Strongly interacting Luttinger liquid and superconductivity in an exactly solvable model

Igor N. Karnaukhov[1,*] and Alexander A. Ovchinnikov[1,2]
[1]*Max-Planck Institut für Physik Komplexer Systeme, Nöthnitzer Straße 38, D-01187 Dresden, Germany*
[2]*Joint Institute of Chemical Physics, Kosygin Street 4, 117334 Moscow, Russia*
(Received 8 May 2002; published 25 September 2002)

A family of exactly solvable one-dimensional models with a hard-core repulsive potential is solved by the Bethe ansatz for an arbitrary hard-core radius. The exact ground state phase diagrams in a plane ''electron-density–on-site interaction'' have been studied for several values of a hard-core radius. It is shown that strongly interacting Luttinger liquid can be realized in superconducting state at a high electron density and high value of repulsive on-site Coulomb interaction.

DOI: 10.1103/PhysRevB.66.104518 PACS number(s): 71.10.Fd, 71.10.Pm

The models of strongly correlated electrons with a bond-charge interaction which conserves a number of double occupied sites, are simple examples of strongly correlated electron systems that exhibit superconductivity.[1–4] The merit of these models is their complete integrability. The phase diagrams on a plane ''electron-density–on-site interaction'' have four phases, two of them exhibit off-diagonal long-range order (ODLRO) and thus are superconducting. The superconducting phase is realized if the value of the on-site interaction less than the critical one U_c and U_c could be even positive at repulsive on-site interaction U. Other phases in which only singly occupied and empty sites are presented called as $U\to\infty$ Hubbard state. In the case of a hard-core repulsive interaction between electrons with a hard core radius which exceeds a half of a lattice spacing the Luttinger liquid state transforms to strongly interacting luttinger liquid (SILL) in a high electron density region.[5] SILL is characterized by a large value of the critical exponent Θ for the momentum distribution function close to the Fermi momentum k_F

$$\langle n_k\rangle \simeq \langle n_{k_F}\rangle - \mathrm{const}|k-k_F|^{\Theta}\mathrm{sgn}(k-k_F), \tag{1}$$

that is defined by a single dimensionless exponent α [or K_ρ, here $\alpha=4K_\rho$ (Ref. 4)] $\Theta=(1/\alpha)(1-\alpha/4)^2$. At $\Theta>1$ the residual Fermi surface disappears. At a high electron density when a hard-core repulsion interaction dominates, SILL do not realized in superconducting state in the framework of the generalized *t-J* and Lai-Sutherland models.[5] The question arises: could SILL state be in superconducting state at positive and finite on-site Coulomb interaction? It turns out that an existence of the Fermi surface is not necessary for the superconducting phase.

In this paper we shall consider a family of integrable models and show that SILL can be realized in superconducting state at a high electron density or at small doping. This superconducting state takes place at a repulsive on-site Coulomb interaction the value of which larger than a band width and depends on the value of the hard-core radius. We shall show also that at zero temperature the ground state phase has the ODLRO and a finite Drude weight. In the models[1–3] the hoppings of single electrons on occupied states are forbidden, whereas the energy of electron pair is finite. In our models we shall use the same hierarchy for the parameters of the interactions, the constants of interactions between single electrons are infinite and define a hard-core radius, the energy of electron pair is finite. We shall consider a new modification of a generalized one-dimensional Lai-Sutherland model for a study of a competition between SILL state and superconducting phase. The model Hamiltonian contains kinetic and interaction terms that combine those of the Hubbard model and the Lai-Sutherland model. The model Hamiltonian includes two terms $\mathcal{H}=\mathcal{H}_{\mathrm{hop}}+\mathcal{H}_{\mathrm{int}}$

$$\mathcal{H}_{\mathrm{hop}}=-t\sum_{\langle i,j\rangle}\sum_{\sigma=\uparrow,\downarrow}[\mathcal{P}_l(1-n_{i-\sigma})c^{\dagger}_{i\sigma}c_{j\sigma}(1-n_{j-\sigma})\mathcal{P}_l - c^{\dagger}_{i\sigma}c_{j\sigma}n_{i-\sigma}n_{j-\sigma}], \tag{2}$$

$$\mathcal{H}_{\mathrm{int}}=J\sum_{j=1}^{L}\sum_{\sigma,\sigma'=\uparrow,\downarrow}(c^{\dagger}_{j\sigma}c_{j\sigma'}c^{\dagger}_{j+1+l\sigma'}c_{j+1+l\sigma} + n_{j\sigma}n_{j+1+l\sigma'})+U\sum_{j=1}^{L}n_{j\uparrow}n_{j\downarrow}, \tag{3}$$

where $c^{\dagger}_{j\sigma}$ and $c_{j\sigma}$ are the creation and annihilation operators of fermions with spin σ, $\sigma\in\{\uparrow,\downarrow\}$, L is a total number of lattice sites, $\langle i,j\rangle$ stands for neighboring sites, the projector $\mathcal{P}_l$ forbids two single electrons at distances less than or equal to l (l is measured in units of the lattice spacing parameter), t is the hopping integral, J is the constant of the exchange interaction. It is important the $\mathcal{P}_l$ operator does not forbid doubly occupied lattice sites, as it takes place in the so-called $U\to\infty$ Hubbard model or the *t-J* model. The last term in Eq. (3) is traditionally the most important term for the Hubbard model, the on-site Coulomb repulsion U separates the energies of single and paired electrons states. The Hamiltonian $\mathcal{H}$ conserves not only the total number of electrons N and also the number of single electrons with spin $\sigma N_{1\sigma}=\Sigma_{j=1}^{L}n_{j\sigma}(1-n_{j-\sigma})$ and the number of electron pairs $N_2=\Sigma_{j=1}^{L}n_{j\uparrow}n_{j\downarrow}, N=\Sigma_\sigma N_{1\sigma}+2N_2$. In the case $l=0$ and $J=0$ the Hamiltonian (2), (3) is reduced to Arrachea-Aligia Hamiltonian[2] and for $l=0$, $U=\infty$, and $J=t$ to the Lai-Sutherland model.[6] For $l>0$ the $\mathcal{P}_l$ operator is equivalent to additional two particle interactions between single electrons $\Sigma_{r=1}^{l}\Sigma_{j=1}^{L}U_r n_{1j}n_{1j+r}$, where $n_{1j}=\Sigma_{\sigma=\uparrow\downarrow}n_{j\sigma}(1-n_{j-\sigma})$ with infinite U_r parameters, according to Eq. (2) $U_{l+1}=J$.

Using this representation we can conclude that the kinetic term of the Hamiltonian (2) is a particle-hole invariant: indeed applying this transformation $c_{j\sigma}^{\dagger} \Rightarrow c_{j\sigma}, c_{j\sigma} \Rightarrow c_{j\sigma}^{\dagger}$, to the Hamiltonian (2), (3) we obtain $\mathcal{H}(t,J,U) \Rightarrow \mathcal{H}(t,J,U) + U(L-N)$. Due to a particle-hole symmetry the phase diagram is symmetrical with respect to a half filling.

We examine the exact ground state phase diagram for the antiferromagnetic coupling $J=t$ (we chose the hopping integral equal to unit then the coupling constants are dimensionless) and different values of the hard-core radius. The results of calculations are compared with the ones for $J=0$—the simplest version of the model . Direct calculations show that the model (2), (3) is an exactly solvable one by the Bethe ansatz method and the set of the quasimomenta $\{k_j\}(j=1,2,\ldots,N_1)$ satisfies the Bethe equations[5]

$$\left(\frac{\lambda_j - i/2}{\lambda_j + i/2}\right)^{L-lN_1} = (-1)^{N_1-1}\exp(-ilP)\prod_{i=1}^{N_1}\frac{\lambda_j-\lambda_i-i}{\lambda_j-\lambda_i+i} \times \prod_{\alpha=1}^{M}\frac{\lambda_j-\chi_\alpha+i/2}{\lambda_j-\chi_\alpha-i/2},$$

$$\prod_{j=1}^{N_1}\frac{\chi_\alpha-\lambda_j+i/2}{\chi_\alpha-\lambda_j-i/2} = -\prod_{\beta=1}^{M}\frac{\chi_\alpha-\chi_\beta+i}{\chi_\alpha-\chi_\beta-i}, \tag{4}$$

where $P=\Sigma_{j=1}{}^{N_1}k_j$ is the momentum, $\lambda_j=\frac{1}{2}\tan(k_j/2)$ and $\chi_\alpha(\alpha=1,2,\ldots,M)$ are the "charge" and "spin" rapidities, M is the number of down spin single electrons.

The eigenvalues and the magnetization are given by

$$E=-2\sum_{j=1}^{N_1}\cos k_j + UN_2, \tag{5}$$

$$S^z=\frac{1}{2}\sum_{\sigma} N_{1\sigma}-M. \tag{6}$$

Let us introduce the partial electron densities $n_1=N_1/L$ is the density of single carriers $(N_1=\Sigma_{\sigma=\uparrow,\downarrow}N_{1\sigma})$, $n_2=N_2/L$ is the density of electron pairs. Clearly $n=n_1+2n_2$, here $n=N/L$ is the total density of electrons.

Numerical exact diagonalization calculations of the model at $J\neq 0$ for small l shown that the ground state is the nongenerated one. Since the Bethe equations (4) we can calculate exactly the ground state phase diagram as a function of the electron density and an on-site interaction for an arbitrary value of the hard-core radius. The densities n_1 and n_2 can be calculated by minimizing the ground state energy per site $\mathcal{E}=E/L$ for a fixed total density of electrons

$$\mathcal{E}=2n_1-2\pi\int_{-Q}^{Q}d\Lambda a(\Lambda)\rho(\Lambda)+Un_2, \tag{7}$$

where $a(\Lambda)=(1/2\pi)[1/(\Lambda^2+1/4)]$.

In the thermodynamic limit the Bethe equations reduce to an integral equation of the Fredholm type for the function of the distribution of λ_j on the real axis

$$\rho(\Lambda)+\int_{-Q}^{Q}d\Lambda' K(\Lambda-\Lambda')\rho(\Lambda')=(1-ln_1)a(\Lambda), \tag{8}$$

with the kernel being

$$K(\Lambda)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\exp(-|\omega|)}{\exp|\omega|+1}\exp(i\omega\Lambda).$$

The Λ-Fermi level denoted as Q controls the band filling, the density of single electrons is defined by

$$n_1=\int_{-Q}^{Q}d\Lambda\rho(\Lambda). \tag{9}$$

$Q=0$ corresponds to an empty subband of single carriers, $n_1=n_0$ for $Q\to\infty$, where $n_0=1/(l+3/2)$ is a "half-filled" density. Equations (7)–(9) are a consequence of the real solutions for "charge" and "spin" rapidities that describe the ground state of the system in the absence of an external magnetic field. In order to define the region of SILL we calculate the critical exponent Θ via the α exponent (1). $\alpha=2\zeta^2(Q)$ is defined by the dressed charge $\zeta(\Lambda)$, where $\zeta(\Lambda)$ is a solution of the following integral equation:

$$\zeta(\Lambda)+\int_{-Q}^{Q}d\Lambda' K(\Lambda-\Lambda')\zeta(\Lambda')=1-ln_1. \tag{10}$$

In the high electron density region $n>2n_2+n_c$ [where n_c is solution of equation $\Theta(n_c)=1$] when the hard-core repulsive potential dominates the behavior of fermions is described as SILL with $\Theta>1$.[5]

We have focused on the calculation of the exact ground state phase diagram in the $n-U$ plane for different values of the hard core radius or l. First we consider peculiarities of behavior of the system using a simple version of the Hamiltonian (2),(3) when $J=0$ and then its transformation for $J=1$. Due to a particle-hole symmetry it is sufficient to discuss the phase diagram for $n\leqslant 1$. For $J=0$ the density of the ground-state energy (6) can be defined analytically $\mathcal{E}=-2[(1-ln_1)/\pi]\sin[\pi n_1/(1-ln_1)]+\frac{1}{2}U(n-n_1)$, therefore a curve that separates a mixed region is defined according to the following equation

$$U(n_1)=4\frac{l}{\pi}\sin\left(\frac{\pi n_1}{1-ln_1}\right)-\frac{4}{1-ln_1}\cos\left(\frac{\pi n_1}{1-ln_1}\right).$$

U varies from -4 at $n_1=0$ to $4(1+l)$ at $n_1=n_{\max}=1/(1+l)$, hence a maximal value $U_c=4(1+l)$. The value of n_c is equal to $n_c=(1-\sqrt{6-4\sqrt{2}})/l$ ($n_c=0.414$ for $l=1$,$n_c=0.207$ for $l=2$, $n_c=0.138$ for $l=3$). For $l=1$ the complete phase diagram is shown in Fig. 1. The lower region (for $U<-4$) is characterized by only doubly occupied (solid circles) and empty (empty circles) sites so $n_1=0$ and $n_2=n/2$. For $-4<U<U(n_1)$ we have a mixed region, the ground state includes both finite densities of single electrons (spheres with dot center) and electron pairs. Note that the pairs are not localized due to exchange between single and double electron states. Both the mixed and the lower phases have a finite ODLRO,[1,7] i.e., $\langle\eta_i^{\dagger}\eta_j\rangle\not\to 0$ for $|i-j|\to\infty$ (here $\eta_j^{\dagger}=c_{j\uparrow}^{\dagger}c_{j\downarrow}^{\dagger}$). However, the second of them is insulator phase

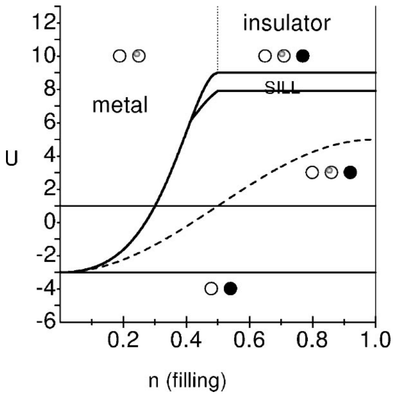

FIG. 1. Ground state phase diagram in the case $J=0$ for $l=1$. Dashed line corresponds to the model (Ref. 2) (or $l=0$), dotted lines separate metallic and insulator phases. The area of a strongly interacting Luttinger liquid state is denoted as SILL.

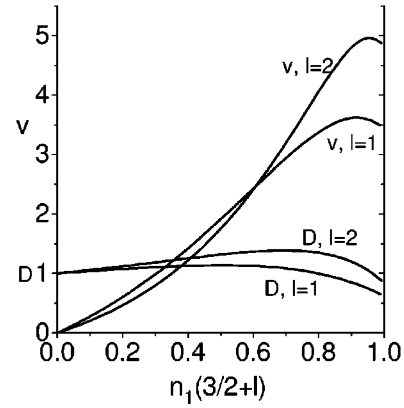

FIG. 3. Normalized charge stiffness $D=D^*/D_c$ $[D_c=(2/\pi)\sin(\pi n_1/2)]$ and the Fermi velocity v as functions of the density of single electrons for $J=1$, $l=1$, and $l=2$.

since in the absence of single electrons pairs are localized (the Drude weight is equal to zero). The mixed phase is a superconducting having normal metallic Drude weight.

At $n>n_c$ in the region denoted as SILL in Figs. 1 and 2 $\Theta>1$ and we deal with SILL superconductor. Note, that SILL state is realized at largest values of a repulsive on-site Coulomb interaction and a high electron density. Comparing the calculations for different l we can conclude that a hard-core repulsive interaction increases a region of SILL superconducting state due to both a larger U_c and smaller n_c. For $U>U(n_1)$ and $n<n_{\max}$ the ground state consists of singly occupied and empty sites; dotted lines separate a metallic phase (at $n<n_{\max}$) and an insulator phase (at $n\geqslant n_{\max}$) with a gap $\Delta\varepsilon=U-U_c$.

An exact solution of the problem enables to study the role of the exchange interaction on the behavior of a strongly interacted electron system. Let us consider a transformation of the exact ground-state phase diagram for $J=1$. We restrict our consideration the case $n\leqslant n_0$. Θ increases monotonically from $\frac{1}{8}$ to $[(3+2l)^2/12][1-3/(3+2l)^2]^2$ with the n_1 density. We should solve equation $\Theta(n_c)=1$ numerically calculating the dressed charge as a function of the electron density n_1 for arbitrary l; for example, $n_c=0.348$ for $l=1$, $n_c=0.192$ for $l=2$, $n_c=0.131$ for $l=3$, $n_c=0.1$ for $l=4$.

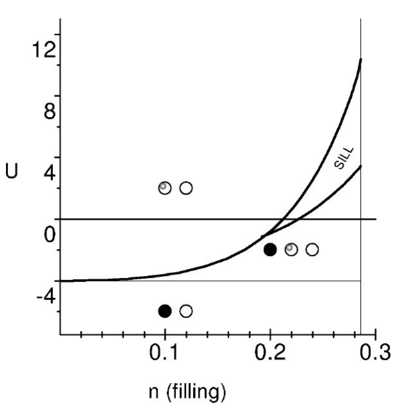

FIG. 2. Ground state phase diagram in the case $J=1$ and $l=2$—similar to that for Fig. 1.

According to the numerical results obtained the critical density n_c is less than n_0. For $l=2$ the ground state phase diagram is given in Fig. 2. All electron states: empty, singly occupied, and doubly occupied sites are presented simultaneously in a mixed region (a closed region in Fig. 2). For $n_c<n<n_0$ two branches of curves separate the Luttinger liquid state and the SILL that is realized between these branches. Comparing the phase diagrams for $J=1$ and $J=0$ calculated for the same value of l we can conclude that the exchange interaction decreases the region of SILL in superconducting state (n_c and U_c decrease slightly). U_c increases with an increasing of hard-core radius.

To demonstrate the superconducting behavior of the mixed phase we calculated the Drude weight of this phase. According to Ref. 8 we will use the expression of the charge stiffness $D^*=(1/4\pi)\alpha v$, where v is the Fermi velocity $v=(d/d\lambda)\ln\rho(\lambda)|_{\lambda=Q}$. The numerical results of the normalized charge stiffness $D=D^*/D_c$ $[D_c=(2/\pi)\sin(\pi n_1/2)]$ and the Fermi velocity for $l=1$ and $l=2$ are presented in Fig. 3.

Taking into account an existence of ODLRO in the mixed phase we can conclude that this phase and SILL state in particular are the superconducting phases. In summary, we have presented a soluble generalization of the Lai-Sutherland model, having the nontrivial Luttinger liquid behavior. The exact solution was obtained by means of the nested Bethe ansatz. We have derived the exact ground-state phase diagram; the latter exhibits an unusual phase state in which strongly interacting Luttinger liquid comes in superconducting state. This phase is realized at high electron density and positive values of the on-site Coulomb interaction. The maximum critical value U_c realized in the model is higher than that of all other exactly solvable models.[2,3] This is important because higher values of U_c expands the region of coexistence of SILL in superconducting state. It has beenshown that the presence of the Fermi level is not necessary for realization of superconducting phase. The results of calculations of one dimensional models do not allow direct application to the real 2D and 3D systems. Nevertheless one can assume that real high-T_c superconductors belong to the family of SILL described above.

I.K. would like to thank Professor E.D. Belokolos for valuable discussions and wishes to thank the support of the Visitor Program of the Max-Planck-Institut für Physik Komplexer Systeme, Dresden, Germany. This work is supported under RFFR Grants No. 00-03-32981, No. 00-15-97334, and the STCU-2354 project.

*Permanent address: Institute of Metal Physics, Vernadsky Street 36 03142 Kiev, Ukraine.

[1] F.H.L. Essler, V.E. Korepin, and K. Schoutens, Phys. Rev. Lett. **68**, 2960 (1992); An.A. Ovchinnikov, J. Phys. Condens. Matter **6**, 11 057 (1994).

[2] L. Arrachea and A.A. Aligia, Phys. Rev. Lett. **73**, 2240 (1994); L. Arrachea, A.A. Aligia, and E. Gagliano, *ibid.* **76**, 4396 (1996); A. Schadschneider, Phys. Rev. B **51**, 10 386 (1995).

[3] F. Dolcini and A. Montorsi, Phys. Rev. B **63**, 121103(R) (2001); F. Dolcini and A. Montorsi, Nucl. Phys. **B592**, 563 (2001).

[4] J. Voit, Rep. Prog. Phys. **58**, 977 (1995).

[5] I.N. Karnaukhov, Europhys. Lett. **52**, 571 (2000); I.N. Karnaukhov and N. Andrei, J. Phys.: Condens. Matter **13**, L891 (2001); I.N. Karnaukhov and A.A. Ovchinnikov, Europhys. Lett. **57**, 540 (2002).

[6] C.K. Lai, J. Math. Phys. **15**, 1675 (1974); Bill Sutherland, Phys. Rev. B **12**, 3795 (1975).

[7] C.N. Yang, Phys. Rev. Lett. **63**, 2144 (1989); C.N. Yang and S. Zhang, Mod. Phys. Lett. B **4**, 759 (1990).

[8] N. Kawakami and S.-K. Yang, Phys. Rev. B **44**, 7844 (1991).