Bose-Einstein condensation on inhomogeneous networks: Mesoscopic aspects versus thermodynamic limit

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We study the filling of states in a pure hopping boson model on the comb lattice, a low-dimensional discrete structure where geometrical inhomogeneity induces Bose-Einstein condensation (BEC) at finite temperature. By a careful analysis of the thermodynamic limit on combs we show that, unlike the standard lattice case, BEC is characterized by a macroscopic occupation of a finite number of states with energy belonging to a small neighborhood of the ground state energy. Such a remarkable feature gives rise to an anomalous behavior in the large distance two-point correlation functions. Finally, we prove a general theorem providing the conditions for the pure hopping model to exhibit the standard behavior, i.e. to present a macroscopic occupation of the ground state only.

I. INTRODUCTION

The most recent experiments on Bose-Einstein condensation (BEC) stimulated a large wealth of theoretical work aimed at a better understanding of the basic properties of such interesting phenomenon.[1] In particular the possibility of confining ultracold bosonic atoms within optical lattices, along with the most recent studies on arrays of Josephson junctions, arouse a great interest in bosonic models defined on Euclidean lattices.[2,3]

Even more interesting from this point of view would be the arrangement of the Josephson junctions, or possibly optical traps, into more complex networks.[4] Indeed it was recently put into evidence[5–7] that, due to topological inhomogeneity, BEC at finite temperature can occur on low dimensional structures, such as the comb lattice,[8] even in the absence of an external potential. Such results were obtained mainly in the thermodynamic limit (i.e., for structures of infinite size), but it is clear that a deeper understanding of the onset of the phenomenon on finite-size structures is needed. This is especially true in view of the experimental research which is currently being developed in this field.[4]

One of the distinctive features of usual BEC (in continuous Euclidean geometry, possibly in the presence of harmonic potentials) is the macroscopic occupation of a single quantum state. More precisely it is possible to prove that the filling of any excited state vanishes in the thermodynamic limit. As for general networks, Refs. 5–7 mainly dealt with the thermodynamic aspect of the problem. In particular it is shown that on a low-dimensional inhomogeneous network, such as the comb lattice, BEC is characterized by the macroscopic occupation of the states belonging to an arbitrarily small energy neighborhood of the ground state. This is a more general condition, since it does not necessarily entail that only the ground state features a macroscopic occupation.

In the following we indeed show that the macroscopic occupation of the ground state does not entirely account for BEC on the comb lattice. Even in the thermodynamic limit, the filling of the structure is completely described only if a finite number of states belonging to a small neighborhood of the ground state is considered. This result is proven by exactly evaluating the occupation of the quantum states of mesoscopic comb lattices, with large but finite sizes. The spectrum of such structures is nearly continuous, but the energy levels are still distinguishable since they pertain to orthogonal wave functions which differ on a macroscopic scale.

The paper is organized as follows. In Sec. II we introduce the pure hopping model[5] which describes the low coupling limit for the Josephson junction arrays or a model of noninteracting atoms confined in an optical lattice. Using the Van Hove spheres, we define the thermodynamic limit for a general infinite discrete structure. Then we recall the definition of *the low energy hidden spectrum*, and we give the general conditions on the graph spectra for the condensation in an arbitrarily small energy region.[7] In Sec. III we consider the problem of the states filling for the pure hopping bosonic model comparing the known results for the three-dimensional lattice with the numerical results obtained on the comb graph. We show that in the first case only the lowest energy state presents a macroscopic occupation, while the comb graph exhibits a different behavior since a macroscopic occupation of many states is present. In Sec. IV we solve analytically the problem of the states filling on the comb graph proving that, in the thermodynamic limit, also the first excited states are filled by a finite fraction of particles. Furthermore we show that the numerical data perfectly fit the analytical predictions. In Sec. V we prove that the macroscopic filling of many states gives rise to an anomalous behavior of the large distance two point correlation functions. In Sec. VI we give the spectral conditions for the pure hopping model to present a macroscopical occupation of a single quantum state, and in Sec. VII we present our conclusions.

II. BOSONIC HOPPING MODELS ON GRAPHS

Graph theory provides the most natural mathematical description of generic discrete networks.[9] A graph is a count-

able set V of sites i connected pairwise by a set E of unoriented links $(i,j)=(j,i)$. Two sites joined by a link are called nearest neighbors. The topology of a graph is fully described by its adjacency matrix A_{ij}, with $A_{ij}=1$ if (i,j) is a link of the graph and $A_{ij}=0$ otherwise. A *walk* in G is a sequence of concatenated links $\{(i,k),(k,h)\ldots,(n,m),(m,k)\}$ and a graph is said to be connected if for any two sites there is always a walk joining them. The length of a walk is the number of links appearing in the sequence, and the length of the shortest walk joining two sites is called the *chemical distance* between them. The latter defines the intrinsic metric every connected graph is endowed with. The Van Hove sphere $S_{r,o}$, of center o and radius r, is the set of sites whose chemical distance from site o is equal or less than r. In the following we will consider only connected graphs with polynomial growth, where the number $N_{r,o}$ of sites within $S_{r,o}$ grows at most as a power of the radius. This choice ensures that the discrete structure can be embedded in a finite dimensional Euclidean space. Since for this class of graphs it is possible to prove that the thermodynamic limit is independent from the choice of the center of the sphere,[10] in the following we will drop the relevant subscript.

Let us recall some basic results obtained for the pure hopping model on a generic graph (a complete description of the subject can be found in Refs. 6 and 7). On a graph G the pure hopping model for bosons is defined by the Hamiltonian[5]

$$H=-t\sum_{i,j\in V}A_{ij}a_i^\dagger a_j,\qquad(1)$$

where $a_i^\dagger$ and a_i are the creation and annihilation operator at site i ($[a_i,a_j^\dagger]=\delta_{ij}$), and A_{ij} is the adjacency matrix of the graph.

To study the behavior of Eq. (1) in the thermodynamic limit, where BEC can arise, we restrict the model to S_r, and analyze its properties as the radius of the Van Hove sphere goes to infinity. The model restricted to S_r is defined by the Hamiltonian

$$H^r=\sum_{i,j}-tA_{ij}^r a_i^\dagger a_j,\qquad(2)$$

where $A_{ij}^r=A_{ij}$ if $i,j\in S_r$, and $A_{ij}^r=0$ otherwise. The corresponding normalized density of states $\rho^r(E)$ is

$$\rho^r(E)=\frac{1}{N_r}\sum_k\delta(E-E_k^r),\qquad(3)$$

where E_k^r are the eigenvalues of $-tA_{ij}^r$ and N_r is the number of sites within the sphere. The function $\rho(E)$ is defined to be the thermodynamic density of states of $-tA_{ij}$ if it satisfies the following condition:

$$\lim_{r\to\infty}\int|\rho^r(E)-\rho(E)|dE=0.\qquad(4)$$

In general the asymptotic behavior at low energies of the thermodynamic density of states is described by a power law of the form
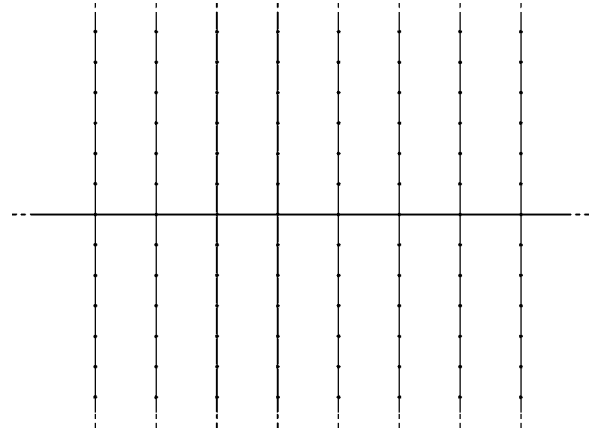

FIG. 1. The comb graph, which is obtained by connecting to each site of a linear chain, called a backbone, a one-dimensional chain called a finger.

$$\rho(E)\sim(E-E_m)^{\alpha/2-1}\quad\text{for}\quad E\to E_m\qquad(5)$$

where $E_m\equiv\mathrm{Inf}(\mathrm{Supp}[\rho(E)])$, and α is an exponent determined by the topology of the graph.[11,12]

On inhomogeneous structures the density of states can present some interesting anomalies, such as the *hidden region* of the spectrum defined in Refs. 6 and 7. A *hidden region* of the spectrum consists of an energy interval $[E_1,E_2]$ such that $[E_1,E_2]\cap\mathrm{Supp}(\rho(E))=\emptyset$ and $\lim_{r\to\infty}N_{[E_1,E_2]}^r>0$, where $N_{[E_1,E_2]}^r$ is the number of eigenvalues of $-tA_{ij}^r$ in the interval $[E_1,E_2]$. Notice that, in general, $N_{[E_1,E_2]}^r$ can diverge for $r\to\infty$ and the eigenvalues can become dense in $[E_1,E_2]$ in the thermodynamic limit. Therefore, the presence of a hidden spectrum is a far more general property than the existence of a discrete spectrum, where a finite number of states fills the spectral region. An interesting example of this kind of behavior is exhibited by the the comb lattice[5,6] (see Fig. 1) which will be studied in detail in the following sections. We now define the lowest energy level for the sequence of densities $\rho_r(E)$, setting $E_0^r=\mathrm{Inf}_k(E_k^r)$ and $E_0=\lim_{r\to\infty}E_0^r$. In general, $E_0\leq E_m$. If $E_0<E_m$, then $[E_0,E_m]$ is a hidden region of the spectrum which will be called low energy hidden spectrum.

The correct way of taking the thermodynamic limit consists in adjusting the population N of the system so that the filling N/N_r is set to a fixed value f as the radius of the Van Hove sphere goes to infinity. In particular, in the macrocanonical ensemble, the equation that determines the fugacity z as a function of $\beta=T^{-1}$, f, and r is

$$f=\int\frac{1}{z^{-1}e^{\beta E}-1}\rho^r(E)dE.\qquad(6)$$

Setting $E_0=0$ yields $0\leq z(f,\beta,r)\leq1$. A system presents BEC at finite temperature if there exists a temperature T_C such that for all $T\leq T_C$ $\lim_{r\to\infty}z(f,\beta,r)=1$.

The general conditions for the occurrence of BEC at finite temperature is strictly related to the properties of $\rho(E)$.[7] In particular it is proven that BEC arises either in models pre-

senting a low energy hidden spectrum or in models where the parameter α appearing in Eq. (5) is larger than two. The comb lattice is an example of the former situation, whereas the latter condition is satisfied by any Euclidean lattice with dimension larger than two, since for such a structure the parameter α coincides with the Euclidean dimension.

III. SINGLE STATE FILLING

BEC on Euclidean lattices is characterized by the nonanalytic behavior of the thermodynamic functions at the critical temperature and by a macroscopic filling of a single quantum state. The definition of BEC we gave in Sec. II is strictly connected with the singular properties at T_C. For example in the thermodynamic limit the average energy per particle is

$$\langle E \rangle = \frac{1}{f} \int \frac{E}{z^{-1} e^{\beta E} - 1} \rho(E) dE, \qquad (7)$$

where $z = 1$ for temperature below the critical temperature, whereas for $T > T_C$ it is a function of T and f determined by Eq. (6). Therefore Eq. (7) presents the typical nonanalytic behavior of the thermodynamic functions of BEC at the critical point.

Let us now consider the filling of the states. Below T_C any arbitrarily small energy region around E_0 is filled by a finite fraction of particles, as it is known by classical results on BEC. More precisely, for all $\epsilon > 0$ the fraction n_ϵ of particles with energy in the region $[E_0, E_0 + \epsilon]$ is always greater than the positive quantity n_{TD},

$$n_{TD} = f - \int \frac{1}{e^{\beta E} - 1} \rho(E) dE, \qquad (8)$$

where $E_0 = 0$ and the subscript TD stands for thermodynamic.

Since E_0 is the only real number belonging to $[E_0, E_0 + \epsilon]$ for any value of ϵ, one would argue that the only possible way to satisfy the condition $n_\epsilon > n_{TD} > 0 \; \forall \epsilon > 0$ is to fill the lowest energy state with a finite fraction of particles. However, if we look carefully at how the thermodynamic limit is performed, the question is not so trivial. Indeed, if the model does not present a gap (this is also the case of the usual condensation on lattices), the energy of the first excited states tends to E_0 in the thermodynamic limit and it belongs to $[E_0, E_0 + \epsilon]$ for all values of ϵ. For instance, in order to determine the filling n_1 of the first excited state E_1, one first has to solve Eq. (6) for $z(r)$ and then evaluate $n_1 = \lim_{r \to \infty} N_r^{-1} (z(r)^{-1} \exp[\beta E_1(r)] - 1)^{-1}$. The result of this limit depends on how $N_r \to \infty$, $E_1(r) \to E_0 = 0$ and $z(r) \to 1$ in the thermodynamic limit.

This limit is trivial on lattices: there the pure hopping model does not present a gap, and it is known that below T_C the lowest energy eigenstate is the only one with a macroscopic occupation.[13] Let us first consider the case of the three-dimensional lattice. The spectrum of a finite lattice of $L^3 = N_r$ sites is given by

$$\sigma_l = \left\{ 6t - 2t \cos\frac{2\pi k}{L} - 2t \cos\frac{2\pi h}{L} \right.$$
$$\left. - 2t \cos\frac{2\pi j}{L} \right\}_{k=1,\ldots,L; h=1,\ldots,L; j=1,\ldots,L}, \qquad (9)$$

and the equation determining the fugacity z for each finite lattice is

$$f = \frac{1}{L^3} \frac{z}{1-z} + \frac{1}{L^3} \sum_{E_k \in \sigma_l}^{E_k \neq 0} \frac{z}{e^{\beta E_k} - z}, \qquad (10)$$

where $n_0 = L^{-3} (z^{-1} - 1)^{-1}$ is the filling of the ground state and $L^{-3} [z^{-1} \exp(\beta E_k) - 1]^{-1}$ is the filling of a state of energy E_k. Below the critical temperature Eq. (10) has a solution in the thermodynamic limit only if $z \to 1$ for $L \to \infty$. In particular one has that

$$z \sim 1 - \frac{\delta}{L^3} \qquad \text{for} \quad L \to \infty. \qquad (11)$$

Substituting Eq. (11) into Eq. (10) one obtains $n_0 = \delta^{-1}$ and

$$n_0 = f - \int \frac{1}{e^{\beta E} - 1} \rho(E) dE. \qquad (12)$$

Hence $n_0 = n_{TD}$ and the lowest energy state is the only macroscopically filled state. As a further check of this result we can explicitly evaluate the filling n_1 of the first excited state in the thermodynamic limit. According to Eq. (9) the energy of the first excited state is $E_1 = t[2 - 2 \cos(2\pi/L)]$ and, from Eq. (11), its filling is

$$n_1 = \lim_{L \to \infty} \frac{1}{L^3} \frac{z}{z^{-1} e^{\beta E_1} - 1} = \lim_{L \to \infty} \frac{1}{L^3} \frac{1}{\beta E_1 + \delta/L^3} = 0 \qquad (13)$$

in the thermodynamic limit.

In Fig. 2 we plot n_{TD} [obtained from Eq. (8) with $f = 1$], n_0 (the filling of the lowest energy state), and n_ϵ for two different values of ϵ. To determine n_0 and n_ϵ we first evaluated the exact spectrum of a finite lattice consisting of $N_r = 125\,000$ sites, then we obtained $z(1, \beta, r)$ by numerical inversion of Eq. (6). This allowed us to evaluate the filling $n_k = N_r^{-1} (z^{-1} e^{\beta E_k} - 1)^{-1}$ of the energy level E_k. From Fig. 2 we have that the differences between the plots are very small and they can be ascribed to finite size effects. These numerical results confirm the known property of BEC on regular lattices of presenting a macroscopic occupation only in the state of lowest energy.

Let us focus on the case of the comb graph proving that on inhomogeneous structures, due to the presence of hidden regions in the spectrum, a macroscopic filling of states of energy arbitrarily near to the ground state is possible. In Fig. 3, n_{TD}, n_0, and n_ϵ for a comb graph consisting of 40 000 sites are shown. In this case it is evident that, since the filling of the ground state (the curve n_0) is lower than n_{TD}, there must be macroscopically filled states other than E_0. From the general result [Eq. (8)] the energies of these states must tend
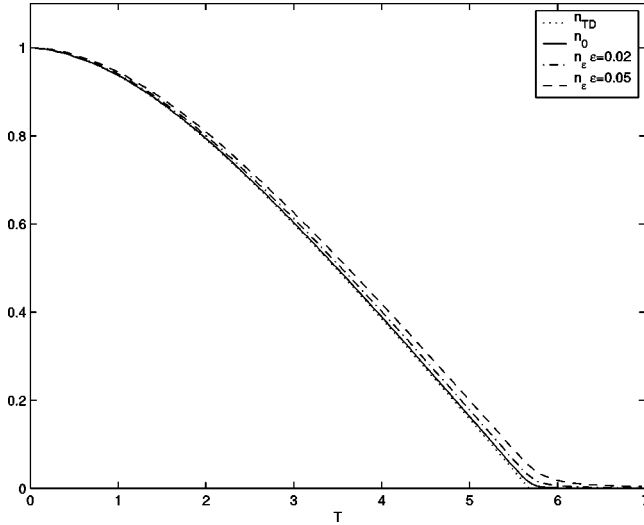
FIG. 2. The state filling in the three-dimensional lattice as a function of the temperature measured in units of t. The dotted line represents the theoretical value of n_{TD} obtained from Eq. (8). The solid line is a numerical evaluation of the filling of the ground state n_0. The dashed and dash-dotted lines are numerical evaluations n_ϵ for two different small values of ϵ: $\epsilon=0.02t$ and $\epsilon=0.05t$. Numerical data are obtained from a lattice of $N_r=125\,000$ sites.

to E_0 as $L\to\infty$. Actually, the curves n_ϵ representing the filling relevant to the energy region $[E_0,E_0+\epsilon]$ is very close to n_{TD} even for small values of ϵ.

IV. STATES FILLING ON THE COMB GRAPH

On the comb graph, the anomalous behavior of the filling can be explained by carefully considering Eq. (6), which
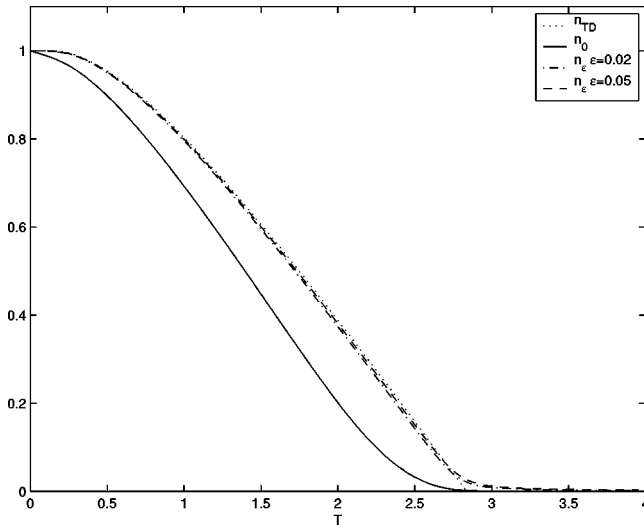

FIG. 3. The state filling in the comb lattice as a function of the temperature measured in units of t. The dotted line represents the theoretical value of n_{TD} obtained from Eq. (8). The solid line is a numerical evaluation of the filling of the ground state n_0. The dashed and dash-dotted lines are numerical evaluations n_ϵ for two different small values of ϵ: $\epsilon=0.02t$, and $\epsilon=0.05t$. Numerical data are obtained from a comb of $N_r=40\,000$ sites.

determines the fugacity for each finite size comb. The energy spectrum of the pure hopping model on a finite comb graph consisting of $N_r=L\times L$ is the union of four sets, $\sigma_P=\{E_0\}\cup\sigma_-\cup\sigma_0\cup\sigma_+$.[6] Setting $E_0=0$:

$$\sigma_-=\left\{E_k=t\left(\sqrt{8}-2\sqrt{1+\cos^2\frac{2\pi k}{L}}\right)\right\}_{k=1,2,[(L-1)/4]}$$

$$\sigma_0=\left\{E_k=t\left(\sqrt{8}-2\cos\frac{2\pi k}{L}\right)\right\}_{k=1,2,L-1}, \quad (14)$$

$$\sigma_+=\left\{E_k=t\right.$$
$$\left.\times\left[\sqrt{8}+2\sqrt{1+\cos^2\left(\frac{2\pi k}{L}+\frac{\pi}{2}\right)}\right]\right\}_{k=1,2,[(L-1)/4]}.$$

The degeneration $d(E_k)$ of the states of energy $E_k\in\sigma_-$ and $E_k\in\sigma_+$ is 2, while for the states of energy $E_k\in\sigma_0$ we have $d(E_k)=L$. Hence, in the thermodynamic limit, σ_- and σ_+ are filled by a vanishing fraction of states $[2(L-1)/(4L^2)]$ and they belong to the hidden spectrum,[6] whereas σ_0 gives rise to the spectral region of measure one, since the relevant fraction of states is $L(L-1)/L^2$. σ_0 reproduce the spectral density of a linear chain and its states are completetely delocalized. On the other hand the states in σ_- and σ_+, which characterize the typical behavior of the comb graph, are localized along the backbone presenting an exponential decay in the direction of the finger. The equation determining the fugacity z for a finite comb is

$$f=n_0(z,L)+n_{\sigma_-}(z,L)+n_{\sigma_0}(z,L)+n_{\sigma_+}(z,L), \quad (15)$$

with $0\leq z\leq 1$. Here $n_0(z,L)$, $n_{\sigma_+}(z,L)$, $n_{\sigma_-}(z,L)$, and $n_{\sigma_0}(z,L)$ represent, respectively, the fraction of particles in the ground state and in the three spectral regions. More precisely,

$$n_\sigma(z,L)=\frac{1}{L^2}\sum_{E\in\sigma}d(E)\frac{z}{e^{\beta E}-z}, \quad (16)$$

where $d(E)$ denotes the degeneracy of the energy state E. Let us analyze these four contributions separately.

In the thermodynamic limit the filling of the ground state,

$$n_0(z)=\lim_{L\to\infty}n_0(z,L)=\lim_{L\to\infty}\frac{1}{L^2}\frac{z}{1-z}, \quad (17)$$

and the filling of the low hidden region,

$$n_{\sigma_-}(z)=\lim_{L\to\infty}n_{\sigma_-}(z,L)$$
$$=\lim_{L\to\infty}\frac{2}{L^2}\sum_{n=1}^{(L-1)/4}[z^{-1}e^{\beta t[\sqrt{8}-2\sqrt{1+\cos^2(2\pi n/L)}]}-1]^{-1},$$
$$(18)$$

are different from zero if and only if $\lim_{L\to\infty}z=1$. The filling of the spectral region of measure 1 is given by

$$n_{\sigma_0}(z) = \lim_{L\to\infty} n_{\sigma_0}(z,L) = \lim_{L\to\infty} \frac{1}{L} \sum_{n=1}^{L-1} \frac{z}{e^{\beta t[\sqrt{8}-2\cos(2\pi n/L)]}-z}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} dk \frac{1}{z^{-1}e^{\beta t[\sqrt{8}-2\cos k)]}-1}$$

$$= \int \frac{1}{z^{-1}e^{\beta E}-1} \rho(E)dE. \tag{19}$$

$n_{\sigma_0}(z)$ is a finite positive number for each value of z, $0 \leq z \leq 1$. For the high hidden region σ_+ we obtain

$$0 \leq n_{\sigma_+}(z) = \lim_{L\to\infty} n_{\sigma_+}(z,L)$$

$$= \lim_{L\to\infty} \frac{2}{L^2} \sum_{n=1}^{(L-1)/4} \frac{z}{e^{\beta t[\sqrt{8}+2\sqrt{1+\cos^2(2\pi n/L)}]}-z}$$

$$< \lim_{L\to\infty} \frac{2}{L} \frac{z}{e^{\beta t(\sqrt{8}+2)}-z} = 0. \tag{20}$$

Hence $n_{\sigma_+}(z,L)$ can be neglected.

Let us consider the behavior of Eq. (15) in the thermodynamic limit. Above the critical temperature it is satisfied by a value of $z = z' < 1$ so that, according to Eqs. (17) and (18), $n_0(z') = n_{\sigma_-}(z') = 0$. On the other hand, for $T < T_C$ Eq. (15) can be solved only letting $z \to 1$ when $L \to \infty$. In particular we obtain a solution of Eq. (15) if and only if

$$z \sim 1 - \frac{\delta}{L^2} \qquad \text{for } L \to \infty \tag{21}$$

Let us study in detail the behavior of $n_0(z)$ and $n_{\sigma_-}(z)$ when z tends to 1 as in Eq. (21). We obtain

$$n_0 = \frac{1}{\delta}, \tag{22}$$

and the fraction of particles in σ_- can be exactly evaluated by summing the corresponding series

$$n_{\sigma_-} = \lim_{L\to\infty} \frac{2}{L^2} \sum_{n=1}^{(L-1)/4} \left[e^{\beta t[\sqrt{8}-2\sqrt{1+\cos^2(2\pi n/L)}]} - 1 + \frac{\delta}{L^2} \right]^{-1}$$

$$= \sum_{n=1}^{\infty} \frac{2}{2\sqrt{2}\beta t \pi^2 n^2 + \delta} = \frac{\mathrm{csch}\ \alpha[\alpha\cosh\alpha - \sinh\alpha]}{\delta}, \tag{23}$$

where

$$\alpha = \frac{1}{2^{\frac{3}{4}}} \sqrt{\frac{\delta}{t\beta}} = \frac{1}{2^{\frac{3}{4}}} \sqrt{\frac{T}{tn_0}}. \tag{24}$$

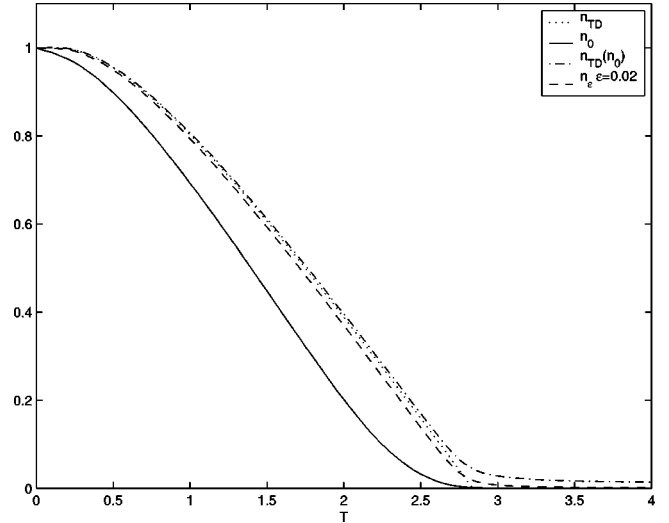Then, below T_C, Eq. (15) for z becomes an equation for the new variable δ,

FIG. 4. Check of Eq. (25). The dotted line represents the theoretical value of n_{TD} obtained from Eq. (8). The solid line is a numerical evaluation of the filling of the ground state n_0. The dashed line is a numerical evaluation n_ϵ for $\epsilon = 0.02t$. The dash-dotted line is obtained evaluating n_{TD} from Eq. (25), where δ and α are given by Eqs. (24) and (22) and n_0 is given by the numerical data of the solid line. The temperature is measured in units of t and numerical data are obtained from a comb of $N_r = 40\,000$ sites.

$$\delta^{-1} + \delta^{-1}\ \mathrm{csch}\ \alpha[\alpha\cosh\alpha - \sinh\alpha] = f - n_{\sigma_0}(1) = n_{TD} \tag{25}$$

which can be numerically solved. Hence, in order to obtain Eq. (25) we had to evaluate the thermodynamic limit of Eqs. (17), (18), (19), and (20). We remark that only in the case of the spectral region of measure one it is possible to replace the sum by an integral, whereas in the other cases the sums have to be explicitly calculated.

Equation (25) explains the main properties of BEC on the comb lattice. First of all, in this case $n_{TD} = f - n_{\sigma_0}(1)$ is the sum of two contributions; the first due to the particles in the lowest energy state $n_0 = \delta^{-1}$, the other due to the particles in the low hidden region, $n_{\sigma_-} = \delta^{-1}\ \mathrm{csch}\ \alpha[\alpha\cosh\alpha - \sinh\alpha]$. Hence, on the comb graph, the filling of the ground state is smaller than n_{TD}, as it has been numerically shown in Fig. 3.

A second point is that only states with arbitrary small energy contribute to n_{TD}. Indeed the result of Eq. (23) does not change if we force the index n to be lower than ϵL, where ϵ is an arbitrarily small fixed parameter. This means that, $\forall \epsilon > 0$, only states with energy smaller than $t[\sqrt{8} - 2\sqrt{1+\cos^2(2\pi\epsilon)}]$ contribute to n_{TD}.

Finally substituting δ^{-1} with n_0 in Eq. (25), we obtain an exact relation between n_0 and n_{TD}. In Fig. 4 we checked that the numerical results for the comb graph presented in Sec. III are perfectly reproduced by the exact calculation. The dashed-dotted line is obtained by first evaluating n_0 and then using Eq. (25) to obtain $n_{TD}(n_0)$. These data are very close both to n_ϵ (filling of a small energy region around E_0) and to the theoretical value of n_{TD}.

V. COHERENCE PROPERTIES OF THE CONDENSATE

The classical Bose-Einstein condensates on homogeneous lattices present important large scale coherence properties due to the macroscopic filling of a single quantum state. One of the most effective ways to put into evidence this relevant feature of BEC is the study of the long distance correlation function $C(i,j)$ defined by

$$C(i,j) = \sum_k \frac{\psi_k^{r*}(i)\psi_k^r(j)}{z^{-1}e^{\beta E_k}-1}, \tag{26}$$

where $\psi_k^r(j)$ is the eigenvector of $-tA_{ij}^r$ (2) of eigenvalue E_k^r. In particular, considering the correlation function between two sites at a macroscopic distance (i.e., $r_{ij} \sim r$, r is the radius of the Van Hove sphere), below the critical temperature in the thermodynamic limit we have

$$C(i,j) \sim N_r n_{TD} \psi_0^{*r}(i)\psi_0^r(j) \quad \text{for } r \to \infty. \tag{27}$$

Equation (27) shows that below T_C there are $N_r n_{TD}$ particles in the ground state $\psi_0^r(j)$. On regular lattices, the wave function of the ground state is constant and the correlation function at large distances does not depend on r_{ij}. In particular for the three-dimensional case $(N_r = L^3)\,C(i,j) \sim (L^3 n_{TD})(L^3)^{-1}$; $(L^3 n_{TD})$ is the asymptotic number of particles of the condensate, and $(L^3)^{-1}$ is the normalizing factor given by the wave function $\psi_0^r(j)$.

The large scale coherence properties of the Bose-Einstein condensate on a comb graph can be studied by calculating the correlation functions between sites i and j of the backbone at a chemical distance $r_{ij}=dL$, (this way if the size of the macroscopic system is defined to be one, we are considering sites at distance d). Let us evaluate the contribution to Eq. (26) of each single spectral region by using the exact wave functions $\psi_k^L(j)$ of the pure hopping model on a finite $L \times L$ periodic comb,[6] with $N_r = L^2$. For E_0 we have

$$C_0(i,j) = C_0(d) = \psi_0^{L*}(i)\psi_0^L(j)\frac{z}{1-z} \sim L^2 n_0(\sqrt{2}L)^{-1},$$

$$\text{for } L \to \infty. \tag{28}$$

$C_0(i,j)$ does not depend on the distance between the sites since the wave function of the ground state is constant along the backbone. In Eq. (28) $L^2 n_0$ represents the filling of the state and $(\sqrt{2}L)^{-1}$ is the normalizing factor of the wave function. In the low energy hidden spectrum we have

$$C_{\sigma_-}(d)$$

$$= \sum_{k=1}^{(L-1)/4} \psi_k^{L*}(i)\psi_k^L(j)[z^{-1}e^{\beta t[\sqrt{8}-2\sqrt{1+\cos^2(2\pi k/L)}]}-1]^{-1}$$

$$\sim \frac{1}{(\sqrt{2}L)}\sum_{k=1}^{\infty}\frac{L^2 2\cos(2\pi kd)}{\sqrt{2}\beta t\pi^2 k^2+\delta} \quad \text{for } L \to \infty. \tag{29}$$

It should be noted that significant contribution to Eq. (29) arise only from states with energy close to E_0, similar to what happens in Eq. (18). Here the wave functions have
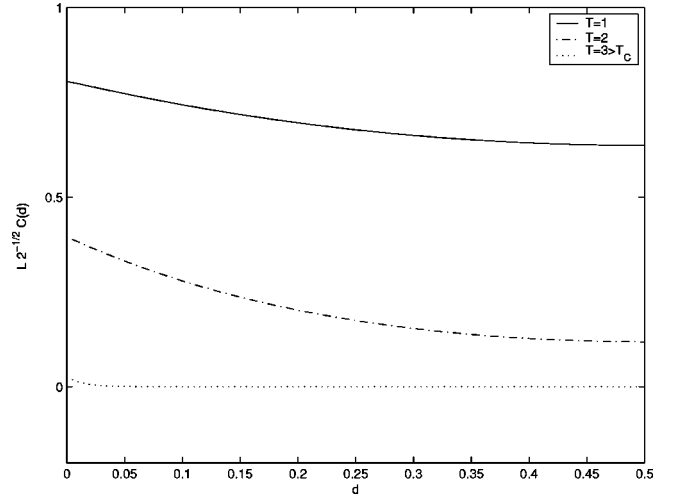

FIG. 5. The correlation function given by Eq. (30) as a function of the macroscopic distance d. Above T_C, $C(d)$ is zero, while at low temperature it depends on d.

macroscopical oscillations along the backbone and the filling of each state $L^2(\sqrt{2}\beta t\pi^2 k^2+\delta)^{-1}$ is multiplied by an oscillating factor $\cos(2\pi kd)$. $(\sqrt{2}L)^{-1}$ is again the normalizing factor of the wave functions. In σ_0 and σ_+ we have $C_{\sigma_0}(i,j)=C_{\sigma_+}(i,j)=0$, since in this case $\psi_k^{L*}(i)\psi_k^L(j)$ is an oscillating factor with diverging frequency giving rise to decoherence effects. The large scale correlation function for the pure hopping model is then

$$C(i,j) = C(d) \sim \frac{L^2}{(\sqrt{2}L)}\left(n_0 + \sum_{k=1}^{\infty}\frac{2\cos(2\pi kd)}{\sqrt{2}\beta t\pi^2 k^2+\delta}\right)$$

$$= \frac{N_r}{(\sqrt{2N_r})}\left(n_0 + \sum_{k=1}^{\infty}\frac{2\cos(2\pi kd)}{\sqrt{2}\beta t\pi^2 k^2+\delta}\right)$$

$$\text{for } r = L \to \infty. \tag{30}$$

The two point correlation function diverges for $L \to \infty$, a well known property of localized condensates. Notice that each state of the condensate provides a different oscillating contribution to Eq. (30). In Fig. 5 we plot the correlation function $C(i,j)$ as a function of the distance d.

We can also evaluate the correlation function between two sites of the same finger. Since all the states of macroscopic filling present the same behavior along this direction (an exponential decay depending on the distance from the backbone), in this case there are no interference effects and one obtains an expression analogous to Eq. (27) ($|i|$ and $|j|$ represent the distance of the sites from the backbone):

$$C(i,j)$$

$$\sim \frac{L^2}{(\sqrt{2}L)}\left(n_0 e^{-\text{arcsh}(1)(|i|+|j|)} + \sum_{k=1}^{\infty}\frac{2e^{-\text{arcsh}(1)(|i|+|j|)}}{\sqrt{2}\beta t\pi^2 k^2+\delta}\right)$$

$$= \frac{N_r n_{TD}}{(\sqrt{2N_r})}e^{-(|i|+|j|)\text{arcsh}(1)} \quad \text{for } r = L \to \infty. \tag{31}$$

$C(i,j)$ presents the diverging behavior for $L \to \infty$ typical of localized condensate.

VI. GENERAL RESULT

As we have seen, in an inhomogeneous structure the occurrence of BEC does not imply the macroscopic occupation of a single quantum state. In this last section we prove the general condition for a bosonic hopping model on a graph to present condensation only on the lowest energy state. In particular we will show that in a model presenting BEC at finite temperature (i.e., with a low energy hidden spectra or with a spectral dimension greater than 2) one has condensation on the single state E_0 if the spectrum $\rho(E)$ satisfies the spectral condition

$$\lim_{r \to \infty} \int \frac{|\rho'^r(E) - \rho(E)|}{|E - E_0|} dE = 0, \tag{32}$$

where

$$\rho'^r(E) = \frac{1}{N_r} \sum_k{}' \; \delta(E - E_k^r). \tag{33}$$

The symbol Σ' means that we are excluding from the sum the contribution given by the ground state. Eq. (32) has a clear mathematical interpretation. Indeed as Eq. (4) is the necessary condition for evaluating the thermodynamic limit of the average value of any bounded function $f(E_k)$ as $\int f(E) \rho(E) dE$, Eq. (32) represents the condition to evaluate by an integral even unbounded functions diverging in E_0 as $(E - E_0)^{-1}$. The general result can be obtained writing Eq. (6) as:

$$f = \frac{1}{z^{-1} - 1} + \int_{E > \epsilon'} \frac{(\rho'^r(E) - \rho(E))}{z^{-1} e^{\beta E} - 1} dE$$

$$+ \int_{E > \epsilon'} \frac{\rho(E)}{z^{-1} e^{\beta E} - 1} dE, \tag{34}$$

where we set $E_0 = 0$ and ϵ' is a generic energy between E_0^r and E_1^r the energy of the first excited state. Let us now take in (34) the thermodynamic limit $r \to \infty$. For the first term $\lim_{r \to \infty} (z^{-1} - 1)^{-1} = n_0$. For the second term we have

$$\int \left| \frac{(\rho'^r(E) - \rho(E))}{e^{\beta E} - 1} \right| dE \leq \lim_{r \to \infty} \int \frac{|\rho'^r(E) - \rho(E)|}{|E|} dE = 0, \tag{35}$$

where we used condition (32). Finally in the last term we let $\epsilon' \to 0$, obtaining

$$n_0 = f - \int \frac{\rho(E) dE}{z^{-1} e^{\beta E} - 1} = n_{TD}, \tag{36}$$

and this implies that only the lowest energy state has a macroscopic filling. The pure hopping model on the comb graph obviously does not satisfy Eq. (32). Notice that Eq. (32) is not a condition on $\rho(E)$, the density of states of the infinite structures, but a prescription on how $\rho^r(E)$ tends to $\rho(E)$. The single state occupation then is determined by the behavior of the model on each large but finite mesoscopic scale.

VII. CONCLUDING REMARKS

In this paper we address the issue of Bose-Einstein condensation for a bosonic hopping model defined on the comb graph, a simple inhomogeneous discrete structure. Such a model is expected to describe the low coupling limit for a comb-shaped network of Josephson junctions (or, possibly, of optical traps confining noninteracting atoms), a realistic system which is currently the subject of experimental research.[4] In particular, we show that the macroscopic occupation of the ground state, which is one of the distinctive features of BEC on (homogeneous) Euclidean lattices, is not sufficient for a complete description of the phenomenon on the comb lattice. Indeed the inhomogeneity of the network forces the macroscopic occupation of a spectral region just above the ground state. This feature is strictly related to the presence of the so-called *hidden regions* in the spectrum of inhomogeneous structures such as the comb graph, and, as we discuss in Sec. V, it is the cause of an anomalous behavior of the large distance two point correlation functions. These results are obtained by exactly evaluating the states filling on mesoscopic combs, where the energy spectrum is nearly continuous, but the levels are still distinguishable, being related to wave functions which differ on a large scale. The conditions restoring the standard feature of the BEC, namely, the macroscopic occupation of a single quantum state, are discussed as well.

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