

Localization from conductance in few-channel disordered wires

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(Received 4 June 2002; published 31 October 2002)

We study localization in two- and three-channel quasi-one-dimensional (1D) systems using multichain tight-binding Anderson models with nearest-neighbor interchain hopping. In the three-chain case we discuss the cases of both free and periodic boundary conditions between the chains. The finite disordered wires are connected to ideal leads and the localization length is defined from the Landauer conductance in terms of the transmission coefficients matrix. The transmission and reflection amplitudes in properly defined quantum channels are obtained from S matrices constructed from transfer matrices in Bloch wave bases for the various quasi-1D systems. Our exact analytic expressions for localization lengths for weak disorder reduce to the Thouless expression for 1D systems in the limit of vanishing interchain hopping. For weak interchain hopping the localization length decreases with respect to the 1D value in all three cases. In the three-channel cases it increases with interchain hopping over restricted domains of large hopping.

DOI: 10.1103/PhysRevB.66.155434

PACS number(s): 73.23.-b, 72.10.-d, 72.15.Rn

I. INTRODUCTION

A wire is a topologically one-dimensional system whose width, of the order of the square root of the cross-sectional area A , is much smaller than its length L , i.e., $\sqrt{A} \ll L$. In a thin wire the motion of electrons in the transverse direction is quantized. The corresponding transverse eigenstates below the Fermi level define a finite number, $N \propto \sqrt{2A}/\lambda_F$ (with λ_F the Fermi wavelength), of quantum channels for the transmission of electrons across the wire. The starting point of the present work is the Landauer two-probe conductance formula^{1,2}

$$g = \frac{2e^2}{h} \text{Tr}(\hat{t}\hat{t}^\dagger), \quad (1)$$

which describes current transport in a disordered wire. Here \hat{t} is the so-called transmission matrix of the N -channel system:

$$\hat{t} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix}. \quad (2)$$

An incoming wave from channel j of an ideal lead at one end of the disordered wire (of length L) has a coefficient $|t_{ij}|^2$ for transmission into channel i of the lead at the other end.

The correctness of the description of the conductance in terms of transmission channels has received striking experimental confirmation³ in the special case of perfectly transmitting channels, where Eq. (1) reduces to

$$g = \frac{2e^2}{h} N. \quad (3)$$

These studies relate to quantum point contacts in the form of narrow conducting two-dimensional strips whose width, and hence the number of discrete transmitting channels, may be varied by varying externally applied gate voltages. Conduc-

tance steps corresponding to increasing values of N in Eq. (3) are clearly observed.³ A further interesting feature of these experiments is that they offer the possibility of realizing physically not only purely one-dimensional systems ($N=1$) but also few-channel systems ($N=2,3,\dots$) such as those studied below. We recall that recent discussions of many-channel mesoscopic systems have been essentially restricted to the case $N \gg 1$,⁴ i.e., real (metallic) wires whose widths are much larger than the Fermi wavelength.

The behavior of the conductance in a disordered wire depends strongly on its length relative to the localization length L_c . In an infinitely long disordered wire of a given cross section all eigenstates are expected to be localized just like in the truly one-dimensional case. Thouless⁵ (also see Ref. 6) indeed showed that for $L > L_c$, where

$$L_c \sim Nl, \quad (4)$$

l being the mean free path, the conductance should fall off exponentially as e^{-L/L_c} , which is a clear manifestation of localization. On the other hand, on the scales of lengths in the domain

$$l < L < L_c = Nl, \quad (5)$$

the eigenstates appear as being delocalized. In fact, it was shown later⁷ that for lengths in the range of Eq. (5) there remains a number $N_{\text{eff}} \sim Nl/L$ of independent ballistic channels leading to a metallic Ohm's law behavior for the conductance (1). Such a diffusive quasimetallic domain does not exist for truly one-dimensional systems where $L_c \equiv L_{1c} \sim l$. Dorokhov⁸ developed a detailed scaling analysis⁹ of localization in a multichannel wire, in which he calculated L_c in terms of a phenomenological mean free path entering as input via Ohm's law at short scales. His final result, which is valid for weak disorder, is

$$L_c = (N+1)l, \quad (6)$$

which coincides with (4) for $N \gg 1$ and suggests that L_c is not directly proportional to the number of channels for few channel systems. In particular, for one-dimensional systems it yields

$$L_{1c} = 2l. \quad (7)$$

We recall that Eq. (7) coincides with the result obtained long ago by Thouless¹⁰ from kinetic transport theory. The same result also follows from the quantum series composition law for conductances $g = |t|^2$ for one-dimensional (1D) conductors, assuming Ohm's law to be valid at short length scales.¹¹ On the other hand, we note that for two-channel systems (6) yields $L_c \equiv L_{2c} = 3l$ which is comparable to the result $L_{2c} = 2l[1 - (1/\pi)]^{-1}$ obtained by Dorokhov¹² in a different scaling treatment.

Shortly before the development of the scaling theory of localization for 1D conductors,⁹ Thouless derived his well-known analytic expression for the quantum localization length in a tight-binding linear chain for weak disorder.¹³ The purpose of the present work is to derive similar exact microscopic expressions for the localization length in quasi-one-dimensional few-channel systems, specifically for two- and three chain systems for weak disorder. This study is of interest in several respects. For example, via the dependence of the localization length on the energy across the energy bands of the pure systems, it provides a first-principles quantum proof of the fact that all states in the quasi-one-dimensional systems are localized. It is also relevant for experimental situations, e.g., for discussing the quasimetallic domain in few-channel systems which are encountered in semiconducting microstructures¹⁴ or may be fabricated artificially, as in the quantum point contact system discussed above.

In Sec. II A we define the tight-binding two- and three-chain systems for modeling few channel wires. In Sec. II B we describe our methodology for studying the localization length. This consists in constructing successively transfer- and transmission-reflection matrices for the tight-binding systems by generalizing well-known methods for 1D systems. It further relies on general results concerning the existence and the properties of a Lyapunov exponent (inverse localization length) describing the asymptotic exponential decay of the conductance [Eq. (1)] in multichannel systems.¹⁵ In Sec. III we present the details of our calculations leading to the final analytic expressions for the transfer and scattering matrices for the two- and three-channel wires for weak disorder. For the case $N=3$, we obtain different results for open and periodic boundary conditions, which correspond to packing the chains on a plane and on a cylindrical surface, respectively. The final analytic expressions for the averaged transmission- and reflection-coefficients and for localization lengths are discussed in Sec. IV. For clarity's sake some details of these calculations are relegated to an Appendix.

II. FEW-CHANNEL WIRES AND LOCALIZATION

A. Anderson models in channel bases

We describe two- ($N=2$) and three- ($N=3$) channel wires by Anderson models for two- and three-coupled chain

systems, respectively. The two chain Anderson model consists of parallel linear chains of N_L disordered sites each (of spacing $a=1$ and length $L=N_L a$) connected at both ends to semi-infinite ideal (nondisordered) leads. It is defined by the tight-binding Schrödinger equation which we write in the matrix form

$$\begin{pmatrix} \varphi_{n+1}^1 + \varphi_{n-1}^1 \\ \varphi_{n+1}^2 + \varphi_{n-1}^2 \end{pmatrix} = \begin{pmatrix} E - \varepsilon_{1n} & -h \\ -h & E - \varepsilon_{2n} \end{pmatrix} \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \end{pmatrix}, \quad (8)$$

where φ_m^i denote the wave-function amplitudes at sites m on the chain i , and h is a constant matrix element for an electron to hop transversally between a site n on chain 1 and its nearest-neighbor site n on chain 2. The site energies ε_{im} are random variables associated with the sites $1 \leq m \leq N_L$ of the disordered chain i , and $\varepsilon_{im} = 0$ on the semi-infinite ideal chains defined by the sites $m > N_L$ and $m < 1$, respectively. The above energies, including E , are measured in units of the constant hopping rate along the individual chains.

The coupled three-chain ($N=3$) Anderson model is defined in a similar way by a set of tight-binding Schrödinger equations, whose actual form depends, however, on inter-chain boundary conditions. For free boundary conditions, which correspond to arranging the parallel equidistant chains on a plane, the tight-binding equations are

$$\begin{pmatrix} \varphi_{n+1}^1 + \varphi_{n-1}^1 \\ \varphi_{n+1}^2 + \varphi_{n-1}^2 \\ \varphi_{n+1}^3 + \varphi_{n-1}^3 \end{pmatrix} = \begin{pmatrix} E - \varepsilon_{1n} & -h & 0 \\ -h & E - \varepsilon_{2n} & -h \\ 0 & -h & E - \varepsilon_{3n} \end{pmatrix} \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \\ \varphi_n^3 \end{pmatrix}, \quad (9)$$

with the sites in the disordered sections of length $L=N_L a$ and in the semi-infinite ideal chain sections labeled in the same way as in the two-chain case. On the other hand, in the case of periodic boundary conditions which correspond to equidistant linear chains on a cylindrical surface the Schrödinger equation is

$$\begin{pmatrix} \varphi_{n+1}^1 + \varphi_{n-1}^1 \\ \varphi_{n+1}^2 + \varphi_{n-1}^2 \\ \varphi_{n+1}^3 + \varphi_{n-1}^3 \end{pmatrix} = \begin{pmatrix} E - \varepsilon_{1n} & -h & -h \\ -h & E - \varepsilon_{2n} & -h \\ -h & -h & E - \varepsilon_{3n} \end{pmatrix} \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \\ \varphi_n^3 \end{pmatrix}. \quad (10)$$

As discussed in Sec. I, a quasi-one-dimensional wire is described by a collection of independent channels for wave transmission. Microscopic models for two- and three-channel wires are obtained from the systems of tight-binding equations (8)–(10) by diagonalizing the interchain coupling terms in the equations describing the ideal leads. This indeed leads to independent quantum channels for the leads defined by amplitudes bases

$$\begin{pmatrix} \vdots \\ \psi_n^i \\ \vdots \end{pmatrix} = \hat{U}^{-1} \begin{pmatrix} \vdots \\ \varphi_n^i \\ \vdots \end{pmatrix},$$

in which the nonrandom parts of the matrices on the right-hand side of Eqs. (8)–(10) are diagonal. For the multichain system above we obtain, respectively:

$$\begin{aligned} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix} = \hat{U}_0 \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \\ \varphi_n^3 \end{pmatrix}, \quad \hat{U}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \end{aligned} \quad (8a)$$

$$\begin{aligned} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix} = (\hat{U}'')^{-1} \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \\ \varphi_n^3 \end{pmatrix}, \quad \hat{U}'' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \end{aligned} \quad (10a)$$

$$\hat{U}_0 \begin{pmatrix} E & -h \\ -h & E \end{pmatrix} \hat{U}_0 = \begin{pmatrix} E-h & 0 \\ 0 & E+h \end{pmatrix},$$

$$(\hat{U}'')^{-1} \begin{pmatrix} E & -h & -h \\ -h & E & -h \\ -h & -h & E \end{pmatrix} \hat{U}''$$

$$\begin{aligned} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix} = \hat{U}' \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \\ \varphi_n^3 \end{pmatrix}, \quad \hat{U}' = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \end{aligned} \quad (9a)$$

$$= \begin{pmatrix} E-2h & 0 & 0 \\ 0 & E+h & 0 \\ 0 & 0 & E+h \end{pmatrix}.$$

$$\hat{U}' \begin{pmatrix} E & -h & 0 \\ -h & E & -h \\ 0 & -h & E \end{pmatrix} \hat{U}' = \begin{pmatrix} E-\sqrt{2}h & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E+\sqrt{2}h \end{pmatrix},$$

Note that \hat{U}_0 and \hat{U}' are unitary, while \hat{U}'' is not.

Finally, in the channel bases defined by Eqs. (8a)–(10a) the tight-binding equations (8)–(10) read, respectively,

$$\begin{pmatrix} \psi_{n+1}^1 + \psi_{n-1}^1 \\ \psi_{n+1}^2 + \psi_{n-1}^2 \end{pmatrix} = \begin{pmatrix} E-h - \frac{1}{2}(\varepsilon_{1n} + \varepsilon_{2n}) & \frac{1}{2}(\varepsilon_{2n} - \varepsilon_{1n}) \\ \frac{1}{2}(\varepsilon_{2n} - \varepsilon_{1n}) & E+h - \frac{1}{2}(\varepsilon_{1n} + \varepsilon_{2n}) \end{pmatrix} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}, \quad (8b)$$

$$\begin{pmatrix} \psi_{n+1}^1 + \psi_{n-1}^1 \\ \psi_{n+1}^2 + \psi_{n-1}^2 \\ \psi_{n+1}^3 + \psi_{n-1}^3 \end{pmatrix} = \begin{pmatrix} E-\sqrt{2}h - \frac{1}{4}(\varepsilon_{1n} + 2\varepsilon_{2n} + \varepsilon_{3n}) & \frac{\sqrt{2}}{4}(\varepsilon_{3n} - \varepsilon_{1n}) & -\frac{1}{4}(\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}) \\ \frac{\sqrt{2}}{4}(\varepsilon_{3n} - \varepsilon_{1n}) & E - \frac{1}{2}(\varepsilon_{1n} + \varepsilon_{3n}) & \frac{\sqrt{2}}{4}(\varepsilon_{3n} - \varepsilon_{1n}) \\ -\frac{1}{4}(\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}) & \frac{\sqrt{2}}{4}(\varepsilon_{3n} - \varepsilon_{1n}) & E + \sqrt{2}h - \frac{1}{4}(\varepsilon_{1n} + 2\varepsilon_{2n} + \varepsilon_{3n}) \end{pmatrix} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix} \quad (9b)$$

$$\begin{pmatrix} \psi_{n+1}^1 + \psi_{n-1}^1 \\ \psi_{n+1}^2 + \psi_{n-1}^2 \\ \psi_{n+1}^3 + \psi_{n-1}^3 \end{pmatrix} = \begin{pmatrix} E-2h - \frac{1}{3}(\varepsilon_{1n} + \varepsilon_{2n} + \varepsilon_{3n}) & -\frac{1}{3}(\varepsilon_{1n} - \varepsilon_{3n}) & -\frac{1}{3}(\varepsilon_{2n} - \varepsilon_{3n}) \\ -\frac{1}{3}(2\varepsilon_{1n} - \varepsilon_{2n} - \varepsilon_{3n}) & E+h - \frac{1}{3}(2\varepsilon_{1n} + \varepsilon_{3n}) & \frac{1}{3}(\varepsilon_{2n} - \varepsilon_{3n}) \\ \frac{1}{3}(\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}) & \frac{1}{3}(\varepsilon_{1n} - \varepsilon_{3n}) & E+h - \frac{1}{3}(2\varepsilon_{2n} + \varepsilon_{3n}) \end{pmatrix} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix}, \quad (10b)$$

which constitute our starting point for deriving transmission- and reflection matrices of the disordered wires in Sec. III. It is seen that the similarity transformation of the disorder matrices by the \hat{U} matrices leads to interchannel coupling in the disordered sections, $1 \leq n \leq N_L$.

B. Localization from conductance

In Sec. IV we will calculate the localization length in the above multichannel wire models from the rate of exponential

decrease (Lyapunov exponent) of the conductance [Eq. (1)] for large L .¹⁵ The transmission matrix in Eq. (2) will be found by constructing a transfer matrix which transforms propagating waves in the multichannel leads [defined by Eqs. (8a)–(10a)] on the left side of the disordered wire into corresponding propagating waves on the right side. The transfer matrix for the wire of length L is expressed, as usual, as a product of N_L transfer matrices for small sections n enclosing only the n th site of each one of the channels. The calculation of the localization length rests on theorems of

Oseledec¹⁶ and of Tutubalin and Vister¹⁷ on the properties of products of a large number of random matrices. Indeed, employing these properties Johnston and Kunz¹⁵ showed that the Lyapunov exponent γ exists for the conductance [Eq. (1)] and is a self-averaging quantity referred to as the inverse localization length. It is defined by the relation

$$\gamma \equiv \frac{1}{L_c} = - \lim_{N_L \rightarrow \infty} \frac{1}{2N_L} \langle \ln g \rangle, \quad (11)$$

where $\langle \dots \rangle$ denotes averaging over the disorder (i.e., the random site energies in the Anderson model).¹⁸ It follows that the asymptotic distribution of the conductance is log normal.

III. DETAILED ANALYSIS

As indicated above the transmission matrices of the form of Eq. (2) for the quasi-1D disordered systems above will be obtained from transfer matrices for wavefunction amplitudes defined from Eqs. (8b)–(10b), respectively. The construction of these transfer matrices proceeds in two steps. First we will define transfer matrices for thin slices enclosing only one site n of each chain in a disordered wire, in a Bloch plane wave basis. Next the transfer matrix of a whole wire of length $L = N_L a$ will be obtained as a product of the transfer matrices for the N_L individual slices composing the wire. We will express it analytically to lowest order in the effect of a weak disorder. Finally we obtain the form of the tight-binding equations describing the transfer of Bloch wave am-

plitudes across a whole disordered wire of length L , which we then cast in the form of scattering equations in order to identify microscopic scattering matrices (S matrices) in terms of elements of the transfer matrices. We recall that the transfer matrix method is well known in the study of one-dimensional disordered systems.^{19–21} Here we generalize and adapt it in the case of few-channel quasi-1D systems.

A. Transfer matrices

Transfer matrices \tilde{X}_{0n} , \tilde{X}'_n , and \tilde{X}''_n for thin slices including a single site n per chain of the quasi-1D systems described by Eqs. (8b)–(10b) are defined by rewriting these equations, respectively, in the forms

$$\begin{pmatrix} \psi_{n+1}^1 \\ \psi_n^1 \\ \psi_{n+1}^2 \\ \psi_n^2 \end{pmatrix} = \tilde{X}_{0n} \begin{pmatrix} \psi_n^1 \\ \psi_{n-1}^1 \\ \psi_n^2 \\ \psi_{n-1}^2 \end{pmatrix}, \quad (12)$$

$$\begin{pmatrix} \psi_{n+1}^1 \\ \psi_n^1 \\ \psi_{n+1}^2 \\ \psi_n^2 \\ \psi_{n+1}^3 \\ \psi_n^3 \end{pmatrix} = \tilde{Y}_n \begin{pmatrix} \psi_n^1 \\ \psi_{n-1}^1 \\ \psi_n^2 \\ \psi_{n-1}^2 \\ \psi_n^3 \\ \psi_{n-1}^3 \end{pmatrix}, \quad \tilde{Y}_n \equiv \tilde{X}'_n, \tilde{X}''_n, \quad (13)$$

where

$$\tilde{X}_{0n} = \begin{pmatrix} E - h - \mu_n & -1 & \nu_n & 0 \\ 1 & 0 & 0 & 0 \\ \nu_n & 0 & E + h - \mu_n & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (14)$$

$$\mu_n = \frac{1}{2}(\varepsilon_{1n} + \varepsilon_{2n}), \quad \nu_n = \frac{1}{2}(\varepsilon_{2n} - \varepsilon_{1n}), \quad (14a)$$

$$\tilde{X}'_n = \begin{pmatrix} E - \sqrt{2}h - \mu'_n & -1 & \nu'_n & 0 & \tau'_n & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \nu'_n & 0 & E - \eta'_n & -1 & \nu'_n & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \tau'_n & 0 & \nu'_n & 0 & E + \sqrt{2}h - \mu'_n & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (15)$$

$$\mu'_n = \frac{1}{4}(\varepsilon_{1n} + 2\varepsilon_{2n} + \varepsilon_{3n}), \quad \nu'_n = \frac{\sqrt{2}}{4}(\varepsilon_{3n} - \varepsilon_{1n}), \quad (15a)$$

$$\tau'_n = -\frac{1}{4}(\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}), \quad \eta'_n = \frac{1}{2}(\varepsilon_{3n} + \varepsilon_{1n}),$$

$$\tilde{X}_n'' = \begin{pmatrix} E-2h-\mu_n'' & -1 & \nu_n'' & 0 & \tau_n'' & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_n'' & 0 & E+h-\eta_n'' & -1 & -\tau_n' & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_n'' & 0 & -\nu_n'' & 0 & E+h-\theta_n'' & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (16)$$

$$\begin{aligned} \mu_n'' &= \frac{1}{3}(\varepsilon_{1n} + \varepsilon_{2n} + \varepsilon_{3n}), & \eta_n'' &= \frac{1}{3}(2\varepsilon_{1n} + \varepsilon_{3n}), & \theta_n'' &= \frac{1}{3}(2\varepsilon_{2n} + \varepsilon_{3n}), \\ \nu_n'' &= -\frac{1}{3}(\varepsilon_{1n} - \varepsilon_{3n}), & \tau_n'' &= -\frac{1}{3}(\varepsilon_{2n} - \varepsilon_{3n}), \end{aligned} \quad (16a)$$

$$\alpha_n'' = -\frac{1}{3}(2\varepsilon_{1n} - \varepsilon_{2n} - \varepsilon_{3n}), \quad \beta_n'' = \frac{1}{3}(\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}).$$

The study of disordered wires in terms of reflection and transmission properties of plane waves requires a determination of plane wave bases in which the transfer matrices for slices n in the leads are diagonal. Such bases are provided by the Bloch wave solutions for the leads, which are defined by

$$\tilde{X}_{00} \begin{pmatrix} \psi_{n,\pm}^1 \\ \psi_{n-1,\pm}^1 \\ \psi_{n,\pm}^2 \\ \psi_{n-1,\pm}^2 \end{pmatrix} = \begin{pmatrix} e^{\pm ik_1} \psi_{n,\pm}^1 \\ e^{\pm ik_1} \psi_{n-1,\pm}^1 \\ e^{\pm ik_2} \psi_{n,\pm}^2 \\ e^{\pm ik_2} \psi_{n-1,\pm}^2 \end{pmatrix}, \quad (17)$$

$$\tilde{Y}_0 \begin{pmatrix} \psi_{n,\pm}^1 \\ \psi_{n-1,\pm}^1 \\ \psi_{n,\pm}^2 \\ \psi_{n-1,\pm}^2 \\ \psi_{n,\pm}^3 \\ \psi_{n-1,\pm}^3 \end{pmatrix} = \begin{pmatrix} e^{\pm ik_1} \psi_{n,\pm}^1 \\ e^{\pm ik_1} \psi_{n-1,\pm}^1 \\ e^{\pm ik_2} \psi_{n,\pm}^2 \\ e^{\pm ik_2} \psi_{n-1,\pm}^2 \\ e^{\pm ik_3} \psi_{n,\pm}^3 \\ e^{\pm ik_3} \psi_{n-1,\pm}^3 \end{pmatrix}, \quad \tilde{Y}_0 \equiv \tilde{X}'_0, \tilde{X}''_0, \quad (18)$$

where \tilde{X}_{00} , \tilde{X}'_0 , and \tilde{X}''_0 denote the transfer matrices for the leads given by Eqs. (14)–(16) with $\varepsilon_{1n} = \varepsilon_{2n} = \varepsilon_{3n} = 0$, respectively. The wave numbers k_i are defined in terms of the energy E by the eigenvalues of Eqs. (17) and (18), respectively. By solving for the eigenvalues we obtain, successively,

$$\begin{aligned} 2 \cos k_1 &= E - h, \\ 2 \cos k_2 &= E + h \end{aligned} \quad (17a)$$

for the two-channel system,

$$2 \cos k_1 = E - \sqrt{2}h,$$

$$2 \cos k_2 = E, \quad (18a)$$

$$2 \cos k_3 = E + \sqrt{2}h$$

for the three-channel system with free boundary conditions, whose leads are described by \tilde{X}'_0 , and, finally,

$$\begin{aligned} 2 \cos k_1 &= E - 2h, \\ 2 \cos k_2 &= 2 \cos k_3 = E + h \end{aligned} \quad (18b)$$

for the three-channel model with periodic lateral boundary conditions. The eigenfunctions in the leads at energy E obtained from Eqs. (12) and (13) and (17) and (18) are of the form

$$\psi_{n,\pm}^j \sim e^{\pm ink_j}, \quad (19)$$

where we choose the wave numbers $k_j, j=1,2,3$, to be positive, $0 \leq k_j \leq \pi$, so that these functions correspond to plane waves traveling from left to right and from right to left, respectively.

The transfer matrices for single site slices in Eqs. (12) and (13) for the leads (i.e., for $\varepsilon_{in} = 0, n < 1$ or $n > N$) are diagonalized in the bases of the Bloch plane wave states (19). The diagonalization matrices which are formed by the eigenvectors of Eqs. (17) and (18), are of the form

$$\hat{V}_0 = \begin{pmatrix} \hat{A}_1 & \hat{O} \\ \hat{O} & \hat{A}_2 \end{pmatrix}, \quad \hat{A}_j = \frac{1}{\sqrt{2i \sin k_j}} \begin{pmatrix} e^{ik_j} & e^{-ik_j} \\ 1 & 1 \end{pmatrix}, \quad (20)$$

for the two-channel quasi-1D model, and

$$\hat{V} = \begin{pmatrix} \hat{A}_1 & \hat{O} & \hat{O} \\ \hat{O} & \hat{A}_2 & \hat{O} \\ \hat{O} & \hat{O} & \hat{A}_3 \end{pmatrix}, \quad (21)$$

with \hat{A}_j defined as in Eq. (20), for the three-channel models. The wave numbers k_j are defined by Eq. (17a) for the $N=2$ case and by Eqs. (18a) and (18b) for the $N=3$ case with free and periodic boundary conditions, respectively. After finding the inverses of \hat{V}_0 and \hat{V} and performing the similarity transformations of \tilde{X}_{0n} , \tilde{X}'_n , and \tilde{X}''_n by \hat{V}_0 and \hat{V} , respectively, we obtain the desired transfer matrices in the Bloch wave representation of the disordered wires. In the two-channel case we find

$$\hat{X}_{0n} \equiv \hat{V}_0^{-1} \hat{X}_{0n} \hat{V}_0 = \begin{pmatrix} e^{ik_1(1+ia_{1n})} & ie^{-ik_1}a_{1n} & -ie^{ik_2}b_n & -ie^{-ik_2}b_n \\ -ie^{ik_1}a_{1n} & e^{-ik_1}(1-ia_{1n}) & ie^{ik_2}b_n & ie^{-ik_2}b_n \\ -ie^{ik_1}b_n & -ie^{-ik_1}b_n & e^{ik_2}(1+ia_{2n}) & ie^{-ik_2}a_{2n} \\ ie^{ik_1}b_n & ie^{-ik_1}b_n & -ie^{ik_2}a_{2n} & e^{-ik_2}(1-ia_{2n}) \end{pmatrix}, \quad (22)$$

where

$$a_{1n} = \frac{\varepsilon_{1n} + \varepsilon_{2n}}{4 \sin k_1}, \quad a_{2n} = \frac{\varepsilon_{1n} + \varepsilon_{2n}}{4 \sin k_2}, \quad (22a)$$

$$b_n = \frac{\varepsilon_{2n} - \varepsilon_{1n}}{4 \sqrt{\sin k_1 \sin k_2}},$$

are real quantities, and k_1 and k_2 are defined by Eq. (17a). For the three-channel systems we write the final transfer matrices \hat{X}'_n and \hat{X}''_n in terms of a generic matrix

$$\hat{Z}_n = \begin{pmatrix} e^{ik_1(1+ia_{1n})} & ie^{-ik_1}a_{1n} & ie^{ik_2}c_n & ie^{-ik_2}c_n & ie^{ik_3}g_n & ie^{-ik_3}g_n \\ -ie^{ik_1}a_{1n} & e^{-ik_1}(1-ia_{1n}) & -ie^{ik_2}c_n & -ie^{-ik_2}c_n & -ie^{ik_3}g_n & -ie^{-ik_3}g_n \\ ie^{ik_1}f_n & ie^{-ik_1}f_n & e^{ik_2}(1+ib_{2n}) & ie^{-ik_2}b_{2n} & ie^{ik_3}d_n & ie^{-ik_3}d_n \\ -ie^{ik_1}f_n & -ie^{-ik_1}f_n & -ie^{ik_2}b_{2n} & e^{-ik_2}(1-ib_{2n}) & -ie^{ik_3}d_n & -ie^{-ik_3}d_n \\ ie^{ik_1}p_n & ie^{-ik_1}p_n & ie^{ik_2}q_n & ie^{-ik_2}q_n & e^{ik_3}(1+ia_{3n}) & ie^{-ik_3}a_{3n} \\ -ie^{ik_1}p_n & -ie^{-ik_1}p_n & -ie^{ik_2}q_n & -ie^{-ik_2}q_n & -ie^{ik_3}a_{3n} & e^{-ik_3}(1-ia_{3n}) \end{pmatrix}, \quad (23)$$

where a_{1n} , a_{3n} , b_{2n} , c_n , g_n , f_n , d_n , p_n , and q_n are real quantities. In the case of free boundary conditions we obtain $\hat{X}'_n = \hat{V}^{-1} \hat{X}''_n \hat{V} \equiv \hat{Z}_n$ with

$$a_{1n} = \frac{\varepsilon_{1n} + 2\varepsilon_{2n} + \varepsilon_{3n}}{8 \sin k_1}, \quad a_{3n} = \frac{\varepsilon_{1n} + 2\varepsilon_{2n} + \varepsilon_{3n}}{8 \sin k_3},$$

$$b_{2n} = \frac{\varepsilon_{1n} + \varepsilon_{3n}}{4 \sin k_2}, \quad c_n = f_n = \frac{\sqrt{2}(\varepsilon_{1n} - \varepsilon_{3n})}{8 \sqrt{\sin k_1 \sin k_2}}, \quad (24)$$

$$d_n = q_n = \frac{\sqrt{2}(\varepsilon_{1n} - \varepsilon_{3n})}{8 \sqrt{\sin k_2 \sin k_3}}, \quad g_n = p_n = \frac{\varepsilon_{1n} - 2\varepsilon_{2n} + \varepsilon_{3n}}{8 \sqrt{\sin k_1 \sin k_3}}.$$

Here k_1 , k_2 , and k_3 are defined by Eq. (18a). On the other hand, for periodic boundary conditions we find $\hat{X}''_n = \hat{V}^{-1} \hat{X}'_n \hat{V} \equiv \hat{Z}_n$ where

$$a_{1n} = \frac{\varepsilon_{1n} + \varepsilon_{2n} + \varepsilon_{3n}}{6 \sin k_1}, \quad a_{3n} = \frac{2\varepsilon_{2n} + \varepsilon_{3n}}{6 \sin k_2},$$

$$b_{2n} = \frac{2\varepsilon_{1n} + \varepsilon_{3n}}{6 \sin k_2}, \quad c_n = \frac{\varepsilon_{1n} - \varepsilon_{3n}}{6 \sqrt{\sin k_1 \sin k_2}},$$

$$g_n = \frac{\varepsilon_{2n} - \varepsilon_{3n}}{6 \sqrt{\sin k_1 \sin k_2}}, \quad (25)$$

$$d_n = \frac{\varepsilon_{3n} - \varepsilon_{2n}}{6 \sin k_2}, \quad f_n = \frac{2\varepsilon_{1n} - \varepsilon_{2n} - \varepsilon_{3n}}{6 \sqrt{\sin k_1 \sin k_2}},$$

$$p_n = \frac{-\varepsilon_{1n} + 2\varepsilon_{2n} - \varepsilon_{3n}}{6 \sqrt{\sin k_1 \sin k_2}}, \quad q_n = \frac{\varepsilon_{3n} - \varepsilon_{1n}}{6 \sin k_2},$$

where k_1 and $k_2 = k_3$ are now given by Eq. (18b). Note, in particular, the diagonalization of the transfer matrices for the leads in the plane wave bases shown in Eqs. (22) and (23).

Finally, we determine the transfer matrices for the disordered wires of length $L = N_L a$ in terms of the transfer matrices of the individual thin slices n . As shown by iteration of the transfer equations (17) and (18) rewritten in the Bloch wave basis above, the matrix transferring an incoming wave at site $n=0$ just outside a disordered wire to the site $N_L + 1$ just beyond its other end is given by a product of transfer matrices of the form

$$\hat{Y}_L = \prod_{n=1}^{N_L} \hat{Y}_n, \quad (26)$$

where \hat{Y}_n and \hat{Y}_L stand for the three pairs of transfer matrices \hat{X}_{0n} , \hat{X}_{0L} , \hat{X}'_n , \hat{X}'_L and \hat{X}''_n , \hat{X}''_L , respectively, which are associated with the wire models above.¹⁹

We shall evaluate the transfer matrices of the disordered wires for weak disorder to linear order in the random site energies. On the other hand, for our explicit calculations of

averages over the disorder below, we assume the site energies to be independent gaussian random variables with zero mean values, and correlation

$$\langle \varepsilon_{in} \varepsilon_{jm} \rangle = \varepsilon_0^2 \delta_{i,j} \delta_{m,n}. \quad (27)$$

In this case the site energies corresponding to different slices n in Eq. (26) are uncorrelated so that it is indeed sufficient to restrict the expansion of the latter expressions to first order in the site energies for determining averages to order ε_0^2 . The transfer matrices \hat{Y}_n given by Eqs. (22) and (23) are sums,

$$\hat{Y}_n = \hat{Y}^{(0)} + \hat{Y}_n^{(1)}. \quad (28)$$

of a zeroth-order diagonal matrix $\hat{Y}^{(0)}$ independent of the site energies (transfer matrix of the leads) and a matrix which is linear in the energies ε_{jn} . By inserting Eq. (28) into Eq. (26) we obtain, to first order,

$$\hat{Y}_L = (\hat{Y}^{(0)})^{N_L} + \sum_{m=1}^{N_L} (\hat{Y}^{(0)})^{m-1} \hat{Y}_m^{(1)} (\hat{Y}^{(0)})^{N_L-m} + \dots \quad (29)$$

Next we insert the slice matrices $\hat{Y}^{(0)}$ and $\hat{Y}_m^{(1)}$ from Eqs. (22) and (23) for our various quasi-1D systems, and obtain, successively,

$$\hat{X}_{0L} = \text{diag}(e^{ik_1 N_L}, e^{-ik_1 N_L}, e^{ik_2 N_L}, e^{-ik_2 N_L}) + \sum_{m=1}^{N_L} \begin{pmatrix} ia_{1m} e^{ik_1 u_1} & ia_{1m} e^{-ik_1 s_1^*} & -ib_m e^{ik_2 w_{21}} & -ib_m e^{-ik_2 v_{21}^*} \\ -ia_{1m} e^{ik_1 s_1} & -ia_{1m} e^{-ik_1 u_1^*} & ib_m e^{ik_2 v_{21}} & ib_m e^{-ik_2 w_{21}^*} \\ -ib_m e^{ik_1 w_{12}} & -ib_m e^{-ik_1 v_{12}^*} & ia_{2m} e^{ik_2 u_2} & ia_{2m} e^{-ik_2 s_2^*} \\ ib_m e^{ik_1 v_{12}} & ib_m e^{-ik_1 w_{12}^*} & -ia_{2m} e^{ik_2 s_2} & -ia_{2m} e^{-ik_2 u_2^*} \end{pmatrix}, \quad (30)$$

for the two-channel case. Here

$$\begin{aligned} s_j &= e^{i(N_L - 2m + 1)k_j}, & u_j &= e^{i(N_L - 1)k_j}, \\ v_{ij} &= e^{i(N_L - m)k_i - i(m-1)k_j}, & w_{ij} &= e^{i(N_L - m)k_i + i(m-1)k_j}, \end{aligned} \quad (31)$$

where k_1 and k_2 are defined by Eq. (17a);

$$\begin{aligned} \hat{X}'_L &= \text{diag}(e^{ik_1 N_L}, e^{-ik_1 N_L}, e^{ik_2 N_L}, e^{-ik_2 N_L}, e^{ik_3 N_L}, e^{-ik_3 N_L}) \\ &+ \sum_{m=1}^{N_L} \begin{pmatrix} ia_{1m} e^{ik_1 u_1} & ia_{1m} e^{-ik_1 s_1^*} & ic_m e^{ik_2 w_{21}} & ic_m e^{-ik_2 v_{21}^*} & ig_m e^{ik_3 w_{31}} & ig_m e^{-ik_3 v_{31}^*} \\ -ia_{1m} e^{ik_1 s_1} & -ia_{1m} e^{-ik_1 u_1^*} & -ic_m e^{ik_2 v_{21}} & -ic_m e^{-ik_2 w_{21}^*} & -ig_m e^{ik_3 v_{31}} & -ig_m e^{-ik_3 w_{31}^*} \\ if_m e^{ik_1 w_{12}} & if_m e^{-ik_1 v_{12}^*} & ib_{2m} e^{ik_2 u_2} & ib_{2m} e^{-ik_2 s_2^*} & id_m e^{ik_3 w_{32}} & id_m e^{-ik_3 v_{32}^*} \\ -if_m e^{ik_1 v_{12}} & -if_m e^{-ik_1 w_{12}^*} & -ib_{2m} e^{ik_2 s_2} & -ib_{2m} e^{-ik_2 u_2^*} & -id_m e^{ik_3 v_{32}} & -id_m e^{-ik_3 w_{32}^*} \\ ip_m e^{ik_1 w_{13}} & ip_m e^{-ik_1 v_{13}^*} & iq_m e^{ik_2 w_{23}} & iq_m e^{-ik_2 v_{23}^*} & ia_{3m} e^{ik_3 u_3} & ia_{3m} e^{-ik_3 s_3^*} \\ -ip_m e^{ik_1 v_{13}} & -ip_m e^{-ik_1 w_{13}^*} & -iq_m e^{ik_2 v_{23}} & -iq_m e^{-ik_2 w_{23}^*} & -ia_{3m} e^{ik_3 s_3} & -ia_{3m} e^{-ik_3 u_3^*} \end{pmatrix}, \end{aligned} \quad (32)$$

for the three-channel case with free boundary conditions where k_1 , k_2 , and k_3 in definitions (24) and (31) are given by Eq. (18a); the transfer matrix \hat{X}'_L for the three-channel model with periodic boundary conditions is given by Eq. (32), using definitions (25) of the slice parameters a_{1m} , a_{3m} , b_{2m} , and c_m ; and definition (18b) of the wave numbers k_1 , k_2 , and k_3 .

B. Scattering matrices

The scattering of plane waves (reflection and transmission) at and between the two ends of the random quasi-1D systems is governed by the S matrix,

$$\hat{S} = \begin{pmatrix} \hat{r}^{--} & \hat{t}^{--} \\ \hat{t}^{++} & \hat{r}^{+-} \end{pmatrix}, \quad (33)$$

where

$$\hat{t}^{\mp\mp} = \begin{pmatrix} t_{11}^{\mp\mp} & t_{12}^{\mp\mp} & \dots \\ t_{21}^{\mp\mp} & t_{22}^{\mp\mp} & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}, \quad (34)$$

and

$$\hat{r}^{\pm\mp} = \begin{pmatrix} r_{11}^{\pm\mp} & r_{12}^{\pm\mp} & \cdots \\ r_{21}^{\pm\mp} & r_{22}^{\pm\mp} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (35)$$

Here $t_{ij}^{++}(t_{ij}^{--})$ and $r_{ij}^{+-}(r_{ij}^{-+})$ denote the transmitted and reflected amplitudes in channel i when there is a unit flux incident from the left (right) in channel j . Left to right and right to left directions are denoted by $+$ and $-$, respectively. The S matrix expresses outgoing wave amplitudes in terms of ingoing ones on either side of the quasi-1D disordered wire via the scattering relations^{11,21}

$$\begin{pmatrix} 0 \\ 0' \end{pmatrix} = \hat{S} \begin{pmatrix} I \\ I' \end{pmatrix}. \quad (36)$$

Here I and I' (0 and $0'$) denote ingoing (outgoing) amplitudes at the left and right sides of the disordered region, respectively. It follows from current conservation that, e.g., for a unit flux which is incident from the right in channel i one has

$$\sum_{j=1}^N (|t_{ij}^{--}|^2 + |r_{ji}^{-+}|^2) = 1. \quad (37)$$

Likewise, one also has

$$\sum_{j=1}^N (|t_{ji}^{++}|^2 + |r_{ij}^{+-}|^2) = 1. \quad (37a)$$

Our task is now to derive microscopic realizations of the S matrix in terms of transfer matrices (30) and (32) describing transfer of Bloch waves across finite quasi-1D disordered systems. Let us remark that this may not always be possible as is seen here in the case of the three channel periodic model. Indeed, in this case we are able to identify proper transmission and reflection amplitudes obeying symmetry relations (37) and (37a) only when assuming the random site energies on chains 1 and 2 to be identical,

$$\varepsilon_{1n} = \varepsilon_{2n}, \quad n = 1, 2, \dots, N_L, \quad (38)$$

rather than allowing the energies at all pairs of sites of the quasi-1D system to be uncorrelated, as in Eq. (27). In the absence of the correlation (38), the obtained transport amplitudes cannot be identified as actual reflection and transmission amplitudes of Bloch waves.

If the amplitude at the n th site in a channel j corresponds to a Bloch wave $\psi_n^j = e^{ink_j}$ [Eq. (19)] then the j, n and $j, n-1$ components of wave amplitude vectors,

$$\hat{W}^{-1} \begin{pmatrix} \vdots \\ \psi_n^j \\ \psi_{n-1}^j \\ \vdots \end{pmatrix}$$

(with $\hat{W} \equiv \hat{V}_0$ or \hat{V}), being transferred by the n th slice, have values $a e^{ink_j}$ and 0 , respectively; on the other hand if $\psi_n^j = e^{-ink_j}$ then the j, n and $j, n-1$ components of the above vectors are 0 and $a' e^{-ink_j}$. Hence, in accordance with our

notation above for transmission and reflection matrix elements, we denote the amplitudes at sites j, n and $j, n-1$, respectively, as $a_{j,n}^+$ and $a_{j,n-1}^-$ since they correspond to amplitudes being transferred by the n slice and propagating in the k_j and $-k_j$ directions, respectively. The transforms of the amplitude vectors

$$\begin{pmatrix} \vdots \\ \psi_n^j \\ \psi_{n-1}^j \\ \vdots \end{pmatrix}$$

and

$$\begin{pmatrix} \vdots \\ \psi_{n+1}^j \\ \psi_n^j \\ \vdots \end{pmatrix}$$

are thus rewritten, respectively, as

$$\hat{W}^{-1} \begin{pmatrix} \psi_n^1 \\ \psi_{n-1}^1 \\ \psi_n^2 \\ \psi_{n-1}^2 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} a_{1,n-1}^+ \\ a_{1,n-1}^- \\ a_{2,n-1}^+ \\ a_{2,n-1}^- \\ \vdots \end{pmatrix}$$

and

$$\hat{W}^{-1} \begin{pmatrix} \psi_{n+1}^1 \\ \psi_n^1 \\ \psi_{n+1}^2 \\ \psi_n^2 \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} a_{1,n}^+ \\ a_{1,n}^- \\ a_{2,n}^+ \\ a_{2,n}^- \\ \vdots \end{pmatrix}. \quad (39)$$

Using a similar notation for wave amplitudes transferred from $n=0$ to $n=N_L$ across a disordered wire of length $L = N_L a$, the wave transfer equations in the Bloch representation, obtained by iterating Eqs. (12) and (13), read

$$\begin{pmatrix} a_{1,L}^+ \\ a_{1,L}^- \\ a_{2,L}^+ \\ a_{2,L}^- \end{pmatrix} = \hat{X}_{0L} \begin{pmatrix} a_{1,0}^+ \\ a_{1,0}^- \\ a_{2,0}^+ \\ a_{2,0}^- \end{pmatrix} \quad (40)$$

and

$$\begin{pmatrix} a_{1,L}^+ \\ a_{1,L}^- \\ a_{2,L}^+ \\ a_{2,L}^- \\ a_{3,L}^+ \\ a_{3,L}^- \end{pmatrix} = \hat{Y}_L \begin{pmatrix} a_{1,0}^+ \\ a_{1,0}^- \\ a_{2,0}^+ \\ a_{2,0}^- \\ a_{3,0}^+ \\ a_{3,0}^- \end{pmatrix}, \quad (41)$$

where \hat{X}_{0L} is given in Eq. (30) and \hat{Y}_L stands for \hat{X}'_L and \hat{X}''_L in Eq. (32) with parameters defined by Eqs. (18a) and (24) and (18b) and (25), respectively.

In order to derive the S matrices for our two- and three-channel wire models we first rewrite Eqs. (40) and (41) in the

forms of equations involving outgoing amplitudes on the left side and incoming ones as the right side, as in Eq. (36). In the notation of Eqs. (40) and (41) we have, e.g., for $N=3$,

$$(I) \equiv \begin{pmatrix} a_{1,0}^+ \\ a_{2,0}^+ \\ a_{3,0}^+ \end{pmatrix}, \quad (I') \equiv \begin{pmatrix} a_{1,L}^- \\ a_{2,L}^- \\ a_{3,L}^- \end{pmatrix},$$

$$(O) \equiv \begin{pmatrix} a_{1,0}^- \\ a_{2,0}^- \\ a_{3,0}^- \end{pmatrix}, \quad (O') \equiv \begin{pmatrix} a_{1,L}^+ \\ a_{2,L}^+ \\ a_{3,L}^+ \end{pmatrix}, \quad (42)$$

and so we rearrange Eqs. (40) and (41) in the form

$$\hat{A}_{2,3} \begin{pmatrix} O \\ O' \end{pmatrix} = \hat{B}_{2,3} \begin{pmatrix} I \\ I' \end{pmatrix}, \quad (43)$$

where the pairs of matrices \hat{A}_2 and \hat{B}_2 and \hat{A}_3 and \hat{B}_3 correspond to two- and three-channel cases, respectively. Using the notation $(\hat{X}_{0L})_{ij} \equiv X_{ij}$ and $(\hat{Y}_L)_{ij} \equiv Y_{ij}$ for the matrix elements in Eqs. (30) and (32), respectively, we find

$$\hat{A}_2 = \begin{pmatrix} -X_{12} & -X_{14} & 1 & 0 \\ -X_{22} & -X_{24} & 0 & 0 \\ -X_{32} & -X_{34} & 0 & 1 \\ -X_{42} & -X_{44} & 0 & 0 \end{pmatrix},$$

$$\hat{B}_2 = \begin{pmatrix} X_{11} & X_{13} & 0 & 0 \\ X_{21} & X_{23} & -1 & 0 \\ X_{31} & X_{33} & 0 & 0 \\ X_{41} & X_{43} & 0 & -1 \end{pmatrix}, \quad (44)$$

$$\hat{A}_3 = \begin{pmatrix} -Y_{12} & -Y_{14} & -Y_{16} & 1 & 0 & 0 \\ -Y_{22} & -Y_{24} & -Y_{26} & 0 & 0 & 0 \\ -Y_{32} & -Y_{34} & -Y_{36} & 0 & 1 & 0 \\ -Y_{42} & -Y_{44} & -Y_{46} & 0 & 0 & 0 \\ -Y_{52} & -Y_{54} & -Y_{56} & 0 & 0 & 1 \\ -Y_{62} & -Y_{64} & -Y_{66} & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{B}_3 = \begin{pmatrix} Y_{11} & Y_{13} & Y_{15} & 0 & 0 & 0 \\ Y_{21} & Y_{23} & Y_{25} & -1 & 0 & 0 \\ Y_{31} & Y_{33} & Y_{35} & 0 & 0 & 0 \\ Y_{41} & Y_{43} & Y_{45} & 0 & -1 & 0 \\ Y_{51} & Y_{53} & Y_{55} & 0 & 0 & 0 \\ Y_{61} & Y_{63} & Y_{65} & 0 & 0 & -1 \end{pmatrix}. \quad (45)$$

The S matrices for the two- and three channel cases are then given by $\hat{S} = \hat{A}_2^{-1} \hat{B}_2$ and $\hat{S} = \hat{A}_3^{-1} \hat{B}_3$, respectively. After inverting \hat{A}_2 and \hat{A}_3 we finally obtain, successively for the two- and three-channel cases,

$$\hat{S} = \frac{1}{\delta} \begin{pmatrix} \delta_1 & \delta_2 & X_{44} & -X_{24} \\ \delta_3 & \delta_4 & -X_{42} & X_{22} \\ X_{11}\delta + X_{21}\delta_5 & X_{13}\delta + X_{23}\delta_5 & -\delta_5 & -\delta_6 \\ +X_{41}\delta_6 & +X_{43}\delta_6 & & \\ X_{31}\delta + X_{21}\delta_7 & X_{33}\delta + X_{23}\delta_7 & -\delta_7 & -\delta_8 \\ +X_{41}\delta_8 & +X_{43}\delta_8 & & \end{pmatrix}, \quad (46)$$

where

$$\delta = X_{22}X_{44} - X_{24}X_{42}, \quad \delta_1 = X_{24}X_{41} - X_{44}X_{21},$$

$$\delta_2 = X_{24}X_{43} - X_{44}X_{23},$$

$$\delta_3 = X_{42}X_{21} - X_{41}X_{22}, \quad \delta_4 = X_{42}X_{23} - X_{22}X_{43},$$

$$\delta_5 = X_{42}X_{14} - X_{12}X_{44}, \quad (47)$$

$$\delta_6 = X_{12}X_{24} - X_{22}X_{14}, \quad \delta_7 = X_{42}X_{34} - X_{32}X_{44},$$

$$\delta_8 = X_{32}X_{24} - X_{22}X_{34},$$

are second order subdeterminants of \hat{X}_{0L} ;

$$\hat{S} = \begin{pmatrix} \hat{S}_1 & \hat{S}_3 \\ \hat{S}_2 & \hat{S}_4 \end{pmatrix}, \quad (48)$$

where

$$\hat{S}_1 = \frac{1}{\Delta} \begin{pmatrix} -\beta_1 Y_{21} - \beta_4 Y_{41} - \beta_7 Y_{61} & -\beta_1 Y_{23} - \beta_4 Y_{43} - \beta_7 Y_{63} & -\beta_1 Y_{25} - \beta_4 Y_{45} - \beta_7 Y_{65} \\ \beta_2 Y_{21} + \beta_5 Y_{41} + \beta_8 Y_{61} & \beta_2 Y_{23} + \beta_5 Y_{43} + \beta_8 Y_{63} & \beta_2 Y_{25} + \beta_5 Y_{45} + \beta_8 Y_{65} \\ -\beta_3 Y_{21} - \beta_6 Y_{41} - \beta_9 Y_{61} & -\beta_3 Y_{23} - \beta_6 Y_{43} - \beta_9 Y_{63} & -\beta_3 Y_{25} - \beta_6 Y_{45} - \beta_9 Y_{65} \end{pmatrix} \quad (48a)$$

$$\hat{S}_2 = \frac{1}{\Delta} \begin{pmatrix} Y_{11}\Delta + Y_{21}\Delta_1 + Y_{41}\Delta_4 + Y_{61}\Delta_7 & Y_{13}\Delta + Y_{23}\Delta_1 + Y_{43}\Delta_4 + Y_{63}\Delta_7 & Y_{15}\Delta + Y_{25}\Delta_1 + Y_{45}\Delta_4 + Y_{65}\Delta_7 \\ Y_{31}\Delta - Y_{21}\Delta_2 - Y_{41}\Delta_5 - Y_{61}\Delta_8 & Y_{33}\Delta - Y_{23}\Delta_2 - Y_{43}\Delta_5 - Y_{63}\Delta_8 & Y_{35}\Delta - Y_{25}\Delta_2 - Y_{45}\Delta_5 - Y_{65}\Delta_8 \\ Y_{51}\Delta + Y_{21}\Delta_3 + Y_{41}\Delta_6 + Y_{61}\Delta_9 & Y_{53}\Delta + Y_{23}\Delta_3 + Y_{43}\Delta_6 + Y_{63}\Delta_9 & Y_{55}\Delta + Y_{25}\Delta_3 + Y_{45}\Delta_6 + Y_{65}\Delta_9 \end{pmatrix} \quad (48b)$$

$$\hat{S}_3 = \frac{1}{\Delta} \begin{pmatrix} \beta_1 & \beta_4 & \beta_7 \\ -\beta_2 & -\beta_5 & -\beta_8 \\ \beta_3 & \beta_6 & \beta_9 \end{pmatrix}, \quad (48c)$$

$$\hat{S}_4 = \frac{1}{\Delta} \begin{pmatrix} -\Delta_1 & -\Delta_4 & -\Delta_7 \\ \Delta_2 & \Delta_5 & \Delta_8 \\ -\Delta_3 & -\Delta_6 & -\Delta_9 \end{pmatrix}, \quad (48d)$$

which involve subdeterminants of second order of \hat{Y}_L ,

$$\begin{aligned} \beta_1 &= Y_{46}Y_{64} - Y_{44}Y_{66}, & \beta_2 &= Y_{62}Y_{46} - Y_{42}Y_{66}, \\ \beta_3 &= Y_{62}Y_{44} - Y_{42}Y_{64}, \\ \beta_4 &= Y_{24}Y_{66} - Y_{64}Y_{26}, & \beta_5 &= Y_{22}Y_{66} - Y_{62}Y_{26}, \\ \beta_6 &= Y_{22}Y_{64} - Y_{62}Y_{24}, & (48e) \\ \beta_7 &= Y_{44}Y_{26} - Y_{24}Y_{46}, & \beta_8 &= Y_{42}Y_{26} - Y_{22}Y_{46}, \\ \beta_9 &= Y_{24}Y_{42} - Y_{22}Y_{44}; \end{aligned}$$

as well as third order subdeterminants of \hat{A}_3 which result from the minors of various elements:

$$\begin{aligned} \Delta_1 &= \min(\hat{A}_3)_{24}, & \Delta_2 &= \min(\hat{A}_3)_{25}, \\ \Delta_3 &= \min(\hat{A}_3)_{26}, & \Delta_4 &= \min(\hat{A}_3)_{44}, & \Delta_5 &= \min(\hat{A}_3)_{45}, \\ \Delta_6 &= \min(\hat{A}_3)_{46}, & \Delta_7 &= \min(\hat{A}_3)_{64}, & \Delta_8 &= \min(\hat{A}_3)_{65}, \\ \Delta_9 &= \min(\hat{A}_3)_{66}, & \Delta &= \det \hat{A}_3. \end{aligned} \quad (48f)$$

The partial matrices \hat{S}_j , $j=1,2,3,4$ in Eq. (48) clearly correspond to reflection and transmission matrices in Eq. (33).

Matrices (46) and (48) (48a)–(48d), are S -matrix expressions for the two- and three-channel quasi-1D systems in terms of characteristic quantum channel wave numbers and of the tight-binding quantities in the transfer matrices (30) and (32) for weak disorder. Identification of these expressions with Eqs. (33)–(35) yields the transmission- and reflection matrices for these quasi-1D disordered models under the proviso that the symmetry relation [Eqs. (37) and (37a)] are obeyed.

IV. RESULTS AND DISCUSSION

The results for the transmission- and reflection-matrices of two- and three-channel tight-binding wires are applied in this section for finding the averaged transmission and reflection coefficients associated with the various channels, for weak disorder. These results allow us, in particular, to explicitly check the symmetry property [Eq. (37)] in the two-channel case, as well as in the three-channel case with free boundary conditions. On the other hand, in the three-channel case with periodic boundary conditions we show that Eq. (37) is obeyed if one restricts the disorder to a correlated site energy disorder with identical site energies [Eqs. (38)] on chains 1 and 2 and independent random energies on chain 3.

The results for the averaged transmission coefficients are used for obtaining the length of the exponential localization from Eqs. (11) and (1). Our exact quantum expressions of localization lengths for weak disorder reduce to the well-known Thouless expression for a 1D chain in the limit of vanishing interchain coupling ($h \rightarrow 0$). It is useful, before presenting our results, to briefly recall the derivation of Thouless' result from the transfer matrix approach.¹⁹ For a single disordered chain of length L the localization length is given by

$$\frac{1}{L_c} = - \lim_{N_L \rightarrow \infty} (2N_L)^{-1} \langle \ln |t|^2 \rangle, \quad (49)$$

where $|t|^2 = |t^{--}|^2 = |t^{++}|^2$ is the transmission coefficient which is related to the two-dimensional transfer matrices \hat{X}_n (in the Bloch wave representation) for thin sections enclosing the n th site by¹⁹

$$\frac{1}{t^{--}} = \left(\prod_{n=1}^{N_L} \hat{X}_n \right)_{22}. \quad (50)$$

This relation follows by transforming the S matrix for scattering states into a transfer matrix whose 22 element is $1/t^{--}$. Expanding the transfer matrix for the whole chain, $\prod_{n=1}^{N_L} \hat{X}_n$, to first order in the uncorrelated site energies and performing the average in Eq. (49), using Eq. (27), yields

$$\frac{1}{L_c} \equiv \frac{1}{\xi} = \frac{\varepsilon_0^2}{8 \sin^2 k}, \quad E = 2 \cos k, \quad (51)$$

which may be related to Thouless' expression for the localization length in a tight-binding chain with a rectangular distribution of site energies of width W , centered at zero mean.¹³ Indeed, by identifying ε_0^2 with the second moment, $W^2/12$, of the rectangular distribution Eq. (51) reduces exactly to Thouless' formula, see Eq. (6.16), p. 31 of Ref. 3.

For convenience of the following discussion for two- and three-channel systems, the explicit forms of the transmission and reflection coefficients obtained by identifying the S matrix [Eqs. (33)–(35)] successively with Eqs. (46) and (48), (48a)–(48d), and replacing the transfer matrix elements entering in these expressions by their explicit forms in Eqs. (30)–(32) are given in the Appendix.

A. Two-channel wires

By averaging the partial transmission- and reflection coefficients given by Eqs. (A1)–(A4) over the disorder, using Eqs. (22a) and (27), we obtain, successively,

$$\langle |t_{11}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{8} \left(\frac{1}{\sin^2 k_1} + \frac{2}{\sin k_1 \sin k_2} \right), \quad (52)$$

$$\langle |t_{22}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{8} \left(\frac{1}{\sin^2 k_2} + \frac{2}{\sin k_1 \sin k_2} \right), \quad (53)$$

$$\langle |t_{12}^{--}|^2 \rangle = \langle |t_{21}^{--}|^2 \rangle = \frac{N_L \varepsilon_0^2}{8 \sin k_1 \sin k_2}, \quad (54)$$

$$\langle |r_{11}^{-+}|^2 \rangle = \frac{N_L \varepsilon_0^2}{8 \sin^2 k_1}, \quad (55)$$

$$\langle |r_{22}^{-+}|^2 \rangle = \frac{N_L \varepsilon_0^2}{8 \sin^2 k_2}, \quad (56)$$

$$\langle |r_{12}^{-+}|^2 \rangle = \langle |r_{21}^{-+}|^2 \rangle = \frac{N_L \varepsilon_0^2}{8 \sin k_1 \sin k_2}. \quad (57)$$

Expressions (52)–(57) are consistent with the symmetry relations [Eqs. (37)] resulting from current conservation.

The inverse localization length for weak disorder is obtained by expanding (11) to lowest order in the random site energies using Eqs. (1) and (52)–(54). It is given by

$$\frac{1}{L_c} \equiv \frac{1}{L_{0c}} = \frac{\varepsilon_0^2}{32} \left(\frac{1}{\sin k_1} + \frac{1}{\sin k_2} \right)^2 \quad (58)$$

for energies E restricted to the Bloch bands $E = h + 2 \cos k_1$ and $E = -h + 2 \cos k_2$. This exact expression for weak disorder reveals three important properties.

(1) It proves microscopically that all states in the Bloch energy bands of two-channel quasi-1D disordered systems are localized.

(2) In the absence of interchain hopping ($h = 0$) it reduces to the localization length [Eq. (51)] for a 1D chain described by the Anderson model. For weak interchain hopping Eq. (58) becomes

$$\frac{1}{L_{0c}} = \frac{\varepsilon_0^2}{8 \sin^2 k} \left[1 + \frac{h^2}{4 \sin^2 k} (1 + 3 \cot^2 k) + O(h^4) \right], \quad (59)$$

which is valid for energies sufficiently close to the band center of the $h = 0$ energy band, $E = 2 \cos k$. This shows that a weak interchain hopping enhances localization in comparison to the purely 1D case, i.e., $L_{0c} < \xi$.

(3) For large interchain hopping rates, i.e., $|h| \gg |E|$ (where E is of the order of the fermi energy) we have $\sin k_{1,2} \approx \sqrt{1 - h^2/4}$, $|h|/2 \leq 1$, which yields

$$\frac{1}{L_c} \approx \frac{\varepsilon_0^2}{2} \frac{1}{4 - h^2} = \frac{1}{\xi_0} \frac{4}{4 - h^2}, \quad (60)$$

where $\xi_0 = \varepsilon_0^2/8$ is the 1D localization length [Eq. (51)] at the band center. Thus at large hopping rates (with $|h| < 2$) localization is also enhanced in comparison to 1D localization.

B. Three-channel wires

1. Free boundary conditions

The evaluation of the disorder averages of the transmission- and reflection-coefficients in Eqs. (A5) and (A6), using Eq. (27) and the explicit expressions in Eqs. (A7)–(A9) with the tight-binding parameters [Eq. (24)] yields, to order ε_0^2 ,

$$\langle |t_{11}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{32} \left(\frac{3}{\sin^2 k_1} + \frac{6}{\sin k_1 \sin k_3} + \frac{4}{\sin k_1 \sin k_2} \right), \quad (61)$$

$$\langle |t_{22}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{32} \left(\frac{4}{\sin^2 k_2} + \frac{4}{\sin k_2 \sin k_3} + \frac{4}{\sin k_1 \sin k_2} \right), \quad (62)$$

$$\langle |t_{33}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{32} \left(\frac{3}{\sin^2 k_3} + \frac{6}{\sin k_1 \sin k_3} + \frac{4}{\sin k_2 \sin k_3} \right), \quad (63)$$

$$\langle |t_{12}^{--}|^2 \rangle = \langle |t_{21}^{--}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{32 \sin k_1 \sin k_2}, \quad (64)$$

$$\langle |t_{13}^{--}|^2 \rangle = \langle |t_{31}^{--}|^2 \rangle = \frac{3N_L \varepsilon_0^2}{32 \sin k_1 \sin k_3}, \quad (65)$$

$$\langle |t_{23}^{--}|^2 \rangle = \langle |t_{32}^{--}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{32 \sin k_2 \sin k_3}, \quad (66)$$

$$\langle |r_{11}^{-+}|^2 \rangle = \frac{3N_L \varepsilon_0^2}{32 \sin^2 k_1}, \quad (67)$$

$$\langle |r_{22}^{-+}|^2 \rangle = \frac{4N_L \varepsilon_0^2}{32 \sin^2 k_2}, \quad (68)$$

$$\langle |r_{33}^{-+}|^2 \rangle = \frac{3N_L \varepsilon_0^2}{32 \sin^2 k_3}, \quad (69)$$

$$\langle |r_{12}^{-+}|^2 \rangle = \langle |r_{21}^{-+}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{32 \sin k_1 \sin k_2}, \quad (70)$$

$$\langle |r_{13}^{-+}|^2 \rangle = \langle |r_{31}^{-+}|^2 \rangle = \frac{3N_L \varepsilon_0^2}{32 \sin k_1 \sin k_3}, \quad (71)$$

$$\langle |r_{23}^{-+}|^2 \rangle = \langle |r_{32}^{-+}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{32 \sin k_2 \sin k_3}. \quad (72)$$

Again, expressions (61)–(72) obey the current conservation property [Eq. (37)] for the three channel case: the reduction of the intrachannel transmission coefficients due to scattering by the disordered wire is exactly compensated by the occurrence of inter-channel transmissions and by reflections.

For the inverse localization length we obtain, from Eqs. (11), (1), and (61)–(66),

$$\begin{aligned} \frac{1}{L_c} = \frac{\varepsilon_0^2}{64} & \left(\frac{1}{\sin^2 k_1} + \frac{4}{3} \frac{1}{\sin^2 k_2} + \frac{1}{\sin^2 k_3} + \frac{4}{3 \sin k_1 \sin k_2} \right. \\ & \left. + \frac{4}{3 \sin k_2 \sin k_3} + \frac{2}{\sin k_1 \sin k_3} \right), \quad (73) \end{aligned}$$

where k_1 , k_2 , and k_3 are defined by Eq. (18a). Like Eq. (58) this expression is exact to order ε_0^2 for weak disorder. It demonstrates that the eigenstates in the Bloch bands of the three-channel quasi-1D system with free boundary condi-

tions are localized. It too reduces to the Thouless result [Eq. (51)] in one dimensional for vanishing interchain hopping. For small values of $|h|$, in particular for $|h| \ll 1$ for $E \rightarrow 0$, Eq. (73) reduces to

$$\frac{1}{L_c} = \frac{\varepsilon_0^2}{8 \sin^2 k_2} \left[1 + \frac{h^2}{\sin^2 k_2} \left(1 + \frac{1}{2} \cos^2 k_2 + \frac{1}{4} \cot^2 k_2 \right) + O(h^4) \right], \quad (74)$$

and for large values ($h^2 \gg E^2/2$), restricted to $|h| < 2$, it becomes

$$\frac{1}{L_c} \approx \frac{\varepsilon_0^2}{8} \frac{1}{2-h^2}, \quad (75)$$

which shows, in particular, that L_c is increased with respect to the 1D value at the band center (ξ_0) for $|h|$ (in units of the interchain hopping rate) less than 1. Thus the domain of interchain hopping rates in which Eq. (73) leads to localization lengths larger than ξ_0 is defined by

$$|E| \ll \sqrt{2}|h| < \sqrt{2}, \quad (76)$$

2. Periodic boundary conditions

We first note that in this case the S matrix constructed from the transfer matrix \hat{X}_L'' in Sec. III is unitary [i.e., Eqs. (37) and (37a) are verified] only in the case where Eq. (38) is obeyed, i.e., when chains 1 and 2 have identical random site energies while the random site energies on chain 3 are arbitrary. This may be readily seen by expressing the averages of the transmission- and reflection-coefficients in Eqs. (A5) and (A6) to second order in the site energies in terms of the averages of Eqs. (A7)–(A9) assuming that site energies belonging to sites $m \neq n$ on the same chain or on different ones are uncorrelated i.e., $\langle \varepsilon_{im} \varepsilon_{jn} \rangle = 0$. In this way we find

$$\sum_{i,j=1}^3 (\langle |t_{ij}^{--}|^2 \rangle + \langle |r_{ij}^{--}|^2 \rangle) = 3 + 2N_L [(\langle c_n^2 \rangle + \langle f_n^2 \rangle - 2\langle c_n f_n \rangle) + (\langle d_n^2 \rangle + \langle q_n^2 \rangle - 2\langle d_n q_n \rangle) + (\langle g_n^2 \rangle + \langle p_n^2 \rangle - 2\langle g_n p_n \rangle)]. \quad (77)$$

Now, for this relation to be compatible with current conservation we require

$$c_n = f_n, \quad d_n = q_n \quad \text{and} \quad g_n = p_n, \quad (78)$$

and from definition (25) it follows that the equalities (78) are fulfilled with Eq. (38) i.e., for identical site energies in the chains 1 and 2. Only under this condition does the S matrix for the periodic 3-channel system represent a true scattering matrix.

We note that (78) and (38) may also be obtained as the condition of simplicity of a transfer matrix for periodic multichannel systems of the form of that defined in Refs. 18. This latter condition is thus an alternative expression of current conservation.

The explicit second-order expressions for the averaged transmission- and reflection-coefficients obtained from Eqs. (A5)–(A9) and (25), for random gaussian site energies [Eq. (27)] in the presence of the correlation [Eq. (38)] are

$$\langle |t_{11}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{36} \left(\frac{5}{\sin^2 k_1} + \frac{8}{\sin k_1 \sin k_2} \right), \quad (79)$$

$$\langle |t_{22}^{--}|^2 \rangle = \langle |t_{33}^{--}|^2 \rangle = 1 - \frac{N_L \varepsilon_0^2}{36} \left(\frac{5}{\sin^2 k_2} + \frac{4}{\sin^2 k_2} + \frac{4}{\sin k_1 \sin k_2} \right), \quad (80)$$

$$\langle |t_{12}^{--}|^2 \rangle = \langle |t_{21}^{--}|^2 \rangle = \langle |t_{13}^{--}|^2 \rangle = \langle |t_{31}^{--}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{36} \frac{1}{\sin k_1 \sin k_2}, \quad (81)$$

$$\langle |t_{23}^{--}|^2 \rangle = \langle |t_{32}^{--}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{36} \frac{1}{\sin^2 k_2}, \quad (82)$$

$$\langle |r_{11}^{--}|^2 \rangle = \frac{5N_L \varepsilon_0^2}{36} \frac{1}{\sin^2 k_1}, \quad (83)$$

$$\langle |r_{22}^{--}|^2 \rangle = \langle |r_{33}^{--}|^2 \rangle = \frac{5N_L \varepsilon_0^2}{36} \frac{1}{\sin^2 k_2}, \quad (84)$$

$$\langle |r_{12}^{--}|^2 \rangle = \langle |r_{21}^{--}|^2 \rangle = \langle |r_{13}^{--}|^2 \rangle = \langle |r_{31}^{--}|^2 \rangle = \frac{N_L \varepsilon_0^2}{36} \frac{2}{\sin k_1 \sin k_2}$$

$$\langle |r_{23}^{--}|^2 \rangle = \langle |r_{32}^{--}|^2 \rangle = \frac{2N_L \varepsilon_0^2}{36} \frac{1}{\sin^2 k_2}, \quad (85)$$

The inverse localization length associated with the conductance [Eq. (1)] obtained from Eqs. (78)–(81), is

$$\frac{1}{L_c} = \frac{\varepsilon_0^2}{216} \left(\frac{5}{\sin^2 k_1} + \frac{14}{\sin^2 k_2} + \frac{8}{\sin k_1 \sin k_2} \right), \quad (86)$$

where k_1 and k_2 are defined by Eq. (18b). This expression which proves localization in periodic three-channel systems, reduces again to the 1D result [Eq. (51)] for $h=0$. For small h it is (with $E=2 \cos k$)

$$\frac{1}{L_c} \approx \frac{\varepsilon_0^2}{8 \sin^2 k} \left[1 + \frac{h^2}{2 \sin^2 k} \left(1 + \frac{70}{27} \cot^2 k \right) \right], \quad (87)$$

which shows enhanced localization for weak interchain hopping. On the other hand for $|h| \gg |E|$ Eq. (86) becomes

$$\frac{1}{L_c} \approx \frac{5 \varepsilon_0^2}{216} \frac{1}{1-h^2}, \quad (88)$$

which leads to increased localization lengths, $L_c > \xi_0$, for $h^2 < 22/27$. Thus the domain where the localization length in the periodic three-channel system is larger than the 1D value is defined by

$$|E| \ll |h| < 0.906 \dots \quad (89)$$

Our exact microscopic results for two- and three-channel systems indicate that in all cases a weak interchain hopping decreases the localization length from its 1D value. For strong interchain hopping a similar decrease persists in the two-channel case while in the three-channel cases the localization length increases with respect to the 1D result over restricted domains of hopping parameters defined by Eqs. (76) and (89). This suggests the possible existence of quasi-metallic domains for the three-channel systems in some ranges of length scales lying between their localization lengths and the 1D localization length [of the order of the mean free path l (Ref. 10)], in the considered ranges of large hopping. In contrast, we recall that for many-channel random wires ($N \gg 1$) the metallic domain exists over a wide range of mesoscopic lengths defined by Eq. (5).

If one wishes to extend the above studies of localization lengths based on Eqs. (1) and (11) to quasi-1D systems with larger numbers of scattering channels one will generally have to resort, e.g., to numerical transfer matrix methods. These techniques together with the difficulties inherent to their application are reviewed in Ref. 18.

APPENDIX

For brevity's sake we only discuss explicit expressions for transmission amplitudes t_{ij}^{--} and reflection amplitudes r_{ij}^{--} relating to outgoing waves to the left of the disordered sample in the scattering equation (36) [see (33) and (42)].

1. Two-channel wires

Identification of Eqs. (33) and (46) yields

$$\begin{aligned} |t_{11}^{--}|^2 &= \frac{|X_{44}|^2}{|\delta|^2}, & |t_{12}^{--}|^2 &= \frac{|X_{24}|^2}{|\delta|^2}, \\ |t_{21}^{--}|^2 &= \frac{|X_{42}|^2}{|\delta|^2}, & |t_{22}^{--}|^2 &= \frac{|X_{22}|^2}{|\delta|^2}, \end{aligned} \quad (A1)$$

and, to lowest order in the random site energies, using Eqs. (47) and (30),

$$\begin{aligned} |r_{11}^{--}|^2 &= |X_{21}|^2, & |r_{12}^{--}|^2 &= |X_{23}|^2, \\ |r_{21}^{--}|^2 &= |X_{41}|^2, & |r_{22}^{--}|^2 &= |X_{43}|^2. \end{aligned} \quad (A2)$$

From the definition of δ in Eq. (47) and the explicit form of the elements X_{ij} in Eqs. (30) and (31), we obtain

$$\begin{aligned} |\delta|^2 &= 1 + \sum_{m,n=1}^{N_L} [a_{1m}a_{1n} + a_{2m}a_{2n} \\ &\quad + 2b_m b_n \cos(m-n)(k_1 - k_2)], \\ |X_{22}|^2 &= 1 + \sum_{m,n} a_{1m}a_{1n}, & |X_{44}|^2 &= 1 + \sum_{m,n} a_{2m}a_{2n}, \end{aligned} \quad (A3)$$

$$|X_{12}|^2 = \sum_{m,n} a_{1m}a_{2n} \cos 2(m-n)k_1,$$

$$|X_{24}|^2 = |X_{42}|^2 = \sum_{m,n} b_m b_n \cos(m-n)(k_1 - k_2),$$

$$|r_{11}^{--}|^2 = \sum_{m,n} a_{1m}a_{1n} \cos 2(m-n)k_1,$$

$$|r_{12}^{--}|^2 = \sum_{m,n} b_m b_n \cos(m-n)(k_1 + k_2),$$

$$|r_{21}^{--}|^2 = \sum_{m,n} b_m b_n \cos(m-n)(k_1 + k_2),$$

$$|r_{22}^{--}|^2 = \sum_{m,n} a_{2m}a_{2n} \cos 2(m-n)k_2.$$

2. Three-channel wires

The identification of Eq. (48c) with the transmission matrix \hat{t}^{--} in Eq. (34) yields

$$\begin{aligned} |t_{11}^{--}|^2 &= \frac{|\beta_1|^2}{|\Delta|^2}, & |t_{12}^{--}|^2 &= \frac{|\beta_4|^2}{|\Delta|^2}, & |t_{13}^{--}|^2 &= \frac{|\beta_7|^2}{|\Delta|^2}, \\ |t_{21}^{--}|^2 &= \frac{|\beta_2|^2}{|\Delta|^2}, & |t_{22}^{--}|^2 &= \frac{|\beta_5|^2}{|\Delta|^2}, & |t_{23}^{--}|^2 &= \frac{|\beta_8|^2}{|\Delta|^2}, \\ |t_{31}^{--}|^2 &= \frac{|\beta_3|^2}{|\Delta|^2}, & |t_{32}^{--}|^2 &= \frac{|\beta_6|^2}{|\Delta|^2}, & |t_{33}^{--}|^2 &= \frac{|\beta_9|^2}{|\Delta|^2}. \end{aligned} \quad (A5)$$

Similarly, the reflection amplitudes r_{ij}^{--} are obtained by identifying \hat{S}_1 in Eq. (48a) with Eq. (35). For analyzing the reflection coefficients for weak disorder it suffices to find the amplitudes to linear order in the site energies. Such linear contributions are associated with β_j terms having a zeroth order contribution (of modulus one). From Eqs. (48a) and (48e) we thus obtain

$$\begin{aligned} |r_{11}^{--}|^2 &= |Y_{21}|^2, & |r_{12}^{--}|^2 &= |Y_{23}|^2, & |r_{13}^{--}|^2 &= |Y_{25}|^2, \\ |r_{21}^{--}|^2 &= |Y_{41}|^2, & |r_{22}^{--}|^2 &= |Y_{43}|^2, & |r_{23}^{--}|^2 &= |Y_{45}|^2, \\ |r_{31}^{--}|^2 &= |Y_{61}|^2, & |r_{32}^{--}|^2 &= |Y_{63}|^2, & |r_{33}^{--}|^2 &= |Y_{65}|^2. \end{aligned} \quad (A6)$$

The explicit expressions of the $|\beta_j|^2$ and $|\Delta|^2$ in Eqs. (A5) [defined in Eqs. (48e) and (48f)] and of the reflection coefficients (A6) in terms of the transfer matrix elements in Eq. (32) are given by

$$\begin{aligned} |\beta_1|^2 &= 1 + \sum_{m,n} [a_{3m}a_{3n} + b_{2m}b_{2n} \\ &\quad + 2d_m q_n \cos(m-n)(k_3 - k_2)], \\ |\beta_5|^2 &= 1 + \sum_{m,n} [a_{1m}a_{1n} + a_{3m}a_{3n} \\ &\quad + 2g_m p_n \cos(m-n)(k_3 - k_1)], \end{aligned}$$

$$\begin{aligned}
|\beta_9|^2 &= 1 + \sum_{m,n} [a_{1m}a_{1n} + b_{2m}b_{2n} \\
&\quad + 2c_m f_n \cos(m-n)(k_2 - k_1)], \\
|\beta_2|^2 &= \sum_{m,n} f_m f_n \cos(m-n)(k_1 - k_2), \\
|\beta_4|^2 &= \sum_m c_m c_n \cos(m-n)(k_1 - k_2), \\
|\beta_3|^2 &= \sum_{m,n} p_m p_n \cos(m-n)(k_1 - k_3), \\
|\beta_7|^2 &= \sum_{m,n} g_m g_n \cos(m-n)(k_1 - k_3), \\
|\beta_6|^2 &= \sum_{m,n} q_m q_n \cos(m-n)(k_2 - k_3), \\
|\beta_8|^2 &= \sum_{m,n} d_m d_n \cos(m-n)(k_2 - k_3),
\end{aligned} \tag{A7}$$

$$\begin{aligned}
|\Delta|^2 &= 1 + \sum_{m,n} [a_{1m}a_{1n} + a_{3m}a_{3n} + b_{2m}b_{2n} + 2g_m p_n \\
&\quad \times \cos(m-n)(k_3 - k_4) + 2d_m q_n \cos(m-n)(k_3 - k_2) \\
&\quad + 2c_m f_n \cos(m-n)(k_2 - k_1)], \\
|r_{11}^{-+}|^2 &= \sum_{m,n} a_{1m}a_{1n},
\end{aligned} \tag{A8}$$

$$\begin{aligned}
|r_{22}^{-+}|^2 &= \sum_{m,n} b_{2m}b_{2n} \cos 2(m-n)k_2, \\
|r_{33}^{-+}|^2 &= \sum_{m,n} a_{3m}a_{3n} \cos 2(m-n)k_3, \\
|r_{12}^{-+}|^2 &= \sum_{m,n} c_m c_n \cos(m-n)(k_1 + k_2), \\
|r_{13}^{-+}|^2 &= \sum_{m,n} g_m g_n \cos(m-n)(k_1 + k_3), \\
|r_{21}^{-+}|^2 &= \sum_{m,n} f_m f_n \cos(m-n)(k_1 + k_2), \\
|r_{23}^{-+}|^2 &= \sum_{m,n} d_m d_n \cos(m-n)(k_2 + k_3), \\
|r_{31}^{-+}|^2 &= \sum_{m,n} p_m p_n \cos(m-n)(k_1 + k_3), \\
|r_{32}^{-+}|^2 &= \sum_{m,n} q_m q_n \cos(m-n)(k_2 + k_3),
\end{aligned} \tag{A9}$$

where the double summations run from $m=1$ to $m=N_L$ and $n=1$ to $n=N_L$. The site-dependent tight-binding parameters a_{1j} , a_{3j} , b_{2j} , c_j , d_j , f_j , g_j , p_j , and q_j in the above expressions are defined by Eq. (24) for free boundary conditions and by Eq. (25) for periodic boundary conditions. Likewise the wave numbers k_1 , k_2 , and k_3 are given by Eqs. (18a) and by (18b) for free and periodic boundary conditions, respectively.

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