathbf{k}}(t,\mathbf{y})=E_{\mathbf{k}}(t)u_{\mathbf{k}}(t,\mathbf{x}), \tag{24}$$

$$\mathcal{H}_{\mathbf{k}}(t,\mathbf{x},\mathbf{y})=\mathcal{H}_{\mathbf{k}-\mathbf{f}(t)}(\mathbf{x},\mathbf{y}), \tag{25}$$

and hence

$$u_{\mathbf{k}}(t,\mathbf{x})=u_{\mathbf{k}-\mathbf{f}(t)}(\mathbf{x}). \tag{26}$$

The solution of Eq. (22) in the adiabatic approximation is

$$\Psi_{\mathbf{k}}(t,\mathbf{x})=\exp\left(i\int_0^t dt'[E_{\mathbf{k}}(t')+\gamma_{\mathbf{k}}(t')]\right)\Phi_{\mathbf{k}}(t,\mathbf{x}),$$

$$\gamma_{\mathbf{k}}(t)=i\int_0^t dt'\left\langle \Phi_{\mathbf{k}}(t')\left|\frac{\partial}{\partial t'}\right|\Phi_{\mathbf{k}}(t')\right\rangle$$
$$=i\int_0^t dt'\left\langle u_{\mathbf{k}}(t')\left|\frac{\partial}{\partial t'}\right|u_{\mathbf{k}}(t')\right\rangle. \tag{27}$$

The reciprocal-lattice vector for the square vortex array is written $\mathbf{G}=(l\mathbf{e}'_x+n\mathbf{e}'_y)(2\pi/d)$ with integers l and n. We note that it is possible to compactify the Hamiltonian as $\mathcal{H}_{\mathbf{k}}(t,\mathbf{x},\mathbf{y})\sim\mathcal{H}_{\mathbf{k}+\mathbf{G}}(t,\mathbf{x},\mathbf{y})$, because they give the equivalent eigensolutions. Also we note that the parameter $\mathbf{f}(t)$ varies on the Brillouin zone [see Eq. (25)]. Therefore when $\mathbf{f}(t)//\mathbf{G}$ we have a period T for a closed loop on the parameter space.[20] For example,

$$T=4\pi/(|\boldsymbol{\nabla}B_z|d) \tag{28}$$

for $\mathbf{f}(t)//\mathbf{e}'_x,\mathbf{e}'_y$ and the Berry phases are defined as $\int_0^T dt\,\gamma_{\mathbf{k}}(t)$ for each case.[1] We introduce the Berry connection, $\mathbf{a}(\mathbf{k})=\langle u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}}\rangle$, which is a gauge field defined on the parameter space.[1] By using Eq. (26), the Berry phases for the parameter $\mathbf{f}(t)//\mathbf{e}'_x,\mathbf{e}'_y$ are written as

$$\Gamma_x(k_y)=i\int_0^{2\pi/d}dk_x a_x(\mathbf{k}) \tag{29}$$

and

$$\Gamma_y(k_x)=i\int_0^{2\pi/d}dk_y a_y(\mathbf{k}), \tag{30}$$

respectively.

When $\mathbf{f}(t)//\mathbf{G}$, we could write down the Berry phase as

$$\Gamma_{\mathbf{f}}=i\oint_{C(\mathbf{f})}d\mathbf{k}\cdot\mathbf{a}(\mathbf{k}), \tag{31}$$

where $C(\mathbf{f})$ is a closed loop on which $\mathbf{f}(t)$ moves. In general, the Berry phase depends on $C(\mathbf{f})$.[1]

V. SPIN QUANTUM HALL EFFECT AND THE BERRY PHASE

Let us calculate the spin current. In ^{3}He-A, the system is invariant under the spin rotation around the z axis $\Phi_{\mathbf{k}}(t)\rightarrow e^{i\theta}\Phi_{\mathbf{k}}(t)$ and $\mathcal{H}(t,\mathbf{x},\mathbf{y})\rightarrow e^{i\theta}\mathcal{H}(t,\mathbf{x},\mathbf{y})e^{-i\theta}$. The spin current $\mathbf{j}^s$ is defined by the spin conservation law,[22] i.e., $\dot{\rho}^s+\boldsymbol{\nabla}\cdot\mathbf{j}^s=0$, where ρ^s is the spin density $(2\pi)^2\rho^s(\mathbf{x})=(1/2)\Sigma_{n\leqslant 0}\int_{\mathrm{BZ}}d^2k\Psi^{\dagger}_{n\mathbf{k}}(\mathbf{x})\Psi_{n\mathbf{k}}(\mathbf{x})$ and we introduce the band index n. The label 0 denotes the zero energy. As we mentioned before, we assume an excitation gap, i.e., there are no partially filled bands. Then, the response of the spin current for the uniform field $\mathbf{f}(t)$ is

$$\langle\mathbf{j}^s(t)\rangle=\frac{1}{2}\sum_{n<0}\int_{\mathrm{BZ}}\frac{d^2k}{(2\pi)^2}\left\langle\Psi^{\dagger}_{n\mathbf{k}}(t)\left|\frac{1}{i}[\mathbf{r},\mathcal{H}(t)]\right|\Psi_{n\mathbf{k}}(t)\right\rangle$$
$$=\frac{i}{2}\sum_{n<0}\int_{\mathrm{BZ}}\frac{d^2k}{(2\pi)^2}\left[\left\langle\dot{u}_{n\mathbf{k}}(t)\left|\frac{\partial u_{n\mathbf{k}}(t)}{\partial\mathbf{k}}\right.\right\rangle-\mathrm{H.c.}\right]$$
$$=-\sigma^s_{xy}[\boldsymbol{\nabla}B_z\times\mathbf{e}_z], \tag{32}$$

$$\sigma^s_{xy}=\frac{1}{8\pi}\sum_{n<0}N^{(n)}_{\mathrm{Ch}},$$

where

$$N^{(n)}_{\mathrm{Ch}}=\int_{\mathrm{BZ}}\frac{d^2k}{2\pi i}[\boldsymbol{\nabla}_{\mathbf{k}}\times\mathbf{a}_n(\mathbf{k})]_z, \tag{33}$$

is the Chern number for the nth band and $\mathbf{a}_n(\mathbf{k})$ is equivalent to the Berry connection for the nth band. The detailed calculation is written, for example, in Ref. 7.

The Chern number takes an integer. The reason is based on the fact that $\mathbf{a}_n(\mathbf{k})$ is defined on the torus (the BZ) and the Chern number becomes finite if and only if $\mathbf{a}_{\mathbf{k}}$ has a nontrivial topology. The nature of the Chern number has been discussed in detail in Ref. 3.

Then, Eq. (32) shows that a stationary spin Hall current flows as an adiabatic spin transport and its conductivity is quantized as an integer multiple of $1/8\pi$. The same result for the conductivity in the vortex state of d-wave superconductors have been obtained.[9] The discrete conductance change is expected to occur when one varies Ω and p_{F}.

Here, we calculated the expectation value of the total spin current directly by using the adiabatic approximation and obtained the spin Hall conductivity. One can see the fact in the calculations of Eq. (32) that this approach is equivalent to calculate the Kubo formula for the spin Hall conductivity argued in Ref. 9.

Finally, we show that σ^s_{xy} could be written in terms of the Berry phase.[4–7] By using Stokes' theorem, we have the relation [see Eqs. (29), (30), (32), and (33)]

$$\sigma^s_{xy}=\frac{-1}{16\pi^2}\sum_{n<0}\left(\int_0^{2\pi/d}dk_x\frac{d\Gamma^n_y(k_x)}{dk_x}-\int_0^{2\pi/d}dk_y\frac{d\Gamma^n_x(k_y)}{dk_y}\right). \tag{34}$$

VI. RELATIONS TO THE OTHER ARGUMENTS

The effect has some similarity to the adiabatic pumping which is originally argued by Thouless and discussed actively at present.[10] In pumping, an adiabatic ac perturbation yields a dc current, and the charge transfer per a cycle is independent of the period of the perturbation. The charge transfer is quantized when the ac perturbation is commensurate with the lattice in one dimension. As we mentioned before, the Hamiltonian in our system $\mathcal{H}_{\mathbf{k}}(t)$ is compactified and moved periodically by the adiabatic parameter $\mathbf{f}(t)$. Then, the Hamiltonian changes ac-like, and the change yields a dc spin Hall current. To make a correspondence to the Thouless arguments, one calculates a spin transfer per period T. Assume that $\mathbf{f}(t)=\mathbf{e}'_y t|\boldsymbol{\nabla} B_z|/2$. The spin Hall current flows along the x' axis (see Fig. 1) and the spin transfer per the boundary of the unit cell along the y' axis is

$$\Delta S_z = d\int_0^T dt \langle j^s_{x'}(t)\rangle = -\sum_{n<0}\frac{N^{(n)}_{\mathrm{Ch}}}{2} \tag{35}$$

[see Eqs. (28), (32), and (33)]. The result does not depend on T . It comes from the fact that both the magnitude of the quantized current and T^{-1} are proportional to $|\boldsymbol{\nabla} B_z|$. We emphasize that the spin transfer is quantized, i.e., *the integral spin transfer occurs*. The result is analogous to the Thouless result.[10]

The value $\Delta S_z/d$ corresponds to the magnetization change per the period. From Eq. (34), the magnetization change is written by the Berry phase. Then, the present result is also similar to the spontaneous polarization of crystalline dielectrics, which is written by the Berry phase introduced by a closed adiabatic change of the Kohn-Sham potential.[11]

Essentially, the similarity comes from the fact that the effects argued here are caused by the closed adiabatic change in the Bloch states with the finite energy gap. A parallel discussion for the present arguments have been made in the Bloch electron systems in the presence of the electromagnetic field with respect to the charge transport.[7]

VII. SUMMARY AND DISCUSSIONS

In summary, we consider Bloch quasiparticles in a vortex state of superfluid ^{3}He-A in two dimensions with a rotation along the z axis. A magnetic field is along the z axis with a weak homogeneous gradient in the rotating frame. The field could be represented by an adiabatically changing vector potential which couples to the spin current. The adiabatic process is defined on a closed loop in the parameter space (the Brillouin zone) and generates a Berry phase. The spin Hall current flows in the process. We calculated the expectation value of the total spin current directly by using the adiabatic approximation and obtain the spin Hall conductivity. This approach is equavalent to calculate the Kubo formular for the spin Hall conductivity.[9] The conductivity is represented by the Chern number and quantized when the quasiparticle has an excitation gap as that in the d-wave vortex state.[9] We have shown that the spin Hall conductivity is written by the Berry phase. The spin transfer per a cycle per the boundary of the unit cell is quantized and related to the Berry phase. The results remind us of the adiabatic pumping, which was introduced by Thouless with respect to the charge transport.[10] The result is also similar to the relation between the spontaneous polarization and the Berry phase in the crystalline dielectrics.[11] Essentially, the similarity comes from the fact that the effects argued here are caused by the closed adiabatic change in the Bloch states with the finite energy gap. With respect to the charge transport, a parallel discussion has been made in the Bloch electron systems in the presence of the electromagnetic field.[7]

As we mentioned before, the spin quantum Hall effect in the vortex state of a $d_{x^2-y^2}$-wave superconductor has been pointed out,[9] but there seems to be some difficulty in making a parallel discussion of the *superconductors*. Because of the Meissner effect, it is not possible to have a magnetic field with a finite homogeneous gradient which is essential to define the adiabatic process on the closed loop in the parameter space. The vortex states in ^{3}He-A are suitable for our arguments because ^{3}He is the fermionic superfluid in which the spin current is well defined, i.e., the spin rotation symmetry around the z axis is retained. In contrast to the $d_{x^2-y^2}$-wave state, the spin quantum Hall effect occurs spontaneously in ^{3}He-A , i.e., one obtains a quantized spin Hall conductivity to calculate the Kubo formula in the absence of the vortices. The effect comes from the broken time-reversal symmetry and the broken parity in the orbital part of the pairing symmetry.[22] But the system does not have the finite spatial periodicity and we cannot make a parallel discussion also in this case.

We would like to comment on the fact that, in our argument, the orbital part of the pairing symmetry is not crucial as long as the quasiparticle spectrum in the vortex state has an excitation gap. The spin part is crucial because the spin rotational symmetry is needed to obtain well defined spin currents.

Several authors have made efforts to find a way to measure spin transport.[23] Some experimental techniques to detect spin transfer are highly desirable.

ACKNOWLEDGMENTS

The authors are grateful to K. Maki, M. Sato, Z. Tešanović, and F. Zhou for useful discussions.

*Present address: Max-Planck Institute for the Physics of Complex Systems, Noethnitzer Street 38, D-01187 Dresden, Germany.

[1] M.V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
[2] D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[3] M. Kohmoto, Ann. Phys. (N.Y.) **160**, 355 (1985).
[4] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983); G.W. Semenoff and P. Sodano, *ibid.* **57**, 1195 (1986).
[5] M. Kohmoto, J. Phys. Soc. Jpn. **62**, 659 (1993).
[6] J. Goryo and M. Kohmoto, J. Phys. Soc. Jpn. **71**, 1403 (2002).
[7] J. Goryo and M. Kohmoto, Phys. Rev. B **66**, 085118 (2002).
[8] M. Franz and Z. Tešanović, Phys. Rev. Lett. **84**, 554 (2000).
[9] A. Vishwanath, Phys. Rev. Lett. **87**, 217004 (2001); O. Vafek, A. Melikyan, and Z. Tešanović, Phys. Rev. B **64**, 224508 (2001).
[10] D.J. Thouless, Phys. Rev. B **27**, 6083 (1983); B.L. Altshuler and L.I. Glatzman, Science **283**, 1864 (1999).
[11] R.D. King-Smith and D. Vanderbilt, Phys. Rev. B **47**, 1651 (1993); R. Resta, Rev. Mod. Phys. **66**, 899 (1994).
[12] A.J. Leggett, Rev. Mod. Phys. **47**, 331 (1975); J.C. Wheatly, *ibid.* **47**, 415 (1975); D. Vollhardt and P. Wólfle, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990); G. E. Volovik, *Exotic Properties of Superfluid* ^{3}*He* (World Scientific, Singapore, 1992).
[13] O.V. Lounasmaa and E. Thuneberg, Proc. Natl. Acad. Sci. U.S.A. **96**, 7760 (1999).
[14] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd edition,

(Butterworth-Heinemann, Washington, DC, 1976).
[15] P.W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).
[16] N.R. Cooper and N.K. Wilkin, Phys. Rev. B **60**, R16 279 (1999).
[17] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Perseus Books, MA, 1966).
[18] J. Zak, Phys. Rev. **134**, A1602 (1964); D.R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[19] O. Vafek, A. Melikyan, M. Frantz, and Z. Tešanović, Phys. Rev. B **63**, 134509 (2001).
[20] J. Zak, Phys. Rev. Lett. **62**, 2747 (1989).
[21] L. Marinelli and B.I. Halperin, Phys. Rev. B **65**, 014516 (2002).
[22] G.E. Volovik and V.M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989); T. Senthil, J.B. Marston, and M.P.A. Fisher, Phys. Rev. B **60**, 4245 (1999); N. Read and D. Green, *ibid.* **61**, 10 267 (2000); J. Goryo and M. Sigrist, Europhys. Lett. **57**, 578 (2002).
[23] See, for example, J.E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).