Competing order in the mixed state of high-temperature superconductors

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We examine the low-temperature behavior of the mixed state of a layered superconductor in the vicinity of a quantum critical point separating a pure superconducting phase from a phase in which a competing order coexists with superconductivity. At zero temperature, we find that there is an avoided critical point in the sense that the phase boundary in the limit $B\rightarrow 0$ does not connect to the $B=0$ critical point. Consequently, there exists a quasi-one-dimensional (1D) regime of the phase diagram, in which the competing order is largely confined to 1D ''halos'' about each vortex core, and in which interactions between neighboring vortices, although relevant at low temperature, are relatively weak.

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Whereas in many well understood metallic compounds over a broad range of compositions and temperatures, the only two phases encountered are the normal (Fermi liquid) and superconducting phases, in the cuprate high temperature superconductors, and other highly correlated electronic systems, there are many ordered phases which appear to compete and sometimes coexist. In addition to the uniform *d*-wave superconducting state, compelling evidence exists of ordered antiferromangetic (Néel), unidirectional charge density wave (''charge-stripe''), unidirectional, colinear, incommensurate spin-density wave (''spin-stripe'') phases, and those with coexisting superconducting and stripe order.[1] Preliminary evidence also exists of possible *d*-density-wave or staggered flux order,[2,3] electron nematic,[4] $d+id$ or $d+is$ superconducting order,[5,6] and various other phases in which more than one of these orders coexist.

One rather direct way to look for competing order parameters in a system which is globally superconducting was recently proposed by Zhang and co-workers[7,8] and others.[9–11] The idea is that if the superconducting order is only slightly favored over a competing order, then where the superconducting order is suppressed in the core of a vortex, the competing order will be manifest. Indeed, recent neutron scattering experiments of Lake *et al.*[12] on $La_{2-x}Sr_xCuO_4$ have revealed a strong enhancement of ''spin-stripe'' order at low energies produced by modest magnetic fields. Similar results have been obtained in La_2CuO_4 by Birgeneau *et al.*[13] Moreover, in scanning tunnelling microscopy (STM) studies by Hoffman *et al.*[14] of near optimally doped $Bi_2Sr_2CaCu_2O_{8+\delta}$ in a 7 T magnetic field, large induced ''halos'' about each vortex core have been imaged where the density of states is modulated with a spatial period $(4a)$ equal to that expected[15] for ''charge-stripe'' order. Additional evidence that there is a substantial degree of local charge stripe order in $Bi_2Sr_2CaCu_2O_{8+\delta}$ comes from the similar patterns of density of states modulation observed in the zero-field STM studies of Howald *et al.*[16] on the same material. (This interpretation is, however, being challenged in a forthcoming paper by Hoffman *et al.*[17])

The notion of a competing order developing an expectation value in a vortex core is basically a mean-field notion. However, in all the examples mentioned above, the competing order is associated with spontaneously broken symmetry, so that fluctuation effects may fundamentally alter the physics. Specifically, in a planar system, the vortex core is a finite size system, and so cannot support a spontaneously broken symmetry, while in three-dimensional superconductors the vortex core is a one-dimensional system, so cannot exhibit any symmetry breaking except at $T=0$, and there only for discrete symmetries.

I. PHASE DIAGRAM

Our principal results are summarized in the schematic zero temperature phase diagram (Fig. 1) for a layered system in which superconducting order, with an order parameter denoted by Ψ, competes with another type of order, whose order parameter is denoted by ϕ. The two axes represent the magnetic induction B and a control parameter ''α,'' such as pressure or doping concentration, with the convention that increasing α disfavors ϕ order. It is assumed that at $B=0$ there exists a continuous quantum phase transition at $\alpha=\alpha_c$ (the heavy circle) separating a pure superconducting phase from a phase with coexisting ϕ and superconducting order.

While the considerations here are rather general, it is useful to have specific realizations in mind. For us, the most important case is that in which ϕ represents[18] stripe orientational (''nematic'') order. In the absence of crystal field effects, the orientation of the stripes is arbitrary and can be parametrized by an angle $0\leqslant\theta<\pi$. Hence, in this case ϕ would then be a *director*, a headless vector, which in two dimensions can be represented by a complex scalar field defined so that $\phi=|\phi|e^{i2\theta}$ corresponds to stripes lying along a preferred direction $\hat{e}_\theta=\hat{x}\cos\theta+\hat{y}\sin\theta$. (Note that $\theta=0$ and $\theta=\pi$ are physically equivalent.) In this limit, ϕ has a continuous XY symmetry. However, in a crystal with appropriate point-group symmetry[19] there are only two preferred stripe

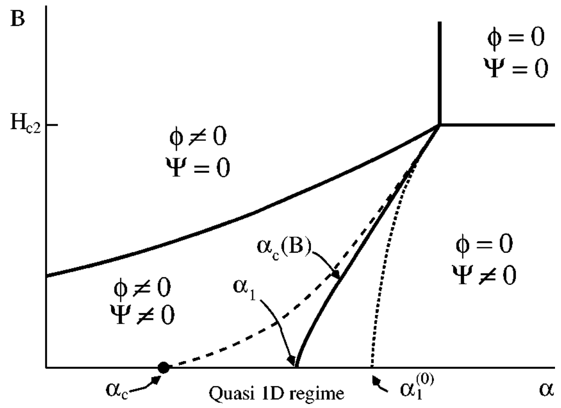

FIG. 1. Schematic zero-temperature phase diagram of a layered superconductor as a function of a control parameter α (described in the text) and the magnetic induction B, i.e., the actual magnetic field which penetrates the system. Here Ψ and ϕ denote, respectively, the expectation values of the superconducting and competing order parameters and it is assumed that for $B=0$ there is a continuous transition at $\alpha=\alpha_c$ between a pure superconducting phase and a phase with a coexisting Ψ and ϕ order. The solid lines denote phase boundaries and the other lines are crossovers (all are described in the text). In particular, $\alpha_c(B)$ represents the boundary between the pure superconducting and the phase with coexisting ϕ and superconducting order, and $\alpha_1 \equiv \lim_{B\to 0}\alpha_c(B)$ marks the point at which the ϕ "halo" about an isolated vortex-line undergoes a transition from a quantum disordered state (for $\alpha > \alpha_1$) to an ordered or quasiordered state for $\alpha < \alpha_1$. ($\alpha_1^{(0)}$ is the mean-field value of α_1.)

orientations. In this case, ϕ reduces to a real scalar field reflecting the Ising character of the symmetry breaking.

If, on the other hand the relevant stripe order is magnetic (incommensurate SDW) order, in addition to orientational symmetry the relevant broken symmetry is spin rotational invariance, so for each stripe orientation, ϕ is a three component real vector field corresponding to Heisenberg symmetry. We shall see that this case is slightly different than the lower symmetry situations. Stripe states can also spontaneously break translational symmetry, but since a vortex core explicitly breaks this symmetry in any case, issues of spatial symmetry breaking are more subtle, and will be discussed elsewhere.[20]

In the phase diagram shown in Fig. 1, the solid lines represent actual phase boundaries, and the broken lines are crossovers. The most striking feature of this phase diagram is the presence[21] of an "avoided critical point," i.e., the phase boundary $\alpha_c(B)$ has a discontinuity at $B=0$:

$$\lim_{B\to 0} \alpha_c(B) \neq \alpha_c. \tag{1}$$

This discontinuity in the phase boundary is a consequence of the fact that the magnetic field is a singular perturbation.

The assumed competition (mutual suppression) between superconducting and ϕ order (the latter assumed to be essentially uncoupled to B) is seen in the fact that the critical line, $\alpha=\alpha_c(B)$ separating the pure and coexistence phases is an increasing function of B up to the point at which superconductivity is completely suppressed by a magnetic field in excess of $B=H_{c2}$. The dashed line marks a crossover from quasi one dimensional to fully three-dimensional order (where ϕ is more or less uniform in space). It essentially coincides with the phase boundary derived by Demler *et al.*[11] for the case in which interactions between layers are neglected.[22] The dotted line represents a crossover associated with the mean-field phase boundary; just to the left of this line, there is a "halo" of the competing order surrounding each vortex core, but each halo fluctuates essentially independently, and there is no true ϕ order.

The arguments that lead to this phase diagram constitute the bulk of the present paper. The reader should be warned that there are some subtleties (which we will discuss below) associated with one or another specific type of competing order that can affect the shape, and even the topology of the phase diagram, as does even weak quenched disorder. For simplicity of discussion, for most of the paper we will assume the extreme type II limit, in which the London penetration depth $\lambda=\infty$, although this assumption is not necessary. Some of the experimental consequences of this phase diagram are discussed in the final section of this paper.

II. THE BASIC PHYSICS

In this section, we sketch the basic physics that leads to the phase diagram in Fig. 1. All actual derivations are deferred to later sections. For simplicity, let us first consider the case in which the broken symmetry associated with the competing order is Ising-like, i.e., ϕ is a real scalar, and there is a symmetry under $\phi \to -\phi$.

A. The Ising case

We start our discussion by considering the structure of a single, isolated vortex in the uniform superconducting state,[23] i.e., $\alpha > \alpha_c$. If α is large, the structure of the vortex is unaffected by the proximity of the ϕ ordered phase, but if α is sufficiently close to α_c, then at mean-field level, the suppression of Ψ in a vortex core region of size ξ_0 (the superconducting coherence length) will result in a halo with radius equal to the critical correlation length $\xi_\phi \gg \xi_0$, in which $\phi \neq 0$. An important aspect of the structure of the vortex, first emphasized in this context by Demler *et al.*,[11] is that the magnitude of the superconducting order parameter does not return to its bulk value exponentially, but rather, for a vortex at the origin and for $r \gg \xi_0$, has a power-law form

$$|\Psi(\vec{r})|^2 \sim \Psi_0^2[1-(\xi_0/r)^2+\cdots]. \tag{2}$$

This $1/r^2$ fall-off is a necessary consequence of the slow decay of the superfluid flow around an isolated vortex, and results in a somewhat larger halo of ϕ order around a vortex than would otherwise occur. In any case, $\xi_\phi \sim |\alpha-\alpha_c|^{-\nu}$ diverges with a critical exponent ν as $\alpha \to \alpha_c$. Since ν is the quantum critical exponent of a system in $d=3$ spatial dimensions, it presumably takes its mean-field value $\nu=1/2$ (up to logarithmic corrections to scaling).

Within a single superconducting plane the halo is a finite size system, and so cannot actually support a broken symmetry state, i.e., quantum fluctuations will cause the system to tunnel between the positive and negative ϕ states with a matrix element $h \equiv h(\alpha)$. Thus, in the absence of interplane couplings, there can be no true ϕ order established until B is large enough, i.e., the vortex density is large enough, that there is significant coupling between neighboring vortices. Because the size of the halo increases with increasing ξ_ϕ, we expect that h vanishes in the limit $\xi_\phi/\xi_0 \to \infty$. We derive an explicit expression for this, below in Eq. (20), from the Landau-Ginzburg theory of competing orders. However, even if the microscopic coupling between the order parameter ϕ on neighboring planes J_0 is weak, the large size of the halo implies a large effective coupling J between the halos on neighboring planes, with $J \sim J_0 \overline{\phi^2 \xi_\phi^2}$, where $\overline{\phi^2}$ is the mean squared value of ϕ in the vortex halo. From the Landau-Ginzburg theory, this coupling is seen to diverge as $\xi_\phi/\xi_0 \to \infty$, although only logarithmically, $J \propto J_0 \ln(\xi_\phi/\xi_0)$, as shown in Eqs. (13) and (19).

Thus, along an isolated vortex, the low energy fluctuations are equivalent to an effective transverse-field Ising model with Ising coupling J and transverse field h. This model has an ordered ground-state provided $J/h > 1$. Since as α approaches α_c, the limit $J/h \to \infty$ is realized, it follows that for α near α_c, an isolated vortex line has an ordered ground state Moreover, in this limit, no matter how small B, the intervortex coupling can never be ignored, and indeed leads to a finite transition temperature. We can estimate this transition temperature using standard methods[24] of quasi-one-dimensional systems: We estimate the intervortex coupling to be $J_{\rm inter} \sim \exp(-r/\xi_\phi) \sim \exp(-A/\sqrt{B\xi_\phi^2})$, where r is the spacing between vortices and A is a number of order 1. At low temperatures, we estimate the susceptibility of an isolated vortex line to be that of the 1D Ising chain $\chi_\phi(T) \sim \exp(A' J/T)$. Finally, we estimate T_c according to $\chi_\phi(T_c) J_{\rm inter} = 1$. This leads to the estimate

$$T_c(B) \sim \xi_\phi J \sqrt{B} \sim B^{1/2} |\alpha - \alpha_c|^{-1/2} \ln[\alpha_c / |\alpha - \alpha_c|], \quad (3)$$

where in the second expression we have adopted the Landau-Ginzburg estimate of J.

Reversing the present logic, it is clear that with increasing α, we will eventually encounter a condition in which $h > J$. Here, the ground state of the isolated vortex line is quantum disordered, and hence even at $T=0$, there is no ϕ ordering until a critical field strength $B_c(\alpha)$ is exceeded, such that the interactions between neighboring vortices is strong enough to induce ordering. The critical value $\alpha_c(B) \to \alpha_1$ in the limit $B \to 0$ marks the quantum critical point of the isolated vortex, where $J(\alpha_1)/h(\alpha_1) = 1$. (For small $|\alpha - \alpha_1|$, the phase boundary is nonanalytic[25]—a subtlety which we have neglected in sketching Fig. 1.)

The remaining phase boundaries that occur at larger values of B are determined by the obvious and conventional physics of competing orders. We now discuss the physics of the two crossover lines shown in Fig. 1.

Where the ϕ halos about the vortices start to overlap strongly, there is a crossover from a quasi-one-dimensional regime in which the magnitude of ϕ is substantially inhomogeneous, to a regime where the variations of ϕ are relatively small. (Indicated by the dashed line in Fig. 1.) At first guess might be that this crossover occurs when the spacing between vortices is of order ξ_ϕ or in other words when $B^* \sim |\alpha - \alpha_c|^{2\nu} = |\alpha - \alpha_c|$ (up to logarithmic corrections to scaling). In fact, as shown by Demler *et al.*,[11] this crossover occurs at somewhat smaller B due to the slow recovery of Ψ away from the vortex core. If we consider the case in which ϕ is homogeneous, then it is as if there were a magnetic field dependent reduction of the effective $\alpha_{\rm eff} = \alpha - \tilde{\gamma}|B| \ln|H_{c2}/B|$ where, as we will see in the next section, $\overline{\gamma}$ is proportional to the coupling strength between the two order parameters. Consequently $B^* \sim |\alpha - \alpha_c| / \ln|\alpha_c/(\alpha - \alpha_c)|$. [Precisely at $\alpha = \alpha_c$, the same line of reasoning leads to the conclusion that $T_c \sim |B \ln(H_{c2}/B)|^{\nu z}$, where z is the dynamical scaling exponent, and hence depends on the dynamics of ϕ.]

For large enough $\alpha > \alpha_1^{(0)}$, the competing order is sufficiently disfavored that, even at mean field level, no halo of nonzero ϕ is induced about the vortex core. Thus, for small B and $\alpha_c(B) < \alpha < \alpha_1^{(0)}$, there are substantial local ϕ correlations along each vortex, but the quantum fluctuations are sufficiently large that the coupling between vortices can be neglected, and no true long-range ϕ order develops. For $\alpha > \alpha_1^{(0)}$, there are no substantial ϕ fluctuations induced by the presence of vortices. [The value of $\alpha = \alpha_1^{(0)}$ at which a nonzero value of ϕ appears at mean-field level is computed from Landau-Ginzburg theory in Eq. (15); roughly it is the point at which $\xi_\phi \sim \xi_0$.]

We note, in passing, that even in 2D the Ising case is quite different than than Heisenberg case considered by Demler *et al.*[11] In particular, as discussed in Ref. 22, in the Ising case the transition to the coexistence phase occurs when the spacing between vortices is parametrically large compared to ξ_ϕ. Consequently, although $\lim_{B\to 0} \alpha_c(B) = \alpha_c$, for α close to $\alpha_c(B)$ and B small, ϕ is not spatially uniform but rather is strongly peaked in halos about individual vortex cores.

B. *XY* and Heisenberg cases

If ϕ is not an Ising variable, but has higher symmetry, the above considerations are somewhat modified. In the case of an XY variable, the physics of the isolated vortex is equivalent to the well-known physics of the 1D quantum rotor model. Again, there is a single site (i.e., intraplane) term, which can be characterized by the energy gap h between the ground state and first excited state of a single rotor; manifestly, h is proportional to the inverse moment of inertia of the rotor and large h favors a quantum disordered state. Ordering, however, is again promoted by an "exchange" interaction J between neighboring "sites," i.e., neighboring planes. As in the Ising case, we estimate the dependence of J on α from the Landau-Ginzburg treatment, below, and with the same result. For a continuous symmetry, h does not involve tunneling, and so its dependence on α is much weaker than in the Ising case; indeed, we will see from the Landau-Ginzburg treatment that h has the inverse dependence as J, i.e., $h(\alpha) \propto 1/J(\alpha)$.

There is still a quantum disordered phase possible if h/J is sufficiently large. However, at smaller h/J the ordered phase of the Ising chain is replaced by a conformally invariant (power-law) phase in the XY chain. The susceptibility in this phase is a power-law in temperature, $\chi_\phi(T) \sim J^{-1}(J/T)^K$, where $K(J/h)$ is an increasing function of J/h. So long as $K>0$, this susceptibility still diverges as $T \to 0$, so the topology of the phase diagram is similar to that in the Ising case. However, now α_1 is determined implicitly from the relation $K(\alpha_1)=0$. For $\alpha_1>\alpha>\alpha_c$, where $K(\alpha)>0$, we can use the same intervortex mean-field theory to estimate the ordering temperature

$$T_c \sim J \exp[-A/(K\xi_\phi\sqrt{B})], \tag{4}$$

i.e., T_c rapidly becomes immeasurably small at small B.

The classical Heisenberg ferromagnet has an ordered ground state, even in one dimension. However, the $O(3)$ quantum rotor model, which represents the low energy theory of the quantum Heisenberg antiferromagnets, does not. In the absence of Berry-phase terms, the one-dimensional $O(3)$ chain possesses[26] a Haldane gap, and hence a nondivergent zero-temperature susceptibility. This means that the topology of the phase diagram is different for the Heisenberg case, and that there is no avoided critical point. However, since the Haldane gap vanishes exponentially for large J/h, $E_{\text{Haldane}} \sim \exp[-A''J/h]$, it follows from simple scaling arguments that $\chi_\phi(T=0) \sim \exp[2A''J/h]$. Thus, following a line of argument similar to the one which in the Ising case led to Eq. (3), intervortex mean-field theory leads to an estimate for the critical B,

$$B_c(\alpha) \sim [\xi_\phi J/h]^{-2} \sim \frac{|(\alpha/\alpha_c)-1|}{\ln^2|(\alpha/\alpha_c)-1|}, \tag{5}$$

where in the second expression we have adopted the Landau-Ginzburg estimates of h and J.

It is also possible to imagine that a single vortex corresponds to a half-integer spin chain, in which case there is a Berry's phase and consequently no Haldane gap. In this case, the situation is essentially equivalent to the XY case, with the susceptibility exponent $K=1$ (up to logarithmic corrections). However, it seems to us that since the effective Heisenberg model in the present case is not sharply defined on the lattice scale, fluctuation effects are likely to smear out the subtle interference phenomena responsible for the special behavior of half-integer spins, leaving us with the physics of the rotor described above.

C. Further subtleties

There are still other subtleties to worry about. In this entire discussion we have assumed that the vortex texture of $\Psi(\vec{r})$ does not lift the symmetries which are spontaneously broken by the ordering of ϕ. However, where one of those broken symmetries is translation invariance, the presence of a vortex core is an explicit symmetry breaking field. If this effect is significant (as it may well be in the case of stripe order), it greatly complicates the analysis. A related issue is that we have assumed that there is no frustration of global ϕ order which arises from the form of the vortex lattice and the nature of the coupling between neighboring halos. We believe that, in the absence of the just mentioned symmetry breaking terms, this assumption is reasonable, but it is manifestly unreasonable in their presence. Finally, especially in the regime where the physics is quasi one dimensional, all results are likely to be extremely sensitive to even tiny amounts of quenched disorder. We have not fully explored the implications of any of these further problems.

III. LANDAU-GINZBURG THEORY

While most features of the problem are largely determined by considerations of order parameter symmetry, it is pedagogically useful to make them explicit by considering the Landau-Ginzburg treatment of two competing order parameters. We are interested in ground-state properties, so we must ultimately analyze a $(D+1)$-dimensional quantum action, where $D=3$ is the spatial dimension. However, we will be analyzing this action semiclassically, in the sense that we will first consider time-independent field configurations which minimize the action, and then analyze quantum fluctuations about this classical ground state. Thus, we start by considering only the classical (static) Landau-Ginzburg free energy density functional in a single plane of a layered system

$$\mathcal{F}[\Psi,\phi] = \mathcal{F}_{SC}[\Psi] + \mathcal{F}_\phi[\phi] + \frac{\gamma}{2}|\Psi|^2|\phi|^2 + \cdots, \tag{6}$$

where Ψ, a complex scalar field, is the superconducting order parameter and ϕ, which may have multiple components, represents the competing order parameter. In the present paper, we will only focus on "competing" orders in the sense that we will always assume that $\gamma>0$. Indeed, we assume that γ is not small—if γ is small it means that the two order parameters hardly interact, as happens, for instance, in conventional superconductors when ϕ and Ψ originate from different pieces of the Fermi surface. The free energy must be invariant under a global U(1) transformation $\Psi(\vec{r}) \to e^{i\alpha_0}\Psi(\vec{r})$ due to gauge invariance and, depending on the nature of the competing order, under an additional set of global transformations $\phi(\vec{r}) \to \mathbf{g}\phi(\vec{r})$, where $\mathbf{g}$ are elements of an appropriate coset space. For instance, if ϕ corresponds to Nèel order, then $\mathbf{g} \in \mathrm{SU}(2)/\mathrm{U}(1)$. In Eq. (6), the superconducting contribution to $\mathcal{F}$ is of the usual form

$$\mathcal{F}_{SC} = \frac{\kappa_0}{2}\left|\left(\frac{\vec{\nabla}}{i} - \frac{2e}{c}\vec{A}\right)\Psi\right|^2 - \frac{\Psi_0^2}{2}|\Psi|^2 + \frac{1}{4}|\Psi|^4 + \cdots, \tag{7}$$

where $\vec{A}$ is the vector potential, $\xi_0 = \sqrt{\kappa_0/2\Psi_0^2}$ is the coherence length, and here, and elsewhere, $\cdots$ refers to higher order terms in powers of the order parameters. The competing order is governed by

$$\mathcal{F}_\phi = \frac{\kappa_\phi}{2}|\vec{\nabla}\phi|^2 + \frac{\alpha}{2}|\phi|^2 + \frac{1}{4}|\phi|^4 + \cdots. \tag{8}$$

The mean-field solution is obtained by solving the Landau-Ginzburg (LG) equations $\delta\mathcal{F}/\delta\Psi|_{\bar{\Psi}} = \delta\mathcal{F}/\delta\phi|_{\bar{\phi}} = 0$. For the most part, we will focus on states deep in the superconducting phase, where we can treat $\bar{\Psi}(\vec{r})$ as a given function, leaving us with the task of computing $\bar{\phi}$. By symmetry, $\bar{\phi}=0$ is always a solution; where a nontrivial solution exists, it leads to a condensation energy

$$E_{\rm cond}=\int d\vec{r}[\mathcal{F}(\bar{\Psi},\bar{\phi})-\mathcal{F}(\bar{\Psi},0)]=-\int d\vec{r}\frac{|\bar{\phi}|^4}{4}. \quad (9)$$

In the spatially uniform case $\bar{\Psi}(\vec{r})=\Psi_0$, there is a nontrivial solution $\bar{\phi}(\vec{r})=\sqrt{(\alpha_c-\alpha)/(1-\gamma^2)}$ for $\alpha<\alpha_c=-\gamma\Psi_0^2$, while $\bar{\phi}=0$ for $\alpha>\alpha_c$. In the ϕ disordered phase, it is easy to see that any inhomogeneous solution of the LG equations will decay exponentially with a correlation length $\xi_\phi=\sqrt{\kappa_\phi/(\alpha-\alpha_c)}$.

In the case of a single vortex at the origin, we look for a solution of the form $\bar{\Psi}(\vec{r})=e^{i\varphi}|\bar{\Psi}(r)|$ where φ is the azimuthal angle. While the exact form of $|\bar{\Psi}(r)|$ is somewhat complicated, at large r it is easily seen to be $|\bar{\Psi}(r)|=\Psi_0\{1-(1/2)(\xi_0/r)^2+\mathcal{O}[(\xi_0/r)^4]\}$. [Of course, if magnetic screening is taken into account, $|\bar{\Psi}(r)|$ approaches Ψ_0 exponentially for $r\gg\lambda$, with the London penetration length; in the high-temperature superconductors, λ is very large, so it is reasonable to approximate it as infinite.] Since none of our results depend critically on the short distance $(r<\xi_0)$ behavior of this solution, we will adopt the approximation

$$|\bar{\Psi}(r)|^2=\Psi_0^2[1-(\xi_0/r)^2] \ \text{ for } \ r>\xi_0 \quad (10)$$

and $|\bar{\Psi}(r)|^2=0$ for $r<\xi_0$. Thus, the LG equation for an isolated vortex is

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{1}{x}\frac{\partial}{\partial x}+\left(\frac{\xi_0}{\xi_\phi}\right)^2-\frac{\tilde{\gamma}}{x^2}+\phi^2(x)\right]\phi(x)=0, \quad (11)$$

where $x=r/\xi_0$, $\tilde{\gamma}=2\gamma\Psi_0^2\xi_0^2/\kappa_\phi$ and $\bar{\phi}(r)=(\sqrt{\kappa_\phi}/\xi_0)\phi(x)$. This equation is valid for $x>1$; for $x<1$, the term $\tilde{\gamma}/x^2$ is replaced by $\tilde{\gamma}$. Manifestly, at large x, $\phi(x)\sim x^{-1/2}\exp[-x(\xi_0/\xi_\phi)]$ falls with the appropriate Ornstein-Zernicke form, so where a nontrivial solution exists, the condensation energy is always finite. (As expected, in unscaled units, the vortex halo has a radius ξ_ϕ.)

At the critical point, $\alpha=\alpha_c$ (i.e., $\xi_0/\xi_\phi=0$), it is easy to see that the solution of Eq. (11) is

$$\phi(x)=\sqrt{1+\tilde{\gamma}}x^{-1} \text{ for } x>1. \quad (12)$$

[In the screened case, for $x>(\lambda/\xi_0)$, the solution is of the same form, but with $\sqrt{1+\tilde{\gamma}}\to 1$.] It thus follows that for large ξ_ϕ/ξ_0, $\bar{\phi}$ looks critical for a large intermediate range, $\xi_\phi>r>\xi_0$. From this, it follows that, as $\alpha\to\alpha_c^+$,

$$\int d\vec{r}|\bar{\phi}|^2\sim 2\pi\kappa_\phi(1+\tilde{\gamma})\ln|A\xi_\phi/\xi_0|,$$

$$\int d\vec{r}|\bar{\phi}|^4\sim 2\pi\left(\frac{\kappa_\phi}{\xi_0}\right)^2(1+\tilde{\gamma})^2A', \quad (13)$$

where A and A' are numbers of order 1. The first of these results is the essential ingredient leading to the existence of an avoided critical point in this problem. Note that the long-range tails of the vortex profile contribute to, but are not essential to this result; even if the vortex is screened, $\bar{\phi}$ at the critical coupling still has a $1/r$ form, so the second moment still diverges logarithmically as the critical coupling is approached, although with a somewhat different prefactor.

For α not so close to α_c, we expect a critical value of $\alpha=\alpha_1^{(0)}$ such that for $\alpha>\alpha_1^{(0)}$ there are no nontrivial solutions to the LG equations in the presence of an isolated vortex. It is relatively straightforward to prove that a necessary and sufficient condition for the existence of a non-trivial solution is that the quadratic kernel in $\mathcal{F}$ have at least one negative eigenvalue, i.e., that

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{1}{x}\frac{\partial}{\partial x}+\gamma|\bar{\Psi}(r)|^2\right]\phi_0(r)=\epsilon\phi_0(r) \quad (14)$$

has a solution with $\epsilon<0$. Thus, $\alpha_1^{(0)}$ is defined implicitly from the condition $\epsilon=0$. For $\tilde{\gamma}=1$ and the approximate form of the vortex profile introduced above,

$$\alpha_1^{(0)}=-0.75\frac{\kappa_\phi}{\xi_0^2} \ \text{ or } \ \xi_\phi=1.97\xi_0. \quad (15)$$

Similar estimates can be obtained for any positive $\tilde{\gamma}$. Finally, it is clear from the asymptotic form of the single vortex solutions that for dilute vortices, the effects of coupling between vortices are of order

$$J_{\rm inter}\sim\exp[-R/\xi_\phi], \quad (16)$$

where $R\gg\xi_\phi$ is the spacing between vortices.

IV. THE EFFECTIVE ACTION

To complete the quantum description of the order parameter fluctuations in the presence of an isolated vortex line, we define an effective Euclidean action which includes interplane couplings and the simplest possible quantum dynamics. We will consider the dynamics only of the slow modes of the order parameter ϕ. However, we will ignore the effects of the (possibly interesting) coupling to superconducting quasiparticles and other low-energy degrees of freedom—at least in a BCS d-wave superconductor they have a vanishingly small density of states and their effects on the dynamics of the order parameters of interest here is likely small. In addition, since the phase mode of the superconducting order parameter Ψ decouples at long wave length from the rest of the degrees of freedom, it is reasonable to suppose that quantum fluctuations of the superconducting order parameter can be integrated out to produce a small renormalization of the effective parameters; we thus consider only static configurations of Ψ, and omit any dependence on $\dot{\Psi}$. For all the cases of competing order ϕ that we are discussing here, the effective Euclidean action takes the form

$$S[\phi,\Psi]=\sum_j \int_0^\beta d\tau \int d\vec{r}\{(M/2)|\dot{\phi}|^2+\mathcal{F}[\phi_j,\Psi] - J_0(\phi_j^\dagger \phi_{j+1}+\text{H.c.})\}, \tag{17}$$

where $\vec{r}$ is a 2D vector in each plane, j labels the planes, $\phi_j(\vec{r})$ is the order parameter field in plane j, and we only consider the case in which $\Psi_j(\vec{r})=\Psi(\vec{r})$ is independent of layer index, i.e., when we consider a vortex core, we assume the vortex line is static and precisely perpendicular to the layers. Other forms of the ϕ dynamics can be considered, but we believe that, for the most part, the results will not be qualitatively different.

Our goal is to obtain explicit expressions for the couplings in the effective Hamiltonian, discussed in Sec. II, above. In particular, wherever there is a nontrivial solution to the mean-field equations, there are clearly a family of equivalent solutions, and important fluctuations which could potentially restore the broken symmetry are those which carry the system from one solution to another.

We begin with the case in which the order parameter ϕ has Ising symmetry. In the Ising case, the classical ground states are $\phi=\sigma\bar{\phi}$, where $\sigma=\pm 1$. Thus, at low temperatures, the relevant states are of the form $\phi_j(\vec{r})=\sigma_j\bar{\phi}(r)$. The effective Hamiltonian has matrix elements, denoted below by h, connecting these classical states with each other via tunneling processes (“spin flip”). However, while a detailed calculation of these matrix elements requires additional analysis to determine the tunneling paths that locally permit the system to fluctuate from one ground state to the other, the end result is quite simple: the resulting effective Hamiltonian must be of the form of a transverse-field Ising model

$$H_{\text{eff}}=h\sum_j \sigma_j^x - J\sum_j \sigma_j^z\sigma_{j+1}^z, \tag{18}$$

where J is given by

$$J/2J_0=\int d\vec{r}|\bar{\phi}(r)|^2. \tag{19}$$

In Eq. (18), h is the tunneling matrix element which therefore depends exponentially on the product of the effective mass M^* times the barrier height E_{cond}. Since the effective mass is renormalized by precisely the same factor $M^*/M = J/2J_0$, it follows that

$$\ln h \sim -A''\sqrt{M^*|E_{\text{cond}}|} \sim -\sqrt{\ln[\xi_\phi/\xi_0]}, \tag{20}$$

where the prefactor A'' is a constant determined by details of the tunneling process. This implies that this problem approaches the classical 1D Ising ferromagnet as $\alpha\to\alpha_c$.

This sort of analysis becomes simpler in the case in which a continuous symmetry is broken by ϕ. For instance, consider the XY and Heisenberg cases in which ϕ is, respectively, a two or three component vector, while $\bar{\phi}$, which is the solution of the LG equations, can be taken to be a scalar. Then, the full set of mean-field solutions can be written as $\phi(\vec{r})=\mathbf{\Omega}\bar{\phi}(r)$, where $\mathbf{\Omega}$ is a unit vector in order-parameter space. Now, for the case of weak interlayer coupling and large “mass” M^* we can ignore “amplitude” fluctuations, and focus our attention on the soft Goldstone modes by restricting our attention to field configurations of the form

$$\phi_j(\vec{r},\tau)=\mathbf{\Omega}_j(\tau)\bar{\phi}(r) \tag{21}$$

so that the effective Hamiltonian is

$$H_{\text{eff}}=\sum_j \frac{|\mathbf{L}_j|^2}{2M^*} - J\sum_j \mathbf{\Omega}_j\cdot\mathbf{\Omega}_{j+1}, \tag{22}$$

where $\mathbf{L}_j$ is the angular-momentum conjugate to $\mathbf{\Omega}$, and J is given by the same expression, Eq. (19), as in the Ising case. This is the Hamiltonian of a 1D array of quantum rotors, as discussed previously. Clearly, it follows from Eq. (13) that as $\alpha\to\alpha_c$, both J and M^* diverge, making the system more and more nearly a classical ferromagnet.

These results flesh out the general physical arguments made in the beginning of this paper. The effect of amplitude fluctuations have not been included in any of the present considerations—since ultimately we are dealing with a 3D quantum system, we believe they are generally less important than the “phase” fluctuations we have explicitly treated. We have not analyzed the further fluctuation corrections to test this supposition.

V. CONCLUDING REMARKS

When there is more than one competing order in a system, and the interactions between the two orders are strong, and a remarkably large and varied set of phase diagrams are possible.[7,27,28] The avoided critical point we have found here is a particularly striking example. In the context of stripe order, the situation is likely to be even more complex, since several distinct stripe orders have been observed in materials with stripy tendencies. Thus, when stripe order competes with superconductivity, the full phase diagram should have multiple, possibly nested versions of the relatively simple phase diagram discussed in the present paper.

There are a few additional observations we would like to make. The existence of vortex halos has many consequences,[29] which we have not explored, for the character of the thermal vortex states. In particular, it gives rise to vortices that effectively have two[30] core radii, ξ_ϕ and ξ_0 rather than a single vortex core radius as in conventional BCS superconductors. In this context, it is interesting to note that Wang *et al.*,[31] from an analysis of the Nernst effect in a variety of high-temperature superconductors, have recently adduced evidence for the existence of well-defined vortex excitations at temperatures well above the superconducting T_c. Moreover, from an analysis of the magnetic field dependence of the Nernst coefficient in $La_{2-x}Sr_xCuO_4$ and $YBa_2Cu_3O_{7-\delta}$, they showed that there were two characteristic magnetic field strengths in the putative vortex liquid state, which they identified as corresponding to two distinct vortex core radii. If this interpretation is accepted, the smaller characteristic field, which they call B^*, is to be associated with the maximal core radius $R^*\equiv\sqrt{\phi_0/2\pi B^*}$, where $\phi_0=hc/2e$ is the superconducting flux quantum. At temperatures well below the zero-field T_c, B^* is typically in the range of 30 T, corresponding to $R^* = 32$ Å. Since this length scale is comparable to the vortex halo radius observed[14] in STM studies of near-optimally doped $Bi_2Sr_2CaCu_2O_{8+\delta}$, it is very tempting to associate R^* with ξ_ϕ. In the Nernst effect experiments, B^* is seen to vary roughly linearly with temperature at low temperatures, so this identification could be tested by looking for similar temperature dependence of the vortex halo radius in STM experiments.

Finally, we note that the picture advocated here has also a "dual" version[12,32,29] in which one or the other form of stripe order is dominant, and superconducting order is subdominant. In this case, a finite density of topological defects of the stripe state, *dislocations*, which could be induced by some form of shear or by disorder, plays the role of the vortex density the problem considered in the present paper. Thus, a similar phase diagram as that in Fig. 1 (with the labels changed) can be constructed for dislocation induced superconductivity in a stripe ordered phase.

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this order, the vortices are so strongly overlapping that ϕ is essentially uniform. They argued that the same is true in the $N=3$ Heisenberg case, as well. A simple argument shows that in the Ising ($N=1$) case, this argument breaks down, at least for α near α_c. An array of votices with spacing $R \gg \xi_\phi$ can be treated as a $(2+1)$-dimensional transverse field Ising model, with one effective Ising spin per vortex halo, an effective exchange coupling which falls off exponentially as $J_{\rm inter}(\alpha,R) \sim \exp(-R/\xi_\phi)$, and a transverse field $h(\alpha)$ for which we can use the same Landau-Ginzburg estimate $h \sim \exp[-\sqrt{\ln(\xi_\phi/\xi_0)}]$ derived in Eq. (20). As a function of increasing magnetic field $R \sim 1/\sqrt{B}$ decreases, and the transition to the ϕ ordered state occurs when $J(\alpha,R) \approx h(\alpha)$. Neglecting any α dependence that is less singular than that in h, we find that this transition occurs when $R/\xi_\phi \sim \sqrt{\ln(\xi_\phi/\xi_0)} \gg 1$. This latter inequality insures the self-consistency of the approach. Consequently, the phase boundary has the very singular magnetic field dependence, $\alpha_c(B) - \alpha_c \propto [B \ln(H_c 2/B)]^{1/2\nu}$, where ν is the correlation length exponent of the $(2+1)$-dimensional Ising model $\nu \approx 0.63$. (For the XY and Heisenberg cases, the α dependence of h is no more singular than that of J, so it is possible that, as is the case in large N, the transition occurs when $R/\xi_\phi \sim 1$.)

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