Stabilization of half-skyrmions: Heisenberg spins on a non-simply connected manifold

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We show that the metastable skyrmion on a plane can be stabilized by removing a disk of radius ρ_0 at the origin. This renders the plane non-simply connected but provides a characteristic length for stabilizing the half-skyrmion which is centered at a radius $\rho_0(1+\sqrt{2})$. We obtain exact solutions for the half-skyrmion and half-skyrmion lattice and demonstrate that these solutions are a limiting case of Heisenberg spins on (i) a truncated cone with half-angle $\alpha \rightarrow \pi/2$ and (ii) an annulus. We also discuss skyrmions in the presence of a perpendicular external magnetic field which may have relevance for polarized electrons, especially in the quantum Hall effect and quantum antidots.

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I. INTRODUCTION

A variety of planar magnetic materials and magnetic surfaces may be modeled as a continuum of classical spins. The continuum limit of the Heisenberg Hamiltonian for classical ferromagnets or antiferromagnets for isotropic spin–spin coupling is the nonlinear σ model. The latter model also arises in the context of a two-dimensional electron gas (2DEG) in a perpendicular magnetic field, specifically in the case of the quantum Hall effect.[1] The nontrivial spin textures in these two cases are solitons and skyrmions, respectively. Recently, an exact correspondence between the quantum Hall skyrmions (interacting by a delta repulsive potential with filling fraction $\nu=1$) and the solitons of the nonlinear σ model has been established.[2] From this perspective the results presented below apply to both the Heisenberg spin solitons and quantum Hall skyrmions.

The skyrmions can also condense into a crystal lattice, i.e., a skyrmion lattice (as discussed below). The latter is observed in a 2DEG in GaAs/GaAl$_x$As$_{1-x}$ heterojunctions.[3] This periodic structure is also important in the context of dense nuclear matter and baryogenesis[4] where it is known as a Skyrme crystal.

Let us consider Heisenberg spins on a plane with homogeneous boundary conditions: spins tend to $\hat{n}_0=$constant at infinity. It is well known[5] that the spin configurations can be classified according to Π_2 (S_2) homotopy classes. Unfortunately, these skyrmion configurations are metastable since for a scale factor λ the total energy is scale invariant,

$$H_\lambda=\int\int(\hat{n}_{\lambda x}^2+\hat{n}_{\lambda y}^2)d(\lambda x)d(\lambda y)=H(\lambda=1),$$

and thus the skyrmions can be shrunk to a point.[6]

In this paper we explore the effect of the support manifold being not simply connected and focus on the simplest case, namely, the plane R^2. Specifically, we consider the plane R^2 with a disk of radius ρ_0 cut out from it. Later we will also consider the spins on $R^2\setminus D^2_{\rho_0}$ in a perpendicular magnetic field **B**. Taking this into consideration and the geometry of the problem we look for cylindrically symmetric solutions only. We introduce a cylindrical coordinate system (ρ,ϕ). The Hamiltonian is given by the nonlinear sigma model,

$$H=2\pi\int_{\rho_0}^{\infty}d\rho\left[\rho\,\theta_\rho^2+\frac{\sin^2\theta}{\rho}\right]. \tag{1}$$

Here $\hat{n}=(\sin\theta\cos\Phi,\sin\theta\sin\Phi,\cos\theta)$, i.e., the spins lie on a unit sphere S^2, and we assume cylindrical symmetry for the spin configurations, $\theta=\theta(\rho)$ and $\Phi=\phi$.

Before discussing the technical details we first summarize the main findings. (i) The non-simply connected geometry of the plane renders the skyrmion stable by introducing a length scale and leads to a sine-Gordon equation. (ii) We obtain a novel *half-skyrmion* solution which centers itself at a distance $\sqrt{2}\rho_0$ from the outer boundary of the hole. (iii) We also find an exact half-skyrmion lattice solution and asymptotic interaction between the half-skyrmions (Sec. II). (iv) These exact solutions can be used to perturb around when the Zeeman and Coulomb interactions are systematically turned on. (v) We show that this problem is topologically equivalent to a truncated cone in the limit that the cone half-angle $\alpha \rightarrow \pi/2$. (vi) We obtain novel single and lattice skyrmion solutions on a truncated cone along with the asymptotic interaction (Sec. III). (vii) For an annulus (or a truncated finite cone) we get *fractional* skyrmions with topological charge less than half (Sec. IV). This geometry is relevant to recent resonant tunneling experiments in the quantum Hall regime measuring fractional charge in quantum antidots.[7] (viii) In the presence of a perpendicular (or axial in the case of the cone) magnetic field we discuss skyrmion solutions of a double sine-Gordon equation with nonconstant coefficients (Sec. V).

We note here that it is *not* possible to consider the usual Belavin–Polyakov[5] solution and apply it to the region $\rho \geqslant \rho_0$ (even if we restrict ourselves only to the cylindrically symmetric configurations). For this region the Belavin–Polyakov configuration belongs to the 0th homotopy class

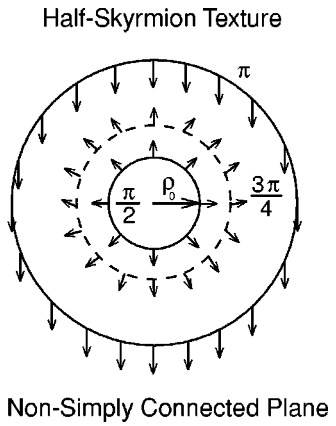

FIG. 1. Cylindrically symmetric $\pi/2 \rightarrow \pi$ sine-Gordon half-skyrmion on an infinite plane with a hole of radius ρ_0.

where the minimum in energy is given by the constant vector field. The Belavin–Polyakov minimum energy configuration is based on topological considerations which are not applicable in our case.

II. NON-SIMPLY CONNECTED PLANE

The configurations with the lowest energy are given by the solutions of the Euler–Lagrange (EL) equation,

$$\theta_\rho + \rho\,\theta_{\rho\rho} = \frac{\sin\theta\cos\theta}{\rho}.$$

Here we consider that $\theta(\rho_0) = \text{constant}$. We define a new radius coordinate $\overline{\rho} = \ln(\rho/\rho_0)$ which allows us to reduce the EL equation to a sine-Gordon equation,

$$\theta_{\overline{\rho}\overline{\rho}} = \frac{\sin 2\theta}{2}. \qquad (2)$$

We find a novel exact half-skyrmion solution (on a nonsimply connected plane depicted in Fig. 1)

$$\theta(\rho,\rho_0) = 2\tan^{-1}\frac{\rho}{\rho_0}. \qquad (3)$$

Clearly, this solution depends on ρ_0 and does not shrink to a point like the usual Belavin–Polyakov skyrmion.[5] It is located at $\rho_c = \rho_0 \cot(\pi/8) = \rho_0(1+\sqrt{2})$ with energy 4π (instead of 8π) and topological charge density,

$$q(\rho) = \frac{\theta_\rho \sin\theta}{4\pi\rho} = \frac{1}{\pi}\frac{\rho_0^2}{(\rho^2+\rho_0^2)^2}.$$

Note that the vector field in Eq. (3) covers a unit sphere exactly half times and thus it is a half-skyrmion. Equivalently, the components of the unit vector field are

$$n^x = \frac{2x\rho_0}{\rho^2+\rho_0^2}, \quad n^y = \frac{2y\rho_0}{\rho^2+\rho_0^2}, \quad n^z = \frac{\rho^2-\rho_0^2}{\rho^2+\rho_0^2}.$$

The corresponding half-skyrmion (radial) lattice solution is given by

$$\cos\theta = n^z = sn\left(\frac{1}{k}\ln\frac{\rho}{\rho_0}, k\right), \quad \rho_m = \rho_0 \exp(2mkK), \qquad (4)$$

where ρ_m denotes the location of mth half-skyrmion (with $\theta = \pi/2$) from the center of the plane. Here $\mathrm{sn}(x,k)$ [and $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ below] are Jacobi elliptic functions with modulus k and $k' = \sqrt{1-k^2}$ denotes the complementary modulus. $K(k)$ [and $E(k)$ below] are the complete elliptic integrals of the first and second kind, respectively. Equivalently,

$$n^x = \frac{x}{\rho}\mathrm{cn}\left(\frac{1}{k}\ln\frac{\rho}{\rho_0}, k\right), \quad n^y = \frac{y}{\rho}\mathrm{cn}\left(\frac{1}{k}\ln\frac{\rho}{\rho_0}, k\right),$$

and the topological charge density per half-skyrmion,

$$q(\rho) = \frac{1}{4\pi k \rho^2}\mathrm{cn}\left(\frac{1}{k}\ln\frac{\rho}{\rho_0}, k\right)\mathrm{dn}\left(\frac{1}{k}\ln\frac{\rho}{\rho_0}, k\right),$$

which is smaller than that for a single half-skyrmion. In the limit $k \rightarrow 1$ these results reduce to those for a single half-skyrmion given above.

The formation energy of the half-skyrmion lattice per half-skyrmion over an "exponential period" is

$$E_s = \frac{4\pi}{k}\left[E(k) - \frac{k'^2}{2}K(k)\right]. \qquad (5)$$

The asymptotic interaction between two half-skyrmions (separated by a large distance ρ_1) is obtained as

$$U(\rho_1) \simeq \pi k'^2 = 16\pi\rho_0/\rho_1.$$

Unlike the exponentially decaying asymptotic interaction between the two solitons on an infinite cylinder,[8,9] this interaction decays inversely with the distance between two half-skyrmions.

Some comments on skyrmion lattices are in order. The existence of ("ferromagnetic") triangular and ("antiferromagnetic") square skyrmion lattices for a 2DEG in the quantum Hall regime at Landau level filling factors near $\nu = 1$ has been shown theoretically[10,11] and structural transitions between these two lattices have been recently studied.[12] Anomalies in specific heat measurements[3] on GaAs heterojunctions are also indicative of condensation of skyrmions into a crystal lattice. We note, however, that due to circular symmetry our skyrmion lattice solution is a radially stretched triangular lattice. It is also interesting to note that using a complex parameterization of the nonlinear σ model[5] Green *et al.*[11] used elliptic functions to represent the square lattice with $w(z) = h\ \mathrm{cn}(z, k = 1/\sqrt{2})$ and the triangular lattice with $w(z) = h\ \mathrm{cn}(z, k = \exp(i2\pi/3))$, where h is a constant.

We also point out that analogous Skyrme crystal states arise in the study of dense nuclear matter using Skyrme's topological excitation model.[13] Specifically, a *half-skyrmion* picture of baryon matter at high density[14] turns out to be very useful in describing a simple cubic to body-centered-cubic *half-skyrmion lattice* structural transition as a function of pressure and density.[4] Recently, based on an analogy with fullerene chemistry and concepts from rational maps be-

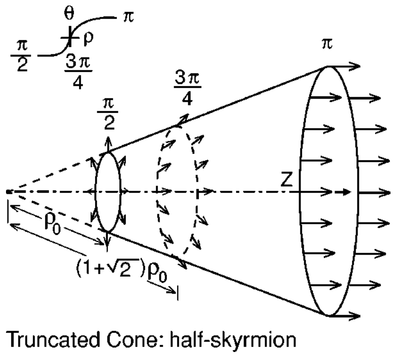

FIG. 2. Cylindrically symmetric $\pi/2 \to \pi$ sine-Gordon half-skyrmion on a truncated right circular cone [Eq. (8)].

tween Riemann spheres, baryon density isosurfaces have been studied using a Skyrme crystal with hexagonal symmetry.[15]

III. TRUNCATED CONE

It is quite instructive to understand these results as a special case of Heisenberg spins on a truncated cone. Consider a right circular cone with apex at $z=0$ and the open angle $=2\alpha$, with the cone opening along the positive z-axis. The cone is truncated at a distance ρ_0 from the apex along the surface (Fig. 2). In this case in cylindrical coordinates $x=\rho(z)\cos\phi$, $y=\rho(z)\sin\phi$, and $z=\rho(z)\cot\alpha$. Note that the distance on the cone surface from apex, ρ, varies with z ($\rho=z\tan\alpha$). The induced metric on a cone,

$$g_{\rho\rho}=1+\cot^2\alpha,\quad g_{\phi\phi}=\rho^2,\quad g_{\phi\rho}=0.$$

Therefore $\sqrt{g}=\rho\sqrt{1+\cot^2\alpha}=\rho/\sin\alpha$, $\sqrt{g}\ g_{\rho\rho}=\rho/\sin^3\alpha$, and $\sqrt{g}\ g_{\phi\phi}=\rho^3/\sin\alpha$.

If we again consider cylindrically symmetric solutions, the Hamiltonian is

$$H=2\pi\int_{\rho_0}^{\infty}d\rho\left[\rho\sin\alpha\,\theta_\rho^2+\frac{\sin^2\theta}{\rho\sin\alpha}\right]. \tag{6}$$

The corresponding Euler–Lagrange equation is

$$\frac{\partial}{\partial\rho}(\rho\cos\alpha\,\theta_\rho)=\frac{\sin 2\theta}{2\rho\cos\alpha\tan^2\alpha}.$$

If we make the coordinate transformation $\overline{\rho}=\sec\alpha\ln(\rho/\rho_0)$, we again get the sine-Gordon equation,

$$\theta_{\overline{\rho\rho}}=\frac{\sin 2\theta}{2\tan^2\alpha}, \tag{7}$$

with the single half-skyrmion solution (shown in Fig. 2)

$$\theta(\rho)=2\tan^{-1}\left(\frac{\rho}{\rho_0}\right)^{1/\sin\alpha}, \tag{8}$$

and the creation energy $E_s=4\pi$ which is independent of ρ_0 as well as the cone half-angle α. The vector field in Eq. (8) also covers a unit sphere exactly half times, thus it also is a half-skyrmion. The associated topological charge density is

$$q(\rho,\alpha)=\frac{\sin\alpha\,\theta_\rho\sin\theta}{4\pi\rho}=\frac{1}{\pi\rho^2}\frac{(\rho\rho_0)^{2/\sin\alpha}}{(\rho^{2/\sin\alpha}+\rho_0^{2/\sin\alpha})^2},$$

and the components of the unit vector field are

$$n^x=\frac{2\cos\phi(\rho\rho_0)^{1/\sin\alpha}}{\rho^{2/\sin\alpha}+\rho_0^{2/\sin\alpha}},\qquad n^y=\frac{2\sin\phi(\rho\rho_0)^{1/\sin\alpha}}{\rho^{2/\sin\alpha}+\rho_0^{2/\sin\alpha}},$$

$$n^z=\frac{\rho^{2/\sin\alpha}-\rho_0^{2/\sin\alpha}}{\rho^{2/\sin\alpha}+\rho_0^{2/\sin\alpha}}.$$

The corresponding half-skyrmion lattice solution is

$$\cos\theta=n^z=\mathrm{sn}\left(\frac{1}{k\sin\alpha}\ln\frac{\rho}{\rho_0},k\right), \tag{9}$$

with periodicity $\rho_m=\rho_0\exp(2mkK\sin\alpha)$. The creation energy per half-skyrmion over an ''exponential period'' is the same as that for the non-simply connected plane. It is independent of both ρ_0 and α. Equivalently,

$$n^x=\mathrm{cn}\left(\frac{1}{k\sin\alpha}\ln\frac{\rho}{\rho_0},k\right)\cos\phi,$$

$$n^y=\mathrm{cn}\left(\frac{1}{k\sin\alpha}\ln\frac{\rho}{\rho_0},k\right)\sin\phi,$$

and the topological charge density is

$$q(\rho)=\frac{1}{4\pi\rho^2k}\mathrm{cn}\left(\frac{1}{k\sin\alpha}\ln\frac{\rho}{\rho_0},k\right)\mathrm{dn}\left(\frac{1}{k\sin\alpha}\ln\frac{\rho}{\rho_0},k\right).$$

In the limit $k\to 1$ these results reduce to the single half-skyrmion results on a cone and for $\alpha\to\pi/2$ we recover results on a non-simply connected plane.

The asymptotic interaction between the two half-skyrmions on a truncated cone is

$$U(\rho_1)\simeq\pi k'^2=16\pi\left(\frac{\rho_0}{\rho_1}\right)^{1/\sin\alpha}.$$

Again, this interaction decays with the separation between two half-skyrmions as a power law with an α-dependent exponent. When $\alpha=\pi/2$, this corresponds to the case of $R^2\setminus D^2_{\rho_0}$ already considered above. The truncated cone interpolates between the plane with the disk D^2 taken out and a semi-infinite cylinder. The transformation $\overline{\rho}=\ln(\rho/\rho_0)$ topologically maps $R^2\setminus D^2_{\rho_0}$ onto an infinite cylinder sending the radius $\rho_0\to-\infty$ and $\rho=\infty$ to $\overline{\rho}\to\infty$. Similarly, the transformation $\overline{\rho}=\sec\alpha\ln(\rho/\rho_0)$ topologically maps a truncated cone onto an infinite cylinder.

The basic philosophy is that when considering sine-Gordon solitons, the underlying geometry may introduce a length in the problem which is picked out by the soliton solutions.[8,9] This is highlighted by the topological equivalence between $R^2\setminus D^2$ and the infinite cylinder. Note that a cylinder and the sine-Gordon equation are intimately related—the cylinder has a constant curvature ($\kappa=1/\rho_0$) while the sine-Gordon equation corresponds to a surface of constant (negative Gaussian) curvature.[16] Note also that shrinking the plane to a finite disk[17] does not stabilize the skyrmion simply because the disk remains topologically equivalent to the infinite plane (and to a finite cone).

IV. FRACTIONAL SKYRMIONS: ANNULUS AND A FINITE TRUNCATED CONE

If we consider an annulus (or equivalently a cone truncated on both sides) with inner radius ρ_0 and the outer radius ρ_1 then we get the *same* form of the solution as for the non-simply connected plane (or the truncated cone in Sec. III). However, now we get a *fractional* skyrmion with topological charge varying between zero and half and the energy varying between zero and 4π. Since $\pi/2<\theta(\rho)<2\tan^{-1}(\rho_1/\rho_0)$, the skyrmion is located on the plane at

$$\rho_c=\rho_0\tan[\pi/8+(1/2)\tan^{-1}(\rho_1/\rho_0)],$$

with a similar (α-dependent) expression for the finite cone. Clearly, as $\rho_0\to 0$, the skyrmion becomes unstable since $\rho_c\to 0$ as in the case of a finite disk.[17] The two limiting cases for the annulus are the finite disk ($\rho_0\to 0$) and the non-simply connected plane ($\rho_1\to\infty$). In the former case the annulus fractional skyrmion becomes unstable whereas in the latter case it is the stable half-skyrmion. Note that an annular geometry has been used recently to model[18] the observed resonant tunneling through a quantum antidot in the quantum Hall regime.[7]

V. EFFECT OF EXTERNAL MAGNETIC FIELD

Next, we consider a magnetic field perpendicular to the plane. In this case the Hamiltonian is given by

$$H=2\pi\int_{\rho_0}^{\infty}d\rho\left[\rho\theta_\rho^2+\frac{\sin^2\theta}{\rho}+\frac{\rho}{\rho_B^2}(1-\cos\theta)\right], \tag{10}$$

where the magnetic length $\rho_B^2=1/g\mu B$. Here B is the applied magnetic field, μ is the magnetic moment, and g denotes the g factor of the electrons in the material. The corresponding Euler–Lagrange equation is

$$\theta_\rho+\rho\theta_{\rho\rho}=\frac{\sin\theta\cos\theta}{\rho}+\frac{\rho}{2\rho_B^2}\sin\theta.$$

Using the transformation $\bar{\rho}=\ln(\rho/\rho_0)$ we obtain a double sine-Gordon equation with *nonconstant* coefficients

$$\theta_{\bar{\rho}\bar{\rho}}=\frac{\sin 2\theta}{2}+\frac{1}{2}\left(\frac{\rho_0}{\rho_B}\right)^2 e^{2\bar{\rho}}\sin\theta. \tag{11}$$

Similarly, if we apply a magnetic field along the axis of the truncated cone then the Hamiltonian is the same as Eq. (10) except that ρ is replaced by $\rho\sin\alpha$ in the square brackets. Using the transformation $\bar{\rho}=\sec\alpha\,\ln(\rho/\rho_0)$ we again obtain a (different) double sine-Gordon equation with nonconstant coefficients,

$$\theta_{\bar{\rho}\bar{\rho}}=\frac{\sin 2\theta}{2\tan^2\alpha}+\frac{1}{2}\left(\frac{\rho_0}{\rho_B}\right)^2 e^{2\bar{\rho}\cos\alpha}\sin\theta. \tag{12}$$

Note that in the case of an infinite cylinder in an axial magnetic field one obtains a double sine-Gordon equation with *constant* coefficients which has exact soliton and soliton lattice solutions.[19] However, in the case of a non-simply connected plane (or a truncated cone) it must be solved numerically. The topological solutions in this case would be skyrmions instead of half-skyrmions.

VI. CONCLUSION

Our main finding is that the metastable skyrmion on a plane can be stabilized by removal of a disk at the origin thus introducing a characteristic length scale in the system. By way of specific transformations of the radius coordinate ρ, that underscore the topological equivalence of Heisenberg spins (with cylindrically symmetric configurations) on a plane with a disk cut out from it, a truncated cone and an infinite cylinder, we obtained a sine-Gordon equation as the Euler–Lagrange equation. The latter provides exact half-skyrmion solutions for the non-simply connected plane and cone geometries.

Above, we considered a *rigid* truncated cone and the half-skyrmion lattice energy [Eq. (5)] was more than 4π due to violation of the self-duality condition[5,20] resulting from the interaction between the skyrmions. For azimuthally symmetric quasi-one-dimensional solutions, if the cone were deformable (i.e., *elastic*) and if we included a magnetoelastic coupling, then the cone would shrink periodically away from the apex to relieve geometric frustration[8,9] and lower its magnetic energy to 4π. These ideas can be directly generalized to both the moving half-skyrmions and other non-simply connected manifolds, e.g., truncated paraboloids and hyperboloids, with the physics of the half-skyrmions remaining essentially quite similar.

Our results may also have some significance for quantum Hall ferromagnets since the skyrmions are already stabilized by the (non-simply connected) geometry—the Zeeman and Coulomb energies would simply modify the radius where the skyrmion is located. However, it would be experimentally very important to probe the predicted properties of skyrmions in a 2D electron gas (e.g., GaAs/$\mathrm{GaAl}_x\mathrm{As}_{1-x}$ heterostructure) with a circular disk cut out, or even polarons[21] in 2D on a non-simply connected manifold. In the former case we do not expect the charge density fluctuations to significantly modify the location of the half-skyrmions because the intrinsic length scale (ρ_0) provides a more dominant effect. We note, however, that in experimental situations involving small scales (e.g., a small disk radius) the application of the nonlinear sigma model (i.e., a long wavelength theory) should be viewed with caution.

Similarly, our results may in general be insightful for quantum mechanical problems in non-simply connected spaces.[22] Finally, we remark that in contrast to a meron,[23] which also represents a half-soliton, the energy of the half-skyrmion is finite. Moreover, in physically relevant systems merons can appear only in pairs.[24]

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