Anomalous quantum diffusion at the superfluid-insulator transition

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We consider the problem of the superconductor-insulator transition in the presence of disorder, assuming that the fermionic degrees of freedom can be ignored so that the problem reduces to one of Cooper pair localization. Weak disorder drives the critical behavior away from the pure critical point, initially towards a diffusive fixed point. We consider the effects of Coulomb interactions and quantum interference at this diffusive fixed point. Coulomb interactions enhance the conductivity, in contrast to the situation for fermions, essentially because the exchange interaction is opposite in sign. The interaction-driven enhancement of the conductivity is larger than the weak-localization suppression, so the system scales to a perfect conductor. Thus, it is a consistent possibility for the critical resistivity at the superconductor-insulator transition to be zero, but this value is only approached logarithmically. We determine the values of the critical exponents η, z, ν and comment on possible implications for the interpretation of experiments.

I. INTRODUCTION

In a perfectly clean system at $T=0$, the free Fermi gas is perched precariously at a critical point. An arbitrarily weak interaction will drive the system superconducting (by the Kohn-Luttinger effect if the interaction is repulsive). In the presence of disorder, however, the *diffusive* Fermi liquid is a stable phase for a finite range of interaction and disorder strengths in dimensions $d>2$. In $d=2$, it remains an open problem whether or not fermions have a stable diffusive metallic phase. Such a phase, if it exists, could not be adiabatically connected to the Fermi liquid[1] since the noninteracting Fermi gas is always insulating in the presence of disorder in $d=2$. In the limit of weak disorder, this can be understood as a quantum interference effect which is singular as a result of the diffusive nature of electron propagation in a disordered system: diffusion at intermediate length scales (longer than the elastic mean-free path) thwarts diffusion at long scales (longer than the localization length).[2] The interacting-electron problem remains unresolved because interactions in the spin-triplet channel are also singular as a result of the languid pace of diffusive motion.[3–6] The upshot of the interplay between these different singularities is unknown (see, however, Refs. 7 and 8).

Consider the critical point separating the insulating and superfluid phases of a perfectly clean system of bosons at $T=0$ in two dimensions. We would like to draw an analogy between it and the free Fermi gas. In the bosonic case, there is a particular value of the chemical potential for which the system has gapless critical modes, loosely analogous to the excitations of the free Fermi gas. For any other value of the chemical potential, the bosons are either in a superfluid state—a superconducting state, if we assume that the bosons are Cooper pairs—or in a gapped insulating state. Suppose we now add disorder to this system. What is the fate of this critical point? On general grounds, we believe that it is unlikely to broaden into a stable diffusive metallic phase, and that the only stable phases are insulating (Mott insulator or Bose glass[9]) or superconducting. Instead, we expect a diffu-

sive metallic critical point with a *universal* conductivity separating the insulating and superconducting phases. The analogy between Fermi and Bose systems is imprecise, but it emphasizes the important point that in both cases there is a ballistic critical point in the clean system which must be usurped by a diffusive fixed point in the disordered one.

Such a fixed point should be amenable to analysis by methods similar to those used for the diffusive Fermi liquid. Conversely, expansion about the pure critical point—which is ballistic, not diffusive—should fail. In considering such a perspective, one is faced with the following question: why do quantum interference effects, which appear to be such an inevitable consequence of diffusive motion, not preclude a finite conductivity at the superfluid-insulator transition? The answer must lie in the effects of interactions, which one might hope to tame since spinless bosons, such as singlet Cooper pairs, do not have a triplet channel—the troublesome, singular one—through which to interact.

In this paper, we present the results of such an analysis. We find that there are two competing effects at a putative two dimensional (2D) diffusive Bose liquid critical point: one resulting from interactions between the bosons; the other, from quantum interference, i.e., weak localization. In the fermionic case, it is advisable to consider quantum interference and interactions on the same footing since they lead to similar logarithmic corrections at the perturbative level. In the bosonic case, one must perforce do so, since quantum interference leads to the existence of localized states even in the weak disorder limit, and bosons would congregate in the lowest energy localized state in the absence of interactions. We find that the effect of interactions is stronger than quantum interference and drives the system to a perfect conductor, thereby explaining how diffusion can remain impervious to localization. This result is congenial to one's intuition that repulsive interactions should disfavor localization. Potential wells due to impurities diminish in attractiveness when they are occupied and, as a result, the random potential is effectively screened. This effect is present for both short-ranged interactions as well as long-ranged Coulomb interactions, but

is stronger in the latter case. The same phenomenon occurs in fermionic systems as well, but it competes with the exchange part of the interaction, which is opposite in sign due to Fermi statistics. If the interaction is short ranged, it is irrelevant for spinless fermions, so it has no effect on the conductivity in the infrared limit. (This is clear in the δ-function limit, where the direct and exchange interactions cancel.) In the case of Coulomb interactions, the exchange interaction between spinless fermions dominates and suppresses the conductivity. In the case of spin-1/2 fermions, the runaway flow of the triplet interaction amplitude indicates that the Hartree interaction begins to prevail over the exchange interaction at longer length scales, thereby leading to an enhanced conductivity. However, the interaction strength diverges before a metallic fixed point is reached, and no conclusion can be drawn about the existence of a metallic state at zero temperature. These difficulties do not arise in the bosonic case. The exchange interaction has the same sign as the direct one, and both enhance the conductivity.

Our result is valid for large conductivities in units of e^2/h. Hence, if the bare conductivity is large—as it can be if the bosons have an anisotropic mass tensor—then the renormalized conductivity is infinite. If the bare conductivity is small, then there are two possibilities. If the conductivity initially flows to sufficiently large values that we can apply our calculation, then it will continue to flow to infinity. However, it is also possible that the system will flow in this case to a different fixed point at which the conductivity if finite. In such a scenario, there would be two different possible universality classes of superconductor-insulator transitions. In either case, we conclude that it is a consistent possibility for the critical point between the superfluid and insulating states of a disordered Bose liquid to be a *perfect conductor*.

We derive these results in a nonlinear σ-model (NLσM) formulation of the problem of diffusing, interacting bosons. Our NLσM is very similar to Finkelstein's model for fermions.[3] However, the NLσM plays a very different role in this problem than in the fermionic problem. There, the NLσM describes the entire metallic phase. In $2+\epsilon$ dimensions, the metal-insulator transition occurs near the metallic fixed point, so the NLσM encompasses it as well. In the bosonic problem which models the superconductor-insulator transition, our NLσM describes the *critical point*. The antiferromagnetic Heisenberg model in $d>2$ provides an enlightening analogy. For isotropic exchange coupling $J_z = J_{x,y}$, the model is ordered and is described by a NLσM. In the ordered phase, continuous symmetries are broken so there are Goldstone modes; this is the analog of our critical point. For $J_z > J_{x,y}$, the model develops Ising order with a gap; this is analogous to our insulating phase. For $J_z < J_{x,y}$, the model develops *XY* order, which is analogous to our superconducting phase.

II. DIRTY BOSONS

Following Ref. 9, we will treat the Cooper pairs in a dirty superconductor as bosons moving in a random potential. We will assume that all fermionic degrees of freedom are gapped or localized and are therefore unimportant. This assumption

has been called into question recently.[10,11] If fermionic degrees of freedom prove to play an important role at the superconductor-insulator transition, then our analysis will need to be modified to include them, but our description of dirty bosons will remain an important component of a richer description of the superconductor-insulator transition.

Note that we are studying here the generic transition[9] between the Bose glass and superfluid phases which occurs at an incommmenusurate boson density. In the special case in which there are an integer number of bosons per lattice site, there may be a direct transition between Mott insulating and superfluid phases which is tuned by varying the ratio of the hopping and interaction parameters.[12]

We begin with a system of interacting bosons moving in a random potential in two dimensions. The derivation which follows goes through in arbitrary dimension with minor changes, but $d=2$ is the most interesting case. The imaginary-time action is

$$S = \int d^2x\,d\tau\,\psi^*\left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu + V(x)\right)\psi$$
$$+ \int d^2x\,d^dx'\,d\tau\,\psi^*(x)\psi(x)u(x-x')\psi^*(x')\psi(x'),$$

$$(1)$$

$u(x-x')$ is the interaction between bosons; we will consider the cases of both short-ranged interactions and Coulomb interactions. $V(x)$ is the random potential; we use the replica trick to average over it, thereby obtaining the action

$$S = \int d^2x\,d\tau\,\psi_a^*(x,\tau)\left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu\right)\psi_a(x,\tau)$$
$$- \int d^2x\,d\tau\,d\tau'\,\frac{1}{2}v_0\psi_a^*(x,\tau)\psi_a(x,\tau)\psi_b^*(x,\tau)\psi_b(x,\tau')$$
$$+ \int d^2x\,d^2x'\,d\tau\,\psi_a^*(x)\psi_a(x)u(x-x')\psi_a^*(x')\psi_a(x'),$$

$$(2)$$

$a = 1,2,\ldots,N$ is a replica index. We have assumed that the potential has the Gaussian white noise distribution $\overline{V(x)V(x')} = v_0\delta(x-x')$.

This action is problematic because it is not positive definite as a result of the second term. To cure this, we will rotate the integration contour in the functional integral, as one does in the noninteracting case. This can be done more conveniently if we work in the Matsubara frequency representation and separate the real and imaginary parts of the Matsubara fields $\psi_{na} = \phi_{na1} + i\phi_{na2}$, where $\epsilon_n = 2\pi n/\beta$. The action can be made positive definite by rotating the fields in the following way: $\phi_{naA} \rightarrow e^{-(\pi/4)i\,\mathrm{sgn}(n)}\phi_{naA}$, $A = 1,2$. We rotate the $n = 0$ mode along with the $n > 0$ modes. The action now takes the form

$$S = \sum_{n,m} \int d^2x \, i \phi_{naA}(x,\tau) \left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu \right) \Lambda_{nm} \phi_{maA}(x,\tau) + \sum_{n,n',m,m'} \int d^2x \frac{1}{2} v_0 \phi_{naA} \Lambda_{nn'} \phi_{n'aA} \phi_{mbB} \Lambda_{mm'} \phi_{m'bB}$$

$$+ \sum_{m_1,\ldots,m_4} \int d^2x \, d^2x' [e^{-\pi i \Sigma \text{sgn}(m_i)/4}] \phi_{m_1 aA}(x) \phi_{m_2 aA}(x) u(x-x') \phi_{m_3 aB}(x') \phi_{m_4 aB}(x') \delta_{m_1+m_2+m_3+m_4}, \qquad (3)$$

where $\Lambda_{mm'} = \text{sgn}(m)\delta_{mm'}$.

In the absence of disorder, repulsive interactions are marginally irrelevant, and the critical behavior of Eq. (3) is controlled by the Gaussian fixed point.[9] Now consider a perturbative treatment of the disorder. In the self-consistent Born approximation, we find a self-energy due to disorder of the form,

$$\Sigma(\epsilon_n) = \frac{m v_0}{2\pi} \left[\ln \left| \frac{\Lambda^2/2m}{i\epsilon_n + \mu + \Sigma(\epsilon_n)} \right| \right.$$

$$\left. + i \tan^{-1} \left(\frac{\epsilon_n + \text{Im}\Sigma(\epsilon_n)}{\mu + \text{Re}\Sigma(\epsilon_n)} \right) \right]. \qquad (4)$$

The random potential shifts the chemical potential and also gives the bosons a finite lifetime τ. As a result of the lifetime τ, single-boson excitations are no longer long-lived degrees of freedom. However, particle-hole pairs are long-lived, as may be seen from the conductivity which, at this level of approximation, is $\sigma = 1/2\pi^3$.

This does *not* preclude critical behavior in the single-particle properties, as has already been seen in the context of interacting fermions[3] and of quasiparticles in a disordered d-wave superconductor where there are density-of-states corrections and also in the context of noninteracting electrons with an extra sublattice symmetry, where the single-particle Green function itself is critical.[16]

The conductivity is small because there are no particle-hole pairs for $\tau = \infty$ (since the transition occurs at the bottom of a quadratic band). A finite lifetime leads to a small density of states $\sim 1/\tau$ for particle-hole pairs, which cancels the factor of the lifetime to which σ is customarily proportional, thereby leading to a conductivity which is $O(1)$. However, we note that a parametrically large conductivity can be obtained in a slight generalization to a model of two species of bosons with anisotropic masses and that mix upon scattering. Suppose that one of them has $m_x = m_1, m_y = m_2$, while the other has masses reversed. Then we find that $\sigma = [\sqrt{m_1/m_2} + \sqrt{m_2/m_1}]/2\pi^3$. For sufficiently large or small ratio m_1/m_2, the conductivity will be large. Such a situation could occur, for instance, in a two-band model in which the two bands of electrons have anisotropic masses, leading to anisotropic masses for the Cooper pairs.

An RG analysis of the dirty boson problem yields the following RG equation in an ϵ expansion about $d=4$:[9]

$$\frac{dv_0}{dl} = (\epsilon + \epsilon_\tau) v_0 + B v_0^2 + \cdots \qquad (5)$$

with $4 - \epsilon - \epsilon_\tau$ spatial dimensions and ϵ_τ time dimensions (the interesting case $d=2$ occurs at $\epsilon = \epsilon_\tau = 1$). $B > 0$, so there is no fixed point at weak coupling; instead, there is a runaway flow to strong disorder. We interpret this as an instability of the pure critical point, at which the critical modes are ballistic, to the diffusive fixed point. To access the latter fixed point, we will construct a nonlinear σ model which is appropriate for physics at length scales longer than the mean-free path. In this regime, transport is diffusive, and we may neglect degrees of freedom, such as the ϕ fields, which are short lived.

III. SADDLE POINTS FOR DIRTY BOSONS

In the absence of the $i\epsilon_n$ term, the noninteracting part of the action (3) has an $O((k+1)N, kN)$ symmetry, where k is a cutoff on the Matsubara frequencies. The key assumption of Finkelstein's theory[3] for fermions is that the elevation of the energies of the diffusion modes by the $i\epsilon_n$ term and the interactions can be neglected compared to the gaps associated with other degrees of freedom; when this condition is satisfied, it is valid to retain only interacting diffusion modes and ignore all other degrees of freedom. We make the same assumption here in our description of the critical point. In the superfluid state, this is clearly not sufficient, and we will have to retain an extra degree of freedom. It may also be necessary to include extra degrees of freedom to properly describe the Bose glass insulating state.

Our treatment of the critical saddle point and NLσM for interacting bosons follows that of Finkelstein for the fermionic case and also that of the bosonic representation of the noninteracting problem. Hence, we will merely give an outline in this section and the following one, emphasizing the important differences. Details are presented in Appendix A.

We begin by using the Hubbard-Stratonovich transformation to decouple the v_0 term with a matrix $Q_{ab,AB}^{mn}$. We then decouple the interaction in two different ways with X, which decouples the direct and exchange channels according to $X \sim \psi^* \psi$, and X_c, which decouples the Cooper channel according to $X_c \sim \psi \psi$. Finally, we decouple the chemical potential term with $\Phi \sim \psi$. In this way, we have a system of noninteracting bosons at zero chemical potential—their critical point—moving in the background fields X, X_c, and Φ. Integrating out the ϕ fields, we obtain the effective action (see Appendix A),

$$S_{\text{eff}}[Q,Y,Z,Z^\dagger,\Phi] = \sum \int \left[\operatorname{tr}\ln\left(i\epsilon_n + \frac{1}{2m}\nabla^2 + Q - i\sqrt{2\Gamma}\,e^{-i(\pi/4)\Lambda}Xe^{+i(\pi/4)\Lambda} - i\sqrt{2\Gamma_c}\,e^{-i(\pi/4)\Lambda}\frac{1}{2}(X_c + X_c{}^\dagger)e^{+i(\pi/4)\Lambda} \right) \right.$$
$$\left. + \frac{1}{2v_0}\operatorname{tr}(Q^2) + \frac{1}{2}\operatorname{tr}(X^2) + \frac{1}{2}\operatorname{tr}(X_c{}^\dagger X_c) + \mu(\Phi^*\Phi)\hat{G}\begin{pmatrix} \Phi^* \\ \Phi \end{pmatrix} \right]. \tag{6}$$

The Green function $\hat{G}$ of the ϕ_As is written as a 2×2 matrix in the final line to emphasize the particle-hole structure. It is the operator inverse of the expression inside the logarithm. For $\mu \leq 0$, it is not even necessary to introduce Φ; we can simply drop the last term of Eq. (6) and insert μ inside the logarithm.

Let us now consider the saddle points of this effective action. For $\mu > 0$, there is a saddle point with $\langle \Phi \rangle \neq 0$. (When we include fluctuations, μ will be renormalized, so the critical value will not be zero.) When Φ develops an expectation value, Q, X_c, and X are forced to follow since they are coupled directly to bilinears in Φ. This is the superfluid phase.

For $\mu \leq 0$, let us consider the noninteracting case $\Gamma = \Gamma_c = 0$. The saddle-point condition is

$$\hat{Q} = -v_0 \int \frac{d^2p}{(2\pi)^2} \frac{1}{i\hat{\epsilon}_n + \frac{1}{2m}\nabla^2 + \mu + \hat{Q}}. \tag{7}$$

Let us absorb the real part of the saddle-point value of Q into a renormalized μ_R and focus on the imaginary part.

For $\mu_R = 0$, the saddle-point solution of Eq. (7) is

$$Q^{m,n}_{ab,AB} = i\frac{mv_0}{2}\operatorname{sgn}(\epsilon_n)\delta_{mn}\delta_{AB}\delta_{ab}. \tag{8}$$

This is the diffusive saddle point for self-consistent Born scattering of critical bosons by impurities. It corresponds to a finite density of states for the bosons at this level of approximation. Notice that this saddle-point solution is taken to be replica symmetric.

Now, for $\mu_R < 0$, there is another translationally invariant saddle point with $Q = 0$. For this solution, a nonzero density-of-states in not generated in the insulating state at this level of approximation; it remains a Mott insulator. We would like to point out two possible mechanisms to generate the finite density of states that occurs in the Bose-glass phase. One is that the correct saddle points are replica symmetry broken mixtures of the $Q = 0$ and Eq. (8) solutions. A possible self-

consistent solution is the one still diagonal in replicas but with zero matrix elements for p replicas and unit matrix elements for $n - p$ replicas. Another possibility is that there are nontrivial instanton saddle points which generate a finite density of states.[13] In the absence of interactions, the bosons will condense into these localized states, so we must consider the corresponding instantons with $\Gamma, \Gamma_c \neq 0$. At present, we do not have a description of the Bose-glass insulator, but this does not affect our ability to describe the critical point between it and a superconductor.

It is useful, in thinking about this theory, to imagine lowering the temperature of a system of dirty bosons. At finite temperature, there will be a finite wedge in the phase diagram—the quantum critical region[14,15]—where the bosons will be effectively critical. In this regime, we may begin by considering noninteracting bosons which are semiclassically scattered by impurities. As we decrease the temperature, we must begin to include the effects of interactions and of quantum interference processes. If we stray too far from the critical μ as we lower the temperature, thereby leaving the quantum critical region, then we cannot include these effects perturbatively. It is clear that they completely destabilize the diffusive saddle point, so they must be included right from the start (e.g., by starting from new saddle points, as we have sketched above) in order to describe the superfluid or insulating phases correctly. However, so long as we remain at criticality, we can hope to account for these effects perturbatively. To such an analysis we turn in the following section.

IV. σ MODEL FOR INTERACTING BOSONS

To go beyond a noninteracting, semiclassical analysis and include the effects of interactions and quantum interference, we construct the NLσM which accounts for fluctuations of Q. We shift Q by $i\epsilon_n + \sqrt{2\Gamma}\,e^{-i(\pi/4)\Lambda}Xe^{+i(\pi/4)\Lambda} + \sqrt{2\Gamma_c}\,e^{-i(\pi/4)\Lambda}\frac{1}{2}(X_c + X_c{}^\dagger)e^{+i(\pi/4)\Lambda}$ to remove these terms from the $\operatorname{tr}\ln[\cdot]$. Then, we expand the $\operatorname{tr}\ln[\cdot]$ about the saddle point and integrate out X, X_c. We obtain an effective action which is essentially the same as Finkelstein's action for the fermionic problem (see Appendixes A and B),

$$S_{\text{eff}}[Q] = \int d^dx \left\{ D\operatorname{tr}(\nabla Q)^2 - 4iZ\operatorname{tr}(\hat{\epsilon}Q) \right.$$

$$+ \Gamma \sum_{n_1,\ldots,n_4} [e^{+i(\pi/4)n_1}Q^{n_1n_2}_{aa,AA'}e^{-i(\pi/4)n_2}]J_{AA'}J_{BB'}[e^{+i(\pi/4)n_3}Q^{n_3n_4}_{aa,BB'}e^{-i(\pi/4)n_4}]\delta_{n_1-n_2+n_3-n_4}$$

$$\left. + \Gamma_c \sum_{n_1,\ldots,n_4} [e^{+i(\pi/4)n_1}Q^{n_1n_2}_{aa,AA'}e^{-i(\pi/4)n_2}]S^+_{AA'}S^-_{BB'}[e^{+i(\pi/4)n_3}Q^{n_3n_4}_{aa,BB'}e^{-i(\pi/4)n_4}]\delta_{n_1+n_2-n_3-n_4} \right\}, \tag{9}$$

where $J_{AB} = (1/\sqrt{2})(\delta_{AB} - \sigma^2_{AB})$ and $S^\pm_{AB} = \sigma^3_{AB} \pm i\sigma^1_{AB}$ express the particle-hole matrix structure for the density-density and Cooper channels, respectively. The parameter Z is 1 in the bare action above; however, this quantity is renormalized, so we have introduced it explicitly here. We have absorbed the density of states into the diffusion constant D (and also the coefficients of the other terms); the resulting quantity is just the bare conductivity and is given by $D = 1/2\pi^3$ in the above model. However, as we noted earlier by considering a model with anisotropic masses, and a sufficiently large or small ratio m_1/m_2, the bare conductivity will be large. The resistivity $g = 1/(2\pi D)$ is the expansion parameter used in our RG equations, so this observation gives us a limit in which they can be applied without apology.

It may strike the reader as strange that we are using a NLσM to describe a critical point; usually NLσMs are used to describe stable phases because they are so highly constrained by symmetry. However, the NLσM of Eq. (9) is not, in fact, so rigidly constrained at all. The interaction terms and the tr$(\hat{\epsilon}Q)$ term explicitly breaks the $O((k+1)N, kN)$ "symmetry" of the model. The latter breaks it in such a way as to push the theory into a diffusive metallic state. However, this symmetry-breaking "field" is small in the low-energy limit, so other symmetry-breaking fields (or anisotropies) can intervene instead. When Φ orders in Eq. (6), Q is forced away from the diffusive "direction" in its saddle point manifold, and into the superfluid "plane," where Q has nonvanishing components which are off diagonal in particle-hole indices. *Thus, we can understand the perturbations which lower the symmetry of the saddle-point manifold as perturbations which drive the system away from criticality.* There are a variety of ways in which one can imagine driving the system into an insulating phase. In the absence of a better understanding of the Bose glass phase, we consider the simplest which is just a "mass" term of the form tr(MQ), with M a constant matrix say in replica space, which breaks the symmetry of the saddle point manifold and leads to an insulating state. Such a perturbation differs only in index strucure with the one imposed by a finite Φ. Such a term is also generated by shifting μ out of the tr ln$[\cdot]$ term when considering replica symmetry broken saddles. Note that none of these possibilities can occur in the noninteracting problem, where the symmetry of the saddle-point manifold is a genuine symmetry.

We parametrize Q about the noninteracting saddle point as

$$Q = \frac{mv_0}{2}\begin{pmatrix} i(1+qq^T)^{1/2} & q \\ q^T & -i(1+q^Tq)^{1/2} \end{pmatrix}, \quad (10)$$

where the block structure is in frequency space, i.e., the matrix q_{nm} is such that $n \geq 0$ and $m < 0$.

The resulting action is very similar to the $O(N)$ sigma model which is appropriate for a system of fermions with spin-orbit scattering. Indeed, one can be transformed into the other by redefining $q \to q, q^T \to -q^T$, and $D \to -D$. The interaction terms look somewhat strange at first glance, but the
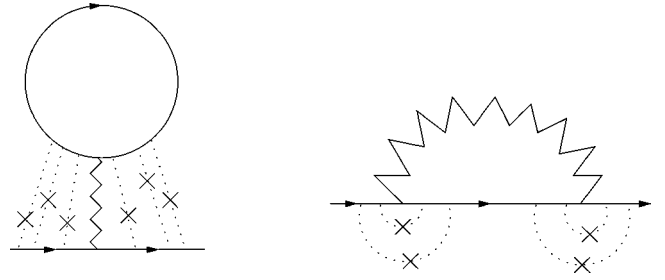

FIG. 1. The Hartree and Fock diagrams for the boson self-energy.

extra i's in Eq. (10) are precisely compensated by the explicit factors of $e^{\pm i(\pi/4)n_i}$ in Eq. (9) (see Appendix C).

V. RG EQUATIONS

Taking advantage of the observation at the end of the preceding section, we can obtain the RG equations for our σ model by flipping $g \to -g$ in the equations for the corresponding fermionic model. Some factors of 2 will be different because our bosons are spinless. More details may be found in Appendix C.

The renormalization group (RG) equation for Γ^2_c is

$$\frac{d\Gamma_c}{dl} = -g\Gamma_c - \Gamma^2_c. \quad (11)$$

Observe that Γ_c flows to zero, even if $g = 0$. Hence, we set Γ_c to its fixed point value of zero and consider the RG equations for g, Γ, and Z in its absence. To order g^2 and all orders in Γ (although, of course, we cannot access nonperturbative effects associated with saddle points which are far from the noninteracting diffusive one), the RG equations are

$$\frac{dg}{dl} = \frac{1}{2}g^2 - g^2\left[2 + 2\left(\frac{Z}{\Gamma} - 1\right)\ln\left(1 - \frac{\Gamma}{Z}\right)\right], \quad (12)$$

$$\frac{dZ}{dl} = g\Gamma, \quad (13)$$

$$\frac{d\Gamma}{dl} = g\Gamma. \quad (14)$$

The physics of these equations is clear from the discussion in the Introduction. Interactions always enhance the conductivity to order g^2 because the exchange term has the same sign as the direct term [they are folded into a single Γ in the bosonic NLσM (9)]. The gist of the effect can be seen from the Hartree and Fock diagrams for the boson self-energy displayed in Fig. 1. In the Hartree diagram, the boson line is repelled by the boson bubble which is a measure of the ground-state density. (In a pure system, this is uniform and cancelled by the neutralizing background.) In a fermionic system, the Fock diagram comes with the opposite sign, so it is an effective attraction. In a bosonic system, however, both diagrams come with the same sign and lead to a repul-
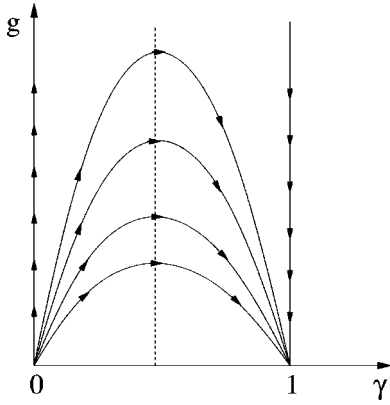
FIG. 2. RG flow for the resistivity g and interaction parameter $\gamma = \Gamma/Z$.

sion of particles from regions of high density—which, of course, are precisely the regions where there are deep wells in the random potential.

The interaction strength, Γ, grows in importance at low energies because it plays a role somewhat analogous to the Pauli exclusion principle: in its absence, all of the bosons would sit in the lowest minimum of the random potential. Z must follow Γ in order to maintain a finite compressibility.

Notice from Eqs. (13) and (14) that $Z - \Gamma$ remains invariant under the RG flow, as a result of Ward identities that originate from charge conservation. It is very useful to introduce the coupling constant $\gamma = \Gamma/Z$, which allows us to rewrite the RG equations in a simpler way,

$$\frac{dg}{dl} = \frac{1}{2} g^2 - g^2 \left[2 + 2 \frac{1-\gamma}{\gamma} \ln(1-\gamma) \right], \quad (15)$$

$$\frac{d\gamma}{dl} = g \gamma (1 - \gamma). \quad (16)$$

For $g > 0$, it follows from Eq. (16) that there are two fixed-point values $\gamma^* = 0, 1$ (a closer analysis rules out the possibility of another value of γ^* with $g = 0$), as shown in Fig. 2. The $\gamma^* = 0$ fixed point is unstable, while the $\gamma^* = 1$ one is stable. Consider the RG equation for g. The first term on the right-hand side is the weak-localization correction, while the second term is the interaction correction. The value $\gamma = 0.423\,16\ldots$ separates the regime where the weak-localization correction dominates over the interaction contribution ($dg/dl < 0$ for $\gamma < 0.423\,16\ldots$ and $dg/dl > 0$ for $\gamma > 0.423\,16\ldots$). Although the entire surface $g = 0$ with arbitrary γ is left invariant under the RG flow, any system with bare $g, \gamma \neq 0$ will necessarily flow into the $g = 0$, $\gamma = 1$ fixed point. This is the case for short-range interactions, where the flow starts with a value $\gamma < 1$. Note that if the bare interaction is weak, $\gamma \ll 1$, then the resistivity will initially increase before eventually decreasing to zero.

Now, consider the case of dynamically screened Coulomb interactions. As in the fermionic case, the Ward identity for charge conservation requires the density-density correlation function to vanish at $\mathbf{q} = 0$. This, in turn, requires the

$\mathbf{q}$-dependent interaction $\Gamma(q)$, which generalizes Γ to the case of Coulomb interactions, to satisfy the identity[3–5]

$$Z - \Gamma(q) = \frac{\partial n}{\partial \mu} \frac{q}{q + 4\pi e^2 (\partial n/\partial \mu)}. \quad (17)$$

Taking the $q \to 0$ limit of Eq. (17), we obtain $Z = \Gamma$. Substituting this identity into Eq. (12), we see that the second term inside the square bracket in Eq. (12) vanishes. Thus, the RG equation for g is $dg/dl = -3g^2/2$, and the resistivity flows logarithmically to zero. The system is controlled by the same infinite-conductivity fixed point as in the short-ranged case.

Before concluding this section, let us write down the asymptotic behavior near the fixed point $g^* = 0$, $\gamma^* = 1$, which we will need later to obtain the critical exponents: $g \sim 2/3l$ and $1 - \gamma \sim \exp(-\int dlg) \sim l^{-2/3}$.

VI. CRITICAL BEHAVIOR

The most striking conclusion about the critical behavior of this system is that the critical resistivity is zero. In other words, the 2D superconductor-insulator transition is broadly similar to the 3D one. This is somewhat unexpected. In models such as the Bose–Hubbard model, which describes a superfluid insulator transition in a clean system, or the $(2+1)$-dimensional XY model, one finds $\sigma^* = ce^2/h$, with c a finite universal number. At our fixed point, $c = \infty$. Another odd feature is the logarithmic approach to the critical resistivity which we find; this logarithm is rather different from the type which are encountered in the lower critical dimension of a phase transition (which happens to be $d = 1$ for the superfluid-insulator transition). Since a logarithmic flow is rather slow, it may not be possible to observe $c = \infty$. Instead, the critical conductivity at a given temperature may actually appear to be a nonuniversal number which depends on the bare conductivity.

Let us also consider the single-boson density of states, $N(\omega)$. This may be studied by introducing a source term for $\mathrm{tr}(\Lambda Q)$ into the effective action and computing its renormalization. In a system with short-ranged interactions, we find

$$\frac{d}{dl} \ln N = -g \gamma \ln(1 - \gamma). \quad (18)$$

Substituting the asymptotic forms of g and γ, we find that the single-particle density of states diverges weakly, $N(\omega) \sim e^{(2/9)[\ln \ln(1/\omega\tau)]^2}$. Since the boson creation operator is the order parameter for the superfluid phase and

$$N(\omega) = \mathrm{Im} \overline{\langle \psi^\dagger(\mathbf{x}, \omega) \psi(\mathbf{x}, -\omega) \rangle} \quad (19)$$

the scaling relation for $N(\omega)$ implies that the critical exponent $\eta = 0$ with logarithmic corrections. However, in the presence of dynamically screened Coulomb interactions, there is a more severe divergence, and we find

$$\frac{d}{dl} \ln N = gl. \quad (20)$$

Consequently, the single-boson density of states diverges at the transition with the power law $N(\omega) \sim \omega^{-2/3}$. This implies that the critical exponents η and z satisfy $\eta/z = -2/3$. Note that we have calculated the density of states at a *metallic* critical point. Thus, we should not expect Coulomb gap physics to suppress it and give $\eta > 0$. In the fermionic case, the suppression of the density of states is due to the dominance of the exchange interaction.

Our NLσM does not explicitly include single-boson operators. We assume that their properties can be deduced from the the density of states. It is certainly possible for single-particle operators to be critical even in a theory in which only collective modes are retained; this is the idea behind bosonization. It is conceivable, however, that our NLσM is incomplete, as regards single-boson properties. This could occur if the critical exponent controlling the correlation function $\langle \psi^\dagger(x,0)\psi(x,0)\rangle$ were unrelated to that controlling $\langle \psi^\dagger(0,\tau)\psi(0,0)\rangle$.

Since Z diverges only logarithmically, the dynamical exponent, $z=2$, as in a noninteracting system. However, in the case of dynamically screened Coulomb interactions, there are actually two different diverging time scales. One, with exponent z, is the scale associated with Z; it controls the scaling of the specific heat and energy diffusion. There is a second exponent, z_c, associated with $Z-\Gamma$, which controls charge diffusion. By the same argument as in a fermionic system,[5] Eq. (17) implies at small q that $Z-\Gamma \sim q$, from which we conclude that $Z-\Gamma \sim \xi^{-1}$, i.e., $z_c=1$. This result was obtained for the superconductor-insulator transition by a closely related argument in Ref. 9. Combining this with our density-of-states calculation, we have $\eta = -2/3$ for Coulomb interactions. Notice that $\eta = -2/3 < 0$ satisfies the lower bound $\eta < 2-d$ of Ref. 9 for $d=2$. The density of states and the dynamical exponent, z_c, are the only quantities which distinguish short-ranged and dynamically screened Coulomb interactions in the infrared limit.

As we discussed in Sec. IV, the leading perturbation of our σ model is a tr(MQ) term, where M is a constant matrix say in replica space, which breaks the replica symmetry of the diffusive saddle-point manifold possibly in the direction of the Bose-glass phase. This is a dimension 2 operator at tree level. (If the matrix M is proportional to the identity in replica space, this operator is instead just a constant at the diffusive saddle point.) Since the coupling constant g flows to zero, we expect a critical exponent $\nu = 1/2$, up to logarithmic corrections. This value of ν—the mean-field value—violates the bound $\nu \geq 2/d$ of Ref. 17. However, the exponent bounded by the theorem of Ref. 17 is, in fact, a finite-size scaling exponent ν_{FS} which can be different from ν. The bound is known to be violated in the depinning of a charge-density wave, where $\nu_{FS} \neq \nu = 1/2$.[18] As we discuss below, some experimental results on the superconductor-insulator transition also violate the bound,[28] as do experiments on helium in aerogel and the metal-insulator transition in doped semiconductors.[19] These violations have been emphasized in Ref. 20, which explains them as arising from a failure of self-averaging, but this interpretation has been challenged in Ref. 21.

VII. DISCUSSION

Diffusion in two dimensions is marginal, and small corrections (in the limit of large conductivity) such as that due to quantum interference or interactions can tip the balance one way or the other. Contrary to conventional wisdom, it is hardly a foregone conclusion which effect will win. After all, weak localization is *weak*. Interactions can easily overpower it, leading to metallic behavior. According to our analysis, this is precisely what occurs at the superconductor-insulator transition. The effect of interactions is so dominant that the universal value of the conductivity at the transition is infinity. Such a diverging conductivity has been found in models with interaction and dissipation, but without disorder.[22]

The possibility of a metallic phase within the Bose-glass phase has been studied recently.[23,24] We focus on the diffusive properties at the critical point, and do not investigate whether saddle-point solutions whithin the Bose glass could lead to nonzero conductivities. However, it is noteworthy that an infinite critical conductivity is consistent with a Bose metal with a diverging conductivity at the transition.[24]

We derive these results in a NLσM approach, in which we discard those critical modes of the clean system which are extraneous and retain only the particle diffusion modes of the disordered system. The resulting NLσM leads to a number of nontrivial predictions: (1) the critical conductivity is infinite; (2) there are two diverging times scales if the interaction is Coulombic, one associated with charge diffusion, which has exponent $z=1$, the other associated with energy diffusion, which has exponent $z=2$; (3) the single-boson density of states diverges as $\omega^{-2/3}$, which implies a critical exponent $\eta = -2/3$ in the case of Coulomb interactions; for short-range interactions, it diverges logarthmically; (4) the correlation-length exponent takes the mean-field value $\nu = 1/2$.

If boson-vortex duality were to hold exactly, then one would expect $g^* = 1$ [in units of $(2e)^2/h$]. Our result appears to imply that duality is violated logarithmically: bosons are more mobile than vortices in the infrared limit. However, it is hard to see how the physics of vortices enters at all into our calculation, so it is possible that we have missed important nonperturbative effects. Our results do not agree with the numerical study of Wallin *et al.*[25] However, the flow to our fixed point is logarithmic, and this may be too slow for a numerical study on a finite-sized system. Alternatively, they may simply be accessing a different fixed point which attracts systems with small bare conductivities. And finally, since their starting point studies phase but no amplitude fluctuations, the two models may simply be in different universality classes. Our results also differ quantitatively from those of Herbut, which are based on an expansion about $d=1$.[26]

The measured critical exponents for the zero-field superconductor-insulator transition, which is accessed by varying the thickness of a thin film,[27,28] are those of classical percolation. This does not agree with our theory, but it also suggests that the experiments are not quite in the asymptotic quantum critical regime, but rather in some higher-temperature classical regime. There is disagreement about

the values of the critical exponents at the magnetic-field-tuned superconductor-insulator transition. One experiment[29] finds percolationlike exponents, while another[28] finds $\nu = 0.7 \pm 0.2$, which includes our theoretical prediction at the edge of its error bar. (All of these experiments find $z \approx 1$, as expected on general grounds,[5,9] and in our theory.) The applicability of our strategy to a magnetic-field-tuned superconductor-insulator transition is a question for future study.

APPENDIX A: DERIVATION OF THE σ MODEL

Here we derive the effective action for the interacting disordered bosons in terms of two fields Φ and Q. We work, in sequence, on the free part, the disorder part, and finally the interaction part of Eq. (3).

1. The free action

We start by introducing a bosonic amplitude Φ to decouple the chemical potential (μ) term. Φ acquires a finite expectation value when bosons condense.

The free part of the action

$$S_{\text{free}}[\phi]$$

$$= \sum_{n,m} \int d^2x\, i\, \phi_{naA}(x) \left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu \right) \Lambda_{nm} \phi_{maA}(x) \tag{A1}$$

is generated upon integration of a decoupling field $\Phi_{na} = \Phi_{na1} + i\Phi_{na2}$ in

$$S_{\text{free}}[\phi, \Phi] = \sum_{n,m} \int d^2x\, i\, \phi_{naA}(x) \left(i\epsilon_n + \frac{1}{2m}\nabla^2 \right)$$

$$\times \Lambda_{nm} \phi_{maA}(x) + \sum_n \int d^2x \frac{1}{2} \Phi_{na}^* \Phi_{na}$$

$$+ \sum_n \int d^2x \sqrt{2\mu}\, \phi_{naA}[e^{-i(\pi/4)\text{sgn}(n)}]\Phi_{naA}. \tag{A2}$$

2. Disorder term

Let us next decouple the four bosons in the disorder term in Eq. (3),

$$S_{\text{rand}} = \sum_{n,n',m,m'} \int d^2x \frac{v_0}{2} \phi_{naA}(x)$$

$$\times \Lambda_{nn'} \phi_{n'aA}(x) \phi_{mbB}(x) \Lambda_{mm'} \phi_{m'bB}(x), \tag{A3}$$

where $\Lambda_{mm'} = \text{sgn}(m)\delta_{mm'}$. The same disorder term is generated upon integration of the Hubbard-Stratonovich matrix field $Q_{ab,AB}^{mn}$,

$$e^{-S_{\text{rand}}[\phi]} = \int DQ \exp\left(-\frac{1}{2v_0} \int d^dx \text{tr} Q^2 \right) e^{-S_{\text{HS}}[Q,\phi]}, \tag{A4}$$

where

$$S_{\text{HS}}[Q,\phi]$$

$$= i \sum_{n,m,m'} \int d^2x\, \phi_{naA}(x) Q_{ab,AB}^{nm}(x) \Lambda_{mm'} \phi_{m'bB}(x). \tag{A5}$$

The matrix Q has indices in three separate spaces, i.e., it is assembled as a direct product in energy n,m, replica a,b, and real-imaginary A,B spaces. The trace of Q^2 corresponds to

$$\text{tr} Q^2 = Q_{ab,AB}^{nm} Q_{ba,BA}^{mm}, \tag{A6}$$

where repeated index summation is carried out in all three spaces. When we write for short $Q_{nn'}$ we mean a matrix whose elements are matrices in replica, and real-imaginary spaces.

3. Interaction term

Let us consider the case of short-range interactions, in the density-density (s) and pairing (c) channel. Once again, we will omit sums over indices for replica and real-imaginary parts, and write explicitly the Matsubara sums,

$$S_{\text{int}} = S_s + S_c, \tag{A7}$$

where

$$S_s = \Gamma_s \sum_{n_1,\ldots,n_4} \int d^d x \exp\left(-i\frac{\pi}{4}\sum_j \mathrm{sgn}(n_j)\right)[\phi_{n_1 aA}(x)J_{AA'}\phi_{n_2 aA}(x)][\phi_{n_3 aB}(x)J_{BB'}\phi_{n_4 aB}(x)]\delta_{n_1-n_2+n_3-n_4},$$
(A8)

$$S_c = \Gamma_c \sum_{n_1,\ldots,n_4} \int d^d x \exp\left(-i\frac{\pi}{4}\sum_j \mathrm{sgn}(n_j)\right)[\phi_{n_1 aA}(x)S^+_{AA'}\phi_{n_2 aA'}(x)][\phi_{n_3 aB}(x)S^-_{BB'}\phi_{n_4 aB'}(x)]\delta_{n_1+n_2-n_3-n_4},$$
(A9)

with the matrices $J_{AB} = (1/\sqrt{2})(\delta_{AB} - \sigma^2_{AB})$ and $S^{\pm}_{AB} = \sigma^3_{AB} \pm i\sigma^1_{AB}$ (the σ^i being Pauli matrices). Notice that the different terms within square brackets above correspond, in terms of the original bosons ψ, to $\psi^*\psi, \psi^*\psi^*$, and $\psi\psi$.

We now introduce two Hubbard-Stratonovich fields X and X_c, X_c^* to decouple the four ϕ interactions,

$$e^{-S_s[\phi]} = \int DX e^{-S_y[X]} e^{-S_{HS/X}[X,\phi]},$$
(A10)

$$S_{HS/X}[X] = i\sqrt{2\Gamma_s}\sum_{n,m} \int d^d x\, \phi_{naA}(x) e^{-i(\pi/4)\mathrm{sgn}(n)} X^{nm}_{ab,AB}(x) e^{-i(\pi/4)\mathrm{sgn}(m)}\phi_{mbB}(x)$$
(A11)

and

$$e^{-S_c[\phi]} = \int DX_c^* DX_c e^{-S_z[X_c]} e^{-S_{HS/X_c}[X_c,\phi]},$$
(A12)

$$S_{HS/X_c}[X_c] = i\sqrt{2\Gamma_c}\sum_{n,m} \int d^d x\, \phi_{naA}(x) e^{-i(\pi/4)\mathrm{sgn}(n)}\frac{1}{2}[X^{nm}_{cab,AB}(x) + X_c^{\dagger nm}_{ab,AB}(x)]e^{-i(\pi/4)\mathrm{sgn}(m)}\phi_{mbB}(x),$$
(A13)

where

$$X^{nm}_{ab,AB} = \mathrm{X}^{n-m}_a \delta_{ab} J_{AB}, \qquad X^{nm}_{cab,AB} = \mathrm{X}^{n+m}_{ca}\delta_{ab}S^+_{AB}.$$

Notice that the matrices $X^{nm}_{ab,AB}$ and $X^{nm}_{cab,AB}$ depend, respectively, only on the energy difference $n-m$ and sum $n+m$. The action for the matrices X and X_c is

$$S_x[X] = \frac{1}{2}\sum_n \int d^d x\, \mathrm{X}^n_a \mathrm{X}^{-n}_a,$$
(A14)

$$S_{x_c}[X_c] = \frac{1}{2}\sum_n \int d^d x\, \mathrm{X}^{*n}_{c\,a}\mathrm{X}^n_{ca}.$$
(A15)

4. Integrating out the ϕ fields

We can summarize all terms discussed in the preceding section,

$$X_c = \int D\phi D\Phi DQ DX DX_c^* DX_c$$

$$\times \exp\left(-\frac{1}{2v_0}\int d^d x \mathrm{tr}Q^2\right)e^{-S_x[X]}e^{-S_{x_c}[X_c]}$$

$$\times e^{-S_{\mathrm{free}}[\phi,\Phi]}e^{-S_{HS}[\phi,Q]}e^{-S_{HS/X}[\phi,X]}e^{-S_{HS/X_c}[\phi,X_c]},$$
(A16)

where we can express $S_{\mathrm{free}}, S_{HS}, S_{HS/X}$, and S_{HS/X_c} in a more concise (matrix) notation as follows:

$$S_{\mathrm{free}}[\phi,\Phi] = \int d^2 x\, i\phi^T(x)\left(i\Omega + \frac{1}{2m}\nabla^2\right)\Lambda\phi(x)$$

$$+ \sqrt{2\mu}\int d^2 x\, \phi^T e^{-i(\pi/4)\Lambda}\Phi,$$
(A17)

$$S_{HS}[\phi,Q] = \int d^2 x\, \phi^T(x) iQ(x)\Lambda\phi(x),$$
(A18)

$$S_{HS/X}[X] = i\sqrt{2\Gamma_s}\int d^d x\, \phi^T(x)e^{-i(\pi/4)\Lambda}X(x)e^{-i(\pi/4)\Lambda}\phi(x),$$
(A19)

$$S_{HS/X_c}[X_c] = i\sqrt{2\Gamma_c}\int d^d x\, \phi^T(x)e^{-i(\pi/4)\Lambda}\frac{1}{2}[X_c(x) + X_c^{\dagger}(x)]e^{-i(\pi/4)\Lambda}\phi(x),$$
(A20)

where the matrix $\Omega_{nm} = \epsilon_n \delta_{nm}$.

Integrating out the boson fields ψ, we obtain

$$X_c = \int D\phi D\Phi DQ DX DX_c^* DX_c \exp\left(-\frac{1}{2v_0}\int d^d x \mathrm{tr}Q^2\right)$$

$$\times e^{-S_x[X]}e^{-S_{z_c}[X_c]}e^{-S_0[Q,\Phi,X,X_c]}$$
(A21)

with

$S_0[Q, \Phi, X, X_c]$

$$= \mu \int d^d x \Phi^T G \Phi + \int d^d x \mathrm{tr} \ln \left[i \left(i\Omega + \frac{1}{2m} \nabla^2 \right) \Lambda \right.$$

$$+ iQ(x)\Lambda + i\sqrt{2\Gamma_s} e^{-i(\pi/4)\Lambda} X(x) e^{-i(\pi/4)\Lambda}$$

$$+ i\sqrt{2\Gamma_c} e^{-i(\pi/4)\Lambda} \frac{1}{2} [X_c(x) + X_c^\dagger(x)] e^{-i(\pi/4)\Lambda} \Big]$$

(A22)

$$= \mu \int d^d x \Phi^T G \Phi + \int d^d x \mathrm{tr} \ln \left[\left(i\Omega + \frac{1}{2m} \nabla^2 \right) \right.$$

$$+ Q(x) - i\sqrt{2\Gamma_s} e^{-i(\pi/4)\Lambda} X(x) e^{+i(\pi/4)\Lambda}$$

$$- i\sqrt{2\Gamma_c} e^{-i(\pi/4)\Lambda} \frac{1}{2} [X_c(x) + X_c^\dagger(x)] e^{+i(\pi/4)\Lambda} \Big]$$

$$+ \mathrm{const.}$$

(A23)

The propagator G depends on Q, X, and X_c, X_c^*.

5. Shifting Q

Let us now shift Q,

$$Q \to \tilde{Q} = Q - i\sqrt{2\Gamma_s} e^{-i(\pi/4)\Lambda} X e^{+i(\pi/4)\Lambda}$$

$$- i\sqrt{2\Gamma_c} e^{-i(\pi/4)\Lambda} \frac{1}{2} (X_c + X_c^\dagger) e^{+i(\pi/4)\Lambda}$$

and

$$\mathrm{tr} Q^2 = \mathrm{tr} \tilde{Q}^2 + i2\sqrt{2\Gamma_s} \mathrm{tr}[\tilde{Q} e^{-i(\pi/4)\Lambda} X e^{+i(\pi/4)\Lambda}]$$

$$+ i2\sqrt{2\Gamma_c} \mathrm{tr} \left[\tilde{Q} e^{-i(\pi/4)\Lambda} \frac{1}{2} (X_c + X_c^\dagger) e^{+i(\pi/4)\Lambda} \right]$$

(A24)

$$-4\Gamma_s 2k \sum_n \int d^d x X_a^n X_a^{-n}$$

$$-4\Gamma_c 2k \sum_n \int d^d x X_c^{*n} X_{ca}^n.$$

(A25)

The Matsubara cutoff k comes from the extra frequency sum in the trace, and there are factors of 2 from the traces over the real-imaginary components, $\mathrm{tr} J = 2$ and $\mathrm{tr} S^+ S^- = 4$.

The next step is to integrate out the X, X_c fields. This generates quadratic in Q terms (we now drop the tildes for notational simplicity). It is useful to define

$$\tilde{\Gamma}_{s,c} = \frac{\Gamma_{s,c}}{v_0^2} \frac{1}{1 - 8\frac{\Gamma_{s,c}k}{v_0}},$$

$$S_{\mathrm{fink}} = \tilde{\Gamma}_{s,c} \sum_{n_1, \ldots, n_4} \int d^d x [e^{+i(\pi/4)\mathrm{sgn} n_1} Q_{aa,AA'}^{n_1 n_2} e^{-i(\pi/4)\mathrm{sgn} n_2}]$$

$$\times \gamma_{AA',BB'}^{s,c} [e^{+i(\pi/4)\mathrm{sgn} n_3} Q_{aa,BB'}^{n_3 n_4} e^{-i(\pi/4)\mathrm{sgn} n_4}]$$

$$\times \delta_{n_1 \mp n_2 \pm n_3 + n_4},$$

(A26)

where the tensors $\gamma_{AA',BB'}$ depend on the channel

$$\gamma_{AA',BB'}^s = J_{AA'} J_{BB'},$$

(A27)

$$\gamma_{AA',BB'}^c = S_{AA'}^+ S_{BB'}^-.$$

(A28)

Summarizing it all, we have an effective action

$$S_{\mathrm{eff}}[Q, \Phi] = \frac{1}{2v_0} \int d^d x \mathrm{tr} Q^2 + \int d^d x \mathrm{tr} \ln \left[\left(i\Omega + \frac{1}{2m} \nabla^2 \right) + Q(x) \right] + \mu \int d^d x \Phi^T G \Phi$$

$$+ \tilde{\Gamma}_s \sum_{n_1, \ldots, n_4} \int d^d x [e^{+i(\pi/4)\mathrm{sgn} n_1} Q_{aa,AA'}^{n_1 n_2} e^{-i(\pi/4)\mathrm{sgn} n_2}] \gamma_{AA',BB'}^s [e^{+i(\pi/4)\mathrm{sgn} n_3} Q_{aa,BB'}^{n_3 n_4} e^{-i(\pi/4)\mathrm{sgn} n_4}] \delta_{n_1 - n_2 + n_3 - n_4}$$

$$+ \tilde{\Gamma}_c \sum_{n_1, \ldots, n_4} \int d^d x [e^{+i(\pi/4)\mathrm{sgn} n_1} Q_{aa,AA'}^{n_1 n_2} e^{-i(\pi/4)\mathrm{sgn} n_2}] \gamma_{AA',BB'}^c [e^{+i(\pi/4)\mathrm{sgn} n_3} Q_{aa,BB'}^{n_3 n_4} e^{-i(\pi/4)\mathrm{sgn} n_4}] \delta_{n_1 + n_2 - n_3 - n_4}.$$

(A29)

APPENDIX B: SEPARATION OF LONGITUDINAL AND TRANSVERSE (DIFFUSIVE) MODES

In this appendix we show that the transverse fluctuations of the Q field correspond to boson diffusion for $\mu = 0$, similarly to the fermionic case. We show that the diffusion term

arises even in the absence of a small parameter $1/E_F \tau$.

Expansion to quadratic order in δQ leads to a term

$$\frac{1}{2} M^{n_1 n_2}(q) \delta Q_{aa',AA'}^{n_1 n_2}(-q) \delta Q_{a'a,A'A}^{n_2 n_1}(q),$$

(B1)

where

$$M^{n_1 n_2}(q) = \frac{1}{v_0} - \int \frac{d^d p}{(2\pi)^d} G(p, n_1) G(p+q, n_2). \quad \text{(B2)}$$

The first term on the right-hand side comes from the $\mathrm{tr}Q^2$ in the action, and the second term has its origin in the $\mathrm{tr}\ln(\cdot)$. $M^{n_1 n_2}(q)$ is selected to be diagonal in and independent of replica a, a' and real-imaginary A, A' indices. The Green's function

$$G(p, n_1) = \frac{1}{i\epsilon_n - E(p) + \dfrac{i}{2\tau}\mathrm{sgn}(\epsilon_n)}. \quad \text{(B3)}$$

For $\mathrm{sgn}n_1 \mathrm{sgn}n_2 > 0$, the real part of the integral vanishes for $q \to 0$, so that $\mathrm{Re}[M^{n_1 n_2}(q)] = 1/v_0$, and we are left with a massive longitudinal mode.

Let us turn to the interesting case $\mathrm{sgn}n_1 \mathrm{sgn}n_2 < 0$. For simplicity, we neglect the $i\epsilon_{n_{1,2}}$ terms in the denominator (these terms can be handled alternatively by shifting the Q field). Expanding the integral in Eq. (B2) in powers of q,

$$\int \frac{d^d p}{(2\pi)^d} G^0(p, n_1) G^0(p+q, n_2)$$

$$= \int \frac{d^d p}{(2\pi)^d} G_+(p) G_-(p)$$

$$+ \frac{1}{2} \frac{q^2}{m}$$

$$\times \int \frac{d^d p}{(2\pi)^d} \left[G_+^2(p) G_-(p) + \frac{4E(p)}{d} G_+^3(p) G_-(p) \right], \quad \text{(B4)}$$

where $G_\pm(p) = [-E(p) \pm i/2\tau]^{-1}$. The integrals over momenta can be transformed into integrals over energy ϵ using the density of states $\nu(\epsilon) = (1/2)[S_d/(2\pi)^d](2m)^{d/2}\epsilon^{d/2-1}$. Define

$$I_{a,b,c} = \int_0^\Omega d\epsilon\, \nu(\epsilon) \epsilon^a [G_+(\epsilon)]^b [G_-(\epsilon)]^c, \quad \text{(B5)}$$

so

$$\int \frac{d^d p}{(2\pi)^d} G^0(p, n_1) G^0(p+q, n_2)$$

$$= I_{0,1,1} + \frac{1}{2} \frac{q^2}{m} \left[I_{0,2,1} + \frac{4}{d} I_{1,3,1} \right] \quad \text{(B6)}$$

(a finite upper frequency cutoff Ω is needed depending on d, a, b, c). It is also convenient to rescale the energies, defining $y = 2\tau\epsilon$, so we can write

$$I_{a,b,c} = \frac{1}{2} \frac{S_d}{(2\pi)^d} (2m)^{d/2} (2\tau)^{-d/2-a+b+c}$$

$$\times \int_0^{2\tau\Omega} dy\, \frac{y^{d/2-1+a}}{(-y+i)^b(-y-i)^c}$$

$$= \mathcal{A}_d (2\tau)^{-a+b+c} \int_0^{2\tau\Omega} dy\, \frac{y^{d/2-1+a}}{(-y+i)^b(-y-i)^c}, \quad \text{(B7)}$$

with $\mathcal{A}_d = (1/2)[S_d/(2\pi)^d](2m)^{d/2}(2\tau)^{-d/2}$.

One can check that once τ is fixed by the saddle point Eq. (7), which can be cast as

$$\mathrm{Im}I_{0,1,0} = -\mathrm{Im}I_{0,0,1} = \frac{1}{2\tau v_0}, \quad \text{(B8)}$$

then it follows trivially that

$$I_{0,1,1} = \frac{1}{v_0}, \quad \text{(B9)}$$

so that the leading order term in $M_\perp(q)$ is of order q^2, which allows us to define the diffusion constant

$$D = \frac{1}{4m} \left[I_{0,2,1} + \frac{4}{d} I_{1,3,1} \right]. \quad \text{(B10)}$$

The last step remaining is to show that D is *purely real*. After simple manipulations, one can show that

$$\mathrm{Im}D = \frac{1}{4m} \frac{S_d}{(2\pi)^d} (2m)^{d/2} (2\tau)^{-d/2+3}$$

$$\times \int_0^{2\tau\Omega} dy \left[\left(\frac{8}{d} - 1 \right) y^2 - 1 \right] \frac{y^{d/2-1}}{(y^2+1)^3}. \quad \text{(B11)}$$

It is trivial to show by integration by parts [splitting the integrands into $f(y) = y/(y^2+1)^3$ and $g(y) = y^\alpha$] that the integral in Eq. (B11) scales as $(\tau\Omega)^{d/2-4}$. Thus the cutoff can be safely taken to infinite for $d < 8$, and $\mathrm{Im}D = 0$.

Notice the difference between the fermionic and bosonic cases. In the fermionic case one can also interchange momentum p integrals for energy ϵ integrals, using the density of states at the Fermi level E_F. The integrals are cutoff by the bottom of the band, $-E_F$ away from the zero energy states. In the bosonic case, one starts from the bottom of the band, and needs to include an energy dependent density of states $\nu(\epsilon)$; the cutoff Ω is introduced only for convergence, and $\Omega \to \infty$ is possible for $d < 8$. In contrast to the fermionic case, where E_F is finite, in the bosonic case for a perfect parabolic spectrum $\Omega \to \infty$. The small parameter for the Fermi case is $(E_F\tau)^{-1}$, whereas for the Bose case it is $(\Omega\tau)^{-1} \to 0$.

APPENDIX C: PARAMETRIZATION OF THE SADDLE AND RELATION TO THE FERMIONIC σ MODEL

As we previously mentioned, we can easily obtain the RG equations for the conductance and interaction couplings by determining a correspondence with the fermionic model. Here we show how this is achieved.

Let us first look at the Finkelstein type terms in the effective action for the Q fields. The Q matrices are parametrized as in in Eq. (10), repeated here for convenience,

$$Q = \frac{mv_0}{2} \begin{pmatrix} i(1+qq^T)^{1/2} & q \\ q^T & -i(1+q^Tq)^{1/2} \end{pmatrix}. \quad (C1)$$

The quantities that appear in the Finkelstein type terms for the bosonic problem are

$$e^{+i(\pi/4)\Lambda}Qe^{-i(\pi/4)\Lambda}$$

$$= \frac{mv_0}{2} \begin{pmatrix} i(1+qq^T)^{1/2} & iq \\ -i(q^T) & -i(1+q^Tq)^{1/2} \end{pmatrix}, \quad (C2)$$

$$\frac{mv_0}{2}i \begin{pmatrix} [1-q(-q^T)]^{1/2} & q \\ (-q^T) & -[1-(-q^T)q]^{1/2} \end{pmatrix}. \quad (C3)$$

Direct comparison with the fermionic saddle point

$$Q_F = \frac{mv_0}{2} \begin{pmatrix} (1-qq^T)^{1/2} & q \\ q^T & -(1-q^Tq)^{1/2} \end{pmatrix} \quad (C4)$$

shows that the terms in the Finkelstein type action for bosons, upon parametrization in terms of q, q^T, are the same as the ones for fermions upon the identification $q \to q$ and $q^T \to -q^T$. The extra factor of i in Eq. (C3), once squared (because the Finkelstein terms are quadratic in Q), makes the sign of the interaction term for bosons and fermions the same.

For discussing the diffusive term $D \int d^d x \, \text{tr} (\nabla Q)^2$, notice that by rewriting

$$Q = \frac{mv_0}{2}i \begin{pmatrix} [1-q(-q^T)]^{1/2} & -iq \\ i(-q^T) & -[1-(-q^T)q]^{1/2} \end{pmatrix} \quad (C5)$$

we again identify it with the fermionic saddle point Q_F, but now the off-diagonal elements have extra factors $i, -i$. These factors will cancel each other in the expansion of the quadratic in Q diffusive term, and hence can be dropped, and once again the fermionic saddle expansion can be used. The overall factor of i has the effect of changing $D \to -D$.

In summary, all the RG equations for the dirty interacting boson problem can obtained from those of the (interacting) fermionic orthogonal ensemble upon replacing $g \to -g$ (or $D \to -D$).

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