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Dynamic localization in continuous ac electric fields

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We present a detailed examination of the properties of ac electric fields that result in the dynamic localization of electrons in periodic potentials. Although exact dynamic localization can only occur if the ac electric fields are discontinuous at all changes in sign, we show that it is surprisingly tolerant to smoothing these discontinuities. We also show that within the nearest-neighbor tight-binding limit, all symmetric ac electric fields are guaranteed to exhibit dynamic localization for some field amplitude provided that the vector potential and electric field are never zero simultaneously. Finally, we derive an area theorem for symmetric electric fields with only two zeros per period.

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I. INTRODUCTION

In recent decades, the study of electrons in a periodic potential in the presence of ac and dc electric fields has attracted consistent attention. This system was discussed for a dc field many years ago by Bloch[1] and Wannier.[2] This early work examined the existence of localized stationary states—the Wannier-Stark ladder (WSL)—and their dynamic analog, Bloch oscillations (BO's). More recently, in combined ac and dc fields, such effects as multiphoton absorption,[3] absolute negative conductance,[4] fractional WSL's,[5] and *dynamic localization* (DL) were predicted.

In this paper we investigate the electric fields that lead to DL. DL refers to the phenomenon whereby the application of a periodic purely ac field results in the periodic localization of an initially localized wave packet. This effect was initially addressed by a number of authors,[6–8] who found that in a one-band, nearest-neighbor tight-binding (NNTB) model, DL occurs when a sinusoidal field with the correct amplitude is applied. We refer to DL in the NNTB limit as *approximate dynamic localization* (ADL). It has also been found that ADL arises for triangular waves;[9] however, the general properties of the fields that lead to ADL are not known.

In some cases, DL arises in a one-band model beyond the NNTB approximation, i.e., for an arbitrary band dispersion. We refer to this type of dynamic localization, which occurs not just in the NNTB limit, as *exact dynamic localization* (EDL). A number of authors[10–12] have shown that EDL can occur for periodic square-wave ac fields. In a recent Letter[12] we derived the conditions for an ac electric field to achieve EDL and showed how such fields may be constructed. A central result of that work is that *EDL can only occur for fields with discontinuities that occur whenever the electric field changes sign*. Due to the requirement of discontinuities, it is not possible to obtain these fields *exactly* in experiment. It is thus important to quantify just how much smoothing of the discontinuities can be tolerated if good dynamic localization is to be achieved.

The WSL and BO were experimentally observed via photoinjection of electron-hole pairs in biased semiconductor superlattices.[13,14] More recently, equivalent phenomena were observed in trapped atomic systems[15,16] and using light propagating in fiber Bragg gratings[17] and coupled optical waveguides.[18,19] However, the experimental evidence for DL is less persuasive. Perhaps the strongest evidence is found in the reduction in the dc current in biased doped superlattices when a THz field of the correct amplitude and frequency is applied.[4] A key difficulty in electronic systems is obtaining direct evidence of the localization when there is no dc field.

One of the most promising systems in which to observe DL is a periodic array of coupled waveguides. The array plays the role that quantum wells play in a superlattice, whereas a thermal gradient,[18] a gradient in the structural parameters of the guides,[19] or a waveguide curvature[20] play the role of the electric field. Once the required waveguide structure has been fabricated, the experiments are fairly straightforward;[18,19] dynamic localization is observed as a return of the optical intensity to the guide into which the light beam was initially injected. An attractive feature of this optical system is that the time variable in the electronic systems maps onto a spatial variable; these are therefore continuous-wave experiments, in which BO's take the form of the light snaking back and forth upon propagation. Any "ac field" can, in principle, be constructed, for example, by a suitable variation of waveguide curvature with propagation distance.[20] The curved-waveguide optical system[18,19] is probably the most promising system for achieving nearly discontinuous "fields," as the discontinuities are obtained simply via an abrupt change in the waveguide curvature.

In this paper we investigate the effect on EDL of smoothing field discontinuities. We also examine the nature of the fields that lead to ADL in systems where the NNTB limit is a good approximation. It has been established[21,22] that the effect of coupling to higher bands on DL is small for modest fields and sufficiently large energy gaps between the bands; we thus use a one-band model, where the band dispersion is arbitrary. We present the theory in the language of electrons in a periodic potential. However, as mentioned, there are optical and atomic systems for which the mathematical treatment is essentially identical.

The paper is organized as follows. In Sec. II, we summarize the theory of EDL,[12] and discuss it in the context of this work. In Sec. III we examine the effects of smoothing the discontinuities in fields that yield EDL. In Sec. IV we quantify the dependence of the electron localization on smoothing. In Sec. V we examine which ac fields are guaranteed to exhibit ADL. In Sec. VI, we develop an area theorem for large-amplitude symmetric continuous fields. Section VII contains our conclusions.

II. EXACT DYNAMIC LOCALIZATION

We consider an electron in a one-dimensional periodic potential of period d, in an external purely ac electric field $E(t)$ of period τ. Using a one-band model, we expand the electronic wave function in the basis of the Wannier functions $|a_n\rangle$ as $|\psi(t)\rangle=\Sigma_n B_n(t)|a_n\rangle$.[10] Using this in the Schrödinger equation, we find for the expansion coefficients after ℓ periods of the ac field[12]

$$B_n(\ell\tau)=e^{-i\varepsilon_0\ell\tau/\hbar}e^{-i\gamma(\tau)}\sum_m A_{n-m}(\ell\tau)B_m(0), \quad (2.1)$$

where ℓ is a positive integer,

$$\gamma(t)=\frac{ed}{\hbar}\int_0^t E(t')dt' \quad (2.2)$$

is the dimensionless vector potential, and

$$A_m(\ell\tau)\equiv\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\,\exp\left[imx-i\sum_{p\neq 0}\frac{\ell\tau\varepsilon_{-p}}{\hbar}S_p e^{ipx}\right], \quad (2.3)$$

where ε_{-p} are the Fourier coefficients of the band dispersion and

$$S_p\equiv\frac{1}{\tau}\int_0^\tau e^{-ip\gamma(t')}dt'. \quad (2.4)$$

It can be shown[12] that the condition for EDL for an arbitrary band structure reduces to $S_p=0$, for $p\neq 0$. Thus, *EDL only occurs if* $S_p=0$ *for all* $p\neq 0$.[12]

In some situations, $|\varepsilon_p|\ll|\varepsilon_1|$ for all $|p|>1$ and it may be sufficient to use the NNTB approximation for the band structure. Under these conditions, we only require $S_1=0$. For instance, the ac field considered most often in discussions of DL is sinusoidal: $E(t)=E_0\sin(\omega t)$, where $\omega\equiv 2\pi/\tau$ and τ is the period. For this field,

$$|S_p|=\left|J_0\left(p\frac{\Omega}{\omega}\right)\right|, \quad (2.5)$$

where $\Omega\equiv edE_0/\hbar$. For exact dynamic localization, we require that $S_p=0$ for all p, but this is impossible for any Ω/ω. The usual argument now is to assume the NNTB approximation for the band structure; the condition $S_1=0$ then reduces to the usual condition for dynamic localization in a sinusoidal field: $J_0(\Omega/\omega)=0$. Thus a sinusoidal field can only yield DL in the NNTB limit.[12,23,24] In Sec. V we show that a similar condition for ADL exists for almost any symmetric ac field.

The condition for EDL can be expressed somewhat differently by defining

$$f(y)\equiv\sum_{p=-\infty}^{\infty}S_p e^{ipy}, \quad (2.6)$$

where $-\pi<y<\pi$. But if $S_p=0$ for all $p\neq 0$, then $f(y)=1$ for $-\pi<y<\pi$. This condition for EDL can be shown to be equivalent to requiring that[12]

$$\sum_{m,j}\frac{1}{|\dot{\gamma}(t_{jm})|}=\frac{\tau}{2\pi}, \quad (2.7)$$

where the t_{jm} are defined by the equation

$$\gamma(t_{jm})=y_{jm}-2\pi m, \quad (2.8)$$

where m is an integer and $0<t_{jm}<\tau$. As discussed previously,[12] this can be used to construct fields that yield EDL. Because the left-hand side of Eq. (2.7) diverges if $\dot{\gamma}(t_{jm})=0$, one key result is that *only electric fields that are discontinuous every time they change sign can yield EDL*.[12] We refer to these as *essential discontinuities*.

We now consider the area under $E(t)$ between neighboring essential discontinuities at times t_i and t_{i+1}:

$$\mathcal{A}=\int_{t_i}^{t_{i+1}}E(t')dt'. \quad (2.9)$$

Because $\gamma(t)$ is the dimensionless area under the electric field curve and essential discontinuities in $\gamma(t)$ only occur when $\gamma(t_i)=2\pi n$,[12] we obtain the following area theorem: *if EDL is to occur, then the area between essential discontinuities must satisfy the condition*

$$\mathcal{A}=\frac{2\pi\hbar}{ed}n, \quad (2.10)$$

where n is an integer.

It is clear that any purely ac field that yields EDL must have at least two discontinuities: one at each sign change of the field. As we wish to minimize the number of discontinuities, henceforth we only consider fields with two discontinuities. There are many such field types yielding EDL: for example,[12]

$$E(t)=\mathcal{E}_n[(1-b)^2+8bt/\tau]^{-1/2} \quad \text{for} \quad 0<t<\tau/2, \quad (2.11)$$

where

$$\mathcal{E}_n\equiv\frac{4\pi n\hbar}{ed\tau}, \quad (2.12)$$

$E(t+\tau/2)=-E(t)$, n is a positive integer, and $|b|<1$. This last condition ensures that the field does not diverge. When $b=0$, this corresponds to a square wave that is symmetric about $t=\tau/4$, with min[E]=max[E]=$\mathcal{E}_n$. The fields given by Eq. (2.11) are shown in Fig. 1 for $n=1$ and different b val-

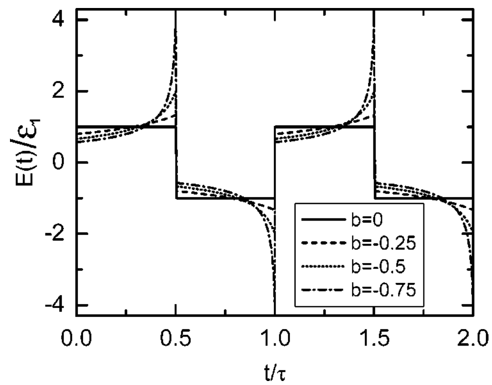

FIG. 1. The $n=1$ family of electric fields from Eq. (2.11) that yield EDL. Different curves correspond to different b as indicated in the legend.

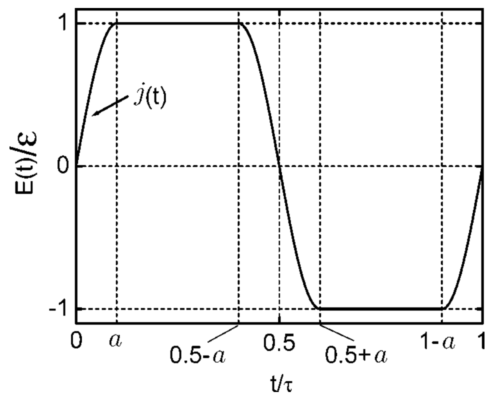

FIG. 2. Smoothed $n=1$ symmetric square wave from Eq. (3.1). The smoothing function is $j(t)=\sin(\pi t/2)$.

ues. The area under the curve between discontinuities is independent of b, but the size of the discontinuity increases monotonically with $|b|$. In fact, it can be shown that all fields leading to EDL have at least one discontinuity with a magnitude that is greater than or equal to that of the $n=1$ symmetric square wave.[12]

III. EFFECT OF FIELD SMOOTHING ON EDL

As discussed in Sec. II, a necessary condition for EDL is that the applied electric field have discontinuities. However, in experiments the field cannot be made truly discontinuous, and it is thus of interest to study the effect of smoothing the discontinuities, thereby making the field continuous. Here we study this issue for the field given by Eq. (2.11) for $n=1$. The $n=1$ square-wave case is the most important, as the sum of the magnitudes of the discontinuities is smallest of all fields with period τ. We obtain analytic results for this square-wave field $(b=0)$ and present numerical results for the fields with $b\neq 0$.

Consider the $n=1$ symmetric square-wave electric field. To describe the smoothing of the discontinuities we introduce a function $j(t)$ with $j(0)=0$ and $j(1)=1$. We scale $j(t)$ to span the discontinuities of the field as shown in Fig. 2. Parameter a measures the degree of smoothing, where $0\leq a\leq 1/4$; note that when $a=1/4$ there is no constant-field section and that when $a=0$ the original square wave is recovered. According to Fig. 2, between 0 and $a\tau$ the original function is replaced by $j(t)$, suitably rescaled, and the other discontinuities are smoothed similarly. The field $E(t)$ thus becomes

$$E(t)=\begin{cases}\mathcal{E}j\left(\dfrac{t}{a\tau}\right), & 0\leq t\leq a\tau,\\[2ex] \mathcal{E}, & a\tau\leq t\leq \dfrac{\tau}{2}-a\tau,\\[2ex] \mathcal{E}j\left(\dfrac{\tau-2t}{2a\tau}\right), & \dfrac{\tau}{2}-a\tau\leq t\leq\dfrac{\tau}{2},\end{cases}\tag{3.1}$$

where $E(t+\tau/2)=-E(t)$ and $\mathcal{E}\neq\mathcal{E}_n$ in general.

To calculate the S_p we determine $\gamma(t)$ using Eqs. (2.2) and (3.1). Then, using Eq. (2.4) we find after some work

$$|S_p|=4a\left|\int_0^1\cos(p\Omega\tau\{a[J(1)-J(t)-1]+1/4\})dt + \frac{4}{p\Omega\tau}\sin[p\Omega\tau(1/4-a)]\right|,\tag{3.2}$$

where $\Omega\equiv ed\mathcal{E}/\hbar$ and

$$J(x)\equiv\int_0^x j(y)dy.\tag{3.3}$$

We now take $\mathcal{E}=\mathcal{E}_1$, and hence $\Omega\tau=4\pi$, deferring the more general case for the moment. As we shall see, this is a good approximation if a is not too large. The expression for $|S_p|$ then reduces to

$$|S_p|=\left|\frac{1}{p\pi}\sin(4\pi pa) -4a\int_0^1\cos\{4\pi ap[J(1)-1-J(t)]\}dt\right|.\tag{3.4}$$

This expression is easy to evaluate for a given $j(t)$ but it is difficult to draw general conclusions from it. Since we are interested in small a, we Taylor expand the right-hand side of Eq. (3.4). It is straightforward to see that this series only contains odd powers of a and that the lowest-order term is of order a^3. Since the next term is of order a^5, this leads only to a small correction to the leading contribution. To lowest order, then, we obtain

$$|S_p|=32\pi^2p^2a^3I,\tag{3.5}$$

where

$$I\equiv\left|\int_0^1[J(1)-1-J(t)]^2dt-\frac{1}{3}\right|.\tag{3.6}$$

This result is remarkable and shows that for modest p and for modest values of a, the $|S_p|$ remain very small. For example, for a smoothed square wave in which each smoothed discontinuity takes 20% of the ac period τ, $a=0.05$, so that $|S_p| \simeq 0.04p^2I$, with, as we discuss shortly, I typically around 0.1. Of course the a^3 dependence causes the $|S_p|$ to decrease quickly with decreasing a. It can be shown that the first term in Eq. (3.6) is never smaller than the second; equality occurs for $j(t)=1$ as required, since in this limit the perfect square wave is obtained.

Though Eq. (3.2) is general, subsequently we took $\Omega\tau = 4\pi$ for simplicity. This corresponds to setting $\mathcal{E}=\mathcal{E}_1$, so the field amplitude is not adjusted when the square wave is smoothed. Here we drop this restriction and take $\Omega\tau = 2(2\pi+\varepsilon)$, where $\varepsilon \ll 1$. This means that the electric field amplitude is adjusted to $\mathcal{E}=\mathcal{E}_1[1+\varepsilon/(2\pi)]$. Taking $\varepsilon \ll 1$, $|S_p|$ can be expanded in powers of ε and a, to obtain an expression of the form

$$|S_p(a,\varepsilon)| = 32\pi^2p^2Ia^3 + \varepsilon/2\pi + C\varepsilon^2 + Da^2\varepsilon + Ga^5 + \cdots, \tag{3.7}$$

where C, D, and G are expansion coefficients, and there is no term in εa. Setting $S_p=0$, we see that to lowest order $\varepsilon = -64\pi^3p^2Ia^3$. We draw two conclusions from this. First, the C, D, and G terms in Eq. (3.7) are of order a^5 or higher. Since $a \ll 1$, we drop these terms and only include the first two. Second, since the first term has a p dependence, we may choose $\mathcal{E}$ to change the values of the $|S_p|$, but we can only set $S_p=0$ for a single p. The natural choice is $\varepsilon = -64\pi^3Ia^3$, for which $S_1=0$ and

$$|S_p| = 32\pi^2(p^2-1)a^3I. \tag{3.8}$$

Although all $|S_p|$ become somewhat smaller, only $S_1=0$. The conclusion that smoothing of the field destroys EDL is of course consistent with the general EDL results.[12]

Thus, for small a, the deviation of the field amplitude from the EDL value $\mathcal{E}=\mathcal{E}_1$ for which $S_1=0$ is very small. Thus, if the smoothing is not too drastic, one can simply use the amplitude specified by the EDL results. This, of course, is not true for strong smoothing, as is discussed in Sec. VI.

The S_p can be evaluated analytically for any $j(t)$ that is a polynomial in t. For example, for $j(t)=t^\nu$, we find

$$I = \frac{1}{(\nu+1)^2}\left(\nu^2 + \frac{2\nu}{\nu+2} + \frac{1}{2\nu+3} - \frac{(\nu+1)^2}{3}\right). \tag{3.9}$$

Thus, for $\nu \to 0$, $I=0$ as expected, whereas when $n\to\infty$, we have $I=2/3$; for $\nu=1$, $I=2/15$. It is easy to construct odd polynomials $j_m(t)$ of degree m such that the first $(m-1)/2$ derivatives of the electric field are zero and thus continuous at $t=1$; we use such polynomials in Sec. VI. For example, for $m=5$,

$$j_5(t) \equiv \frac{15}{8}t - \frac{5}{4}t^3 + \frac{3}{8}t^5, \tag{3.10}$$

we have $j_5'(1)=j_5''(1)=0$ and $I=50/1287 \simeq 0.039$.

The results of S_p calculations are given in Fig. 3 for two smoothing function $j(t)\equiv j_1(t)=t$ and $j(t)=j_{\sin}(t) \equiv \sin(\pi t/2)$. Results are given for both the series expansion and the numerical evaluation of Eq. (2.4). There is clearly

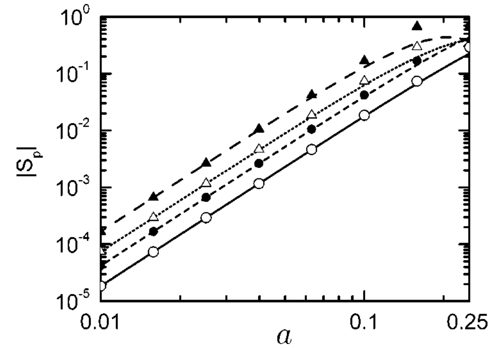

FIG. 3. $|S_p|$ vs a for the smoothed $n=1$ square wave. Analytic result for $j(t)=t$ are given by solid circles $(p=1)$ and by solid triangles $(p=2)$; associated numerical results from Eq. (2.4) are given by short-dashed $(p=1)$ and long-dashed $(p=2)$ curves. Analytic result for $j(t)=\sin(\pi t/2)$ are given by open circles $(p=1)$ and open triangles for $(p=2)$; associated numerical results are given by the solid $(p=1)$ and dotted $(p=2)$ curves.

excellent agreement for small a, but for $a \gtrsim 0.1$ the power series result is somewhat inaccurate, as expected. For $j_{\sin}(t)$, we find that $I \simeq 0.0586$ and thus S_p is smaller than for $j_1(t)$. It can be shown that this is associated with the fact that for the sine function, the first derivative of the field is continuous. This is also consistent with the even smaller value of I found for $j_5(t)$.

Figure 4 gives numerical results for the fields of Eq. (2.11) with $b\neq 0$. As a decreases, the S_p initially decrease as a^3, consistent with Eq. (3.5). However, for small a all S_p have the same modulus, which decreases linearly with a. This behavior appears to be typical when $b\neq 0$. As $b\to 0$, the value of a where the crossover occurs becomes increasingly smaller as does the value of $|S_p|$. We thus conclude that the square wave $(b=0)$ has advantages over all other members of family (2.11) in their behavior under smoothing. This is not surprising since, as discussed in connection with Fig. 1, the discontinuities are smallest for the square wave.

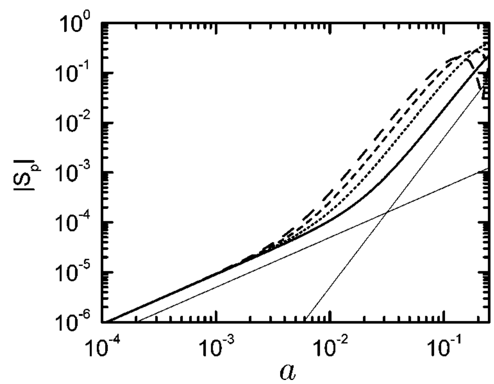

FIG. 4. $|S_p|$ vs a for the smoothed field in Eq. (2.11) with $n=1$ and $b=0.39$ and smoothing function $j(t)=\sin(\pi t/2)$. Thick lines are for $p=1$ (solid line), $p=2$ (dotted line), $p=3$ (short-dashed line), and $p=4$ (long-dashed line). Thin lines are reference curves: $y(a)=0.005a$ and $y(a)=5a^3$.

IV. LIMITS ON ACCEPTABLE SMOOTHING

Now that we have the dependence of the S_p on the degree of smoothing, we determine the dependence of the electron localization on the S_p. We use this to determine the degree to which we may depart from the ideal discontinuous fields before DL is effectively destroyed.

For simplicity and without loss of generality, we take the initial condition $B_n(0)=\delta_{n,0}$, so the electron is initially in one Wannier state. Then, at time t, the probability of the electron being in the nth Wannier state is $P_n(t)=|A_n(t)|^2$ [see Eq. (2.3)]. We use this to determine the probability of the electron being in a state other than $n=0$ at time $t=\ell\tau$. Considering only fields that nearly yield EDL, we know that $|\tau\varepsilon_{-p}S_p/\hbar|\ll 1$. A first-order expansion of Eq. (2.3) in this small quantity then gives

$$A_m(\ell\tau)\simeq\delta_{m,0}-i\sum_{p>0}\frac{\ell\tau\varepsilon_p}{\hbar}\{S_p\delta_{m,-p}+S_p^*\delta_{m,p}\}, \quad (4.1)$$

where we have used the fact that ε_p is real and $\varepsilon_p=\varepsilon_{-p}$. If we define $P(n\neq 0)$ to be the probability of finding the electron not in the $n=0$ Wannier state at time $t=\ell\tau$, we obtain

$$P(n\neq 0)\simeq 2\sum_{p>0}\left[\frac{\varepsilon_p\ell\tau}{\hbar}\right]^2|S_p|^2. \quad (4.2)$$

This can be used to determine the S_p to achieve a specified degree of DL. In fact, if our field is a smoothed square wave with amplitude such that $S_1=0$ [Eq. (3.8)], then Eq. (4.2) gives

$$P(n\neq 0)\simeq 2\sum_{p>1}\left[\frac{\varepsilon_p\ell\tau}{\hbar}32\pi^2 I(p^2-1)a^3\right]^2. \quad (4.3)$$

From this, we can determine the acceptable limits on the smoothing parameter a for any band structure and evolution time $\ell\tau$. As an example, we consider a $GaAs/Ga_{0.7}Al_{0.3}As$ superlattice with a well width of 8.5 nm, a barrier width of 1.5 nm, a conduction-band offset of 400 meV, and an electron mass of 0.067 free electron masses.[12] For this superlattice, the first three band-dispersion Fourier coefficients are $\varepsilon_1\simeq -5.75$ meV, $\varepsilon_2\simeq 0.54$ meV, and $\varepsilon_3\simeq 0.083$ meV. Because $|(p+1)^2\varepsilon_{p+1}|/|p^2\varepsilon_p|\ll 1$, only the first few terms in Eqs. (4.2) and (4.3) are important. This inequality holds generally for symmetric superlattices. Requiring that there be at most a 10% probability of finding the electron outside its initial state after 5 ps—which is considerably longer than the usual dephasing time—we obtain the requirement $|S_2|<0.056$. Though this might seem rather stringent, for $j_{\sin}(t)$, this corresponds to $a<0.1$. This represents a large degree of smoothing, as up to 40% of the period may be taken up by the rise and fall of the field. In contrast, for a pure sine wave, when $S_1=0$, $|S_2|=0.238$, and hence for the same structure we find $P(n\neq 0)\simeq 1.9$. Clearly the probability is now so large that expression (4.2) breaks down. The poor performance of the sine field was demonstrated numerically in Fig. 1 of Dignam and de Sterke.[12]

V. DL IN THE NNTB APPROXIMATION: ADL

We noted in Sec. II that *continuous* ac fields cannot yield EDL. However, such fields can yield localization in the NNTB approximation (ADL); the sinusoidal field from Sec. II is an example. Here we examine the types of fields for which ADL occurs and derive the general properties of continuous ac electric fields that yield ADL.

We write the field as $E(t)=E_0 g(t)$, where $E(t)$ has period τ and E_0 is the field amplitude. Thus from Eq. (2.2) we have $\gamma(t)=\Omega h(t/\tau)$, where $\Omega\equiv edE_0/\hbar$ and

$$h(x)\equiv\int_0^x g(\tau x')dx'. \quad (5.1)$$

We choose $g(t)$ and E_0 such that $|h(x)|\leqslant 1$ for all x, with $\max[|h|]=1$. Using these definitions, we obtain

$$S_1=\int_0^1 e^{-i\Omega h(x)}dx. \quad (5.2)$$

We now determine the electric fields that guarantee that $S_1=0$ for some amplitude Ω. Evaluating S_1 exactly requires knowledge of the precise functional form of $h(x)$. However, using the method of stationary phases,[25] we can study the general properties of S_1 and in many cases obtain an excellent approximation for it. The details of the application of this method of evaluating S_1 are given in the Appendix and we just summarize the results here.

Let us assume that $h'(x)=0$ for $x=\{x_\ell\}$, where $\ell=1,\ldots,N$ and $0<x_\ell\leqslant 1$, and let n_ℓ be the order of the lowest-order nonzero derivative of h at $x=x_\ell$. Now, classify the zeros according to whether the (n_ℓ)th derivative $h^{(n_\ell)}(x_\ell)$ is positive or negative. This derivative is positive at the points $x_\ell^>$ $(\ell=1,\ldots,N_1)$ and negative at the points $x_\ell^<$ $(\ell=N_1+1,\ldots,N)$. We then obtain the asymptotic expansion

$$S_1\sim\left\{\sum_{\ell=1}^{N_1}\frac{2\Gamma\left(\frac{1}{n_\ell}\right)\exp[-i\Omega h(x_\ell^>)]R_{n_\ell}(e^{i\pi/2n_\ell})}{n_\ell|\Omega h^{(n_\ell)}(x_\ell^>)|^{1/n_\ell}}+\sum_{\ell=N_1+1}^{N}\frac{2\Gamma\left(\frac{1}{n_\ell}\right)\exp[-i\Omega h(x_\ell^<)]R_{n_\ell}(e^{-i\pi/2n_\ell})}{n_\ell|\Omega h^{(n_\ell)}(x_\ell^<)|^{1/n_\ell}}\right\}, \quad (5.3)$$

where $\Gamma(x)$ is the gamma function and

$$R_{n_\ell}(z)\equiv\begin{cases} z & \text{if } n_\ell \text{ is even,}\\ \mathrm{Re}(z) & \text{if } n_\ell \text{ is odd.}\end{cases} \quad (5.4)$$

Attaining the condition $S_1=0$ for some Ω for a general function $h(x)$ is difficult since the real and imaginary parts of S_1 must vanish simultaneously using only one degree of freedom (Ω). Thus, let us restrict ourselves to functions $F(t)$ for which S_1 is purely real (or can be made real) for all Ω. It is easily shown that S_1 is real if $E(t)$ is symmetric about some time t_0, i.e. $E(t_0+t)=E(t_0-t)$; we refer to these as *symmetric fields*. The sine field is an example, with $t_0=\tau x_0=\tau/4$. For these symmetric fields, we have $\gamma(t_0+t)=\gamma(t_0)-\gamma(t_0-t)$, and thus using the periodicity of $\gamma(t)$, for every maximum in $h(x)$ at $x_\ell^>$ there is a corresponding minimum at $x_\ell^<=\mathrm{mod}[2x_0-x_\ell^>,1]$ with $h(x_\ell^>)=h(x_0)-h(x_\ell^<)$. Using periodicity, we are free to choose $t_0=0$, and since $E(t)$ is the physical field, we can choose $\gamma(t)$ such that $\gamma(t_0)=0$. Then, $h(x_\ell^>)=-h(x_\ell^<)$, and we obtain for a general symmetric field

$$S_1\sim 2\sqrt{2\pi}\mathrm{Re}\left\{\sum_{\ell=1}^{N/2}\frac{R_{n_\ell}(e^{i\pi/2n_\ell})}{n_\ell|\Omega h^{(n_\ell)}(x_\ell^>)|^{1/n_\ell}}\exp[-i\Omega h(x_\ell^>)]\right\}. \tag{5.5}$$

Using this result, we now show that for most symmetric fields we find $S_1=0$ for some Ω. Equation (5.5) can be written as

$$S_1\sim\sum_{\{n_\ell\}}\frac{1}{\Omega^{1/n_\ell}}G_{n_\ell}(\Omega), \tag{5.6}$$

where the sum is over the zeros with different derivative orders (n_ℓ) and

$$G_m(\Omega)\equiv\sum_{\ell=1}^{2N_m}C_\ell^m\exp[-i\Omega\omega_\ell^m], \tag{5.7}$$

where $\omega_\ell^m=h(x_\ell^{m,>})$ with $x_\ell^{m,>}$ being the position of the N_m maxima for which the first nonzero derivative of $h(x)$ is order m and

$$C_\ell^m\equiv\frac{\sqrt{2\pi}R_m(e^{-i\pi/2m})}{m|h^{(m)}(x_\ell^>)|^{1/m}}\quad\text{for}\quad 1\leqslant\ell\leqslant N_m, \tag{5.8}$$

with $C_{\ell+N_1}^m=C_\ell^{m*}$. The C_ℓ^m are bounded complex coefficients and $-1\leqslant\omega_\ell^m\leqslant 1$.

From the mean-value theorem of almost periodic functions,[26] the mean value of $G_m(\Omega)$ is given simply by $M[G_m]=2\,\mathrm{Re}\{C_0^m\}$, where C_0^m is the coefficient of the exponential with frequency $\omega_0^m=0$ (if this exists) and the mean is defined by

$$M[G_m]\equiv\lim_{T\to\infty}\frac{1}{T}\int_q^{q+T}G_m(\Omega)d\Omega, \tag{5.9}$$

which is independent of q.[26] Thus for functions $G_m(\Omega)$ for which $C_0^m=0$ $[\gamma(\tau x_\ell^{m,>})\neq 0]$ for all m, we have that $M[G_m]=0$. In addition, from Parceval's theorem for almost periodic functions,[26] we know that

$$M[|G_m|^2]=2\sum_{\ell=1}^{N_m}|C_\ell^m|^2>0. \tag{5.10}$$

Furthermore, since $G_m(\Omega)\leqslant 2\Sigma_{\ell=1}{}^{N_1}|C_\ell|$, we know that the $G_m(\Omega)$ are bounded. Now, because the mean value [Eq. (5.9)] is independent of q, we see from Eq. (5.6) that for large Ω, only the G_m with the largest m (denoted by m') contributes significantly to S_1. Thus, for large Ω,

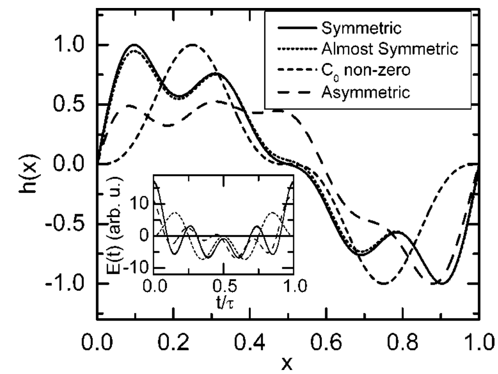

FIG. 5. Function $h(x)$ for four different fields: symmetric (solid line), almost-symmetric (dotted line), symmetric with $C_0^2\neq 0$ (short-dashed line), and asymmetric (long-dashed line). Inset shows the associated electric fields.

$$S_1\sim\frac{1}{\Omega^{1/m'}}G_{m'}(\Omega). \tag{5.11}$$

We know that $M[G_{m'}]=0$ *and* $M[|G_{m'}|^2]>0$. This can only be true if $G_{m'}(\Omega)$ is positive over some ranges of Ω and negative over others. Since $G_{m'}(\Omega)$ is continuous, $G_{m'}(\Omega)=0$ for some $\Omega>0$. From Eq. (5.11) we therefore also know that $S_1=0$ for some $\Omega>0$. Thus, we have the central result of this section: *For any continuous symmetric ac field, ADL occurs for at least one value of the field amplitude E_0 as long as the vector potential γ is nonzero whenever the electric field is zero.*

To demonstrate the differences in the likelihood of finding amplitudes that yield ADL for various symmetric and asymmetric electric fields, we consider S_1 for the four fields shown in Fig. 5. One of the fields is symmetric, another is almost symmetric, a third is symmetric but has one of the $C_0^2\neq 0$, and a fourth has no symmetry.

In Fig. 6(a) we plot S_1 and the asymptotic expansion for S_1 versus Ω for the symmetric field. Note that for a symmetric field there are many field amplitudes that yield ADL. In

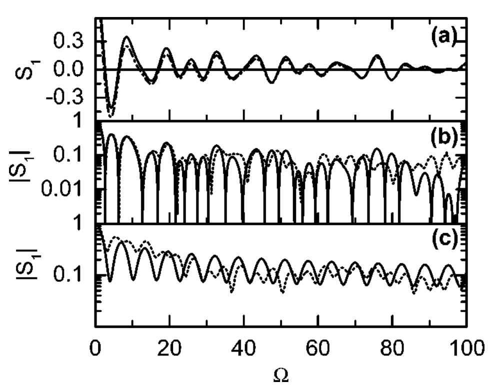

FIG. 6. Dependence of S_1 on Ω for the fields in Fig. 5: (a) S_1 for the symmetric field using the exact evaluation (solid line) and the asymptotic expansion (dot-dashed line); (b) $|S_1|$ for the symmetric (solid line) and almost symmetric (dotted line) fields; (c) $|S_1|$ for the asymmetric (solid line) and symmetric field with $C_0^2\neq 0$ (dotted line).

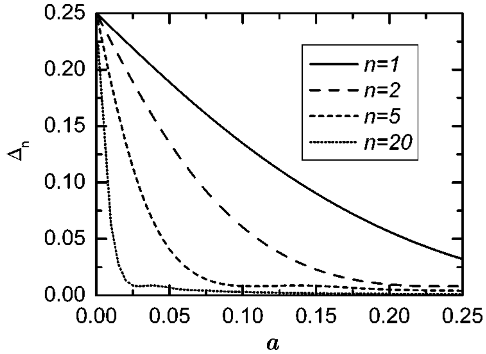

FIG. 7. Area deviation Δ_n vs a for the smoothed symmetric square wave with $j(t)=j_5(t)$. Different curves correspond to different n as indicated.

addition, these amplitudes do not generally have to be much larger than $\mathcal{E}_1$. The asymptotic expansion gives good estimates for the zero crossings, once $\Omega \gtrsim 20$. The agreement is even better for functions that are less steep such as the sine wave. In Sec. VI, we discuss the reason for this.

In Fig. 6(b), we plot $|S_1|$ for the symmetric and almost-symmetric fields. Note that the slight asymmetry has a minor effect for small Ω: though the $|S_p|$ no longer go exactly to zero, their minima occur at essentially the same Ω as the zeros of the symmetric field and for the first three minima $|S_1|<0.024$. Using Eq. (4.2) we see that for a system with a bandwidth of 10 meV, within NNTB, we only require $|S_1|<0.037$ to have a 90% of finding the electron in its initial state after 5 ps.[27] However, the asymmetry has a significant effect for large Ω, reducing greatly the number of Ω for which $|S_1|<0.037$. Thus, having a small deviation from a symmetric field does not greatly affect the ADL for low-amplitude fields (small n), but this is not true for large-amplitude fields.

Finally, in Fig. 6(c), We plot $|S_1|$ for the asymmetric field and the field for which $C_0^2 \neq 0$. For both fields, $|S_1|$ never drops below 0.04 for the Ω plotted. The first asymmetric-field curve demonstrates the importance of the symmetry of the field. The results for the $C_0^2 \neq 0$ field demonstrates the importance of the condition that the γ not vanish when the field does.

VI. AREA THEOREM FOR SYMMETRIC FIELDS WITH TWO ZEROS

We now consider an important class of electric fields: those that are symmetric and that have only two sign changes per period. An example is $E(t)=E_1\sin(\omega t)$, discussed in Sec. II. We can now say much more about which of these fields lead to ADL, and we can derive an approximate area theorem for ADL. This area theorem is very powerful in that, unlike the area condition for EDL, there are no other conditions on the field.

Let us take the zeros of the ac field to occur at times t_1 and t_2 where $\gamma(t_1)>0$ and of course $\gamma(t_2)=-\gamma(t_1)$. Let the order of the first nonzero derivative of $\gamma(t)$ at the zero crossings be n_1. From Eq. (5.5), we see that to have $S_1=0$ and for large field amplitude we require

$$\mathrm{Re}\{\exp[-i\gamma(t_1)]R_{m_1}(e^{i\pi/2n_1})\}=0, \tag{6.1}$$

which yields

$$\gamma(t_1)=\pi\left[n-\frac{1}{2}+\frac{\delta_{n_1}}{2n_1}\right], \tag{6.2}$$

where n is a positive integer and

$$\delta_m \equiv \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \tag{6.3}$$

Defining the area between consecutive zeros as

$$\mathcal{A}_n \equiv \left| \int_{t_x}^{t_y} E(t)\,dt \right|, \tag{6.4}$$

we find from Eq. (6.2) that for large Ω

$$\mathcal{A}_n \sim \mathcal{A}_n(\infty) \equiv \left[n-\frac{1}{2}+\frac{\delta_{n_1}}{2n_1}\right]2\pi\hbar/ed. \tag{6.5}$$

We thus obtain the area theorem for large-amplitude continuous ac fields that have only two zeros: *In the limit of large field amplitude, DL only occurs if the area under the total field curve between zeros is given by Eq. (6.5).*

This area theorem differs from the area theorem for EDL, Eq. (2.10), in two important ways. First, the present area theorem does not require additional stringent constraints on the functional form of $E(t)$, as is the case for EDL, the only constraint being that it be symmetric and have two zeros. Second, the area between zeros that leads to ADL is different from the area of $2\pi n\hbar/ed$ between discontinuities to achieve EDL. The difference between the ADL and EDL areas is smallest when $n_1=2$, for which $\mathcal{A}_n(\infty)=[n-\frac{1}{4}]2\pi\hbar/ed$.

The validity of ADL area theorem (6.5) is easily seen for sinusoidal fields, for which, according to Eq. (2.5), ADL occurs when Ω/ω is a root of the zeroth-order Bessel function. For large Ω these zeros are given by $\Omega/\omega=(n-\frac{1}{4})\pi$, which gives an area between zero crossings of $2\pi(n-\frac{1}{4})$; this is precisely our area theorem for ADL for the sine function (for which $n_1=2$).

We now turn to reconciling the EDL and ADL area theorems. To do so, we define the area difference between the exact result and asymptotic ADL result:

$$\Delta_n \equiv \frac{ed}{2\pi\hbar}[\mathcal{A}_n-\mathcal{A}_n(\infty)]. \tag{6.6}$$

In Fig. 7 we plot this area deviation as a function a for the symmetric square-wave function smoothed by the fifth-order polynomial $j_5(t)$ given by Eq. (3.10). This field was chosen because its derivatives are continuous up to second order and because a can be used to vary the steepness of the field near the zero crossings. Note that in all cases, as a becomes small, the areas tend towards the EDL area ($\Delta_n=1/4$), whereas for large a, the areas tend to the ADL area ($\Delta_n=0$). This is as expected: for small a we recover the square wave, whereas

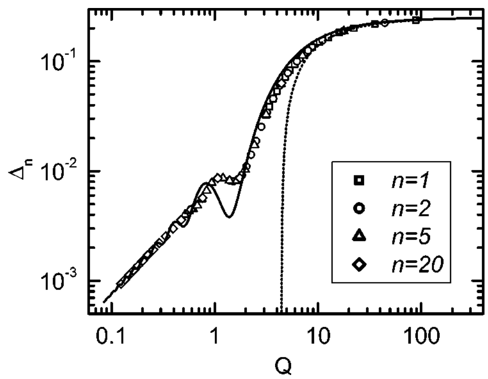

FIG. 8. Area deviation Δ_n for the smoothed symmetric square wave vs Q. The symbols correspond to the results for $j(t)=j_5(t)$; different symbols are for different n as indicated in the legend. The solid curve is for $j(t)=\sin(\pi t/2)$. The dotted curve is the small-a expansion (6.9) for $j(t)=j_5(t)$.

for large a we obtain a very smooth field. We note in particular that for $a \geqslant 0.2$, all the areas except $n=1$ are within 0.01 of the asymptotic value.

From the Appendix, we see that parameter Q, defined in Eq. (A9), is a more natural parameter than a to measure the steepness of the field near the zero crossings. For any smoothing function [such as $j_5(t)$], where the first derivatives of $\gamma(t)$ that are nonzero when $E(t)=0$ are the second and fourth ($n_1=2$ and $m_1=4$ in the language of the Appendix), we have

$$Q=\frac{|E'''(0)|}{|E'(0)|^2}\frac{5\mathcal{E}_1\tau}{3}. \tag{6.7}$$

Thus Q is a measure of the deviation from linear of the time dependence of the electric field at the zero crossings. In Fig. 8 we plot the area deviation Δ_n as a function of Q. We first note that, remarkably, the points for different n all fall onto the same curve. For small Q, Δ_n goes to zero linearly with Q and we see that the asymptotic expansion result that $\Delta_n=0$ is very accurate for $Q<1$ as expected.

When Q is large, we can use the results of Sec. III to get an approximate result. As is seen in that section, if a is not too large, then $\mathcal{E}\simeq\mathcal{E}_n$. With this approximation, we have, for the $j_5(t)$ function, $Q\simeq 8/(9an)$. Furthermore, it is easily seen from Eq. (3.1) that

$$\mathcal{A}_n=2a\tau\mathcal{E}(J(1)-1)+\mathcal{E}\tau/2. \tag{6.8}$$

Putting these results together we obtain

$$\Delta_n\simeq\frac{1}{4}-\frac{10}{9Q}, \tag{6.9}$$

which is shown as the dotted curve in Fig. 8. Note that it captures the exact behavior quite well for $Q>10$. The failure of Eq. (6.9) when Q is small demonstrates that the field amplitude must deviate significantly from the EDL value of $\mathcal{E}=\mathcal{E}_n$ when the product an is large.

We also plot in Fig. 8 the area deviation Δ_n as a function of Q for the symmetric square field with the smoothing function $j_{\sin}(t)\equiv\sin(\pi t/2)$. As can be seen, the behavior of this curve is very similar to that of the $j_5(t)$ smoothing function in the limits of small and large Q. For example, the maximum difference between the curves found for $j_5(t)$ and $j_{\sin}(t)$ is less than 0.016. Thus, using, say, the $\Delta_n(Q)$ curve for the $j_5(t)$ smoothing function would lead to at most an error of 1.6% for the amplitude of field that gives ADL for the $j_{\sin}(t)$ smoothing function.

For a pure sinusoidal field $E(t)=E_0\sin(\omega t)$, $n_1=2$, $m_1=4$, and $Q=20\pi\omega/6\Omega$. Now, ADL occurs for this function when Ω/ω are the zeros of the zeroth-order Bessel function. It is thus found that $Q\ll 1$ for field amplitudes associated with the fourth zero or higher. This is why the asymptotic expression for the location of the zeros is so good for the sinusoidal field as discussed earlier.

We finally note that the curve, $\Delta_n(Q)$, given in Fig. 8 is essentially a universal curve for all functions with $n_1=2$ and $m_1=4$ for which the fifth derivative of the electric field at the zero crossings is small; this is why it works well for square-wave fields with monotonic smoothing functions. However, it is not a universal curve for all symmetric fields. For example, because of their potentially larger fifth-order derivatives, electric fields that have multiple minima and maxima do not in general yield ADL areas that are well approximated by the solid line in Fig. 8. However, we find that the linear dependence of $\Delta_n(Q)$ for small Q in Fig. 8 is universal for all symmetric electric fields with $n_1=2$ and $m_1=4$.

VII. CONCLUSIONS

We have presented a detailed examination of the properties of the electric fields that yield dynamic localization. We began with a systematic discussion of ADL and EDL in periodic potentials in an applied periodic ac electric field. We reiterated the recent results of Dignam and de Sterke,[12] according to which it is straightforward to find an indefinite number of electric fields that yield EDL. All of these fields have the properties that (a) the electric field is discontinuous when the field changes sign and that (b) the electric field area between consecutive zero crossings is an integer multiple of $2\pi\hbar/ed$.

We examined the tolerance of EDL to smoothing of the electric field discontinuities. Localization arising from square-wave ac electric fields was found to be surprisingly tolerant to smoothing, as indicated by Fig. 3. Other functions leading to EDL, which have larger discontinuities, were found to be less tolerant. We also found that even a square wave with very significant smoothing results in much better localization than a pure sine wave.

We also considered ADL—that is, DL in the NNTB limit—and found a number of general results. First we showed that all symmetric ac electric fields are guaranteed to exhibit ADL for some value of the field amplitude as long as the vector potential and electric field are never zero simultaneously. This demonstrates that the earlier results showing that ADL can occur for sinusoidal fields[23,24] and triangular fields[9] are not isolated cases, but can be considered examples of a more general result.

We also showed that for symmetric, large-amplitude electric fields with only two zeros per period, the area under the electric field curve between consecutive sign changes tends to $2\pi\hbar(n-1/4)/ed$. Though the result is asymptotic—i.e., applies strictly only in the limit of large field strengths ($n\to\infty$)—it is in fact already remarkably accurate when n is small for many smooth functions, such as the commonly used sinusoidal field.[23,24]

We have ignored dephasing and coupling to higher bands. Although this is valid for many systems of interest, we plan to investigate their effects on DL in future work. We have determined general trends and principles governing DL rather than present an exhaustive examination of systems of current interest. Although much of what is presented here can be applied to any system, the details of the electric field are of course dependent on the method of generation and the system of interest. For such systems, one could use the principles outlined here to determine the general type of field that should yield EDL and then perform exact calculations of the S_p for the experimentally attainable field to optimize it.

ACKNOWLEDGMENTS

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APPENDIX: ASYMPTOTIC EXPANSION

Here we present some details of the asymptotic expansion of the S_p. We begin with the definition $\gamma(t)=\Omega h(t/\tau)$. Using this we obtain

$$S_1=\int_0^1 e^{-i\Omega h(x)}dx. \tag{A1}$$

We assume that $h'(x)=0$ for $x=\{x_\ell\}$, where $\ell=1,\ldots,N$ and $0<x_\ell\leqslant 1$. Now, we employ the method of stationary phase[25] by noting that for large Ω, the only contributions to the integral occur in small regions about the points $x=x_\ell$. Away from these, the phase in the integrand changes so rapidly so as to average to zero. Thus, we expand $h(x)$ about the point x_ℓ as

$$h(x)=h(x_\ell)+\sum_{n=2}^{\infty}\frac{1}{n!}h^{(n)}(x_\ell)(x-x_\ell)^n, \tag{A2}$$

where $h^{(n)}(x_\ell)$ is the nth derivative of h evaluated at $x=x_\ell$. Thus, in the limit of large Ω, S_1 can now be approximated by

$$S_1\sim\sum_{\ell=1}^{N} e^{-i\Omega h(x_\ell)}\int_{-\infty}^{\infty}\exp\left\{-i\Omega\sum_{n=2}^{\infty}\frac{h^{(n)}(x_\ell)}{n!}(x-x_\ell)^n\right\}dx, \tag{A3}$$

where we have extended the integration over an infinite interval by noting that the integrand only contributes when x is near the x_ℓ.

Let n_ℓ be the order of the lowest nonzero derivative of h at $x=x_\ell$. Now we classify the zeros according to whether $h^{(n_\ell)}(x_\ell)$ is positive or negative. Assume that this derivative is positive at the points $x_\ell^{>}$ ($\ell=1,\ldots,N_1$) and negative at the points $x_\ell^{<}$ ($\ell=N_1,\ldots,N$). Then, performing the integration,[28] we obtain after some work the expression given by Eq. (5.3) in Sec. V. This can be rewritten in terms of the electric field as

$$S_1\sim\sum_{\ell=1}^{N_1}\frac{2\Gamma\left(\frac{1}{n_\ell}\right)\exp[-i\gamma(t_\ell^{>})]R_{n_\ell}(e^{i\pi/2n_\ell})}{n_\ell[ed\tau^{n_\ell}|E^{(n_\ell-1)}(t_\ell^{>})|/\hbar]^{1/n_\ell}}+\sum_{\ell=N_1+1}^{N}\frac{2\Gamma\left(\frac{1}{n_\ell}\right)\exp[-i\gamma(t_\ell^{<})]R_{n_\ell}(e^{-i\pi/2n_\ell})}{n_\ell[ed\tau^{n_\ell}|E^{(n_\ell-1)}(t_\ell^{<})|/\hbar]^{1/n_\ell}}, \tag{A4}$$

where $t_\ell^{>,<}=\tau x_\ell^{>,<}$ and $E^{(n)}(t)$ is the nth derivative of the electric field with respect to time. In the usual case, where all second-order derivatives of $h(x)$ at the extrema are nonzero, this reduces to

$$S_1\sim\sqrt{\frac{2\pi}{\Omega}}\left\{\sum_{\ell=1}^{N_1}\frac{\exp\{i[-\Omega h(x_\ell^{>})+\pi/4]\}}{\sqrt{|h''(x_\ell^{>})|}}+\sum_{\ell=N_1+1}^{N}\frac{\exp\{i[-\Omega h(x_\ell^{<})-\pi/4]\}}{\sqrt{|h''(x_\ell^{<})|}}\right\}. \tag{A5}$$

It is useful to estimate the range of the validity of these asymptotic results. Rather than pursue the general case, we instead consider only symmetric electric fields with only two zeros per period. Let n_1 be the order of the lowest nonzero term in the expansion about the point x_1 and let m_1 be the order of the next lowest nonzero term. Thus, we can obtain a somewhat better approximation to S_1 than Eq. (5.3) by evaluating

$$S_1\sim 2\,\mathrm{Re}\left\{e^{-i\Omega h(x_1)}\int_{-\infty}^{\infty}\exp\left[-i\frac{\Omega}{n_1!}h^{(n_1)}(x_1)(x-x_1)^{n_1}-i\frac{\Omega}{m_1!}h^{(m_1)}(x_1)(x-x_1)^{m_1}\right]\right\}dx. \tag{A6}$$

We now consider the parameter that characterizes whether the second term in this expansion is necessary. The first term in the expansion makes a significant contribution to the integrand as long as

$$\frac{\Omega}{n_1!}|h^{(n_1)}(x_1)(x-x_1)^{n_1}|<2\pi s, \tag{A7}$$

where s is a factor that is on the order of 4–10. For larger values of $|x-x_1|$ the integrand oscillates very rapidly and makes negligible contributions. For the next term in the expansion to be negligible, we require that

$$\frac{|h^{(m_1)}(x_1)|n_1!|x-x_1|^{m_1-n_1}}{|h^{(n_1)}(x_1)|m_1!}\ll 1. \tag{A8}$$

Using Eq. (A7) in Eq. (A8), and expressing the result in electric field derivatives, we obtain the condition $Q \ll 1$, where

$$Q \equiv \left[\frac{20\pi\hbar}{ed} \right]^{(m_1-n_1)/n_1} \frac{|E^{(m_1-1)}(t_1)/m_1!|}{|E^{(n_1-1)}(t_1)/n_1!|^{m_1/n_1}}, \quad \text{(A9)}$$

where $t_1 = \tau x_1$ and we have chosen $s=10$. Thus, not surprisingly, the asymptotic expression is good as long as the curvature of the electric field curve at the zero crossings is small. Parameter Q is useful as a variable to quantify the deviation from the asymptotic expansion as is discussed in Sec. VI.

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