Mesoscopic circuits with charge discreteness: Quantum current magnification for mutual inductances

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Current magnification is studied for a system of two rings with external magnetic flux; we have considered, in addition to self-inductance, a mutual inductance between the rings. The system is studied using a method recently proposed by Li and Chen, which takes into account the charge quantization of the system, allowing for a simplified description. We find that, for some values of the external flux enclosed by one of the rings, quantum current magnification exists in the other ring. This magnification effect is a purely quantum phenomenon, which is given here an alternative explanation, different from the detailed quantum-mechanical explanation.

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I. INTRODUCTION

Mesoscopic physics deals with the frontiers between classical and quantum physics. Phenomena such as persistent currents, Coulomb blockage, magnetoresistance fluctuations, and others, are currently studied in this area. In a recent paper, Li and Chen[1] considered explicitly the effects of charge discreteness in mesoscopic circuits, finding a simple way to obtain many of the results found in mesoscopic physics. The main point of their theory is the inclusion of charge discreteness as an essential element of the discussion. In Ref. 2, an application and extension of the above theory was considered; in this work, quantum transmission lines and their modes of propagation, called *cirquitons*, were studied. Cirquitons are found to be closely related to charge discreteness, and are described by non linear equations. The inclussion of electrical resistance,[2] very important for electrical circuits, was also considered, using an analogy with the Caldirola-Kanai[3–5] theory of quantum dissipation.

The study of the persistent current in metallic rings enclosing magnetic flux[6] is the paradigm of mesoscopic physics. These currents are related to coherence effects (elastic scattering) within the ring; the existence of these persistent currents has been verified experimentally.[7–9] A surprising effect related to these currents is the so-called quantum current magnification.[10–13] In this case, two rings with current are coupled magnetically through the magnetic flux, where the current induced in one of the rings is larger than the current in the other. In fact, that system is described by a ring with a bubble (second ring). The ring encloses an external magnetic flux, while the bubble does not enclose any external flux; for some values of the external flux, quantum current magnification exists in the bubble. As pointed out in Refs. 10–13, this is a purely quantum effect which has no classical analog, since it depends on the charge discreteness and the Planck constant.

In this work we consider two metallic rings enclosing external fluxes (see Fig. 1). The rings have self-inductances L_1 and L_2 and mutual inductance *M*. Moreover, discreteness of the charge is considered in a similar manner as it is done in previous work.[1,2] Current magnification is obtained as a consequence of charge discreteness and quantum effects.

Circuit quantization with continuous charge[14] is straightforward. In fact, if Q denotes the charge flowing through a section of a circuit and ϕ is the internal magnetic flux through the inductance, then quantization follows from the usual rules, i.e.,

$$Q \to Q; \quad \phi \to i\hbar \frac{\partial}{\partial Q}. \tag{1}$$

However, when charge discreteness is assumed, the quantization process must be reconsidered. In this case, the charge operator assumes discrete eigenvalues of the form $q_l = l q_e$, where l is an integer and q_e is the elementary charge which is a fixed parameter. So, roughly speaking, the derivative must be adequately replaced by a finite difference of step size q_e (see Ref. 1 for a rigorous treatment). To be explicit, consider

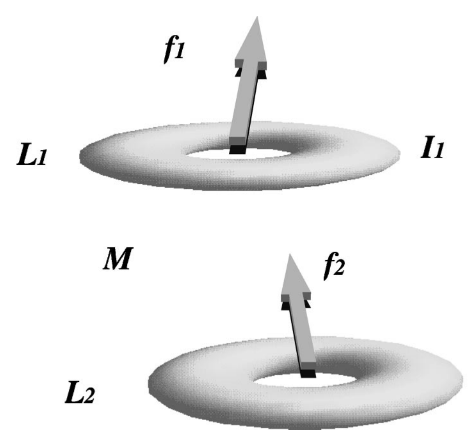

FIG. 1. A drawing of a simple two ring system. The first ring has self-inductance L_1, and it encloses an external flux f_1. The second ring has self-inductance L_2 and encloses a flux f_2. The mutual inductance is *M*.

an inductance L in the continuous charge case [Eq. (1)] the magnetic energy operator $\hat{T}$ is given by

$$\hat{T}=\frac{1}{2L}\hat{\phi}^2, \tag{2}$$

then, from Refs. 1, 2 charge discreteness is obtained from the formal change

$$\hat{\phi}\rightarrow\frac{2\hbar}{q_e}\sin\left(\frac{q_e}{2\hbar}\hat{\phi}\right), \tag{3}$$

in the kinetic energy operator [Eq. (2)]. We remark that this change allows us to keep the usual commutation rule $[\hat{Q},\hat{\phi}]=i\hbar$. Using Eqs. (2) and (3) we compute the current in the inductance $d\hat{Q}/dt=[\hat{T},\hat{Q}]/i\hbar$, obtaining

$$\frac{d\hat{Q}}{dt}=\frac{\hbar}{Lq_e}\sin\left(\frac{q_e}{\hbar}\hat{\phi}\right), \tag{4}$$

in agreement with Refs. 1, 2. It is important to note that in Eq. (3), the operator $\hat{\phi}$ is not the flux operator, since that role is played here by the operator $L d\hat{Q}/dt$; therefore, to avoid misunderstanding, in this work we will refer to $\hat{\phi}$ as the pseudoflux operator. Moreover, it is easy to show that the commutator between charge and current is not proportional to the identity operator because of charge discreteness.[1,15]

For definiteness, for a ring enclosing an external, time-dependent flux $f(t)$, the Hamiltonian with charge discreteness is

$$\hat{H}=\frac{2\hbar^2}{Lq_e^2}\sin^2\left(\frac{q_e}{2\hbar}\hat{\phi}\right)+\frac{df}{dt}\hat{Q}, \tag{5}$$

where, as stated previously, $[\hat{Q},\hat{\phi}]=i\hbar$. In this case, the evolution equation for the pseudoflux $\hat{\phi}$, obtained from this Hamiltonian becomes

$$\frac{d}{dt}\hat{\phi}=\frac{df}{dt}. \tag{6}$$

From Eqs. (4) and (6), the current in the ring becomes

$$\frac{d\hat{Q}}{dt}=\frac{\hbar}{Lq_e}\sin\left(\frac{q_e}{\hbar}[f(t)+\hat{\phi}^0]\right), \tag{7}$$

where $\hat{\phi}^0$ is the initial condition and it is related to the pseudoflux operator in the Schrödinger picture.

II. CLASSICAL TWO RING SYSTEMS

As stated previously, we consider two rings with self-inductance L_1, L_2, and mutual inductance M, under the influence of external magnetic fields, so that the external magnetic fluxes are $f_1(t)$ and $f_2(t)$. If Q_1 and Q_2 are the respective charges across a section of each ring at time t, then the Lagrangian $\mathcal{L}$ of the system is given by

$$\mathcal{L}=\frac{1}{2}L_1\dot{Q}_1^2+\frac{1}{2}L_2\dot{Q}_2^2+M\dot{Q}_1\dot{Q}_2-(\dot{f}_1Q_1+\dot{f}_2Q_2), \tag{8}$$

where the terms within the bracket represent the electromotive force (emf) induced by the external flux in each ring, and the dot over a quantity represents its time derivative. The above expression defines the canonical variable ϕ conjugate to the charge Q by the expression $\phi=(\partial\mathcal{L}/\partial\dot{Q})$. Then the Hamiltonian $\mathcal{H}$ of the system becomes

$$\mathcal{H}=\frac{1}{2L_1'}\phi_1^2+\frac{1}{2L_2'}\phi_2^2+\frac{1}{M'}\phi_1\phi_2+(\dot{f}_1Q_1+\dot{f}_2Q_2), \tag{9}$$

where the "momentum" ϕ physically represents the induced flux in each ring. The equations of motion derived from the above Hamiltonian may be obtained easily, and are expressed in terms of the effective inductances L', given by

$$L_1'=(L_1L_2-M^2)/L_2=(1-k^2)L_1,$$

$$L_2'=(L_1L_2-M^2)/L_1=(1-k^2)L_2,$$

$$M'=(L_1L_2-M^2)/M=(1-k^2)\sqrt{L_1L_2}/k,$$

where we have defined the parameter $k^2=M^2/(L_1L_2)\leqslant 1$, as it is customary in electrical circuit theory.

III. QUANTUM TWO RING SYSTEMS WITH CHARGE DISCRETENESS

In the case of continuous charge, the quantization of the above Hamiltonian is straightforward, since ϕ and Q become canonical operators. However, after Li and Chen,[1] when charge discreteness is considered, the charge operator has discrete eigenvalues nq_e where n is an integer and q_e is the elementary charge and, as said before, a fixed parameter. Charge discreteness modifies the flux operator as pointed-out in Sec. I. Then, from Eqs. (3) and (9) the quantized Hamiltonian of the system, which properly accounts for charge discreteness must be given by

$$\hat{\mathcal{H}}=\frac{2\hbar^2}{L_1'q_e^2}\sin^2\left(\frac{q_e}{2\hbar}\hat{\phi}_1\right)+\frac{2\hbar^2}{L_2'q_e^2}\sin^2\left(\frac{q_e}{2\hbar}\hat{\phi}_2\right)+\frac{4\hbar^2}{M'q_e^2}\sin\left(\frac{q_e}{2\hbar}\hat{\phi}_1\right)\sin\left(\frac{q_e}{2\hbar}\hat{\phi}_2\right)+(\dot{f}_1\hat{Q}_1+\dot{f}_2\hat{Q}_2). \tag{10}$$

The above Hamiltonian is the basis of our calculations. Our system has parameters L_1, L_2, and M, assumed known; while q_e represents the elementary charge of the theory which corresponds to the electronic charge. In the limit $q_e\rightarrow 0$ the operator (10) becomes the (usual) quantum version of the classical Hamiltonian operator [Eq. (9)], as expected. This limit defines the parameters L_1', L_2' and M in a well-defined fashion.

From the quantum Hamiltonian [Eq. (10)], and using the Heisenberg equations of motion, the dynamical equations for the current and the pseudoflux operators become

$$\frac{d}{dt}\hat{Q}_1=\frac{\hbar}{q_eL_1'}\sin\left(\frac{q_e}{\hbar}\hat{\phi}_1\right)+\frac{2\hbar}{q_eM'}\cos\left(\frac{q_e}{2\hbar}\hat{\phi}_1\right)\sin\left(\frac{q_e}{2\hbar}\hat{\phi}_2\right),$$

$$\frac{d}{dt}\hat{Q}_2=\frac{\hbar}{q_eL_2'}\sin\left(\frac{q_e}{\hbar}\hat{\phi}_2\right)+\frac{2\hbar}{q_eM'}\cos\left(\frac{q_e}{2\hbar}\hat{\phi}_2\right)\sin\left(\frac{q_e}{2\hbar}\hat{\phi}_1\right),$$

$$\frac{d}{dt}\hat{\phi}_1=\left(\frac{d}{dt}f_1\right)\hat{I},\quad\text{and}\quad\frac{d}{dt}\hat{\phi}_2=\left(\frac{d}{dt}f_2\right)\hat{I},\tag{11}$$

which in the limit $q_e\to 0$ have the expected form.

Consider now the case in which only the first ring encloses a nonzero external magnetic field flux, i.e., we let $f_1=f_1(t)$ and $f_2=0$, then, from Eq. (11), the total enclosed fluxes become

$$\hat{\phi}_1=f_1(t)\hat{I}+\hat{\phi}_1^0\quad\text{and}\quad\hat{\phi}_2=\hat{\phi}_2^0,\tag{12}$$

where $\hat{\phi}_1^0$ and $\hat{\phi}_2^0$ are the initial pseudoflux conditions corresponding to the pseudoflux operators in the Schrödinger picture. The equations for the current operators become

$$\frac{d}{dt}\hat{Q}_1=\frac{\hbar}{q_eL_1'}\sin\left(\frac{q_e}{\hbar}(f_1\hat{I}+\hat{\phi}_1^0)\right)+\frac{2\hbar}{q_eM'}\cos\left(\frac{q_e}{2\hbar}(f_1\hat{I}+\hat{\phi}_1^0)\right)\sin\left(\frac{q_e}{2\hbar}\hat{\phi}_2^0\right),\tag{13}$$

$$\frac{d}{dt}\hat{Q}_2=\frac{\hbar}{q_eL_2'}\sin\left(\frac{q_e}{\hbar}\hat{\phi}_2^o\right)+\frac{2\hbar}{q_eM'}\cos\left(\frac{q_e}{2\hbar}\hat{\phi}_2^0\right)\times\sin\left(\frac{q_e}{2\hbar}(f_1\hat{I}+\hat{\phi}_1^0)\right).\tag{14}$$

For the sake of simplicity, consider a state $|\psi\rangle$, so that the average of the pseudoflux operators, computed for this state, are $\langle\hat{\phi}_1^0\rangle=\langle\hat{\phi}_2^0\rangle=0$. With all this the equations of motion for the averaged current operator Eqs. (13) and (14) simplify to

$$\left\langle\frac{d}{dt}\hat{Q}_1\right\rangle=\frac{\hbar}{q_eL_1'}\sin\left(\frac{q_e}{\hbar}f_1\right),\tag{15}$$

$$\left\langle\frac{d}{dt}\hat{Q}_2\right\rangle=\frac{2\hbar}{q_eM'}\sin\left(\frac{q_e}{2\hbar}f_1\right).\tag{16}$$

Consider the case where the external magnetic field is constant, then the external flux $f_1=f^0$. The surprising fact is that when $q_ef^0/\hbar=l\pi$, for an integer l, the current in the second ring, where the external flux vanishes, is at an extremum, and the current in the first ring vanishes. This is known as quantum current magnification, or quantum current enhacement, and it is clearly a purely quantum effect since its existence depends on Plank's constants $\hbar$ and q_e being both nonzero. In Fig. 2 we show a plot of the currents within the two rings, as a function of the (constant) external flux f^0. Flux variation changes the sign of the current (paramagnetic or diamagnetic), and one sees that the current in the second ring is maximal when $q_ef^0/\hbar=\pi$ and the current in the first ring is zero (quantum magnification).

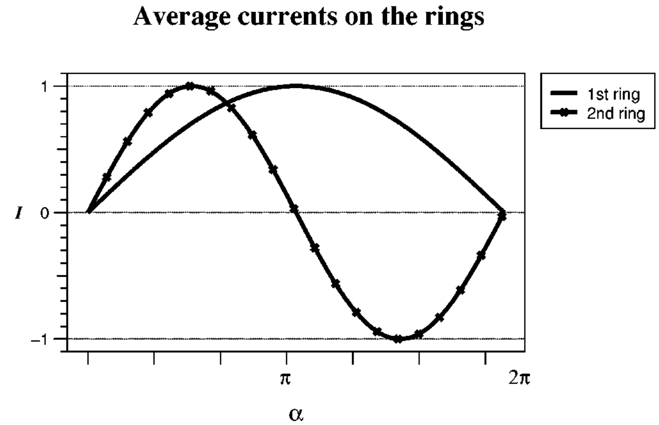

FIG. 2. Plot of the currents in both rings, divided by their respective maximum values, as a function of the argument $\alpha=q_ef_1/\hbar$, for $0\leq\alpha\leq 2\pi$, when $f_2=0$. Showing the quantum magnification, at $\alpha=\pi$.

A final point respect to Eqs. (15) and (16). The classical limit may formally be obtained if we take the limit $q_e\to 0$ in these equations. So, classically, when the current is zero in the first ring it is zero also in the second ring.

IV. DISCUSSION AND CONCLUSIONS

The equations for the currents [Eqs. (15) and (16) above], have a sinusoidal nonlinearity which has the quantum magnification effect as a consequence. It is also interesting to consider the effect of sinusoidal time-varying perturbations. Levy *et al.*[7] have studied the effect of a slowly time-varying $\phi(t)=\phi_{dc}+\phi_{ac}\sin(\omega t)$, externally applied flux, over a dilute statistical sample of 10^7 noninteracting copper rings on a saphire substrate at low temperatures, and measured the induced magnetic moments μ, as a function of the field strength. They compute the Fourier amplitudes of the magnetic moments, as

$$\mu_{2n}=2\mu_a(T)J_{2n}(\delta)\sin(\theta),\tag{17}$$

$$\mu_{2n+1}=2\mu_a(T)J_{2n+1}(\delta)\cos(\theta),\tag{18}$$

where $\delta=2\pi\phi_{ac}/\phi_p$ and $\theta=2\pi\phi_{dc}/\phi_p$, and ϕ_p is the assumed flux periodicity of the average moment.

Starting from Eqs. (15) and (16), for our two ring system, we obtain expressions for the (average) induced currents wich are of the same general form as Eqs. (17) and (18). To be precise, the Fourier components of the current $\langle\dot{Q}_1(t)\rangle$, may be written as $I_n^{(1)}$

$$I_{2n}^{(1)}=2I_0^{(1)}J_{2n}(\delta_1)\sin(\theta_1),$$

$$I_{2n+1}^{(1)}=2I_0^{(1)}J_{2n+1}(\delta_1)\cos(\theta_1),$$

where $I_0^{(1)}=\hbar/(q_eL_1')$, $\delta_1=2\pi\phi_{ac}/\phi_p^{(1)}$, and $\theta_1=2\pi\phi_{dc}/\phi_p^{(1)}$, and $\phi_p^{(1)}=2\pi\hbar/q_e$; while for the current $\langle\dot{Q}_2(t)\rangle$ in the other ring, the Fourier components $I_n^{(2)}$, are

$$I_{2n}^{(2)}=2I_0^{(1)}J_{2n}(\delta_2)\sin(\theta_2),$$

$$I_{2n+1}^{(2)}=2I_0^{(1)}J_{2n+1}(\delta_2)\cos(\theta_2),$$

where $I_0^{(2)}=2\hbar/(q_eM')$, $\delta_2=2\pi\phi_{ac}/\phi_p^{(2)}$, and $\theta_2=2\pi\phi_{dc}/\phi_p^{(2)}$, and $\phi_p^{(2)}=2\phi_p^{(1)}$. Note also that, in our expressions for the current magnification, the flux is a periodic function of the external flux, with period $\Phi_0=2\pi\hbar/q_e$ or $2\Phi_0$. We point out here that that the ''quantum of charge'' q_e may be identified here with the electronic charge, but the theory outlined here is unable to predict its value independently. Therefore, one may regard parameters such as L, M, and Φ_0, as well as the currents $I_0^{(1)}$ and $I_0^{(2)}$, as experimentally defined.

There are other ways to drive a system such as this; for example, one may apply an external electromotive force, i.e., a slightly different coupling, but the main result described here should remain valid. Note that, in this case, the corresponding experimental setup would be different than in the case considered in this work.

We have also considered the case of a continuous-charge ring interacting with a discreteness-charge ring. We let the second ring enclose an external flux, with no external flux on the first ring. We obtain the somewhat surprising result that the magnification effect is still present.

In conclussion, quantum current magnification exist for a system of two ring with mutual inductance and charge discreteness when one of them encloses an external flux (15), (16).

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