Long-time tails and anomalous slowing down in the relaxation of spatially inhomogeneous excitations in quantum spin chains

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Exact analytic calculations in spin-1/2 XY chains show the presence of long-time tails in the asymptotic dynamics of spatially inhomogeneous excitations. The decay of inhomogeneities for $t\to\infty$, is given in the form of a power law $(t/\tau_Q)^{-\nu_Q}$, where the relaxation time τ_Q and the exponent ν_Q depend on the wave vector Q, characterizing the spatial modulation of the initial excitation. We consider several variants of the XY model (dimerized, with staggered magnetic field, with bond alternation, and with isotropic and uniform interactions), that are grouped into two families, whether the energy spectrum has a gap or not. Once the initial condition is given, the nonequilibrium problem for the magnetization is solved in closed form, without any other assumption. The long-time behavior for $t\to\infty$ can be obtained systematically in a form of an asymptotic series through the stationary phase method. We found that gapped models show critical behavior with respect to Q, in the sense that there exist critical values Q_c where the relaxation time τ_Q diverges and the exponent ν_Q changes discontinuously. At those points, a slowing down of the relaxation process is induced, similarly to phenomena occurring near phase transitions. Long-lived excitations are identified as incommensurate spin density waves that emerge in gapped systems, as a consequence of both approximate nesting of the spectrum and the degeneracy of some stationary points. In contrast, gapless models do not present the above anomalies as a function of the wave vector Q.

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I. INTRODUCTION

The spin-1/2 XY chain and its variants are among the most widely used and quoted models in theoretical investigations on spin systems. This popularity among theorists is due to two facts. On the one hand, this family of models allows for exact theoretical description of many static as well as dynamic properties(see Ref. 1–33), and on the other hand, in some important cases, it provides a good description of real systems.[34–37] As an example, we note that the dimerized chain, with alternating antiferromagnetic bonds, has been studied in relation with the spin-Peierls transition,[38] and it is thought to represent the spin degrees of freedom of organic compounds that undergo the so called *Peierls distortion*. The present state of the art in fabricating low-dimensional systems, with the material science technology developed after the synthesis of the superconducting cuprate oxides, may now tailor compounds that reveal a wealth of new magnetic phenomena, including random spin chains, spin ladders, and doped magnetic chains among other systems with exotic behaviors. For theorists, these systems are fascinating, with no parallel in classical or three-dimensional physics. Being systems of low dimensionality with low values of the spin, they are dominated by quantum effects.[39] Compounds such as $PrCl_3$, $PrEtSO_4$ (Pr ethyl sulfate), and Cs_2CoCl_4 are among the quasi-one-dimensional systems, whose low-temperature properties are thought to be described by the XY model.

A great deal of theoretical information has been gathered concerning equilibrium properties, with calculations of quantities such as the specific heat, the magnetic susceptibility, and equal-time spin-spin correlation functions.[1,3,8,13] As for the dynamic properties, the quantities most thoroughly studied are the time-dependent spin-spin correlation functions (TDCF)

$$\langle S_j^\mu(t)S_l^\mu\rangle=\left\langle \exp\left(\frac{i}{\hbar}Ht\right)S_j^\mu\exp\left(-\frac{i}{\hbar}Ht\right)S_l^\mu\right\rangle, \qquad (1)$$

where H is the Hamiltonian of the system, μ is an index for the spin component ($\mu=x,y,z$), and $\langle\cdots\rangle$ is the equilibrium average.[3,7,14,23] They are important for the description of such dynamic phenomena as magnetic resonance, magnetic neutron scattering, spin diffusion, and other relaxation properties.[24,25] It should be noted however, that their application is restricted to situations near the equilibrium state, where the linear response theory is valid.[40,41] In particular, for the uniform XY model, an old calculation for the time-dependent autocorrelation $\langle S_0^z(t)S_0^z(0)\rangle$ showed the absence of spin diffusion in the limit $t\to\infty$.[3] This behavior was thought to be accidental and specific of the XY model for spin 1/2 (see, for example, Ref. 25). However, recent research has shown that this surprising property is shared by a whole family of integrable models, and is attributed to the existence of a macroscopic number of conservation laws.[42]

This remarkable result has been probed in magnetic resonance experiments in some one-dimensional (1D) systems.[36] We will turn to this point later.

In spite of the wide range of their applicability, we note that TDCF do not give a direct description of the time evolution of the nonequilibrium process.[24,41] This description is achieved through a different method, which we briefly discuss below.

This second direction in investigating the dynamics relies on calculation of averages of the type

$$\langle A\rangle_t = \mathrm{Tr}[\rho(t)A], \tag{2}$$

where $\rho(t)$ is the density matrix that satisfies Liouville equation

$$i\hbar\frac{\partial}{\partial t}\rho(t) = [H,\rho(t)]. \tag{3}$$

Averages of such a kind give a direct account of the nonequilibrium evolution of the physical observable A, independently of how far the initial state is from the equilibrium state or from a stationary one. Unfortunately, calculations of these quantities are much more involved than the calculation of TDCF (1) for linear response, and few exact results are known.[3,4,6,26–33]

Whether the dynamical process is near or far from equilibrium, most works in the literature deal with cases where the initial state is *spatially uniform*. This premise is assumed in an explicit or a nonexplicit way, and the methods developed to solve the problem are heavily based on it. In contrast, the study of spatially nonuniform excitations is practically new, in spite of its interest, both theoretical and experimental. This problem is important for a deeper understanding of dynamical processes in many-particle spin systems with strong exchange interactions. Inhomogeneous initial states can be prepared in real systems by external actions, for instance, strong inhomogeneous magnetic fields or acoustic waves. On the other hand, from the theoretical side, exact results on the dynamics (as in the case of the XY model), can elucidate details of spin-spin relaxation processes in more complicated systems.

In the present contribution, we will adopt the method based on formulas (2) and (3), and will analyze in detail the long-time evolution of the magnetization in three versions of the XY model. The initial excitation is always prepared in the form of a spatially inhomogeneous magnetization (SIM), and the calculation is done in exact analytic form. The three variants of the XY model that we consider are enumerated below:

(I) Isotropic dimerized XY model with Hamiltonian:

$$H_1 = -\mu_B g h\sum_{j=1}^{N} S_j^z - \sum_{m=1}^{M}[J_1(S_{2m-1}^x S_{2m}^x + S_{2m-1}^y S_{2m}^y) + J_2(S_{2m}^x S_{2m+1}^x + S_{2m}^y S_{2m+1}^y)]. \tag{4}$$

(II) Isotropic XY model in a staggered magnetic field with Hamiltonian:

$$H_2 = -\mu_B g\sum_{m=1}^{M}(h_1 S_{2m-1}^z + h_2 S_{2m}^z) - J\sum_{j=1}^{N}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y). \tag{5}$$

(III) XY model with bond alternation without magnetic field:

$$H_3 = -\sum_{m=1}^{M}(I_1 S_{2m-1}^x S_{2m}^x + I_2 S_{2m-1}^y S_{2m}^y + I_2 S_{2m}^x S_{2m+1}^x + I_1 S_{2m}^y S_{2m+1}^y). \tag{6}$$

We will refer to them as models I, II, III, respectively. In expressions (4)–(6), $N=2M$, μ_B is the Bohr magneton, g is the gyromagnetic ratio, h,h_1,h_2 are magnetic fields, J,J_1,J_2,I_1,I_2 are exchange integrals ($J_1 \geqslant J_2 > 0$ and $J > 0$), and cyclic boundary conditions are assumed. The above restriction on J_1,J_2, and J is not important in our problem and is introduced for convenience.

To investigate SIM dynamics in the above models, we will use a previously developed method given in Ref. 31. This method was applied earlier by one of the authors[32,33] to the same study for the isotropic XY chain in a homogeneous magnetic field, with Hamiltonian:

$$H_4 = -\mu_B g h\sum_{j=1}^{N} S_j^z - J\sum_{j=1}^{N}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y). \tag{7}$$

We will refer to this model as model IV subsequently. We remark that highly nontrivial results were obtained in Refs. 32 and 33. In particular, it was shown that in the $t\to\infty$ limit, some of the spatially inhomogeneous excitations do not disappear and are still time dependent. The time evolution of SIM's (and also their spatial distribution) is probed through the computation of the Fourier components $\langle S_Q^z(t)\rangle$ of the magnetization as a function of the Q-wave number. The calculation yields a relaxation process in the form of a power law

$$\left(\frac{t}{\tau}\right)^{-\nu}, \tag{8}$$

where the exponent ν depends on the initial state. In Ref. 33 the anisotropic XY model was also studied in the limit of strong anisotropy. It was found that the exponent ν in power law (8) changes discontinuously at some critical values Q_c of the wave vector Q. The values Q_c depend solely on the parameters of the Hamiltonian and are not connected with the preparation of the initial state, nor the particular component of the magnetization that is relaxing. Moreover, in the limit $Q\to Q_c$, the relaxation time τ of power law (8) diverges to $\tau\to\infty$, with the corresponding slowing down of the process. This phenomenon is at variance with some conventional views for the time evolution of physical quantities in many particle systems, according to which all quantities must be temporally independent in the limit $t\to\infty$. The concept of spin temperature, widely used in nonequilibrium magnetic

phenomena[43] is based on the assumption that the spin-spin relaxation is much faster than the final relaxation to the lattice degrees of freedom.

Model I was also preliminary investigated by some of the authors, and similar conclusions were attained for the dimerized XY model[44] in the sense that a critical-like slowing down of the relaxation process takes place at special points $\pm Q_c$ of the Q space. In other words, for $|Q|\to Q_c$, the inverse time scale τ^{-1} for some of the oscillating components of SIM, goes to zero following the power law $\tau^{-1} \sim |(|Q|-Q_c)|$. Such behavior is not surprising for the case $Q_c=0$, for which the corresponding value of the total magnetization S^z is a constant of motion, but is unusual for $Q_c \neq 0$, where the corresponding Fourier component S^z_Q is not conserved.

We note that $Q=0$ is the only critical point for the uniform isotropic XY model (model IV and limit $J_1=J_2$ for model I and $h_1=h_2$ for model II). The $t\to\infty$ behavior is dominated by one oscillating component, and this component has no critical properties. The present paper is devoted to elucidate this remarkable difference within a more general context, by the extensive study of a whole family of models. We suggest that dissimilar behaviors are due to the presence of a gap in the spectrum of the energy excitations, the uniform isotropic model being gapless.

Our paper is organized as follows: In Sec. II, we outline the main steps in the analytic calculation. All the models are diagonalized by means of a modified Jordan-Wigner transformation, which maps the spin model into an equivalent fermion Hamiltonian. Then, the average $\langle S^z_Q\rangle_t$ is calculated in exact closed form, and its asymptotic behavior for $t\to\infty$ is obtained after taking the thermodynamic limit $N\to\infty$. In Sec. III, we give a detailed analysis of the results, including all models and the long-time behavior of the magnetization. Section IV closes the paper with final discussions.

II. MAIN STEPS OF SOLUTION

The quantity of our interest is the time-dependent z component of SIM,

$$\langle S^z_Q\rangle_t \equiv \mathrm{Tr}[\rho(t)S^z_Q], \tag{9}$$

where $\rho(t)$ is the density matrix of the system, and

$$S^z_Q=\sum_{j=1}^{N} S^z_j \exp(iQj) \tag{10}$$

is the Fourier transform of the magnetization. In Eq. (9), Q is the wave vector characterizing the spatial inhomogeneity of the initial state ($Q=2\pi n/N, n=-N/2+1,\ldots,N/2$). To calculate $\langle S^z_Q\rangle_t$ one can use the identity:

$$\langle S^z_Q\rangle_t=\langle S^z_Q(t)\rangle_0 \equiv \mathrm{Tr}[\rho(0)S^z_Q(t)], \tag{11}$$

where $\rho(0)$ is the initial density matrix, and $S^z_Q(t)$ is the spin operator in the Heisenberg representation

$$S^z_Q(t)=\exp\left(\frac{i}{\hbar}Ht\right)S^z_Q\exp\left(-\frac{i}{\hbar}Ht\right).$$

To obtain exact results for $\langle S^z_Q\rangle_t$, for each of the models considered, we follow the steps:

(i) Diagonalization of the corresponding Hamiltonian using the fermion representation of the Jordan-Wigner transformation. At the end, we get a free-fermion system.

(ii) Transformation of S^z_Q in terms of fermionic operators to get the time-dependent $S^z_Q(t)$.

(iii) Averaging of $S^z_Q(t)$ with the initial density matrix $\rho(0)$.

The last step depends on the form of initial matrix $\rho(0)$. As it was shown in Ref. 31, exact solutions of this problem may be obtained in the case when $\rho(0)$ is a functional of only one component of the spin operator:

$$\rho(0)=F(S^\gamma) \quad (\gamma=x,y,z). \tag{12}$$

Let us note that such initial state can actually be prepared in real systems at low temperature, with a strong nonhomogeneous magnetic field directed along the coordinate axis γ.

A. Diagonalization

Methods for diagonalization of Hamiltonians (4)–(6) are well known (see for example Refs. 9, 11, and 13). Minor differences are specific for the version of the model to be solved. In our case we have employed the following procedure:

(a) Use of the Jordan-Wigner transformation to change from spin to Fermi operators $(b^\dagger_j, b_j)$:

$$S^x_j=L_j(b^\dagger_j+b_j)/2, \quad S^y_j=L_j(b^\dagger_j-b_j)/2i,$$

$$S^z_j=b^\dagger_j b_j-1/2, \tag{13}$$

where

$$L_j=\prod_{l=1}^{j-1}(2b^\dagger_l b_l-1), \quad L_1\equiv 1, \quad (L^2_j=1). \tag{14}$$

(b) Introduction of two types of Fermi operators for even and odd sites:

$$c^\alpha_m=b^\alpha_{2m}, \quad d^\alpha_m=b^\alpha_{2m-1}, \quad m=1,2,\ldots,M, \quad \alpha=\pm 1, \tag{15}$$

and of their Fourier transforms c^α_k, d^α_k:

$$c^\alpha_k=\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\exp(-i\alpha km)c^\alpha_m,$$

$$d^\alpha_k=\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\exp[-i\alpha k(m-1/2)]d^\alpha_m, \tag{16}$$

$k=2\pi m_1/M$, with $m_1=-M/2+1,\ldots,M/2$, where we have adopted the compact notation

$$c^-=c, \quad c^+=c^\dagger,$$

$$d^-=d, \quad d^+=d^\dagger. \tag{17}$$

(c) After the above steps, Hamiltonians (4)–(6) are transformed into quadratic forms in terms of operators (c_k^α, d_k^α). Diagonalization of these quadratic Hamiltonians is standard, but depends on the specific model under consideration. For Hamiltonians (4) and (5) it can be done with the help of the Bogoliubov transformation given below:

$$c_k^\alpha = u_k \eta_k^\alpha + v_k^\alpha \beta_k^\alpha, \quad d_k^\alpha = v_k^{-\alpha} \eta_k^\alpha - u_k \beta_k^\alpha, \tag{18}$$

where $(\eta_k^\alpha, \beta_k^\alpha)$ are new Fermi operators, $v_k^{-\alpha} = (v_k^\alpha)^*$, and u_k is a real function of k. We are using the same convention (17) for $\alpha = \pm$. For Hamiltonian (4), the functions (u_k, v_k) are given by

$$u_k = \frac{1}{\sqrt{2}}, \quad v_k^\alpha = \frac{1}{\sqrt{2}} \exp(i\alpha\theta_k), \tag{19}$$

where

$$\tan\theta_k = \frac{1-\delta}{1+\delta} \tan(k/2), \quad \delta = J_2/J_1 \leqslant 1.$$

For Hamiltonian (5), one gets

$$u_k = [(P_k - \omega_d)/2P_k]^{1/2},$$

$$v_k = [(P_k + \omega_d)/2P_k]^{1/2} \mathrm{sgn}(\cos k/2), \tag{20}$$

where v_k^α is real (independent of α) and

$$\omega_d = \mu_B g(h_1 - h_2)/2\hbar,$$

$$P_k = (\omega_d^2 + \omega_e^2 \cos^2 k/2)^{1/2}, \quad \omega_e = J/\hbar. \tag{21}$$

(d) Following Ref. 11, diagonalization of Hamiltonian (6) is achieved through

$$c_k^\alpha = (-\eta_k^\alpha + \eta_{-k}^\alpha + \beta_k^\alpha + \beta_{-k}^\alpha)/2,$$

$$d_k^\alpha = (\eta_k^\alpha + \eta_{-k}^\alpha + \beta_k^\alpha - \beta_{-k}^\alpha)/2. \tag{22}$$

As a final result, all the three Hamiltonians can be represented in the diagonal form

$$H_i = \hbar \sum_k [\omega_k^{(i)} \eta_k^\dagger \eta_k + \Theta_k^{(i)} \beta_k^\dagger \beta_k], \qquad (i = 1,2,3) \tag{23}$$

except for constant terms that are not important for the dynamics. The index $i = 1,2,3$ refers to models (I), (II), and (III), respectively. The dispersion relations of Eq. (23) are given by

$$\omega_k^{(1)} = -\omega_0 + \omega_1 R_k, \quad \omega_k^{(2)} = -\omega_s + P_k,$$

$$\omega_k^{(3)} = -(I_2/\hbar)\cos(k/2),$$

$$\Theta_k^{(1)} = -\omega_0 - \omega_1 R_k, \quad \Theta_k^{(2)} = -\omega_s - P_k,$$

$$\Theta_k^{(3)} = (I_1/\hbar)\cos(k/2),$$

where

$$R_k = \sqrt{1 + \delta^2 + 2\delta\cos k}, \quad P_k = \sqrt{\omega_d^2 + \omega_e^2 \cos^2(k/2)},$$

$$\omega_0 = \mu_B g h/\hbar, \quad \omega_s = \mu_B g(h_1 + h_2)/2\hbar,$$

$$\omega_1 = J_1/2\hbar.$$

Note that here and in the following, we neglect, as usually done,[1,4,9,13] the boundary term of the order $1/N$ that appeared in a chain with cyclic boundary conditions. The thermodynamic limit $N \to \infty$ will be taken at the end of the calculation.

B. Calculation of $\langle S_Q^z \rangle_t$

Using formulas (10) and (13)–(16), one can easily obtain the following expression,

$$S_Q^z = -M\delta_{Q,0} + \sum_k (c_p^+ c_q + d_p^+ d_q), \tag{24}$$

where $p = k - Q, q = k + Q$, with definitions common for all models. To obtain the time dependence of S_Q^z, we first express relation (24) in terms of the canonical η's and β's through transformations (18) for models I and II, and relation (22) for model III. Then, we insert the time evolution of such operators

$$\eta_k^\alpha(t) = \eta_k^\alpha \exp(i\alpha\omega_k^{(i)} t),$$

$$\beta_k^\alpha(t) = \beta_k^\alpha \exp(i\alpha\Theta_k^{(i)} t), \quad i = 1,2$$

to get

$$\begin{aligned} S_Q^z(t) = -M\delta_{Q,0} + \sum_k \{ & (u_p u_q + v_p^* v_q) \exp[i(\Omega_i^-(k,Q)t] \\ & \times \eta_p^+ \eta_q + (u_p u_q + v_p v_q^*) \exp[-i\Omega_i^-(k,Q)t] \beta_p^+ \beta_q \\ & + (u_p v_q^* - u_q v_p^*) \exp[i\Omega_i^+(k,Q)t] \eta_p^+ \beta_q \\ & + (u_q v_p - u_p v_q) \exp[-i\Omega_i^+(k,Q)t] \beta_p^+ \eta_q \} \end{aligned} \tag{25}$$

for models I and II, where

$$\Omega_i^+(k,Q) = \omega_p^{(i)} - \Theta_q^{(i)},$$

$$\Omega_i^-(k,Q) = \omega_p^{(i)} - \omega_q^{(i)}, \quad i = 1,2$$

and functions (u_p, v_p) are defined by formulas (19) and (20) respectively.

In the same way, for model III, we obtain

$$\begin{aligned} S_Q^z(t) = \sum_k \{ & \beta_{-q}^+ \eta_p^+ \exp[i(\omega_p^{(3)} + \Theta_q^{(3)})t] \\ & + \eta_{-p} \beta_q \exp[-i(\omega_p^{(3)} + \Theta_q^{(3)})t] \}. \end{aligned} \tag{26}$$

To calculate the time dependent average $\langle S_Q^z \rangle_t$, we use the identity (11), which means that, equivalently, one can calculate the average of operator $S_Q^z(t)$ in the Heisenberg picture in relation to the initial density matrix $\rho(0)$.

From Eqs. (25) and (26), one sees that this problem reduces to calculation of averages of the type $\langle \eta_p^\dagger \eta_q \rangle_0$, $\langle \beta_p^\dagger \beta_q \rangle_0, \ldots$, including all the combinations of canonical operators. This is achieved going back to the spin represen-

tation, in order to employ the initial condition (12). One is then led to computation of spin averages including the ''string'' operators (14). This technique has been previously used by some of the authors in Refs. 32, 33, and 44, and is described in detail in Ref. 45. Putting everything together in Eq. (25) for models I and II, and taking the continuum limit $N\to\infty$, we get expressions for $\langle S^z_Q\rangle_t$ in the form of integrations over the Brillouin zone:

$$\langle S^z_Q\rangle_t=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{F^+(k,Q)\cos[t\Omega_i^+(k,Q)] + F^-(k,Q)\cos[t\Omega_i^-(k,Q)]\}dk, \tag{27}$$

where $i=1,2$, for the two models treated here. Quantities in Eq. (27) are

$$F^{\pm}(k,Q)=\pm 4u_q(u_p^2-|v_p|^2)[i(e^{-ip/2}+e^{iq/2})\langle\varepsilon^x_Q\rangle_0 \mathrm{Im}v_q + (e^{-ip/2}-e^{iq/2})\langle\varepsilon^x_{Q+\pi}\rangle \mathrm{Re}v_q]$$

for $\gamma=x$, and

$$F^{\pm}(k,Q)=A^{\pm}(k,Q)\langle S^z_Q\rangle_0+B^{\pm}(k,Q)\langle S^z_{Q+\pi}\rangle_0,$$

$$A^{\pm}(k,Q)=1\mp[1-2u_p^2u_q^2-2|v_pv_q|^2-4u_pu_q\mathrm{Re}(v_p^*v_q)],$$

$$B^{\pm}(k,Q)=\pm 4iu_pu_q\mathrm{Im}(v_p^*v_q),$$

for $\gamma=z$.

Substituting the specific values (19) and (20) for (u_p,v_p) into formula (27) one obtains the corresponding expressions for models I and II. In the same way, analogous results may be obtained for model III.

Below, we will summarize these results together with the ones obtained in Ref. 33 for model IV. We note that exact solutions can be obtained for the special forms of the initial density matrix $\rho(0)$ given by Eq. (12).

(1) In the case $\gamma=x$ (the case $\gamma=y$ is the same by symmetry), we have

$$\langle S^z_Q\rangle_t=0 \tag{28}$$

for models I, III, IV, and

$$\langle S^z_Q\rangle_t=-\frac{i}{\pi}\omega_e\omega_d\sin(Q/2)e^{iQ/2}\langle\varepsilon^x_{Q+\pi}\rangle_0\int_{-\pi}^{\pi}\frac{\sin^2(k/2)}{P_pP_q} \times\{\cos[t\Omega_2^+(k,Q)]-\cos[t\Omega_2^-(k,Q)]\}dk, \tag{29}$$

for model II, where

$$\langle\varepsilon^x_k\rangle_0=\sum_{j=1}^{N}\langle S^x_jS^x_{j+1}\rangle_0\exp(ikj) \tag{30}$$

is the Fourier transform of the short-range order correlation of the x component.

(2) In the case $\gamma=z$, we have different expressions for all the cases.

(a) For model I, we get the formula

$$\langle S^z_Q\rangle_t=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{F^+(k,Q)\cos[t\Omega_1^+(k,Q)] + F^-(k,Q)\cos[t\Omega_1^-(k,Q)]\}dk, \tag{31}$$

with

$$F^{\pm}(k,Q)=A^{\pm}(k,Q)\langle S^z_Q\rangle_0+B^{\pm}(k,Q)\langle S^z_{Q+\pi}\rangle_0,$$

$$A^{\pm}(k,Q)=1\mp\frac{(1+\delta^2)\cos Q+2\delta\cos k}{R_pR_q},$$

$$B^{\pm}(k,Q)=\pm\frac{i(1-\delta^2)\sin Q}{R_pR_q}. \tag{32}$$

(b) For model II, we have the more explicit form

$$\langle S^z_Q\rangle_t=\frac{1}{4\pi}\langle S^z_Q\rangle_0\int_{-\pi}^{\pi}dk(-[\omega_e^2\cos(p/2)\cos(q/2)/P_pP_q] \times\cos[t\Omega_2^+(k,Q)]+\{1+[\omega_d^2+\omega_e^2\cos(p/2) \times\cos(q/2)]/P_pP_q\}\cos[t\Omega_2^-(k,Q)]). \tag{33}$$

(c) For model III, we have the compact expression:

$$\langle S^z_Q\rangle_t=\frac{1}{2\pi}\langle S^z_Q\rangle_0\int_{-\pi}^{\pi}\cos[t\Omega(k,Q)]dk, \tag{34}$$

where

$$\Omega(k,Q)=v_Q\cos(k/2+\alpha_Q),$$

$$\tan\alpha_Q=\frac{I_1+I_2}{I_1-I_2}\tan(Q/2),$$

$$\hbar v_Q=\sqrt{I_1^2+I_2^2-2I_1I_2\cos Q}. \tag{35}$$

Let us note that at the specific value $Q=\pi$, the time dependence of $\langle S^z_Q\rangle_t$ is given by the simple formula:

$$\langle S^z_\pi\rangle_t=\langle S^z_\pi\rangle_0\ J_0\left(\frac{I_1+I_2}{\hbar}t\right),$$

where $J_0(x)$ is the Bessel function of first kind.

(d) For model IV, the result

$$\langle S^z_Q\rangle_t=\langle S^z_Q\rangle_0J_0(\Omega_Qt), \tag{36}$$

was found in Ref. 33 with $\Omega_Q=2(J/\hbar)\sin Q/2$.

III. ANALYSIS OF RESULTS

In this section, we first discuss general properties of the time evolution of the magnetization, which can be inferred directly from Eqs. (27)–(36). In addition, we will study in detail the asymptotic behavior at long times, in the limit $t\to\infty$. We will show that the evolution of SIM's displays

interesting unusual features, depending on the wave vector Q characterizing the inhomogeneous initial state.

A. General properties of $\langle S^z_Q\rangle_t$

Let us first consider model I in the case $\gamma=z$ for the initial density matrix, since it presents a specific feature that is absent in the other models studied here. From Eqs. (27), (31), and (32), one sees that the value of the magnetization $\langle S^z_Q\rangle_t$ at $t>0$, depends on the initial values of the two Fourier components $\langle S^z_Q\rangle_0$ and $\langle S^z_{Q+\pi}\rangle_0$. This means that, if only one component of S^z_Q exists at $t=0$, for example, for wave vector Q_0, i.e.,

$$\langle S^z_Q\rangle_0=\langle S^z_{Q_0}\rangle_0\delta_{Q,Q_0},$$

then a twofold response develops for $t>0$, with components $\langle S^z_{Q_0}\rangle_t$ and $\langle S^z_{\pi+Q_0}\rangle_t$, whose time evolution is described by Eq. (27), with $F^{\pm}(k,Q)=A^{\pm}(k,Q_0)\langle S^z_{Q_0}\rangle_0\delta_{Q,Q_0}$ and $F^{\pm}(k,Q)=B^{\pm}(k,Q_0+\pi)\langle S^z_{Q_0}\rangle_0\delta_{Q,Q_0+\pi}$, respectively. In contrast, for models II–IV, if there is only one initial component $\langle S^z_{Q_0}\rangle_0$, only one component $\langle S^z_Q\rangle_t$ of the same wave vector $Q=Q_0$ exists at subsequent times, $t>0$.

When the initial condition is prepared with $\gamma=x$, the average $\langle S^z_Q\rangle_t$ vanishes identically for models I, III, and IV, for any $t>0$. This result may be understood intuitively, without any calculation, if one notes that the part of the Hamiltonian that describes the coupling of spins with the magnetic field commutes with the other one describing the exchange interactions [in the case of model III, this statement is trivial for we only have the exchange part]. Denoting the latter as $H_{\rm ex}$, we get the time evolution

$$\langle S^z_Q\rangle_t=\mathrm{Tr}\left[\rho(0)\exp\left(\frac{i}{\hbar}H_{ex}t\right)S^z_Q\exp\left(-\frac{i}{\hbar}H_{ex}t\right)\right], \quad (37)$$

which is governed by the exchange only. Inside the trace in Eq. (37), we now perform a unitary transformation consisting in a rotation by π around the x axis. Under this operation, the spin operators transform as follows:

$$S^x_j\to S^x_j, \quad S^y_j\to -S^y_j, \quad S^z_j\to -S^z_j.$$

The exchange Hamiltonian H_{ex} is invariant under this transformation because it is a sum of products of the form $S^x_jS^x_{j\pm1}$ and $S^y_jS^y_{j\pm1}$. The initial density matrix $\rho(0)$ is also invariant since it only depends on the S_x component of the spins ($\gamma=x$). So, the right-hand side of Eq. (37) will change its sign, yielding $\langle S^z_Q\rangle_t=-\langle S^z_Q\rangle_t$ i.e., $\langle S^z_Q\rangle_t=0$. Another important consequence of Eq. (37) is that the time evolution of SIM's is independent of the value of the external field h (valid for any γ), and is determined solely by the exchange term.

In the case of model II, the exchange part does not commute with the total Hamiltonian H_2. This leads, in general, to a nonvanishing $\langle S^z_Q\rangle_t$, for $t>0$, which gives results proportional to $\langle\varepsilon^x_{Q+\pi}\rangle_0$, where $\langle\varepsilon^x_k\rangle_0$ is the Fourier transform of the spin-spin correlation function $\langle S^x_jS^x_{j+1}\rangle_0$ evaluated at the initial condition [see relation (30)]. If the initial state is prepared with a single Q vector in the form $\langle\varepsilon^x_k\rangle_0=\langle\varepsilon^x_{Q_0}\rangle_0\delta_{k,Q_0}$, then $\langle S^z_{\pi+Q_0}\rangle_t$ is the only component that exists for $t>0$.

For $Q=Q_0=0$ (homogeneous initial state), and for any γ, we get the (evident) result

$$\langle S^z_{Q=0}\rangle_t=\langle S^z_{Q=0}\rangle_0,$$

for models I, II, IV. This is a direct consequence of the conservation of the total z component of the magnetization. In the case of model III, when $\gamma=z$, we get the closed result:

$$\langle S^z_{Q=0}\rangle_t=\langle S^z_{Q=0}\rangle_0J_0(\nu_0t),$$

where $\hbar\nu_0=|I_1-I_2|$ and J_0 is the Bessel function of zero order.

Special important limits are $\delta=1(J_1=J_2)$ in model I and $h_1=h_2$ in model II, that map onto model IV. In both of these cases, the corresponding formulas for $\langle S^z_Q\rangle_t$ reduce to Eq. (36), which was obtained in Ref. 33.

Unfortunately, it is difficult to get more information from formula (27), (29), (32), and (33), because they are rather involved. So, in the following, we will consider the limiting case $t\to\infty$, which will be studied using the stationary phase method.

B. Long-time behavior of $\langle S^z_Q\rangle_t$ for the initial condition $\gamma=z$

1. Model I

According to the stationary phase method,[46] the long-time evolution of $\langle S^z_Q\rangle_t$ is dominated by the contribution of the stationary points of the functions $\Omega^{\pm}_1(k,Q)$. The number of these points depends on the value of the wave vector Q. For convenience of further discussion, let us introduce here the so-called *critical* values of Q:

$$Q_{c_1}=\arccos\delta, \quad Q_{c_2}=\pi-\arccos\delta, \quad Q_{c_3}=0. \quad (38)$$

The above values are determined in a standard manner through the stationary phase method, being the locus where stationary points become degenerate.[46] Let us first examine the role of $\Omega^+_1(k,Q)$. It is easy to show that it has five nondegenerate stationary points for Q in the interval $Q_{c_1}<|Q|<Q_{c_2}$:

$$k_1=0, \quad k_2=-k_3=\pi,$$

$$k_4=-k_5=\varphi_Q, \qquad \varphi_Q=\arccos(-\delta^{-1}\cos Q). \quad (39)$$

In turn, for $Q_{c_2}<|Q|\leqslant\pi$ and $Q_{c_3}<|Q|<Q_{c_1}$, one gets three nondegenerate points at

$$k_1=0, \quad k_2=-k_3=\pi.$$

Exactly at the critical values, we obtain:

(i) For $Q=\pm Q_{c_1}$, one nondegenerate point $k_1=0$ and two degenerate ones $k_2=-k_3=\pi([\Omega^+_1(\pm\pi,Q)]''=[\Omega^+_1(\pm\pi,Q)]'''=0$, $[\Omega^+_1(\pm\pi,Q)]^{\rm IV}\neq0$, where derivations are taken with respect to k).

(ii) For $Q=\pm Q_{c_2}$, two nondegenerate points $k_2=-k_3=\pi$, and one degenerate at $k=k_1=0([\Omega_1^+(0,Q)]''=[\Omega_1^+(0,Q)]'''=0,[\Omega_1^+(0,Q)]^{\mathrm{IV}}\neq 0)$.

(iii) For $Q=Q_{c_3}=0$, there is no time evolution, since the magnetization $\langle S^z_{Q=0}\rangle_t$ is a constant of the motion

$$\langle S^z_{Q=0}\rangle_t=\langle S^z_{Q=0}\rangle_0 .$$

The function $\Omega_1^-(k,Q)$ vanishes at $Q=Q_{c_3}=0$ and $Q=\pm\pi$. For any other wave number Q, it has two nondegenerate stationary points:

$$k_6=-k_7=\psi_Q , \quad \psi_Q=\arccos(-\delta\cos Q).$$

This finishes the analysis of the stationary points that dominate the long-time behavior of Eq. (31). The corresponding asymptotic development of $\langle S^z_Q\rangle_t$ for $t\to\infty$, can be represented as a sum of several oscillating components with Q-dependent frequencies and amplitudes:

$$\langle S^z_Q\rangle_t\sim\sum_i S_i(Q,t), \tag{40}$$

$$S_i(Q,t)=\sum_{l=0}^{\infty} a_{i,l}(Q)\left(\frac{t}{\tau_i}\right)^{-\nu_i(2l+1)}\cos[t\Omega_i(Q)+\alpha_{i,l}], \tag{41}$$

where $\tau_i\equiv\tau_i(Q)$ are functions of Q and the exponent ν_i assumes the values 1/2 or 1/4 (the latter value will be discussed in detail below). The number of components $S_i(Q,t)$ depends on Q. For $Q_{c_1}<|Q|<Q_{c_2}$ (excluding $Q=\pm\pi/2$, which is a degenerate point where Ω_1 and Ω_2 coincide), there are four components. For $0\leqslant|Q|<Q_{c_1}$ and $Q_{c_2}<|Q|<\pi$ (excluding accidental degeneracies), there are only three terms in relation (40). The frequencies $\Omega_i(Q)$ and the inverse of the characteristic times τ_i^{-1} of Eq. (41) are given below:

$$\Omega_1(Q)=\omega_1(1+\delta^2+2\delta\cos Q)^{1/2},$$

$$\tau_1^{-1}=\omega_1\delta(1+\delta\cos Q)\frac{|\delta+\cos Q|}{R_Q^3}, \quad Q\neq\pm Q_{c_2}$$

$$\Omega_2(Q)=\omega_1(1+\delta^2-2\delta\cos Q)^{1/2},$$

$$\tau_2^{-1}=\omega_1\delta(1-\delta\cos Q)\frac{|\delta-\cos Q|}{R_{Q+\pi}^3}, \quad Q\neq\pm Q_{c_1}$$

$$\Omega_3(Q)=\omega_1\left(\frac{R_{\psi_Q-Q}-R_{\psi_Q+Q}}{2}\right),$$

$$\tau_3^{-1}=\omega_1\frac{\delta}{2(1-\delta^2)}(R_{\psi_Q-Q}+R_{\psi_Q+Q})|\sin Q|\sin\psi_Q ,$$

$$Q\neq Q_{c_3}=0$$

$$\Omega_4(Q)=\omega_1\left(\frac{R_{\varphi_Q-Q}+R_{\varphi_Q+Q}}{2}\right),$$

$$\tau_4^{-1}=\omega_1\frac{\delta}{2(1-\delta^2)}(R_{\varphi_Q-Q}-R_{\varphi_Q+Q})\sin Q\sin\varphi_Q ,$$

$$Q_{c_1}<|Q|<Q_{c_2}. \tag{42}$$

We display typical Q dependences of the frequencies $\Omega_i(Q)$ and of the inverse of the relaxation times τ_i^{-1} in Figs. 1 and 2, respectively, for particular values of δ. Slowing down of the relaxation process occurs at points where the τ_i^{-1}'s vanish.

At a generic Q point, the first nonvanishing terms in Eq. (41) have the form

$$S_i(Q,t)\approx a_{i,0}(Q)\left(\frac{t}{\tau_i}\right)^{-1/2}\cos[t\Omega_i(Q)+\alpha_{i,0}](i=1,\ldots,4), \tag{43}$$

with the amplitudes

$$a_{i,0}(Q)=a^c_{i,0}(Q)\langle S^z_Q\rangle_0+ia^s_{i,0}(Q)\langle S^z_{Q+\pi}\rangle_0 ,$$

showing that the value at Q depends on the two initial components $\langle S^z_Q\rangle_0$ and $\langle S^z_{Q+\pi}\rangle_0$. Explicit formulas for the coefficients $a^c_{i,0}(Q)$ and $a^s_{i,0}(Q)$, as well as the phases $\alpha_{i,0}$, are given in Ref. 45.

The dynamical process described by relations (40) and (41), with the explicit formulas (42), is remarkable, since it exhibits long-time tails in the relaxation of SIM's to the spatially homogeneous state. In contrast to the exponential relaxation, which is characterized by a single parameter that yields the time scale or the relaxation rate, the power law relaxation given by relations (41) and (43) is characterized by two parameters: τ_i^{-1}, which determines the inverse time scale of the process, and the exponent ν_i, which determines the relaxation rate. In general, no conservation laws or long-lived hydrodynamic modes seem to be associated with the above long-time tails.

In the neighborhood of critical points, the relaxation of some of the components of relation (40) begins to stop, with the corresponding relaxation time τ_i diverging, as shown in Fig. 2. Exactly in the limit, the corresponding exponent ν_i jumps discontinuously from 1/2 to 1/4. This slowing down of the relaxation process is very similar to the critical slowing down found in phase transition phenomena.[47]

Let us summarize below the singular behavior of the relaxation time at special points (they are displayed in Fig. 2):

(a) $\tau_1^{-1}\to 0$ when $Q\to\pm Q_{c_2}$, for any δ ;

(b) $\tau_2^{-1}\to 0$ when $Q\to\pm Q_{c_1}$, for any δ;

(c) $\tau_4^{-1}\to 0$ when $Q\to\pm Q_{c_1},\pm Q_{c_2}$, for any δ;

(d) $\tau_3^{-1}\to 0$ when $Q\to 0,\pm\pi$, for any δ.

All the τ_i^{-1} vanish according to the law:

$$\tau^{-1}\sim|(|Q|-Q_c)|\text{for}|Q|\to Q_c .$$

We now give the behavior of components of relation (40) at the critical points:

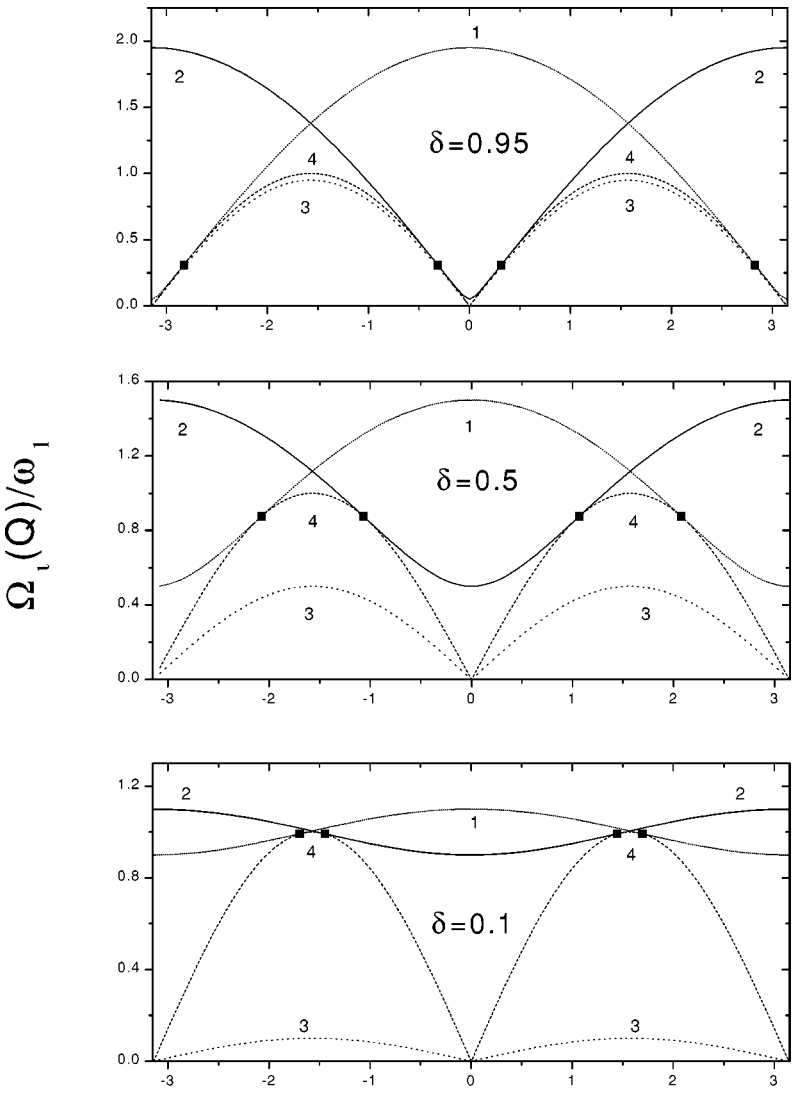

FIG. 1. Frequencies of the asymptotic components of the magnetization in Eq. (41), as functions of the wave number Q, for several values of the parameter δ. The different branches Ω_i, for $i=1,2,3,4$, are indicated by the numbers. We also display the critical points $\pm Q_{c_1}$ and $\pm Q_{c_2}$ by solid squares. Note that the Ω_4 branch is tangent to the Ω_1 and Ω_2 branches at Q_{c_2} and Q_{c_1}, respectively, but it is only defined between critical points, $Q_{c_1}<|Q|<Q_{c_2}$. The complete Ω_4 curve is shown as a guide for the eye.

(I) In the limit $Q\to\pm Q_{c_1}$, the components $S_2(Q,t)$ and $S_4(Q,t)$ merge into a single component with $\nu=1/4$:

$$S_2(\pm Q_{c_1},t)=a_{2,0}(\pm Q_{c_1})\left(\frac{t}{\tau_c}\right)^{-1/4}\cos[t\Omega_2(Q_{c_1})+\pi/8], \tag{44}$$

with

$$a_{2,0}(\pm Q_{c_1})=\frac{\sqrt[4]{24}}{8\pi}\Gamma(1/4)[(1+\delta)\langle S^z_{\pm Q_{c_1}}\rangle_0 + i\sqrt{1-\delta^2}\langle S^z_{\pi\pm Q_{c_1}}\rangle_0],$$

$$\tau_c^{-1}=3\omega_1\delta^2/(1-\delta^2),$$

where $\Gamma(x)$ is the gamma function. The other components follows the law (43) with $Q=\pm Q_{c_1}$.

(II) Analogously, for $Q\to\pm Q_{c_2}$, the components $S_1(Q,t)$ and $S_4(Q,t)$ join together with $\nu=1/4$ and the time dependence:

$$S_1(\pm Q_{c_2},t)=a_{1,0}(\pm Q_{c_2})\left(\frac{t}{\tau_c}\right)^{-1/4}\cos[t\Omega_1(Q_{c_2})+\pi/8],$$

where

$$a_{1,0}(\pm Q_{c_2})=\frac{\sqrt[4]{24}}{8\pi}\Gamma(1/4)[(1-\delta)\langle S^z_{\pm Q_{c_2}}\rangle_0 + i\sqrt{1-\delta^2}\langle S^z_{\pi\pm Q_{c_2}}\rangle_0].$$

The other components follows the law (43) with $Q=\pm Q_{c_2}$.

(III) The component $S_3(Q,t)$ is critical at $Q=Q_{c_3}=0$, with the relaxation time $\tau_3\sim 1/|Q|$ diverging in the vicinity of $Q=0$. Note that the frequency Ω_3 vanishes in the limit

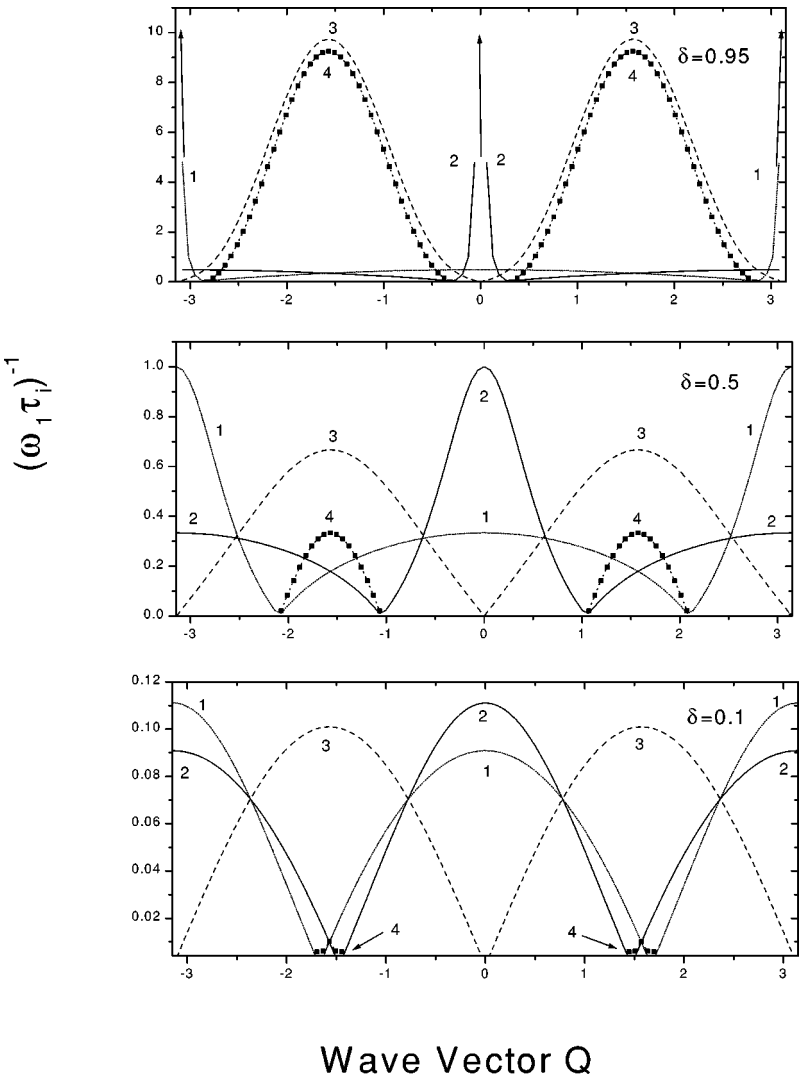

FIG. 2. Inverse of the relaxation times $(\omega_1\tau_i)^{-1}$ of the power-law decay of Eq. (41), as functions of the wave number Q for the same values of the parameter δ displayed in the previous figure. The numbers label the different branches, in close correspondence with Fig. 1. Solid squares are used to display $(\omega_1\tau_4)^{-1}$, which vanishes at the critical points. The arrows in the upper figure indicate values out of the scale.

$Q\to 0$. The criticality of Q_{c_3} has a different connotation, since it is related to the conservation of the total z component of the spin,

$$S^s_{Q=0}=\sum_{j=1}^{N} S^z_j,$$

which is valid for all the model with axial symmetry along the z axis. The divergence of τ_3 near $Q\approx 0$ is connected to the stability of hydrodynamic excitations of long wavelength.

(IV) Note the exception of $Q\to\pm\pi$ for the component $S_3(Q,t)$, because the amplitude $a_{3,0}(Q)$ goes to zero in this limit.

2. Model II

When applying the same method to get the asymptotic behavior of Eq. (33) for long times, one sees that the work is facilitated by the development of the preceding section. In fact, the frequencies $\Omega_2^{\pm}(k,Q)$ and the functions P_k that appeared in Eq. (33) of model II, can be mapped onto the corresponding $\Omega_1^{\pm}(k,Q)$ and $\omega_1 R_k$ of model (I), with the substitution:

$$\delta\to\frac{\sqrt{\omega_d^2+\omega_e^2}-|\omega_d|}{\sqrt{\omega_d^2+\omega_e^2}+|\omega_d|},\qquad \omega_1\to\frac{1}{2}\sqrt{\omega_d^2+\omega_e^2}+\frac{|\omega_d|}{2}. \tag{45}$$

To avoid numerous definitions, we use the previous symbols δ and ω_1 throughout this section, with the meaning given above in relation (45). With these variables, the stationary and critical points, the inverse characteristic times τ_i^{-1} and the frequencies $\Omega_i(Q)$ coincide with those of model I.

The expression for $\langle S^z_Q\rangle_t$ has the form (41) with the following values of the first nonvanishing amplitudes $a_{i,k}(Q)$ (at a generic Q point):

$$a_{1,1}(Q)=\langle S^z_Q\rangle_0\frac{\delta(1-\delta)^2(1-\cos Q)}{R_Q^4\sqrt{8\pi}},$$

$$a_{2,0}(Q)=\langle S^z_Q\rangle_0\sqrt{\frac{2}{\pi}}\frac{\delta(1-\cos Q)}{R^2_{Q+\pi}},$$

$$a_{3,0}(Q)=\langle S^z_Q\rangle_0\sqrt{\frac{2}{\pi}}\frac{1+\delta\cos Q}{1+\delta},$$

$$a_{4,0}(Q)=\langle S^z_Q\rangle_0\sqrt{\frac{2}{\pi}}\frac{\delta+\cos Q}{1+\delta},$$

Note that the component with frequency $\Omega_1(Q)$ decays to zero as $t^{-3/2}$ (exponent $\nu=1/2$ and $l=1$), i.e., more rapidly if compared with the similar component of model I. We now summarize the behavior at the critical points:

(i) At the critical point $Q=\pm Q_{c_1}$, the components $S_2(Q,t)$ and $S_4(Q,t)$ are degenerate and decay to zero with exponent $\nu=1/4$, following the law (44) with

$$a_{2,0}(\pm Q_{c1})=\frac{\sqrt[4]{24}}{2\pi}\Gamma(1/4)\frac{\delta}{1+\delta}.$$

(ii) At $Q=\pm Q_{c_2}$, the components $S_1(Q,t)$ and $S_4(Q,t)$ merge into a single component with the evolution [$\nu=1/4$ and $l=1$ in Eq. (41)]

$$S_1(\pm Q_{c_2},t)=a_{1,1}(\pm Q_{c_2})\left(\frac{t}{\tau_c}\right)^{-3/4}\cos[t\Omega_1(Q_{c_2})+\pi/8], \tag{46}$$

with

$$a_{1,1}(\pm Q_{c_2})=\frac{1}{2\pi}(3/2)^{3/4}\Gamma(3/4)\frac{\delta}{1+\delta}.$$

3. Model III

The long-time behavior of Eq. (34) is much simpler than those displayed by models I and II, for the phase function $\Omega(k,Q)$ in Eq. (35) has only one stationary point at $k=k_1=-2\alpha_Q$. In the asymptotic limit $t\to\infty$, it may be represented in the form

$$\langle S^z_Q\rangle_t=\langle S^z_Q\rangle_0\sqrt{\frac{2}{\pi}}(t/\tau_Q)^{-1/2}\cos(t\upsilon_Q-\pi/4),$$

where υ_Q is given by formula (35) and $\tau_Q^{-1}=\upsilon_Q$.

C. Long-time behavior of $\langle S^z_Q\rangle_t$ for the initial condition $\gamma=x$

As it was mentioned in Sec. II B, $\langle S^z_Q\rangle_t$, for the initial condition $\gamma=x$, vanishes identically for models I, III, IV, and only has a nonzero value for model II. For the latter, the asymptotic expression of $\langle S^z_Q\rangle_t$ for $t\to\infty$, has the form of Eq. (41) with the quantities Ω_i and τ_i defined through formulas (42) and (45). The coefficients $a_{i,l}(Q)$ for the first nonvanishing amplitudes in Eq. (41), at generic Q points, are

$$a_{1,1}(Q)=\langle\varepsilon^x_{Q+\pi}\rangle_0\frac{\sqrt{\delta}(1-\delta)}{2\sqrt{2\pi}}(1-e^{iQ})/R^2_Q,$$

$$a_{2,0}(Q)=\langle\varepsilon^x_{Q+\pi}\rangle_0\frac{2\sqrt{\delta}(1-\delta)}{\sqrt{2\pi}}(1-e^{iQ})/R^2_{Q+\pi},$$

$$a_{3,0}(Q)=-\langle\varepsilon^x_{Q+\pi}\rangle_0\frac{2\sqrt{\delta}}{\sqrt{2\pi}(1+\delta)}(1-e^{iQ})(1+\delta\cos Q),$$

$$a_{4,0}(Q)=\langle\varepsilon^x_{Q+\pi}\rangle_0\frac{2\sqrt{\delta}}{\sqrt{2\pi}(1+\delta)}(1-e^{iQ})(1+\delta^{-1}\cos Q),$$

where $\langle\varepsilon^x_k\rangle_0$ is the correlation function defined in Eq. (30). We remark that the first nonvanishing term for the component with frequency Ω_1 has $\nu=1/2$ and $l=1$, yielding a $t^{-3/2}$ power law for the time decay.

We make a summary of the critical behavior at singular points:

(i) At the critical point $Q=\pm Q_{c_1}$, the components $S_2(Q,t)$ and $S_4(Q,t)$ merge into one component with time evolution given by Eq. (44), with $\nu=1/4$ and $l=0$, with the amplitude

$$a_{2,0}(\pm Q_{c_1})=\frac{\sqrt[4]{24}}{2\pi}\Gamma(1/4)\frac{\sqrt{\delta}}{1+\delta}(1-\delta\mp i\sqrt{1-\delta^2}).$$

(ii) In analogous form, at $Q=\pm Q_{c_2}$ the components $S_1(Q,t)$ and $S_4(Q,t)$ become degenerate with the time evolution of Eq. (46), with $\nu=1/4$ and $l=1$, and the amplitude

$$a_{1,1}(\pm Q_{c_2})=\frac{1}{2\pi}(3/2)^{3/4}\Gamma(3/4)\sqrt{\delta}\left(1\mp i\sqrt{\frac{1-\delta}{1+\delta}}\right).$$

IV. FINAL DISCUSSION

We have studied the relaxation to the homogeneous state, of an initial excitation that has been prepared with a nonhomogeneous magnetization profile along the magnetic chain (SIM). The time evolution of this excitation is probed through the calculation of the Fourier component $\langle S^z_Q(t)\rangle$ of the magnetization, which is analyzed as a function of the wave vector Q. We use periodic boundary conditions for the spin Hamiltonian and take the thermodynamic limit ($N\to\infty$) before studying asymptotic long times. As remarked in Ref. 4, the order of the limits is very important. All the models treated here show long-time tails in the relaxation of $\langle S^z_Q(t)\rangle$, which is apparent from the asymptotic study for $t\to\infty$, developed in the previous sections. This behavior manifests itself in the form of a power law decay in the long-time evolution of SIM's. This slow relaxation is a remarkable result for its own sake and is probably due to the absence of dissipation in the models. Since the systems are isolated at zero temperature, the dynamics is exclusively driven by quantum fluctuations.

Our calculation also shows striking differences, when one compares the behaviors of models I and II on the one hand, with models III and IV, on the other. The dissimilarity lies in the presence of critical values Q_{c_i} for the wave vector Q, where we get a slowing down of the relaxation process. Near

critical points, the time scale τ_i, giving the relaxation of some specific oscillating components of $\langle S^z_Q(t)\rangle$, diverge with the law $\tau^{-1}\sim|(|Q|-Q_{c_i})|$, for $|Q|\to Q_{c_i}$, reducing the damping of these excitations. Such behavior is not surprising for the case $Q_{c_3}=0$, for which the corresponding value $S^z_{Q=0}$ is an integral of the motion, but is unusual for the case $Q_{c_i}\neq 0$, when S^z_Q is not conserved. In this sense, the criticality at $\pm Q_{c_1}$, $\pm Q_{c_2}$ has different implications in the theory. The position of the critical values $\pm Q_{c_1}$ and $\pm Q_{c_2}$ depends only on parameters of the Hamiltonian, and in the limits $Q\to \pm Q_{c_1}, \pm Q_{c_2}$, the exponent ν jumps discontinuously, the relaxation rate becoming slower than for excitations with $|Q|\neq Q_{c_{1,2}}$. This means that the critical components will be the only surviving ones at long enough times. These long-lived excitations may be pictured as spin density waves (charge density waves for the fermion model) with incommensurate wave vector $Q_{c_{1,2}}$, which develop in models that undergo a Peierls instability.[38] In contrast, the dynamics of models III and IV does not have such anomalies. There is no gap in the spectrum, and in the language of particles, the systems are always metallic and display no critical points (except the point $Q=0$). In the asymptotic regime $t\to\infty$, only one oscillating component exists which displays no critical behavior.

Peculiar properties of the energy spectrum seem to determine the critical relaxation phenomena. In fact, models I and II display an energy gap between the ground and excited states, whose size is monitored by the parameter δ [for model II through the transformation (45)]. The existence of critical points is directly related to the presence of the gap, since the latter changes the curvature of the dispersion relation for the energy. If one looks at formula (25) for the Fourier component $S^z_Q(t)$ of the magnetization, one realizes that processes that contribute with the $\pm\Omega_i^+$ frequency (the one that is critical) come from transitions between both branches of the spectrum (interband transitions), where a particle with momentum $q=k+Q$ is destroyed in a given band, and a particle with momentum $p=k-Q$ is created in the other one, with a net momentum transfer $\Delta k=\pm 2|Q|$. So the criticality at $Q_{c_{1,2}}$ is not directly related to the one-particle spectrum, but to a big density of states for those interband transitions that exchange the same momentum $\Delta k=\pm 2|Q|$ coherently (constructive interference). The above is a consequence of a *nesting* effect, which is achieved when $Q=Q_{c_{1,2}}$, with the onset of a spin density wave for the S^z magnetization. In fact, the spectrum can be linearized in the neighborhood of the inflection points, yielding parallel linear branches for the upper and lower bands, which are connected by $p=k-Q_c$ and $q=k+Q_c$. This is accompanied by the degeneracy of some stationary points.

We have found that the above anomalous behavior in the relaxation is not an exclusive property of the Peierls transition. In fact, other 1D gapped spin models, as the XY chain with anisotropic interactions, present the same slowing down of the relaxation at special points (Ref. 33 treats the Ising limit). A general connection between this critical behavior and the spectral properties is currently under research.[48]

Our results are all exact, with closed analytical forms, from where the asymptotic regime for $t\to\infty$ is obtained. This has been done using the fermion representation of the Jordan-Wigner approach for spin chains. Other procedures to solve the same systems are available, the most conspicuous being the Bethe ansatz method, which applied to the XY model might hint at ''hidden'' conservation laws of the several versions of the model treated here. However, it is difficult to conceive a similar analytic calculation of the relaxation properties, using Bethe ansatz techniques.

We remark that integrable models display other unusual dynamical phenomena, with anomalies in the transport properties.[42] For the Heisenberg model, the absence of spin diffusion has been probed by bosonization techniques[49] and numerical calculations.[50] Reference 50 presents exact numerical computations for the spin-1/2 XXZ chain at infinite temperature ($T=\infty$), which include, as particular cases, the isotropic Heisenberg model and the XY regime, which are gapless, and the Heisenberg-Ising model, with a gap in the spectrum and long-range order for the ground state. Numerical data are inconclusive due to finite-size effects, but there are strong hints of a crossover from nondiffusive to diffusive behavior when we go from the gapless region to the gapped one. Concerning the low-temperature properties, a calculation by Sachdev and Damle[51] in the gapped region, shows that the diffusive behavior holds in the presence of long-range order. It is however puzzling to admit that low-energy properties of the spectrum may affect the $T=\infty$ behavior.

From a more fundamental point of view, we have studied the time evolution of a closed quantum many-body system (at zero temperature), which is prepared in an arbitrary nonhomogeneous initial state. The dynamics is solely given by the Schrödinger equation for the wave function or by the Liouville equation for the density operator (which includes the more general situation of a mixed ensemble for the initial state). In the thermodynamic limit and for long times ($N\to\infty$ first than the limit $t\to\infty$), we get irreversibility in the form of a power law relaxation for the magnetization, despite the dynamics being unitary. No internal interactions are present, since all the Hamiltonians are reduced to the free-particle form. The irreversibility can be entirely ascribed to interference effects, which in our calculation are handled using the stationary phase method at asymptotic long times. The infinite number of degrees of freedom precludes quantum recurrences, which may be observed in finite chains or systems with a discrete spectrum.[52,53] Due to constructive interference, gapped models develop long-lived collective excitations with the texture of spin density waves (charge density waves for the fermion versions) that persist for longer times. We believe that those structures are universal features of gapped one-dimensional models.

Finally, our simple quantum systems are good candidates for *aging* effects, a concept that has been coined to specify nonequilibrium dynamics that depends on the initial conditions and relaxes very ''slowly.''[54] But a complete characterization of quantum aging requires the calculation of two-time correlation functions.[53]

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