Thermally assisted quantum vortex tunneling in the Hall and dissipative regime

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Quantum vortex tunneling is studied for the case where the Hall and the dissipative dynamics are simultaneously present. For a given temperature, the magnetization relaxation rate is calculated as a function of the external current and the quasiparticle scattering time. The relaxation rate is solved analytically at zero temperature and obtained numerically at finite temperatures by the variational method. In moderately clean samples, we have found that a minimum in the relaxation rate exists at zero temperature, which tends to disappear with an increase in the temperature.

I. INTRODUCTION

The depinning properties of vortices in high-temperature superconductors (HTSC's) have generated a good deal of interest over the past decade.[1],[2] Yeshurun and Malozemoff[3] reported on the existence of a giant flux creep which arises from the thermally activated motion of vortices from one metastable state to a neighboring one. The probability for such a process is proportional to $\exp(-U/k_B T)$, where U is the height of the energy barrier which depends on the pinning strength and the external current.[4] At an extremely low temperature the exponent diverges and the vortex can no longer move out of the pinning sites. Hence the dynamical magnetization relaxation rate Q, defined as $k_B T/U$, is expected to vanish at $T=0$. However, many experiments[5–10] demonstrated that the relaxation rate does not disappear at sufficiently low temperatures, which leads to the existence of quantum tunneling of vortices trapped in the pinning potential.

In general, quantum vortex creep is well described by the dynamics of two major forces: the Hall force and the dissipative force. Within the collective pinning theory, Blatter *et al.*[11] considered quantum vortex tunneling for the case where the dissipative term is dominant in the motion of vortices. On the other hand, Feigel'man *et al.*[12] proposed that Hall tunneling is dominant in clean superconductors by estimating the low-lying level spacing in the vortex core and the transport relaxation time of the charge carriers. Many experimental results have been interpreted within the two frameworks. Recently, however, van Dalen *et al.*[8] observed experimentally that the vortex tunneling in HTSC's may occur in an intermediate regime between the purely dissipative tunneling and the superclean Hall tunneling. Feigel'man *et al.*[12] and Morais Smith *et al.*[13] studied the problem in the two regimes, but they only obtained the qualitative results based on the scaling analysis of the action. The main difficulty of the problem is in the fact that there is the time nonlocality caused by the dissipative dynamics. Recently, the present authors[14] treated the problem quantitatively by using the variational method and presented the numerical results for the magnetic relaxation rate at zero temperature in the intermediate regime. Later, Melikidze[15] studied a similar problem by considering the quadratic Hamiltonian of the vortex coupled to the envi-

ronment. Through an analytic diagonalization, he obtained the dynamical magnetization relaxation rate at zero temperature as a function of the Hall and dissipative coefficients and found the minimum feature in the intermediate regime. Previous works treated the problem only at zero temperature, but we extend it to finite temperature in this work. Based on the instanton approach, we have obtained numerical results for the relaxation rate at finite temperatures and its analytic expression around the crossover temperature between thermal activation and quantum tunneling. Using the functional dependence of the relaxation rate on the Hall and dissipative coefficients at the crossover temperature, we have also obtained an analytic expression of the relaxation rate at zero temperature.

This paper is organized as follows. In Sec. II, we introduce the general formulation for the vortex tunneling rate in the presence of the Hall and the dissipative dynamics based on the instanton method, and discuss the Ohmic dissipation formulated by Caldeira and Leggett.[16] In Sec. III, we calculate the magnetic relaxation rate by taking into account the pinning potential barrier generated by impurities. Writing the action and the corresponding classical equations in Fourier space, we analytically calculate the relaxation rate as a function of the external current and the Hall and the dissipative coefficients at zero temperature. We also discuss the minimum of the relaxation rate in the intermediate regime. In Sec. IV, we numerically calculate the finite-temperature relaxation rate based on the variational method. It is found that the minimum in the relaxation rate tends to disappear with the increase in the temperature. We conclude in Sec. V.

II. BASIC FORMULATION

We consider the pancake vortex in the xy plane with length L_c along the z axis. L_c is the collective pinning length which can be expressed in terms of the mass anisotropy parameter $\varepsilon_a^2 = m/M < 1$, the coherence length ξ, the depairing current density j_0, and the critical current density j_c: $L_c \simeq \varepsilon_a \xi (j_0/j_c)^{1/2}$ within the weak collective pinning theory.[1] L_c is obtained by minimizing the energy density which includes the elastic energy of the vortex string, the energy gain from the random pinning potential and the contribution from the Lorentz force. Thus each segment of the length L_c of the

vortex is pinned by the collective action of all the defects within the collective pinning volume $V_c \simeq \xi^2 L_c$.

To study the quantum tunneling of the pancake vortex at a finite temperature, we consider the path integral representation of the partition function given by

$$Z(\beta\hbar) = \oint D[\mathbf{u}(\tau)]\exp(-S_E/\hbar), \qquad (1)$$

where $\beta = 1/k_B T$, and S_E is the Euclidean action. The path sum includes all the periodic paths $\mathbf{u}(\tau) = \mathbf{u}(\tau + \beta\hbar)$, where $\mathbf{u}$ is the displacement vector of the vortex in the xy plane. The Euclidean action S_E includes the Euclidean version of the Lagrangian L_E:

$$S_E[\mathbf{u}(\tau)] = \int_0^{\beta\hbar} d\tau L_E[\mathbf{u}(\tau)]. \qquad (2)$$

The tunneling rate Γ in the semiclassical limit, with an exponential accuracy, is given by[17]

$$\Gamma \propto \exp[-S_E^{\min}(T)/\hbar]. \qquad (3)$$

We study $S_E^{\min}(T)$ which gives the trajectory with the period $\beta\hbar$ that minimizes the Euclidean action.[18] Considering the situation where the inertia term is not relevant and the vortex dynamics is dominated by the Hall and the dissipative forces, we write the Euclidean action as

$$S_E = \int_0^{\beta\hbar} d\tau \left\{ L_c \left[-i\alpha \frac{du_x}{d\tau} u_y + V(u_x, u_y) \right] \right.$$
$$+ \sum_k \left[\frac{1}{2} m_k (\dot{x}_k^2 + \dot{y}_k^2) + \frac{1}{2} m_k \omega_k^2 \left(\left(x_k - \frac{C_k}{m_k \omega_k^2} u_x \right)^2 \right.\right.$$
$$\left.\left.\left. + \left(y_k - \frac{C_k}{m_k \omega_k^2} u_y \right)^2 \right) \right] \right\}, \qquad (4)$$

where α is the Hall coefficient and $V(u_x, u_y)$ is the pinning potential per unit length which includes the contribution from the Lorentz force. The last term of Eq. (4) represents the dissipative environment of the vortex consisting of a set of harmonic oscillators as formulated by Caldeira and Leggett.[16] The effect of the dissipative environment is characterized by the spectral function

$$J(\omega) = \frac{\pi}{2} \sum_k \frac{C_k^2}{m_k \omega_k} \delta(\omega - \omega_k). \qquad (5)$$

With the oscillators integrated out, the Euclidean action takes the form

$$S_E = \int_0^{\beta\hbar} d\tau \left\{ L_c \left[-i\alpha \frac{du_x}{d\tau} u_y + V(u_x, u_y) \right] + \frac{1}{2} \int_{-\infty}^{\infty} d\tau' \right.$$
$$\left. \times K_0(\tau - \tau')[(u_x(\tau) - u_x(\tau'))^2 + (u_y(\tau) - u_y(\tau'))^2] \right\}, \qquad (6)$$

where the nonlocal influence function is expressed as

$$K_0(\tau) = \frac{1}{2\pi} \int_0^{\infty} d\omega J(\omega)\exp(-\omega|\tau|). \qquad (7)$$

III. QUANTUM TUNNELING OF A VORTEX

In order to study the motion of a vortex, we need to first analyze the structure of the model potential $V(u_x, u_y)$. Since the external current j along the y direction brings the system into a metastable state by tilting the potential, the vortex has a chance to move out of the pinning potential. Let us define u_{x_i} as the critical position of the vortex at which the barrier vanishes at the critical current j_c. In the limit $j \rightarrow j_c$, u_{x_i} and j_c satisfy

$$\left[\frac{\partial V}{\partial u_x} \right]_{u_x = u_{x_i}} = \left[\frac{\partial^2 V}{\partial u_x^2} \right]_{u_x = u_{x_i}} = 0. \qquad (8)$$

With $V(u_x, u_y) = V_p(u_x, u_y) - \phi_0 j u_x/c$, u_{x_i} and j_c are given by the relations $(\partial V_p/\partial u_x)_{u_x = u_{x_i}} = \phi_0 j_c/c$ and $(\partial^2 V_p/\partial u_x^2)_{u_x = u_{x_i}} = 0$, where $\phi_0 = hc/2e$ is the flux quantum. For the pinning potential, we choose an appropriate model potential describing a typical tunneling situation: $V_p(u_x, u_y)$ should exhibit a local minimum and should be connected via a saddle to the free space along one direction (we choose this direction as the x axis.). A model potential satisfying the requirement is

$$V(u_x, u_y) \simeq \frac{1}{2} V_0 \left[c_1 \epsilon \left(\frac{u_x}{R} \right)^2 - \frac{2}{3} c_2 \left(\frac{u_x}{R} \right)^3 + \left(\frac{u_y}{R} \right)^2 \right], \quad (9)$$

where $c_1 = R^2 \sqrt{2\phi_0 j_c/(cV_0^2)} |(\partial^3 V/\partial u_x^3)_{u_x = u_{x_i}}|^{1/2}$, $c_2 = R^3 |(\partial^3 V/\partial u_x^3)_{u_x = u_{x_i}}|/(2V_0)$, and $\epsilon = \sqrt{1 - j/j_c} \ll 1$. In Eq. (9), V_0 and R are the height and range of the pinning potential, respectively, and for a typical weak pinning potential $V_0 \simeq (\phi_0/4\pi\lambda_{xy})^2$ and $R(\geq \xi)$. $c_{1,2}$ are the dimensionless coefficients of the order of 1 and λ_{xy} is the bulk-planar penetration depth.

To consider the tunneling of a vortex in the two regimes, we investigate the behavior of the Euclidean action [Eq. (6)]. In order to estimate the order of magnitude of each term in the action and to simplify the calculation for $\epsilon \ll 1$, we introduce the dimensionless variables

$$\bar{u}_x = \left(\frac{2c_2}{c_1 \epsilon R} \right) u_x, \quad \bar{u}_y = \left(\frac{2c_2}{c_1^{3/2} \epsilon^{3/2} R} \right) u_y, \quad \bar{\tau} = \left(\frac{\sqrt{c_1} \epsilon V_0}{\sqrt{2} R^2 \alpha_0} \right) \tau, \qquad (10)$$

where $\alpha_0 = \pi\hbar n_s$ and n_s is the number density of the electrons in the condensate.

Assuming the Ohmic dissipation where the frictional force acting on the vortex is linear to the vortex velocity,[1,19,20] the spectral density becomes $J(\omega) = \eta\omega$, where $\eta = (\pi/2)\Sigma_i(C_i^2/m_i\omega_i^2)\delta(\omega - \omega_i) = \text{const.}$[16] With this choice, we have the influence function

$$K_0(\tau) = \frac{\eta}{2\pi|\tau|^2}, \qquad (11)$$

which leads to the Euclidean action

$$S_E = \left(\frac{\sqrt{2}c_1^{5/2}}{4c_2^2}\right)(L_c\alpha_0 R^2)\epsilon^{5/2}I_{HD}, \qquad (12)$$

where

$$I_{HD} = \int_0^\Lambda d\bar{\tau} \left\{ -i\alpha_1\frac{d\bar{u}_x}{d\bar{\tau}}\bar{u}_y + \frac{1}{2}\bar{u}_y^2 + \frac{1}{2}\bar{u}_x^2 - \frac{1}{6}\bar{u}_x^3 + \frac{1}{4}\eta_1\int_{-\infty}^\infty d\bar{\tau}_1 \right.$$
$$\left. \times \frac{[\bar{u}_x(\bar{\tau}) - \bar{u}_x(\bar{\tau}_1)]^2 + c_1\epsilon[\bar{u}_y(\bar{\tau}) - \bar{u}_y(\bar{\tau}_1)]^2}{|\bar{\tau} - \bar{\tau}_1|^2} \right\}, \qquad (13)$$

where $\Lambda = \beta\hbar V_0\sqrt{\epsilon c_1}/(\sqrt{2}\alpha_0 R^2)$. The dimensionless Hall $[\equiv \alpha/(\sqrt{2}\alpha_0)]$ and dissipation coefficients $[\equiv \eta/(L_c\sqrt{2}\pi^2 c_1\epsilon\alpha_0)]$ are given by[20]

$$\alpha_1 = \frac{1}{\sqrt{2}}\frac{(\omega_0\tau_r)^2}{1+(\omega_0\tau_r)^2}, \qquad (14)$$

$$\eta_1 = \frac{\sqrt{2}}{2\pi\sqrt{c_1\epsilon}}\frac{\omega_0\tau_r}{1+(\omega_0\tau_r)^2}. \qquad (15)$$

Here ω_0 is the level spacing of the quasiparticle bound states inside the vortex core and $\tau_r(=m/ne^2\rho_n)$ is a quasiparticle scattering time, where n is the number density of the charge carriers and m and ρ_n are their effective mass and resistivity, respectively. As can be seen in Eqs. (14) and (15), the Hall coefficient α is reduced from its pure value $\pi\hbar n_s$ due to the dissipative effect. Although Ao *et al.* suggested that the Hall coefficient originates from the topological property and is thus not renomalized,[21] it seems that at least some aspects of the experimental behavior[8] can be understood on the basis of the renomalization of the Hall coefficient. Therefore, it is meaningful to take α and η to be two parameters determined by the magnitude of $\omega_0\tau_r$.

A. Action in the Fourier space

When the Hall and the dissipative dynamics are simultaneously present, the classical trajectories of $\bar{u}_x$ and $\bar{u}_y$ satisfy

$$i\alpha_1\frac{d\bar{u}_y}{d\bar{\tau}} + \bar{u}_x - \frac{\bar{u}_x^2}{2} - \eta_1\int_{-\infty}^\infty d\bar{\tau}_1\left(\frac{d\bar{u}_x}{d\bar{\tau}_1}\right)\frac{1}{\bar{\tau}_1 - \bar{\tau}} = 0, \quad (16)$$

$$-i\alpha_1\frac{d\bar{u}_x}{d\bar{\tau}} + \bar{u}_y - \eta_1 c_1\epsilon\int_{-\infty}^\infty d\bar{\tau}_1\left(\frac{d\bar{u}_y}{d\bar{\tau}_1}\right)\frac{1}{\bar{\tau}_1 - \bar{\tau}} = 0. \quad (17)$$

The substitution $\bar{\tau} \to -\bar{\tau}$ in Eqs. (16) and (17) shows the invariance of the equations by taking $\bar{u}_x(-\bar{\tau}) = \bar{u}_x(\bar{\tau})$ and $\bar{u}_y(-\bar{\tau}) = -\bar{u}_y(\bar{\tau})$. We will keep c_1 in the ensuing equations, although we will take $c_1 = 1$ for the numerical calculations. Denoting $\bar{\mathbf{u}}(\bar{\tau}) \equiv [\bar{u}_x(\bar{\tau}), \bar{u}_y(\bar{\tau})]$, we have $\bar{\mathbf{u}}(\bar{\tau}+\Lambda) = \bar{\mathbf{u}}(\bar{\tau})$ at

a finite temperature. A simple analysis shows that $\bar{u}_x(\bar{\tau})$ is real and $\bar{u}_y(\bar{\tau})$ pure imaginary, so they can be expanded into the Fourier series

$$\bar{u}_x(\bar{\tau}) = \sum_{n=-\infty}^\infty u_n \exp(i\bar{\omega}_n\bar{\tau}), \qquad (18)$$

$$\bar{u}_y(\bar{\tau}) = -i\sum_{n=-\infty}^\infty v_n \exp(i\bar{\omega}_n\bar{\tau}), \qquad (19)$$

where $\bar{\omega}_n = 2\pi n/\Lambda$ ($n = 0, 1, 2 \dots$). Substituting these into Eqs. (16) and (17), we have

$$\left(1 + \pi\eta_1|\bar{\omega}_n| + \frac{\alpha_1^2\bar{\omega}_n^2}{1+\pi\epsilon\eta_1 c_1|\bar{\omega}_n|}\right)u_n = \frac{1}{2}\sum_{m=-\infty}^\infty u_{n+m}u_m, \qquad (20)$$

$$v_n = -\frac{i\alpha_1\bar{\omega}_n}{1+\pi\epsilon\eta_1 c_1|\bar{\omega}_n|}u_n, \qquad (21)$$

where $u_{-n} = u_n$ and $v_{-n} = -v_n$. Although Eq. (20) is a one-dimensional problem with respect to u_n, its solution becomes complicated by the presence of the nonlocal term arising from the cubic potential. For general $\omega_0\tau_r$ and ϵ, we have numerically solved Eqs. (20) and (21) via the variational method. The trial function for the variational method has been taken by a combination of the analytic solutions in the two extreme limits as follows. At zero temperature, $\bar{u}_x(\bar{\omega})$'s are of the form $\bar{\omega}/\sinh(\alpha_1\pi\bar{\omega})$ in the Hall limit ($\omega_0\tau_r \to \infty$) and $\exp(-\pi\eta_1|\bar{\omega}|)$ in the dissipative limit ($\omega_0\tau_r \to 0$),[14] so a natural choice for the trial function at finite temperatures is

$$u_n = \frac{p_1\bar{\omega}_n}{\sinh(p_2\bar{\omega}_n)} + p_3\exp(-p_4|\bar{\omega}_n|), \qquad (22)$$

where p_i's ($i = 1,2,3,4$) are free parameters to be determined by the variational method. It turns out that the numerical variational method with the trial function works very successfully.

Using the Fourier series in Eqs. (18) and (19), we write I_{HD} as

$$I_{HD} = \Lambda\sum_{n=-\infty}^\infty \left[-\alpha_1\bar{\omega}_n u_n v_n + \frac{1}{2}u_n^2 - \frac{1}{2}v_n^2 \right.$$
$$\left. -\frac{1}{6}u_n\left(\sum_{m=-\infty}^\infty u_{n+m}u_m\right) + \frac{\pi}{2}\eta_1|\bar{\omega}_n|(u_n^2 - c_1\epsilon v_n^2) \right], \qquad (23)$$

which further reduces to

$$I_{HD} = \frac{1}{6}\Lambda\sum_{n=-\infty}^\infty \left(1 + \pi\eta_1|\bar{\omega}_n| + \frac{\alpha_1^2\bar{\omega}_n^2}{1+\pi\epsilon\eta_1 c_1|\bar{\omega}_n|}\right)u_n^2. \qquad (24)$$
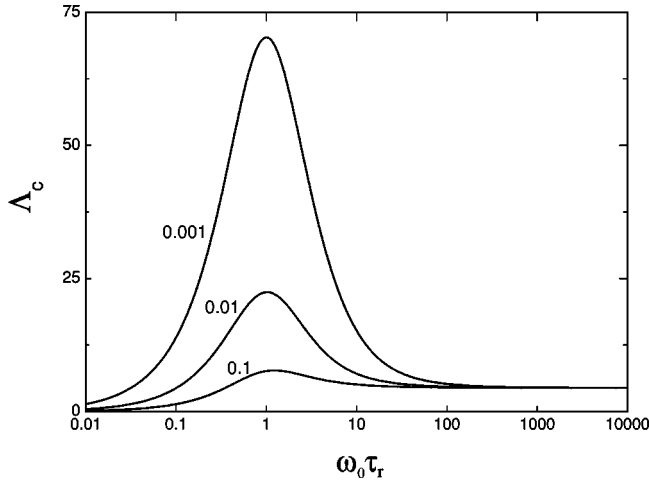
FIG. 1. Λ_c vs $\omega_0\tau_r$ where $c_1=1$. Note that the crossover temperatures become independent of ϵ in the Hall regime ($\omega_0\tau_r\to\infty$).

B. Quantum relaxation near the crossover temperature and at zero temperature

At T_c, the crossover temperature between thermal activation and quantum tunneling, the classical trajectories become independent of $\bar{\tau}$, i.e., $\bar{u}_x(\bar{\tau})=2$ and $\bar{u}_y(\bar{\tau})=0$. When $\Lambda(T)$ is slightly greater than $\Lambda_c[\equiv\Lambda(T_c)]$, we take only the first Fourier harmonics for the solution because the next harmonics are smaller near T_c:

$$\bar{u}_x(\bar{\tau})=u_0+2u_1\cos\left(\frac{2\pi}{\Lambda}\bar{\tau}\right), \qquad (25)$$

$$\bar{u}_y(\bar{\tau})=2v_1\sin\left(\frac{2\pi}{\Lambda}\bar{\tau}\right). \qquad (26)$$

Exploiting the fact that u_n's are zero except for u_0 and u_1 in Eq. (20), we obtain

$$u_0=1+\frac{2\pi^2\eta_1}{\Lambda}+\frac{4\pi^2\alpha_1^2}{\Lambda(\Lambda+2\pi^2c_1\epsilon\eta_1)}, \qquad (27)$$

$$u_1^2=u_0-\frac{1}{2}u_0^2. \qquad (28)$$

Setting $u_0=2$ in Eq. (27) and solving for $\Lambda=\Lambda_c$, we have

$$\Lambda_c=\frac{4\pi(\alpha_1^2+\pi^2\eta_1^2c_1\epsilon)}{\sqrt{\pi^2\eta_1^2(1+c_1\epsilon)^2+4\alpha_1^2}-\pi\eta_1(1-c_1\epsilon)}. \qquad (29)$$

Using the relations in Eqs. (14) and (15), we plot Λ_c against $\omega_0\tau_r$ for different values of ϵ in Fig. 1. The maximal values of the crossover periods are more pronounced in the limit of smaller ϵ, and Λ_c's converge to $2\pi\alpha_1(=\sqrt{2}\pi)$ in the Hall regime and to $2\pi^2\eta_1(=\sqrt{2}\pi\omega_0\tau_r/\sqrt{\epsilon})$ in the dissipative regime. The reduced action integration near the crossover temperature can also be simply obtained by summing only $n=0$ and $n=1$ contributions
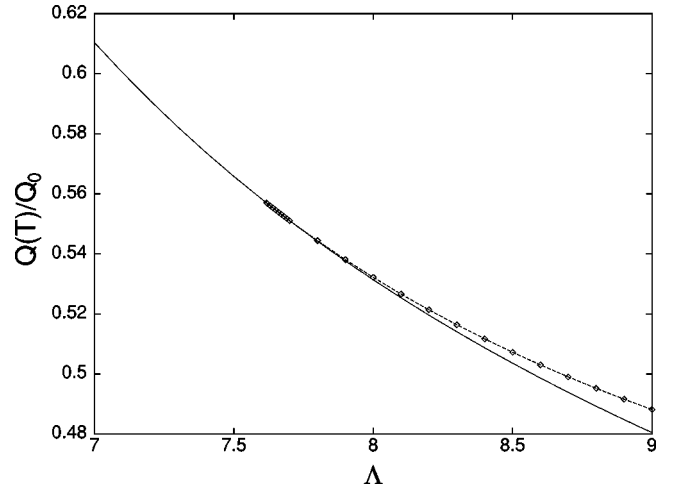

FIG. 2. The relaxation rate $Q(T)/Q_0$ vs Λ near the crossover temperature for $\epsilon=0.1$ and $\omega_0\tau_r=1$, where $\Lambda_c\sim7.6$. The solid line represents the analytical curve from Eq. (31), and the dotted line with diamonds indicates the result of the numerical calculation.

$$I_{HD}=\frac{1}{6}\Lambda\left[u_0^2+2\left(1+\frac{2\pi^2\eta_1}{\Lambda}+\frac{4\pi^2\alpha_1^2}{\Lambda(\Lambda+2\pi^2c_1\epsilon\eta_1)}\right)u_1^2\right], \qquad (30)$$

which is reduced to

$$I_{HD}=\frac{1}{6}\Lambda u_0^2(3-u_0) \qquad (31)$$

by using Eqs. (27) and (28).

The action integration I_{HD} obtained by the numerical variational method around T_c and the one by Eq. (31) are compared in Fig. 2. The two curves in the figure perfectly join at the crossover period, which implies that our numerical method gives the correct solution. The dynamical magnetization relaxation rate Q is given by $Q=\hbar/S_E$. In real experiments Q is extracted from the magnetization $M(t)=M_0[1-Q\ln(t/t_0)]$.[3] From Eq. (12), we have $Q(T)/Q_0=2\sqrt{2}/I_{HD}$, where $Q_0=(\pi n_s L_c R^2\epsilon^{5/2})^{-1}$ by taking $c_1=c_2=1$. Then, at the crossover temperature T_c, since $I_{HD}=2\Lambda_c/3$, we have

$$\frac{Q(T_c)}{Q_0}=\frac{3\sqrt{2}}{\Lambda_c}. \qquad (32)$$

In Fig. 3, $Q(T_c)/Q_0$ and $Q(0)/Q_0$ are plotted with respect to $\omega_0\tau_r$. It is interesting that the shape of $Q(T_c)/Q_0$ for each ϵ is close to that of $Q(0)/Q_0$ at zero temperature.[14] This fact can be understood by considering the following features of the relaxation rate. Since the tunneling rate Γ is expected to be almost temperature independent for $T\lesssim T_c$ and $\Gamma\sim\exp(-U/k_BT)$ for $T\gtrsim T_c$, the crossover temperature is approximately given by the relationship $U/(k_BT_c)\simeq S_E(T=0)/\hbar$. And $Q(0)=\hbar/S_E(T=0)\simeq(k_BT_c)/U=Q(T_c)$. Hence we obtain the analytic expression for the relaxation rate at zero temperature using Eqs. (29) and (32),
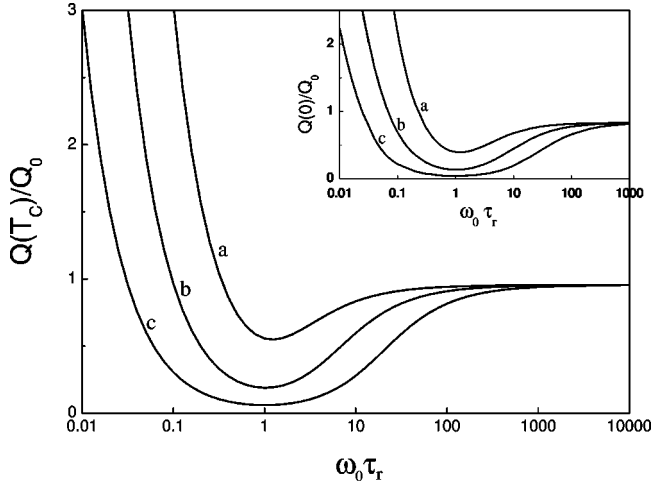
FIG. 3. The relaxation rate evaluated at the crossover temperature: $Q(T_c)/Q_0$ vs $\omega_0\tau_r$ where $\epsilon = 0.1$ (a), 0.01 (b), and 0.001 (c). Inset: the relaxation rate $Q(0)/Q_0$ at zero temperature with $\epsilon = 0.1$ (a), 0.01 (b), and 0.001 (c).

$$\frac{Q(0)}{Q_0} \simeq \frac{3}{2\pi} \frac{\sqrt{(1+\epsilon)^2 + 4(\omega_0\tau_r)^2\epsilon} - (1-\epsilon)}{(\omega_0\tau_r)\sqrt{\epsilon}}, \quad (33)$$

by taking $c_1 = 1$ and using $\alpha_1/\eta_1 = \omega_0\tau_r\pi\sqrt{\epsilon}$. Equation (33) agrees with the result of Ref. 15 up to a numerical factor. Since the analytic form for $Q(0)/Q_0$ is known to be 5/6 in the limit of $\omega_0\tau_r \to \infty$,[14] the correct prefactor of Eq. (33) is 5/12 instead of $3/(2\pi)$. Although $Q(0)/Q_0$ goes to infinity as $\omega_0\tau_r \to 0$ in Eq. (33), it actually does not diverge in that limit, because, by including the inertia term not considered in this work, the approximate form of the classical action in the limit becomes $S_E/\hbar \sim L_c\sqrt{m_vV_0}\xi\epsilon^{5/2}$, which is independent of $\omega_0\tau_r$, where m_v is the inertia mass of a vortex.[1,22,23] In general, the mass term is relatively small in the Hall and dissipative regime and can usually be neglected.

In a moderately clean regime, for small values of ϵ each curve in Fig. 3 has a minimum around $\omega_0\tau_r = 1$, which is interested. In fact, from Eq. (33) we can see that the position of the minimum at zero temperature is $\omega_0\tau_r = (1+\epsilon)/(1-\epsilon)$. As ϵ becomes smaller, i.e., as $j \to j_c$, the minimum becomes much more pronounced with its location moving toward $\omega_0\tau_r = 1$ at the same time. The existence of such minima can be understood by considering the following qualitative features of the relaxation rates in the two regimes. Since $\bar{u}_x(\omega)$ is proportional to $\alpha_1\omega/\sinh(\alpha_1\pi\omega)$ in the Hall limit,[14] the classical trajectories with $|\omega| \lesssim 1/\alpha_1$ contribute to the Euclidean action mostly. From Eqs. (12) and (24) the correction to the Hall action $S_E^{(H)}$ by the small dissipation is given by $(1+\eta_1/\alpha_1)S_E^{(H)} \sim (1+1/\omega_0\tau_r)S_E^{(H)}$, which leads to the relaxation rate given by $Q(0) \sim \omega_0\tau_r/(1+\omega_0\tau_r)$. So the relaxation rate Q decreases with a decrease in $\omega_0\tau_r$ from ∞, which is also physically clear because the classical action increases by inclusion of the dissipation. In the opposite limit the correction to the purely dissipative action $S_E^{(D)}$ by the small Hall contribution is $[1+(\alpha_1/\eta_1)^2]S_E^{(D)} \sim \omega_0\tau_r[1+(\omega_0\tau_r)^2]$, leading to the relaxation rate $Q(0) \sim 1/[\omega_0\tau_r(1$
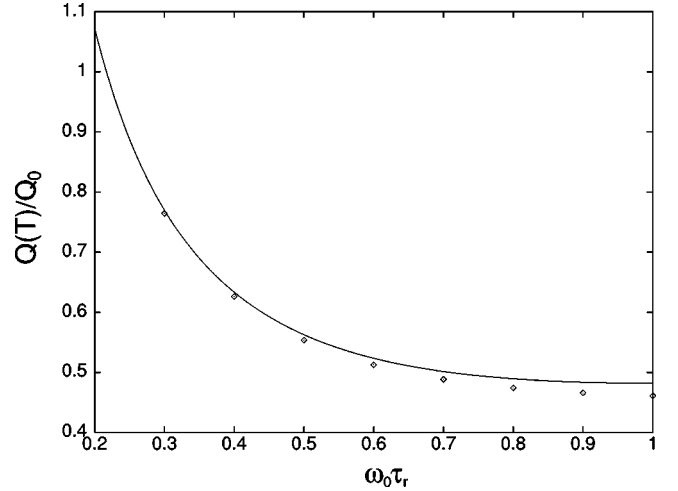

FIG. 4. The relaxation rate in the dissipative limit when $\epsilon = 0.1$ and $\Lambda = 10$. The solid line represents the evaluation of Eq. (35), and the diamonds are the numerical results from the variational method.

$+(\omega_0\tau_r)^2)]$. In this case, $Q(0)$ decreases with increase in $\omega_0\tau_r$. Therefore, a minimum in $Q(0)$ should exist in the intermediate regime, which suggests the existence of the strong pinning in the moderately clean samples.

C. Quantum relaxation in the dissipation regime

In the dissipative limit, we take $\alpha_1 \to 0$ in the action integration of Eq. (24). The reduced action then becomes

$$I_{HD} = \frac{1}{6}\Lambda \sum_{n=-\infty}^{\infty} (1 + \pi\eta_1|\bar{\omega}_n|)u_n^2 \equiv I_D. \quad (34)$$

Noting that u_n in the dissipative limit is given by $u_n = u_0\exp(-b|n|)$ where $u_0 = 4\pi^2\eta_1/\Lambda$ and $b = \tanh^{-1}(2\pi^2\eta_1/\Lambda)$,[24] we obtain the reduced action given by

$$I_D = \Lambda_c\left[1 - \frac{1}{3}\left(\frac{T}{T_c}\right)^2\right], \quad (35)$$

where $\Lambda_c = 2\pi^2\eta_1$ and $k_BT_c = \hbar\sqrt{c_1\epsilon}V_0/(2\sqrt{2}\pi^2R^2\alpha_0\eta_1)$. In Fig. 4, the relaxation rate using Eq. (35) is compared with the one obtained from the numerical solution: the two curves match quite well asymptotically in the region of small $\omega_0\tau_r$ values.

D. Quantum relaxation in the Hall regime

In the Hall limit we take $\eta_1 \to 0$ in Eq. (24), which leads to the reduced action given by

$$I_{HD} = \frac{1}{6}\Lambda \sum_{n=-\infty}^{\infty} (1 + \alpha_1^2\bar{\omega}_n^2)u_n^2 \equiv I_H, \quad (36)$$

where u_n satisfies

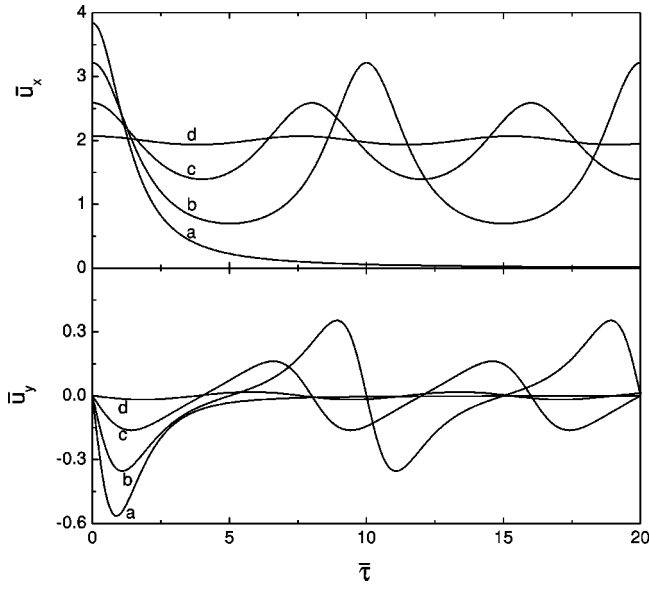$$(1 + \alpha_1^2\bar{\omega}_n^2)u_n = \frac{1}{2}\sum_{n=-\infty}^{\infty} u_{n+m}u_m. \quad (37)$$

FIG. 5. Typical instanton solutions with different periods: $\bar{u}_x(\bar{\tau})$ (top) and $-i\bar{u}_y(\bar{\tau})$ (bottom) for $\epsilon = 0.1$ and $\omega_0\tau_r = 1.0$, where the periods are ∞ (a), 10 (b), 8 (c), and 7.617 (d).

While the instanton solution can be obtained analytically in the dissipative regime, the solution of Eq. (37) can be found numerically. We use Eqs. (16) and (17) rather than Eq. (37) and obtain the reduced differential equation

$$2\alpha_1^2 \frac{d^2\bar{u}_x}{d\bar{\tau}^2} - 2\bar{u}_x + \bar{u}_x^2 = 0. \qquad (38)$$

We then integrate for I_{HD} with $\eta_1 = 0$ in Eq. (13) using the solution for $\bar{u}_x$ in Eq. (38), and obtain the reduced action I_H. As in the case of the dissipative regime, we have found that the relaxation rate agrees with the one obtained from the variational procedure in the limit of $\omega_0\tau_r \rightarrow \infty$.

IV. DISCUSSION AT FINITE TEMPERATURE

We now consider the problem at the finite temperature in the whole regime, i.e., when the Hall and dissipative dynamics are simultaneously present. The solutions for $\bar{u}_x(\tau)$ and $\bar{u}_y(\tau)$ in Eqs. (18) and (19) are obtained through u_n's and v_n's of Eqs. (20) and (21) which are numerically obtained by the variational method. In Fig. 5, we show $\bar{u}_x(\bar{\tau})$ and $\bar{u}_y(\bar{\tau})$ for various periods Λ which exhibit the typical trend of the classical trajectories as the period is successively shortened. The peak-to-valley amplitudes of $\bar{u}_x(\bar{\tau})$ and $\bar{u}_y(\bar{\tau})$ decrease as the period becomes shorter, and eventually become flat at T_c, i.e., $\bar{u}_x(\bar{\tau}) = 2$ and $\bar{u}_y(\bar{\tau}) = 0$. We subsequently calculate the reduced action [Eq. (13)] via Eq. (24) and the corresponding relaxation rate. The three-dimensional plot of $Q(T)/Q_0$ versus $\omega_0\tau_r$ and Λ for $\epsilon = 0.01$ is shown in Fig. 6. We have also plotted $Q(T)/Q_0$ against Λ for the different values of $\omega_0\tau_r$ in Fig. 7. As can be seen in the figure, $\omega_0\tau_r = 1$ is the boundary of the two different behaviors of the
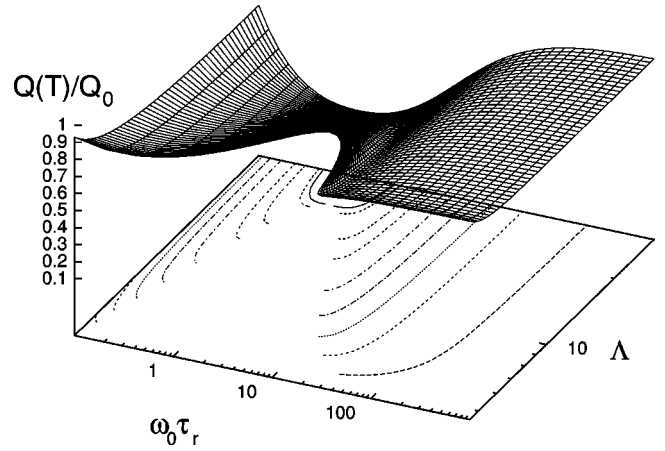

FIG. 6. The relaxation rate $Q(T)/Q_0$ for $\epsilon = 0.01$ against $\omega_0\tau_r$ and Λ. In order to show the curve for $Q(T_c)/Q_0$, we have omitted the purely thermal relaxation rate. See Fig. 8 for details.

relaxation rate: for $\omega_0\tau_r < 1$, the relaxation rate increases with decreasing $\omega_0\tau_r$, whereas for $\omega_0\tau_r > 1$ it increases with increasing $\omega_0\tau_r$. The dependence of $Q(T)/Q_0$ on $\omega_0\tau_r$ is shown in Fig. 8 for the different values of the period.

What is interesting is the behavior of the relaxation rate in the intermediate region of $\omega_0\tau_r$. We focus our attention on the four different temperature regimes, as indicated in Fig. 8(b). If the temperature is sufficiently low so that $\Lambda > \Lambda_c^{(m)}$, the line yields no intersection points, and there exists a quantum relaxation in the whole regimes of $\omega_0\tau_r$. If the temperature is sufficiently high so that $\Lambda < \Lambda_c^{(H)}$, quantum relaxation occurs only in the dissipative regime [$\omega_0\tau_r < (\omega_0\tau_r)_D$]. In the temperature range $\Lambda_c^{(H)} < \Lambda < \Lambda_c^{(m)}$, on the other hand, quantum relaxation exists either in the Hall regime or in the dissipative regime, and purely thermal relaxation occurs in the crossover region between the two regimes. The values for $\Lambda_c^{(m)}$, $\Lambda_c^{(H)}$, and $(\omega_0\tau_r)_D$ can readily be computed from the position of the minimum and using Eq. (29) with $c_1 = 1$:
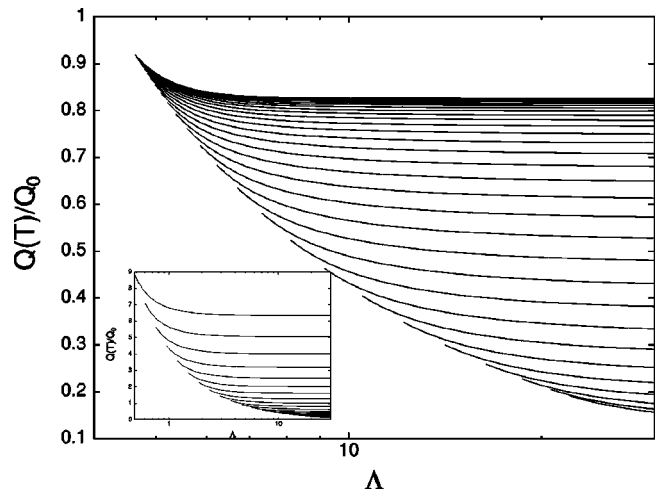

FIG. 7. The relaxation rate $Q(T)/Q_0$ vs Λ for the different values of $\omega_0\tau_r$ when $\epsilon = 0.01$. $\omega_0\tau_r$ increases from the bottom ($\omega_0\tau_r = 1$) and approaches ∞. Inset: the case for $\omega_0\tau_r \lesssim 1$. $\omega_0\tau_r$ decreases from the bottom.
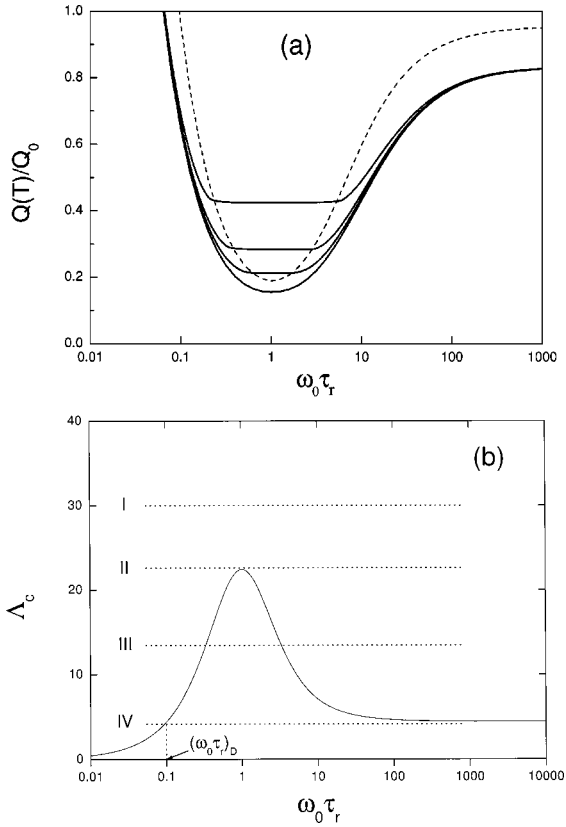
FIG. 8. (a) $Q(T)/Q_0$ vs $\omega_0\tau_r$ for different periods when ϵ $= 0.01$. The periods are 30, 20, 15, and 10 (from the bottom). The curve with the period of 30 is already very close to that of an infinite period, which corresponds to zero temperature. The dotted curve is $Q(T_c)/Q_0$ of Fig. 3. In the region above the dotted curve, purely thermal relaxations exist along the horizontal lines. (b) A schematic diagram of Λ_c vs $\omega_0\tau_r$ with the lines of constant temperatures. Four cases are considered : (I) $\Lambda > \Lambda_c^{(m)}$, (II) $\Lambda = \Lambda_c^{(m)}$, (III) $\Lambda_c^{(H)} < \Lambda < \Lambda_c^{(m)}$, and (IV) $\Lambda \lesssim \Lambda_c^{(H)}$. Note that $\Lambda_c^{(m)} = 22.4$, $Q(T_c)/Q_0 = 0.189$, and $(\omega_0\tau_r)_D = 0.101$.

$\Lambda_c^{(m)} = \pi(1+\epsilon)/\sqrt{2\epsilon}$, $\Lambda_c^{(H)} = \sqrt{2}\pi$, and $(\omega_0\tau_r)_D = \sqrt{\epsilon}/(1-\epsilon)$. The corresponding relaxation rates are given by $Q(\Lambda_c^{(m)})/Q_0 = 6\sqrt{\epsilon}/[\pi(1+\epsilon)]$ and $Q(\Lambda_c^{(H)})/Q_0 = 3/\pi$. The minimum of $Q(T)/Q_0$ in the intermediate regime is then

noted. As the temperature becomes lower, the quantum relaxation rate is more developed in the intermediate regime, and at an extremely low temperature it has a minimum at $\omega_0\tau_r \sim 1$. This feature is more pronounced for smaller ϵ and larger Λ. Correspondingly, in such a regime the quantum depinning of a vortex is expected to be smaller at lower temperatures in the regime.

Before concluding, we illustrate our results with specific numbers. In the experiment of Ref. 8, the relaxation rate is $Q(0)/Q_0 \sim 2.3$ (2.0) in the $YBa_2Cu_3O_7(BiSr_2CaCu_2O_8)$ system. In this case, $\omega_0\tau_r \sim 0.29$ (0.37) for YBCO (BiS-CCO) which corresponds to the Hall angle $\Theta_H = \arctan(\omega_0\tau_r) \sim 16°$ (20°), which depends on the oxygen content. The numbers imply that the samples are moderately clean. However, the regime which was considered in Ref. 8 was $\omega_0\tau_r \lesssim 1$, where the onset of the minimum just takes place. In order to observe the minima, the experiment should be extended to the region $\omega_0\tau_r \gg 1$.

V. CONCLUSIONS

In conclusion, we have considered the quantum tunneling of a vortex in the presence of the Hall and dissipative dynamics. We have derived an analytic expression for the relaxation rate at zero temperature and obtained numerical solutions by a variational method at finite temperatures. The relaxation rate is constant in the Hall limit and proportional to $1/(\omega_0\tau_r)$ in the dissipative limit, and, consequently, a minimum exists at $\omega_0\tau_r = 2(j_c/j)(1+\sqrt{1-j/j_c}) - 1$. Therefore, the strongest pinning is expected in the moderately clean sample at zero temperature. At finite temperatures, the quantum relaxation rate tends to vanish in the intermediate regime where both the Hall and dissipative terms contribute to the dynamics of a vortex. At sufficiently low temperatures, quantum vortex tunneling occurs in the whole regime and the corresponding relaxation rate has a minimum at $\omega_0\tau_r \sim 1$. These features are expected to be observed in future experiments.

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