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# Quantum Griffiths effects in metallic systems

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Elementary analytical extremal statistics arguments are used to analyze the possibility of quantum Griffiths effects in nearly critical systems with overdamped dynamics, such as arise in conventional theories of metallic quantum criticality. The overdamping is found to strongly suppress quantum tunneling of rare regions, leading to superparamagnetic rather than quantum Griffiths behavior. Implications for theories of non-Fermi-liquid behavior in heavy-fermion materials are discussed.

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## I. INTRODUCTION

The interplay of disorder and quantum criticality is a long standing and still open problem in condensed-matter theory. One aspect of this problem which has received considerable recent attention is the ''quantum Griffiths'' behavior which has been shown to occur near quantum critical points in certain model systems.[1–5] The model systems in which quantum Griffiths behavior has been unambiguously demonstrated all possess a crucial common feature, namely that in the absence of disorder the critical degrees of freedom exhibit dissipationless, Hamiltonian spin dynamics (indeed typically characterized by dynamical exponent $z=1$). However, many systems of experimental importance involve magnetic degrees of freedom coupled to conduction electrons,[6–9] and therefore *overdamped* dynamics implying a pure-system critical behavior characterized by $z>1$. Extension of the theory of quantum Griffiths behavior to this case is therefore an important issue. In a series of papers[10–12] Castro-Neto and Jones have argued from various points of view that such overdamped systems exhibit quantum Griffiths behavior similar to that exhibited by undamped systems, and they and others have further argued that this phenomenon is at the heart of the ''non-Fermi-liquid'' behavior observed in many heavy-fermion materials.[9,13]

In this paper we examine the issue of quantum Griffiths behavior in nearly critical systems exhibiting overdamped dynamics, finding that it is essentially nonexistent, being replaced instead by ''superparamagnetic'' behavior. The essence of our analysis is as follows: in undamped models quantum Griffiths effects arise from an interplay between the low probability of nucleating magnetic ''droplets'' in the paramagnetic state and a low but non-negligible quantum tunneling of these droplets. In a metallic, dissipative environment there is a strong suppression of tunneling by dissipation, so that the droplets which dominate the susceptibility behave more or less classically, leading to superparamagnetic behavior rather than quantum Griffiths behavior.

Our results amount to an implementation of ideas outlined in Ref. 14 and to a generalization, to a nonvanishing density of defects, of a previously reported analysis[15] of the ''magnetic droplet'' produced by a single, spatially localized defect, and rely heavily on the results of this previous work.

The method used to analyze the dynamics of a droplet is similar to that used in Ref. 11 and the broad qualitative features of the results we obtain are very similar to those obtained in that work. However, the specifics and the physical implications seem rather different. The issue is discussed in more detail in the conclusion.

The outline of this paper is as follows. In Sec. II we present the model and the method used in our analysis. In Sec. III we show that the approach reproduces results previously obtained in the dissipationless case. Section IV presents our new results concerning Griffiths-like behavior in systems with overdamped dynamics. Section V is a summary, comparison to other work, and conclusion, and is written so that readers uninterested in the details of the derivations may obtain from it the essence of our results.

## II. MODEL AND METHOD OF SOLUTION

### A. Model

The canonical quantum Griffiths problem concerns the effect of weak disorder added to a ''pure'' (nondisordered) system which possesses an Ising symmetry and is tuned to be near a quantum critical point. We consider a system in imaginary time and three spatial dimensions (differences occurring for two spatial dimensions warrant a separate treatment, which will be presented elsewhere). The model is described by the action

$$S=S_{\text{static}}+S_{\text{dyn}}+S_{\text{disorder}}, \tag{1}$$

with

$$S_{\text{static}}=\frac{E_0}{8\pi}\int_0^\beta d\tau\int\frac{d^3x}{\xi_0^3}\left[\frac{\xi_0^2}{\xi^2}\phi^2(x,\tau)+\xi_0^2[\nabla\phi(x,\tau)]^2+\frac{1}{2}\phi^4(x,\tau)\right]. \tag{2}$$

Here $\phi$ is a dimensionless scalar order parameter, $E_0$ is the basic energy scale of the theory (perhaps of the order of the mean Kondo temperature for a heavy-fermion system), $\xi_0$ is the basic length scale (typically of the order of a lattice constant), $\xi$ is the magnetic correlation length, and $\beta$ is the inverse temperature. It is convenient to define a parameter

 

$r=(\xi_0/\xi)^2>0$ which measures distance from criticality. We consider only parameters such that the pure system is in the paramagnetic phase.

We take the disorder to couple to the square of the order parameter via

$$S_{\text{disorder}}=\frac{E_0}{8\pi}\int_0^\beta d\tau\int\frac{d^3x}{\xi_0^3}V(x)\phi^2(x,\tau),\tag{3}$$

and assume it to be Gaussian distributed with correlator ($\langle\cdots\rangle$ represents average over configurations of the disorder)

$$\langle V(x)V(y)\rangle=V_0^2K\left(\frac{x-y}{\xi_0}\right),\tag{4}$$

where the kernel $K(u)$ decays on the scale $u\sim 1$ and satisfies $\int d^3uK(u)=1$. Because we are interested only in length scales $x-y>\xi_0$, we will take $K$ to be a $\delta$ function. The dimensionless quantity $V_0$ parametrizes the strength of the disorder. Weak disorder corresponds to $V_0\ll 1$.

The dynamic term $S_{\text{dyn}}$ is crucial to the quantum criticality described by Eq. (1) and to our subsequent discussions. We consider two cases:

(i) *dissipationless*, $z=1$ *dynamics*, as is usually assumed in studies of quantum Griffiths behavior, with

$$S_{\text{dyn}}^{(z=1)}=\frac{E_0}{8\pi}\int_0^\beta d\tau\int\frac{d^3x}{\xi_0^3}\left(\frac{\xi_0}{c}\right)^2\left(\frac{\partial\phi(x,\tau)}{\partial\tau}\right)^2.\tag{5}$$

Here $c$ is a characteristic velocity of the undamped excitations, such that $c/\xi_0$ is an energy presumably of the order of $E_0$.

(ii) *Hertz antiferromagnet*, $z=2$ *dynamics*, corresponding to the generic antiferromagnetic transition in a Fermi liquid,

$$S_{\text{dyn}}^{(z=2)}=S_{\text{dyn}}^{(z=1)}+\frac{T}{E_0}\sum_{\omega_n}\frac{|\omega_n|}{\Gamma}\int\frac{d^3x}{\xi_0^3}|\phi(r,\omega)|^2,\tag{6}$$

where

$$\phi(r,\omega_n)=E_0\int_0^\beta d\tau\,\phi(r,\tau)\,e^{i\omega_n\tau},\tag{7}$$

and $S_{\text{dyn}}^{(z=1)}$ provides an upper frequency cutoff if needed.

In these conventions the dynamics are dissipative (i.e., dominated by the $\Gamma$-term) if $\omega<\omega^*\equiv c^2/(\xi_0\Gamma)$ and nondissipative at higher frequencies. One expects in most systems (and finds, for example, in a weakly coupled Fermi liquid or in the slave boson theory of the Kondo lattice) that all scales are roughly equal: $E_0\approx c/\xi_0\approx\Gamma$.

## B. Method

### *1. Overview*

The dissipative term in Eq. (6) corresponds to a long-ranged interaction in time and renders available numerical methods prohibitively difficult to apply. To analyze the model defined by Eq. (1) we use simple analytical arguments modeled on those of Ref. 1. We note that the effective dimensionality of the model defined by Eq. (1) is $d_{\text{eff}}=d+z$. In this paper we consider only the spatial dimension $d=3$ so we are concerned only with models at and above the upper critical dimension $d_c=4$, so that quantal and thermal fluctuations of the order parameter in a fixed disorder configuration can be treated by an essentially mean-field approximation. The usual fluctuation analysis which justifies the mean-field approximation for $d_{\text{eff}}>d_c$ involves a translation-invariant model and fluctuations for which momentum is a good quantum number. Here we must deal with fluctuations in a system whose translation invariance is broken. These were investigated in Refs. 16 and 15 and were found not to affect the structure of the static mean-field solution when $d_{\text{eff}}\geqslant d_c$ (except for some insignificant changes in some constants).

As noted for example by Ref. 14, in the presence of the random potential, the crucial feature of the mean-field solution is the presence of *droplets*: regions in which the order parameter is locally nonvanishing. Quantum Griffiths effects then arise from dynamical fluctuations of these droplets; to study them one must estimate the droplet density and tunneling rate. We use statistical arguments and mean-field analysis to estimate the density and an adaptation to the present case of the analysis presented for a droplet produced by a single point defect in Ref. 15 to estimate the tunneling rate.

### *2. Probability for the existence of a droplet*

The assumption that the model is at or above its upper critical dimension means that mean-field theory is a good starting point.[15,16] We therefore consider static configurations, $\phi(x)$, which minimize the combination of Eqs. (2) and (3). These satisfy

$$\xi_0^2\nabla^2\phi(x)+r\phi(x)+\phi(x)^3=-V(x)\phi(x).\tag{8}$$

If $V(x)=0$, then because we assume $r>0$ the minimum corresponds to $\phi(x)=0$; however, regions in which $V(x)<0$ can lead to $\phi(x)\neq 0$. In the regions where $V(x)=\text{const.}<0$, $\phi(x)$ is roughly constant whereas in between these regions $\phi(x)$ decays exponentially. We refer to the regions where $\phi$ is not exponentially small as droplets. If the droplets are reasonably dilute, one may set $\phi=0$ in the exponential tail regions[16] and estimate the density of droplets of a given size and mean amplitude.

To motivate our estimate we first consider solving Eq. (8) if $V(x)=\overline{V}$ for $|x|<R$ and $V_0=0$ otherwise. A previous paper[15] considered a special case of this equation, with $V_0(x)=V\delta^{(d)}(x)$ and the solutions found in that work may easily be modified for the present case. In $d=3$ one finds that the solution is, roughly (and neglecting unimportant logarithmic factors in the $x$ dependence)

$$\phi(x)=\begin{cases}\phi_0 & \text{for}\quad x<R\\ \dfrac{\phi_0R}{x}e^{-(x-R)/\xi} & \text{for}\quad R<x.\end{cases}\tag{9}$$

In other words, the magnetic order induced by the region of attractive is roughly constant inside the region and decays outside it, as $1/x$ in a shell of width set by the correlation length and exponentially for larger lengths. Inserting the above ansatz, Eq. (9), into Eqs. (2) and (3), and minimizing the resulting action with respect to $\phi_0$ yields

$$-\overline{V}=\frac{\xi_0^2}{\xi^2}a(R/\xi)+\phi_0^2 b(R/\xi), \tag{10}$$

with $a(x)=1+3/x+3/x^2$ and $b(x)=1+3/(1+x)$. These particular forms for $a$, $b$ depend on the specific potential configuration studied (here $\overline{V}=\text{const}$ for $x<R$ and $\overline{V}=0$ otherwise) and on the variational approximation used; but we argue that a generic droplet is described by a similar equation with $a,b$ functions that vary on the scale $R/\xi\sim 1$ and which tend to unity as $R/\xi\to\infty$. Also in three dimensions $a(x)\sim 1/x^2$ as $x\to 0$ while $b(x)$ tends to a constant for $x<1$.[15] The precise forms of $a,b$ affect only nonuniversal details such as widths of crossover regions. In this paper we shall assume

$$a(x)=1+3x^{-2} \tag{11a}$$

$$b(x)=1 \tag{11b}$$

where the 3 arises from the difference in integrating a constant or $1/r^2$ over $r^2dr$.

One sees from Eq. (10) that in order to obtain a solution at all the average potential, $\overline{V}$, must be smaller than a (negative) $R$-dependent critical value,

$$V_c=-\frac{\xi_0^2}{\xi^2}a(R/\xi), \tag{12}$$

which tends to $\xi_0^2/\xi^2$ as $R\to\infty$ and to a number of order 1 as $R\to\xi_0$. As is evident from these formulas, the natural scale of the droplets is the correlation length $\xi$ that diverges as the quantum critical point is approached.

Equations (10) and (12) thus imply that one obtains a droplet in a region of linear dimension $R$ only if the average value $\overline{V}$ of the potential in that region is larger than a value of the order of $V_c(R/\xi)$ ($V_c$ is not an exact estimate because it pertains to the idealized disorder configuration discussed above). The standard estimate of the probability of a region of linear dimension $R$ with mean potential $\overline{V}$ is

$$P(R^3,\overline{V})\sim\frac{(R/\xi_0)^{3/2}}{\sqrt{\pi}V_0}\exp\left[-\left(\frac{R}{\xi_0}\right)^3\left(\frac{\overline{V}}{V_0}\right)^2\right], \tag{13}$$

and we therefore argue that the density $N(R^3,\phi_0^2)$ of droplets of amplitude $\phi_0^2$ and core size $R$, must be proportional to $1/V_0\exp(-R^3(\phi_0^2+V_c(R/\xi))/V_0^{22})$. This argument does not determine the preexponential factors (which involve, e.g., the issue of whether the region of size $R$ considered in Eq. (13) is part of a larger region which can sustain a droplet and numerical factors arising from the difference between idealized disorder configuration and typical one, which we have absorbed into $V_0$ and $V_c$). Because some of our subsequent considerations will require an estimate of the preexponential factors, we present the following arguments to fix them.

We begin by making a rough estimate of the fraction of sites contained in droplets (i.e., the fraction of sites having a $\phi_0^2>0$) as a function of distance from criticality. To do this we coarse grain the theory to the scale $\xi$. A given correlation volume $\xi^3$ will have a nonvanishing $\phi_0^2$ if the potential averaged over the droplet volume, $V_{\text{ave}}$, is larger than $V_c(\xi)\approx a(1)\xi_0^2/\xi^2$. From Eq. (13) we see that the probability $P_\phi$ that a given correlation volume will have a nonvanishing $\phi_0$, i.e., a $V<V_c(\xi)$, is (recall $V_c<0$)

$$P_\phi=\frac{1}{2}\left\{1-\text{erf}\left[\left(\frac{\xi}{\xi_0}\right)^{3/2}\frac{V_c(\xi)}{V_0}\right]\right\}, \tag{14}$$

where erf is the error function.

Clearly, a picture of independent droplets must break down if $P_\phi$ exceeds the percolation probability $P_{\text{perc}}$ at which the set of correlation volumes with nonvanishing $\phi_0$ percolate. Use of Eq. (14) and the estimate for three-dimensional cubic lattices $P_{\text{perc}}\approx 0.2$ shows that percolation will have occurred by the time $\xi$ exceeds $\xi_{\text{perc}}\approx 2.8a(1)^2/V_0^2$. ($\xi_{\text{perc}}$ is an underestimate because droplets larger than $\xi$ may occur). These estimates also show that the natural scale for $\xi$ is $V_0^{-2}$ and strongly suggest that the probability that a given site is in a droplet (of any size) is a function only of the combination $\xi V_0^2$.

We therefore argue that the prefactors in the droplet density must be such that the total probability of finding a site in a droplet, $P_{\text{tot}}=\xi_0^{-6}\int dR^3 d\phi_0^2 R^3 N(R^3,\phi_0^2)$ must be a function only of $\xi V_0^2$ and must be of the order of $P_{\text{perc}}$ when $\xi$ is of the order of $\xi_{\text{perc}}$. This implies

$$N[R^3,\phi_0^2]=\frac{R^{-9/2}}{C_{V_0}V_0}\exp\left(-\frac{R^3(\phi_0^2+V_c(R/\xi))_2}{V_0^2}\right), \tag{15}$$

where the factor of $R^{-9/2}$ ensures the correct scaling with $\xi$ and the numerical factor $C_{V_0}\approx 11.25$ ensures that when $\xi=\xi_{\text{perc}}$, we have $P=P_{\text{perc}}$. We emphasize that these formulas are phenomenological and must, in particular, break down when $\xi$ approaches $\xi_{\text{perc}}$.

It is convenient to adopt a dimensionless system of units in which

$$R=y\xi, \tag{16}$$

$$\phi_0=f\frac{\xi_0}{\xi}, \tag{17}$$

$$\xi=u\frac{\xi_0}{V_0^2}, \tag{18}$$

for which

$$N(y^3,f^2)dy^3df^2=\frac{y^{-9/2}}{C_{V_0}u^{1/2}\xi^3}\times\exp\left(-\frac{y^3[f^2+a(y)]^2}{u}\right)dy^3df^2. \tag{19}$$

The factor of $\xi^{-3}$ expresses the fact that if the probability of a given site being in a droplet is a function only of $u$, then the density of droplets must be smaller by an extra factor of the typical droplet volume $\xi^3$.

#### *3. Tunneling of the droplet for undamped, z=1, dynamics*

We now estimate the rate $\omega_{\text{tun}}$ at which a droplet characterized by the mean amplitude $\phi_0$ and length scale $R$ tunnels in the case of undamped, $z=1$ dynamics by performing a variational instanton calculation using Eqs. (2) and (5), and the solution Eq. (9). In the simplest estimate one assumes that the droplet maintains its shape while collapsing and reforming. To estimate the action associated with this process we write the droplet solution as

$$\phi(x,\tau)=\phi(x)\,\eta(\tau). \tag{20}$$

Substitution into Eqs. (2) and (5) leads to

$$S_{\text{inst}}=S_{\text{kin}}+S_{\text{barrier}}, \tag{21}$$

$S_{\text{kin}}$ involves the integral of $(\partial_t\phi)^2$ over the droplet and as noted in Ref. 15 involves the $1/r$ "tail" of the droplet in a crucial manner; in contrast, the cost $S_{\text{barrier}}$ of creating the instanton does not. One obtains

$$S_{\text{kin}}^{(z=1)}=C_{\text{kin}}\xi f^2y^3a'(y)\int\frac{d\tau}{E_0}\left(\frac{\partial\eta}{\partial\tau}\right)^2, \tag{22}$$

$$S_{\text{barrier}}=C_{\text{barrier}}\xi^{-1}f^4y^3b'(y)\int E_0d\tau[-2\eta(\tau)^2+\eta(\tau)^4]. \tag{23}$$

Here $C_{\text{kin}}$ and $C_{\text{barrier}}$ are nonuniversal constants. $C_{\text{kin}}$ involves the square of the ratio $E_0/(c/\xi_0)$ of the basic energy scale to the kinetic (or zone-boundary phonon) energy and $C_{\text{barrier}}$ is just a number. In the approximation we have employed, $C_{\text{kin}}=E_0^2\xi_0^2/C^2$ and $C_{\text{barrier}}=1$. The functions $a'=\int d^3x\,\phi_0(x)^2$ and $b'=\int d^3x\,\phi_0(x)^4$ are functions with behavior similar to $a,b$; in our explicit calculations we set $a'=a/3$ and $b'=1/3$ for simplicity; again different choices affect only nonuniversal details.

The action associated with one instanton may now be determined by a standard minimization of Eqs. (22) and (23) and is

$$S_{\text{inst}}^{(z=1)}=S_1d(y)f^3y^3. \tag{24}$$

For the present model in the present approximation the nonuniversal constant $S_1=\sqrt{C_{\text{kin}}C_{\text{barrier}}/3}$ and $d(y)=3\sqrt{a'(y)b'(y)}=\sqrt{a(y)}$, where the last equality follows from our simplifying assumptions $b'=1/3$, $a'=a/3$. The value of $S_1$ controls the width of the crossover regime before the universal behavior is reached, and is linearly proportional to $E_0\xi_0/C$.

The tunneling rate is then given by

$$\omega_{\text{tun},z=1}=\omega_0e^{-S_{\text{inst}}(\tau_0)}. \tag{25}$$

Here, $\omega_0$ is an attempt frequency presumably of order $E_0$ whose value is beyond the scope of this theory.

To conclude this section we briefly estimate the action associated with a different tunneling mechanism, namely, nucleation of a domain wall. For small droplets ("core size" $R$ less than $\xi$) the important process was shown to be collapse and reformation of the entire droplet.[15] We therefore need consider only the case $R\gg\xi$. We observe that by expanding about the static uniform solution one obtains a domain wall with width $W\sim\phi_0^{-1}$. The kinetic term associated with the domain-wall motion, therefore, has one fewer factor of the small quantity $\phi_0\sim f/\xi\sim V_0^2f$, leading to a larger action and hence a smaller rate, in the weak disorder, near criticality limit. We note in passing that for $\phi_0\sim 1$ the powers of $R$ will be the same as we have considered, but the extra factor of $\phi_0^{-1}$ will work in the other direction, favoring domain-wall motion.

#### *4. Tunneling of the droplet for overdamped, z=2, dynamics*

For overdamped dynamics two important differences occur. First, as shown in Refs. 15,17 the damping changes the action associated with a single instanton, strongly suppressing the *bare* tunneling rate relative to that found for undamped dynamics. Essentially, the tunneling is limited by the droplet's ability to move through a viscous medium rather than by its ability to climb over a barrier. Second, and much more important, the overdamped dynamics leads to a long-ranged (in time) instanton-instanton interaction, which reduces the tunneling rate further and indeed drives it to zero if the damping exceeds a critical value, as noted by previous authors.[11,15]

To calculate the effects of damping we insert the ansatz, Eq. (20) into Eq. (6). The new term arising from the overdamped dynamics is

$$S_{\text{diss}}=\frac{\gamma}{4}\int d\tau d\tau'\frac{d\eta}{d\tau}\frac{d\eta}{d\tau'}\ln\frac{(\tau-\tau')^2+\tau_m^2}{\tau_m^2}, \tag{26}$$

with $\tau_m$ a "microscopic" time of the order of $E_0^{-1}$. The net dissipative coefficient $\gamma$ is given for the Hertz antiferromagnet by[15]

$$\gamma=\frac{E_0}{4\pi\Gamma}\int\frac{d^3x}{\xi_0^3}\phi_0(x)^2=c_\gamma f^2y^3a''(y)\xi/\xi_0. \tag{27}$$

The approximations employed in the previous section imply that the nonuniversal constant $c_\gamma=E_0/\Gamma$ and $a''=a'$. In a generic system one expects all scales to be of the same order so that in particular $c_\gamma$ is expected to be of order unity.

The estimate of $\gamma$ is subject to the important caveat that the electron bath which causes the dissipation can penetrate the entire droplet. A reasonable estimate of the penetration depth $L_p$ may be obtained by dividing the electron velocity $v_F$ by the magnitude of the order parameter; in rescaled units $L_p/\xi \sim v_F/(E_0 f)$. We shall see below Eq. (48) that the parameters are such that the electrons can penetrate the entire droplet.

We have not been able to solve analytically for the instanton; instead we estimate the action by inserting the variational ansatz

$$\frac{d\eta}{d\tau}=\frac{2\Theta(\tau_0^2-4\tau^2)}{\tau_0} \tag{28}$$

into Eqs. (6),(23) obtaining $S=S_{\rm kin}+S_{\rm diss}+S_{\rm barrier}$ with

$$S_{\rm kin}=\frac{2C_{\rm kin}\xi y^3 a'(y) f^2}{E_0\tau_0\xi_0}, \tag{29}$$

$$S_{\rm diss}=2c_\gamma\frac{\xi}{\xi_0}y^3 a'(y) f^2 \ln(c_d\tau_0/\tau_m), \tag{30}$$

$$S_{\rm barrier}=\frac{2}{15}C_{\rm barrier}\frac{\xi_0}{\xi}y^3 b'(y) f^4 E_0\tau_0, \tag{31}$$

where $\ln(c_d)=\int_{-1/2}^{1/2}dx dy \ln[1+(x-y)^2]\approx 0.1152\ldots$.

Minimization over the instanton duration then leads to

$$1=\frac{c_\gamma}{C_{\rm kin}}\tau_0 E_0+\frac{C_{\rm barrier}}{15C_{\rm kin}}\frac{b' f^2\xi_0^2}{a'\xi^2}(\xi_0/\xi)^2(E_0\tau_0)^2. \tag{32}$$

As previously remarked, we expect the ratios of the various dimensional parameters to be of the order of unity; also, as we shall see below, in this problem the important droplets have $f\sim\xi^{-1/2}$, so that provided $\Gamma$ is less than a number of the order unity times $\xi_c/f\xi_0\sim\xi^{3/2}$ (within our approximations the precise numerical factor is $\sqrt{15}$) the $\tau_0^2$ term is negligible and one has

$$\tau_0=\frac{\Gamma\xi_0^2}{c^2} \tag{33}$$

and thus

$$S_{\rm inst}^{(z=2)}=c_\gamma C_2 f^2\xi\xi_0^{-1}y^3 a(y), \tag{34}$$

where $C_2$ ($=2.283\ldots$ in the present approximations) is a numerical factor of the order of unity arising from combining the factors in Eqs. (29) and (30).

We observe that for the value of $\tau_0$ given in Eq. (33), the term written in Eq. (34) is larger than $S_{\rm barrier}$ [Eq. (23)] by two powers of the correlation length (provided that the quantity $f$ is of order unity or less, as is the case for the situations considered here). Thus, in the metallic case and near to criticality, the difficulty in tunnelling arises from moving through the viscous medium, not climbing over the barrier. This result was noted previously.[15]

The bare tunnelling amplitude is thus

$$\omega_{\rm bare}^{(z=2)}=\omega_0 e^{-S_{\rm inst}^{(z=2)}} \tag{35}$$

and is much smaller than in the dissipationless case, because of the factor $f\xi$ in the argument of the exponential.

The standard macroscopic quantum tunnelling arguments[11,15,17] imply that the instanton-instanton interaction renormalizes the bare tunnelling rate so that if $\gamma<1$ then the $T=0$ tunnelling rate is

$$\omega_{\rm tun}=\omega_0\left(\frac{\omega_{\rm bare}}{\omega_0}\right)^{1/(1-\gamma)} \tag{36}$$

whereas if $\gamma>1$ tunnelling stops at $T=0$. We see from Eq. (27) that $\gamma$ is a strong function of the droplet size and amplitude; droplets which may tunnel (i.e., have $\gamma<1$) have a very weak amplitude even in rescaled units: $f\sim\xi^{-1/2}$.

Equation (36) is a zero-temperature result. At $T>0$ the "Caldeira-Leggett" renormalization is temperature dependent. The key question for this paper is the temperature at which $\omega_{tun}(T)<T$. If $\gamma>1$ then $\omega_{tun}(T)<T$ at all $T<E_0$, implying that the droplet behaves classically at all $T$. If $\gamma<1$ then the usual arguments shows that $\omega_{tun}(T)$ drops below $T$ when $T$ becomes greater than $\omega_{tun}(T=0)$, so that Eq. (36) gives the temperature scale separating a high-$T$ region, in which the droplet behaves classically, from the low-$T$ region, in which it behaves quantum mechanically.

## III. ESTIMATE OF QUANTUM GRIFFITHS BEHAVIOR

### A. Overview

The standard Griffiths estimate is that a droplet of magnetic moment $M_d=\int d^3r\, e^{i\vec{Q}\cdot\vec{r}}\phi_0(r)$ ($\vec{Q}$ is the ordering vector) and tunnelling frequency $\omega_{\rm tun}[R,\phi_0^2]$ gives rise to a susceptibility $\chi_d$ proportional to $M_d^2/(\omega_{\rm tun}+T)$. The susceptibility of a system with a distribution of droplets is then given by

$$\chi(T)=\int d^3R\, d^2\phi_0\frac{N(R^3,\phi_0^2)\, M_d^2[R,\phi_0]}{\omega_{\rm tun}[R,\phi_0^2]+T}. \tag{37}$$

For a droplet in an antiferromagnetic system, we find $M_d$ is a random function with magnitude $\phi_0 R$—the term proportional to $R$ comes from the boundary of the droplet, where the order parameter amplitude is dropping and the cancellation over one unit cell of the antiferromagnetic order is not complete. A different dependence would change prefactors but not affect our results crucially.

It is convenient to introduce an explicit integral over frequency, writing

$$\chi(T)=\xi^{-3}\int d\omega\frac{I(\omega)}{\omega+T}, \tag{38}$$

so that after conversion to dimensionless units we have

$$I(\omega)=\int dy^3 df^2[\xi^3 N(y^3,f^2)]f^2y^2\delta(\omega-\omega_{\rm tun}(y,f)). \tag{39}$$

The prefactor $\xi^{-3}$ in $\chi$ arises because each droplet has magnetic moment of the order of unity and the density of droplets is $\xi^{-3}$. The quantity $\xi^3 N$ has no explicit dependence on $\xi$ [see Eq. (19)].

We will use the $\delta$ function to eliminate the $f$ integral in $I$ and perform the integration over y either numerically or via an extremal value argument.

### B. z=1

Using Eq. (25) yields

$$f(\omega,y)=\left(\frac{\ln\left(\frac{\omega_0}{\omega}\right)}{S_1 d(y)}\right)^{1/3}\frac{1}{y} \tag{40}$$

Substituting this result into Eq. (35) yields

$$I(\omega)=\frac{2\ln^{1/3}(\omega_0/\omega)}{\omega S_1^{4/3}}\int_0^{\infty}\frac{\xi^3 N(y^3,f^2(\omega,y))dy}{d(y)^{4/3}}, \tag{41}$$

where $N(y,f(\omega,y))$ is $N(y,f)$ [Eq. (15)] with $f$ given by Eq. (36).

In the limit of very low frequency one may use asymptotic methods to analyze the integral in Eq. (37); the extremum is at

$$y_{\max}=\frac{\ln^{1/3}\left(\frac{\omega_0}{\omega}\right)}{\sqrt{3}S_1^{1/3}} \tag{42}$$

Substitution leads to

$$\chi(T)\sim\frac{1}{\xi^3 C_{V_0}}\frac{1}{T^{1-d_{\text{asympt}}}} \tag{43}$$

with (restoring units)

$$d_{\text{asymp}}(\xi)=\frac{16}{3\sqrt{3}S_1}\frac{1}{\xi V_0^2}. \tag{44}$$

This is the familiar quantum Griffiths result: if one is sufficiently close to the pure-system critical point $[d(\xi)<1]$ then the susceptibility diverges, with degree of divergence characterized by an exponent which approaches unity proportional to one power of the inverse correlation length.

Note that the prefactor in Eq. (39) rapidly vanishes as criticality is approached, so although the susceptibility diverges more strongly, the amplitude of the divergence decreases. Note further that in the asymptotic limit, $f\approx 1$ so that the mean order-parameter density (integrated order parameter divided by droplet volume) is of the order of $\xi^{-1}$. Thus the picture that emerges is of large, weak droplets.

We have evaluated

$$d_{\text{eff}}(\omega)=1+\frac{d\ln[I(\omega)]}{d\ln(\omega)} \tag{45}$$

via a numerical computation of Eq. (37). Figure 1 shows $d_{\text{eff}}(\xi,\omega)$ as a function of $\xi$ for $\omega=0.001\omega_0$ and several different values of the nonuniversal parameter $S_1$ (solid lines) along with the asymptotic limit estimates from Eq. (40). We observe that for these low frequencies and not too long $\xi$ the asymptotic limit provides a reasonable (but not perfect) estimate of the effective exponent: relative corrections are of the order of $[\xi/ln(\omega_0/\omega)^{2/3}]$. We see also that depending on the value of the nonuniversal parameter $S_1$, the effective exponent may remain above the critical value of unity (corresponding to a nondivergent susceptibility) until $\xi$ becomes of the order of $\xi_{perc}$. For $\xi$ of the order of $\xi_{perc}$ the standard quantum Griffiths approximation (independent droplets) breaks down, and one must deal instead with the critical singularities appropriate to a phase transition in a disordered system; in other words, with the still unsolved problem of the mixing of quantum critical and quantum Griffiths singularities.

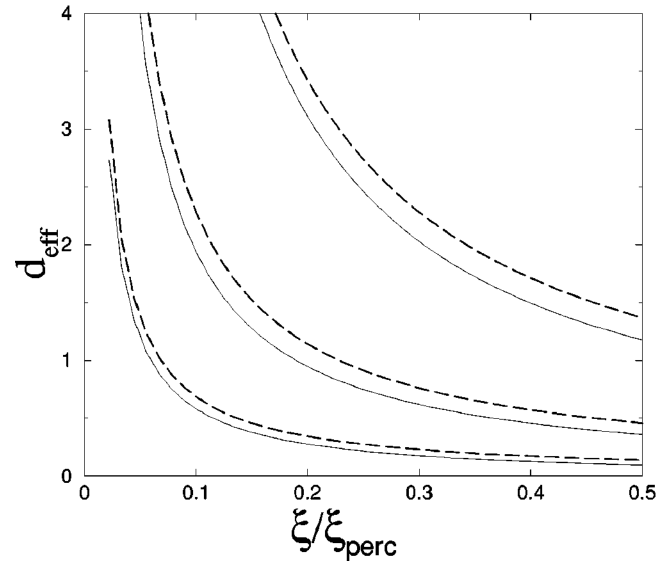


FIG. 1. Solid lines: calculated effective Griffiths exponent for undamped ($z=1$) case defined in Eq. (41) plotted vs correlation length normalized to correlation length $\xi_{perc}$ at which droplets percolate , with (from top to bottom) nonuniversal coefficient $S_1=.1,.3,1$ and frequency $\omega=10^{-3}\omega_0$. Dashed lines: asymptotic result [Eq. (40)] for same parameters.

### C. z=2

For overdamped dynamics, some droplets will have $\gamma>1$, and therefore will not tunnel at all at $T=0$. The function $I(\omega)$ will thus have a contribution proportional to $\delta(\omega)$ leading to the $1/T$ behavior expected of classical droplets. For those droplets which do tunnel we must use Eq. (32) in Eq. (35). We write

$$I(\omega)=I_0\delta(\omega)+I_{rest}(\omega), \tag{46}$$

with $I_{rest}$ given by Eq. (39) and $I_0$ by

$$I_0=\int dy^3 df^2 f^2 y^2 N(y^3,f^2)\Theta(\gamma(y,f)-1) \tag{47}$$

From Eq. (38) we see that if $I_0$ is appreciable, then $\chi\sim 1/T$: this is the superparamagnetism expected from essentially classical droplets.

We begin by estimating $I_0$. The $\Theta$ function limits the $f$ integration to

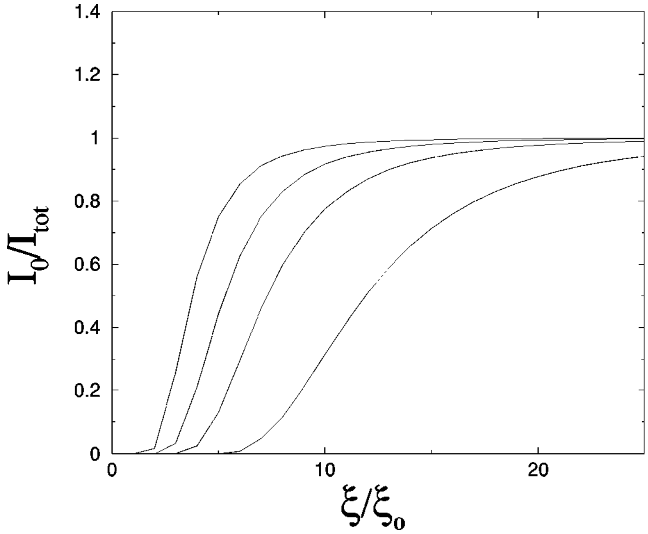


FIG. 2. Ratio of density of magnetization of nontunneling droplets $I_0$ [Eq. (43)] to total density of droplets $I_{tot}=\int d\omega I(\omega)$ for overdamped case and nonuniversal constant $c_\gamma=0.1$ (larger values of $c_\gamma$ lead to an $I_0/I_{tot}\approx 1$ even for much smaller values of $\xi$, as a function of correlation length *(not normalized to disorder strength)* for dimensionless disorder strength $V_0=1$, (top curve) .7,.5,.3. Note that for all reasonable parameters a non-negligible fraction of droplets do not tunnel at all.

$$f^2>f^2_{\min}(y)=\frac{\xi_0}{c_\gamma \xi y^3 a(y)} \tag{48}$$

Note that for large $\xi$, $f_{\min}\ll a(y)$. Further, the typical scale for $f$ is $\xi^{-1/2}$ so that the penetration depth $L_p$ of electrons into the droplet is large: $L_p/\xi\sim\xi^{1/2}$ so the assumption that electrons penetrate the droplet is indeed self-consistent.

Use of Eq. (48) in Eq. (47) gives

$$I_0(\xi)=\frac{3\sqrt{\pi}}{2C_{V_0}}\int_0^\infty y^{-2}dy\left\{\frac{\sqrt{u}\exp[-y^3(f^2_{\min}(y)+a(y))^2/u]}{\sqrt{\pi}y^{3/2}}\right.$$
$$\left.+a(y)\left[\operatorname{erf}\left(\frac{y^{3/2}(f^2_{\min}(y)+a(y))}{u^{1/2}}\right)-1\right]\right\}, \tag{49}$$

$I_0$, normalized to the total weight in $I$, $\int d\omega I(\omega)$ is plotted in Fig. 2 as a function of $\xi$ for different values of the disorder strength $V_0$. We see that the factor of $\xi^{-1}$ in Eq. (44) means that as criticality is approached, almost all of the weight in the droplet probability distribution is in droplets which do not tunnel.

For the droplets which are able to tunnel at frequency $\omega$, we find from Eqs. (34) and (36) that

$$f^2_{z=2}(y)=\frac{\xi_0}{c_\gamma \xi y^3 a(y)}\,\frac{\ln\left(\frac{\omega_0}{\omega}\right)}{C_2+\ln\left(\frac{\omega_0}{\omega}\right)}. \tag{50}$$

Note that in contrast the expression for $f$ in the $z=1$ case shown in Eq. 39, in the $z=2$ case, $f$ does *not* diverge as $\omega\to 0$. As in the $z=1$ case considered above, one obtains an expression for $I_{\rm rest}(\omega)$ by substituting the result for $f$ into Eq. (35) yielding

$$I_{\rm rest}(\omega)=\frac{3\xi_0^3}{\omega C_V c_\gamma^3 \xi^3}\,\frac{C_2\ln\left(\frac{\omega_0}{\omega}\right)}{\left[C_2+\ln\left(\frac{\omega_0}{\omega}\right)\right]^3}$$
$$\times\int\frac{dy y^{-1/2}\exp\left[-\frac{y^3(f^2_{z=2}(y)+a(y))^2}{u}\right]}{[y^3a(y)]^2} \tag{51}$$

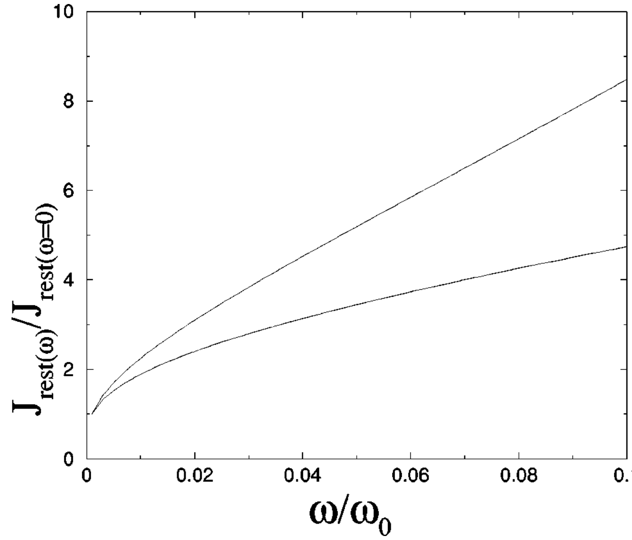


FIG. 3. Contribution $J_{rest}(\omega)=\omega I_{rest}(\omega)$ [Eq. (47)] of tunneling droplets to susceptibility integral, plotted vs frequency for nonuniversal constants $V_0=0.5$, $C_2=1$, and $c_\gamma=0.1$ at $\xi=5$ (top curve) and $\xi=20$ (bottom curve).

The resulting expression is to good accuracy proportional to $1/\omega$ times logarithms. The physics is that even the average of droplets which are able to tunnel is dominated by those droplets on the verge of freezing, leading again to a superparamagnetic contribution to the susceptibility.

We have numerically evaluated the integral in Eq. (51) for parameters such that $I_0$ is not too large. Sample results are shown in Fig. 3, which plots the quantity $J_{rest}=\omega I_{rest}$. The frequency dependence is a consequence of the logarithmic factors in Eq. (51); the nonvanishing intercept as $\omega\to 0$ means that up to logarithmic corrections the contribution to the susceptibility arising from this term is $\sim 1/T$.

## IV. CONCLUSION

This paper presents an investigation of the possibility of quantum Griffiths effects in three-dimensional metallic system near an antiferromagnetic quantum critical point with Ising symmetry. For comparison we present also a parallel investigation of quantum Griffiths effects in a model of an insulating system near a similar critical point. The key feature of metallic systems is the dissipative dynamics arising from the particle-hole continuum of electrons; in the model insulating system the dynamics are undamped. Comparison of the two calculations shows that dissipation suppresses quantum Griffiths effects completely, leaving instead an effectively superparamagnetic behavior.

A simple precis of our results follows. Quantum Griffiths effects are a consequence of randomness: essentially, in a random system which is on average in the paramagnetic phase, regions (''droplets'') may occur in which the randomness pushes the system locally to the ordered side of the phase diagram, so that local formation of an order parameter is favored. In certain circumstances (first noted by McCoy[2]) these droplets may dominate the response. In this situation one may approximately write the susceptibility $\chi$ as an average over droplets times a susceptibiltiy for each droplet, i.e.,

$$\chi=\int_{\text{droplets}} P(\text{droplet})\chi_{\text{droplet}}. \tag{52}$$

We have used simple extremal statistics arguments (similar to those used by Thill and Huse[1]) to estimate the droplet probability distribution $P(\text{droplet})$ and an extension of earlier work which studied a particular class of droplets[15] to obtain the susceptibility $\chi_{\text{droplet}}$ of a given droplet. We were then able to perform the average over droplets and obtain an estimate for the susceptibility.

This method reproduces the essential features of the standard results for quantum Griffiths effects in undamped (insulating) systems, namely, that the low $T$ behavior of the susceptibility is governed by a new exponent $d_{\text{eff}}$ given by the product of the inverse correlation length $\xi^{-1}$ and inverse mean-square disorder amplitude $V_0^{-2}$ and a nonuniversal number (which we estimate for the particular model we consider). A divergent susceptibility results when $d_{\text{eff}}$ becomes less than unity, and the results are functions only of $\xi V_0^2$. We note one additional interesting finding. The standard arguments which produce the standard quantum Griffiths results are based on a picture of dilute droplets and apply only if the $\xi$ is not too large (otherwise the droplets percolate, and an isolated droplets picture fails). For the model we consider we obtain an estimate for the critical value of $\xi$, and find that depending on the value of the nonuniversal factor in $d_{\text{eff}}$, droplets may reach the percolation point before the Griffiths exponent drops below unity. In other words, in the models we consider the existence of a quantum Griffiths regime (which one may somewhat imprecisely define as a divergent susceptibility arising from isolated droplets) is not guaranteed—it may or may not occur depending on the value of a nonuniversal coefficient. Sufficiently near a critical point a regime of divergent susceptibility does of course occur, but the proper theory of this regime would have to go beyond the model of isolated droplets and treat correctly the mixing of critical and Griffiths singularities.

We also found that for systems near antiferromagnetic critical points the *amplitude* of the divergent term in the susceptibility vanishes rapidly as criticality is approached, indeed as $\xi^{-3}$, essentially because each relevant droplet has a magnetic moment of the order of unity and as criticality is approached the droplets get larger in size but fewer in number.

The main new result of our work, however, pertains to metallic systems with overdamped (dissipative) dynamics. For these systems (i.e., for quantum critical phenomena in metals) the answer is entirely different. The physics in the undamped case is a balance between the probability of a droplet occurring (which vanishes rapidly as the droplet size or amplitude increases) and $\chi_{\text{droplet}}$, which is of the order of the inverse of the quantum tunneling rate of the droplet and diverges rapidly as the droplet size or amplitude increases. The effect of dissipation is to strongly decrease the tunneling rate, and indeed to drive it to zero for droplets larger than a particular, amplitude dependent, size. For relevant parameters we find that a nonvanishing density of droplets does not tunnel at $T=0$; these give rise to a superparamagnetic ($\chi \sim 1/T$) susceptibility rather than a quantum Griffiths (continuously varying exponent) behavior. For those droplets which do behave quantum mechanically, the effect of dissipation on the tunneling rate is found to change the balance between probability and $\chi_{\text{droplet}}$ dramatically. We find that even considering only the droplets which can tunnel quantum mechanically, those which dominate the integral for $\chi$ are those which are right on the edge of classical (nontunneling) behavior, leading again to superparamagnetism rather than to quantum Griffiths behavior. We also find that the dependence on parameters is different: in the undamped case, apart from prefactors the mean-square disorder strength $V_0^2$ and the correlation length enter via the combination $\xi V_0^2$. In the damped case additional factors of $\xi$ occur which drive the system more rapidly to classical behavior.

Our results cast doubt on the claims[9–12] that quantum Griffiths effects are important in heavy-fermion materials, which are precisely three-dimensional metals with Ising symmetry, typically near antiferromagnetic quantum critical points. Ref. 9 contains a phenomenological description of data. If the theoretical results presented here are accepted, then these data require a different, non-Griffiths interpretation. Reference 10 argued that a disordered system near a quantum critical point could be mapped onto the *disspationless* Ising model in a transverse field; the results of the present paper and of Ref. 15 indicate on the contrary that dissipation is essential.

Reference 12 uses a novel variant of a technique introduced by Dotsenko[18] to study essentially the same model as is studied here but obtaining a rather different result, namely, that quantum Griffiths effects can be important in a reasonable range of the phase diagram even in the metallic case. We outline the differences between the results found here and those of Ref. 12. The method introduced by Dotsenko[18] and used by Ref. 12 begins from a *classical* theory defined by a functional integral with action given by the static term in Eq. (1) and evaluates the disorder average by the replica method. Whereas other workers[19,16] then used the replicated field theory to derive scaling equations for variables including the mean disorder strength, Dotsenko argued that one should look for spatially localized energetically *unstable* configurations of the replicated field theory, which correspond to local maxima of the replicated action and are to be identified with the droplets discussed above. Dotsenko shows that the leading nonanalytic contribution to the free energy in the vicinity of an assumed $T>0$ critical point comes from droplets with size of the order of the magnetic correlation length, $\xi$; we refer to these henceforth as typical droplets. The authors of Ref. 12 assume that the $T\to 0$ limit of this classical theory may be straightforwardly taken, and then add to this theory estimates of the dynamics of typical droplets. The results reported in Ref. 12 disagree in a number of specific details (for example, concerning the tunnelling rates of droplets of given size) with the results presented here. The most important difference, however, is in the interpretation of the results. Ref. 12 argues that one should identify the boundary of the Griffiths region with the value of $\xi^{-2}$ at which a typical droplet ceases to tunnel. Our analysis, which involves averaging over all droplets, indicates that independent of whether the typical droplet (however, defined) may tunnel, the susceptibility is dominated by droplets which are at or beyond the edge of ceasing to tunnel; these give an essentially superparamagnetic ($\chi\sim 1/T$) behavior, instead of the continuously varying exponent characteristic of quantum Griffiths behavior.

Reference 11 presents a detailed analysis of a different model in which spins are added to a pure system which itself is far from any critical point. In this model the phase transition is *disorder driven:* it occurs when the density of added spins is high enough that these order; whereas our interest here has been in models in which even the nondisordered system is near a critical point. Furthermore, in the model studied in Ref. 11, the way the disorder is introduced means that the local spin amplitude $\phi_0$ [cf. our Eq. (9)] is always of order unity, whereas in our treatment the local spin amplitude may be considerably smaller. An approximate mapping between the model considered in Ref. 11 and the one considered here may be obtained by setting our parameters $\phi_0$ and $\xi$ equal to unity and considering the behavior as the disorder strength $V_0$ is increased (whereas we consider a fixed $V_0$ and study the behavior as $\xi$ is increased).

Although specific details differ, in a broad qualitative sense results obtained in Ref. 11 are similar to those obtained here. In particular, Ref. 11 states that at sufficiently low temperature dissipation will suppress the quantum Griffiths behavior. However, Ref. 11 argued that an extremely wide temperature regime could exist in which behavior characteristic of the undamped system occurs, whereas in the model we consider, for any reasonable parameters there is no such temperature regime. A crucial point is that Ref. 11 focused on model parameters such that the damping coefficient was extremely weak (i.e., in our notations [see below Eq. (27)] they took $c_\gamma \ll 1$). In this limit, it is plausible that there is a temperature regime in which behavior characteristic of the undamped model may occur, before finally a crossover occurs to a regime (similar to the one we considered) in which damping is important. Important avenues for future investigation include more detailed studies of the crossovers between the weak-damping and order unity damping cases and between the disorder driven–criticality effects studied in Ref. 11 and the pure system criticality–driven effects studied here, as well as determination of the damping coefficient values appropriate to the heavy fermion materials of interest.

Our work has the following implications for experiment. First, the canonical quantum Griffiths effects are due to weak disorder added to a pure critical point. We have shown that in the limit of weak disorder and a pure critical point described by the Hertz theory,[20,21] the dissipation characteristic of metallic systems changes the quantum Griffiths singularities into a kind of superparamagnetic behavior. In other words, as a matter of principle the canonically defined quantum Griffiths behavior should *not* be observable in metals near magnetic quantum critical points. This suggests that claims[9] to have observed quantum Griffiths behavior in heavy-fermion systems should be treated with caution (at least for systems with Ising symmetry). Further, we showed that the droplets that dominate the susceptibility can tunnel only when the system is not close to criticality, and in these cases the droplet size is not much larger than the basic scale of the theory. Thus if the susceptibility is dominated by the tunneling of droplets, the picture that emerges is more similar to the Kondo disorder picture of[22,23] than it is to the conventionally defined quantum Griffiths picture. Indeed, the experimental claims involve heavy-fermion systems where the interaction which favors a nonmagnetic phase is the Kondo effect. As noted by many authors,[23] the fact that Kondo temperatures are exponentially sensitive to system parameters means that a slight variation in system parameters can lead to a wide variation in Kondo temperatures. The canonical assumption of weak disorder which we and others[1,4,12,2] have made, may not be valid for these systems. The interplay between quantum criticality and a broad distribution of disorder should be treatable by the methods introduced here, and seems worth examining.

A second point is that the very slow dynamics of the droplets makes it much easier for them to order. Further, in a metallic system the droplet-droplet interactions are of long range (see, e.g., Ref. 24 for a discussion in the context of the two-dimensional metal insulator transition). For this reason we expect that in the presence of disorder the actual phase transition at which long ranged order sets in is an essentially classical affair, in which droplets lock together when the temperature becomes lower than some droplet-droplet coupling.

A third point, perhaps relevant beyond the present context, is that (as seen for example in Eq. (30)] dissipation can have a crucial effect on bare tunneling rates: in the metallic problem we considered the cruical impediment to tunneling of a droplet was found to be the viscosity of the medium, not the energy barrier that had to be surmounted.

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