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Monte Carlo renormalization group study of the dynamic scaling of hysteresis in the two-dimensional Ising model

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Extending the Monte Carlo renormalization group technique to the two-dimensional Ising model, we find from first principles that the dynamic scaling behavior of hysteresis originates from a rate exponent that characterizes the response of the system to the sweep rate of the field that leads to the hysteresis. The static hysteresis is determined and a scaling law attained. The effect of the temperature is discussed.

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Hysteresis is a ubiquitous phenomenon in nature.[1–4] Magnetic hysteresis, ferroelectric hysteresis, superconducting hysteresis, adsorption hysteresis, and optical hysteresis are just a few common examples of it. Glass transition essentially results from it; and a characteristic feature of first-order phase transitions goes to it. In spite of such theoretical and technological significances, its nonlinear and nonequilibrium nature renders itself elusive to most scientific treatments. The past decade has been seeing, however, increasing theoretical[5–18] and experimental[19–25] activities focusing on a new perspective upon its dynamic scaling behavior. Yet, most results are essentially empirical, lacking a firm theoretical ground. Here, applying a Monte Carlo (MC) renormalization group (RG) technique to a two-dimensional (2D) Ising model, we find from first principles that the dynamic scaling of hysteresis originates from a new exponent that characterizes the transformation property of the sweep rate. This scaling behavior contrasts with that near the equilibrium critical point in that the system's character shows up from the behavior of its outside environment. Our results should thus provide convincing evidence for the scaling of hysteresis, and for the applicability of the RG theory to such first-order-like dynamic phenomena. They should also shed light to the study of hysteresis and its related phenomena and other far-from-equilibrium systems that involve an external driving.

Consider the Ising model with a Hamiltonian

$$\mathcal{H}=-\frac{J}{k_BT}\sum_{\langle i,j\rangle}S_iS_j-\frac{H}{k_BT}\sum_i S_i\,, \qquad (1)$$

where the classical spin $S_i=\pm1$, the first summation is over all different nearest-neighbor pairs $\langle i,j\rangle$ and the second is over all sites i of a 2D lattice of length L. For simplicity, we shall set the Boltzmann's constant k_B and the coupling constant J to one. Periodic boundary conditions are applied throughout. We start with all spins pointing down ($S_i=-1$), and let the magnetic field $H=Rt$ with a sweep rate R and the time t, which is measured in units of MC steps per spin. Each step includes L^2 random attempts to flip the spins using the standard Metropolis algorithm.[26] The system then evolves with time to the "up" state (all $S_i=1$) with a hysteresis depending on R. It has been found previously that when the field sweeps cycles by a sinusoidal[5–12,18] or saw-tooth[13–17] manner, the area A enclosed in the magnetization-field loop may well be fitted to

$$A=A_0+A_1R^{\alpha} \qquad (2)$$

over a large order of magnitude of R, where A_0 and A_1 are constants independent of R, and the area exponent α depends only slightly on temperature below the critical temperature T_C. A_0 is the static hysteresis[8,12,14,15,17,21] that is important to set the area exponent right and sometimes is controversial.[12,17] A previous momentum-space RG investigation has clearly shown analytically that the scaling is determined by the zero-temperature fixed point, and the underlying invariance is probably the dynamical similarity associating with different rates of the driving.[16] But that is on a somewhat unrealistic $O(N)$ vector model with an infinite number of vector components N, i.e., the large-N model, which involves transverse Goldstone modes that can circumvent the energy barrier between the up and the down states. The Ising model, on the other hand, has been found to belong to a different universality class,[15] and has been used as a primary model to fit the experimental results from magnetic thin films.[17,19,23–25] So, whether the RG theory is applicable to this and other more realistic classes of models is still an important issue in pursuing the scaling of hysteresis. In this Communication, we shall make an attempt along this line by extending the MCRG approach.

The MCRG technique was introduced by Ma,[27] and developed and extended by Swendsen,[28] to analyze the critical phenomena. It has since been applied successfully to the study of, among others, critical dynamics,[29] and phase ordering.[30,31] The method consists in matching correlation functions on different-sized lattices at different levels of renormalization. The renormalization was obtained by the usual majority rule. Ties are broken by random assignments of ±1 to the coarse-grained spin $S^{(m)}$, where m is the level of blocking. As renormalization, the system flows with respect to the fixed point. As a result, one assumes that the renormalized $t'=tb^{-z}$, $R'=Rb^{n}$, for example, where b is the length rescaling factor, z a dynamical exponent, and n a new rate exponent. One expects that after the irrelevant variables are iterated away, the system becomes invariant, so that any quantity determined after m blockings of an $L\times L$ lattice should be identical to that after $m+1$ blockings of another

TABLE I. Results of successive renormalizations of E.

	$R=0.005$, $L=2048$, $T=1.8$			$R=0.01$, $L=1024$, $T=1.5$		
m	n	z $H_0\neq 0$	z/n $H_0\neq 0$	n	z $H_0\neq 0$	z/n $H_0\neq 0$
1	2.49(2)[a]	1.656(8)	0.666(5)	2.06(2)	1.46(1)	0.707(7)
		1.544(8)	0.621(5)		1.37(1)	0.663(7)
2	2.78(1)	1.856(7)	0.667(4)	2.40(2)	1.71(1)	0.713(6)
		1.735(7)	0.624(3)		1.61(1)	0.671(6)
3	3.04(4)	2.03(2)	0.667(7)	2.61(3)	1.86(3)	0.713(9)
		1.90(2)	0.626(7)		1.75(2)	0.672(8)
4	3.06(5)	2.04(3)	0.666(14)	2.73(7)	1.95(5)	0.714(25)
		1.91(3)	0.624(13)		1.84(5)	0.673(24)
5	2.63(33)	1.75(20)	0.67(11)	2.46(31)	1.75(22)	0.71(13)
		1.65(21)	0.63(11)		1.65(21)	0.67(12)

[a]The standard deviations are estimated by the propagation of errors.

$Lb\times Lb$ lattice in anticipation of possible finite-size effects, and continue to track each other upon further blockings.

Specifically, consider the nearest-neighbor correlation

$$E^{(m)}(L,R,t)=\frac{b^{2m}}{L^2}\left\langle \sum_{\langle i,j\rangle} S_i^{(m)}S_j^{(m)}\right\rangle, \tag{3}$$

and the spatial correlation function

$$C^{(m)}(L,R,t,r)=\left\langle \sum_i S_i^{(m)}S_{i+r}^{(m)}\right\rangle. \tag{4}$$

They both develop a valley as the system transitions from the down state to the up one. Then assuming exact matching, one finds from a series of runs at different rates on a small lattice of size L $t_s^{(m)}$ (the time at the minimum of $E^{(m)}$ or $C^{(m)}$. The subscript indicates the small lattice.) and $R_s^{(m)}$ at which the minimum of $E^{(m)}$ or $C^{(m)}$ equals the corresponding minimum at $t^{(m+1)}=t^{(m)}b^{-z}$ and $R^{(m+1)}=R^{(m)}b^n$ run on a large lattice of size Lb at the same temperature. Accordingly,

$$z=\ln(t_s^{(m)}/t^{(m)})/\ln(b),$$

$$n=\ln(R_s^{(m)}/R^{(m)})/\ln(b). \tag{5}$$

We have neglected the effect of the temperature, which will be taken up towards the end of the text. To be consistent, intrinsic properties associate with the fixed point must start to be invariant after some blockings that have iterated away the irrelevant variables.

In practice, as the number of renormalization m increases, the minimum of E and C approach 0, and become almost independent of the rates after a few blockings, beyond which the present method fails. Also the bigger the m, the stronger the fluctuations due to the random assignments and so the larger the standard errors estimated by the propagation of errors. In order to reach a reasonable result, each rate has to be run for a number of times for the thermal average. Generally, the bigger the size of the lattices, the less the fluctuations and so the fewer the number of the runs needed for good statistics. However, the results are nearly independent of the size of the lattices used.

Table I collects the results for the large lattices of 2048×2048 and 1024×1024 run at $R=0.005$ and $T=1.8$ and $R=0.01$ and $T=1.5$ with a length rescaling factor $b=2$, respectively. The most remarkable result is that z/n is invariant after the first blocking, which generates a very close value as well. Although the spatial correlation function (we used average along both x and y axes) at longer distances offers fewer blockings, similar results are also obtained when matching its minima. This result is not because of our interpolation by a power law to find $t_s^{(m)}$ and $R_s^{(m)}$. Other methods only yielded a variation within the statistical errors. We also checked the results for $b=3$, in which case fewer blockings may be performed. Another appreciable outcome is that for large rates and high temperatures, the dynamical exponent z also reaches the fixed-point value 2 within statistical errors, which accords with that found from phase ordering,[30] where systems are governed by the zero-temperature fixed point.[16,30] For small rates and low temperatures, whether this is true has yet to be determined.

The reason for the invariance of z/n for a given rate can be seen from Fig. 1, showing the effects of renormalization in terms of $H^{(m)}=R^{(m)}t^{(m)}$ (the H value at E's minimum) versus $R^{(m)}$. $H^{(0)}$ and $R^{(0)}$ represent the values of the unrenormalized large lattice; other $m\neq 0$ values correspond to those matching values of the small lattice. It is notable that all but a few initial blockings converge to a single "renormalized trajectory" that fits nicely to $H^{(m)}=H_0+A_2R^{(m)\beta}$, with $H_0=0.070(2)$, $\beta=0.379(2)$, and $A_2=2.69(2)$ for $T=1.8$ and $H_0=0.116(2)$, $\beta=0.327(2)$, and $A_2=2.82(1)$ for $T=1.5$ for $m\geqslant 3$, independent of the size of the lattices. However, z/n varies with R. For example, $z/n=0.75(4)$ and 0.76(6) at $T=1.8$ and 1.5 for $R=0.0001$, respectively, bigger than those listed in Table I albeit with larger errors. This implies that the invariance obtained in this way is only locally in rate. The reason is that the static hysteresis[8,12,14,15,17,21] that may associate with the dynamic transition[18] has been neglected so far.

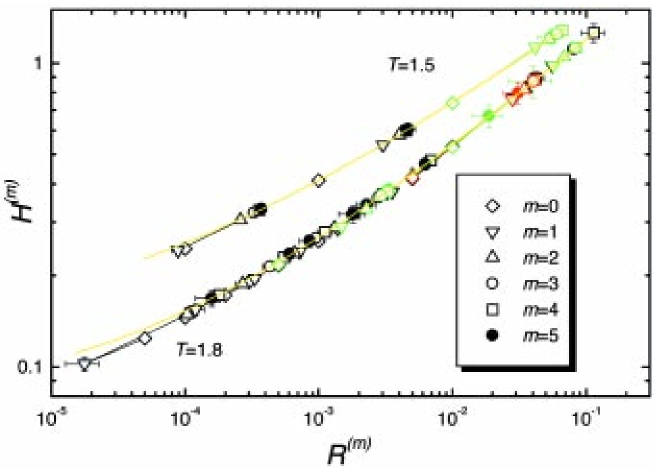

FIG. 1. (Color) Flows of the renormalized magnetic field $H^{(m)}$ and the rate $R^{(m)}$ of the minimum of the correlation function E with the number m of renormalizations. Thin lines connecting the data show the flows. The yellow thick lines are fits to data with $m \geqslant 3$. Black lines and symbols indicate results of 512×512 lattices, green of 1024×1024, and red of 2048×2048.

It has already been seen in Fig. 1 that the renormalized trajectory extrapolates down to $R=0$ with a finite H_0, the static hysteresis. However, H_0 so determined relies on the smallest rate used, so we find it by adjusting it in the way such that all relevant curves collapse onto a single one after rescaling (see below), similar to the determination of T_C in MCRG analysis,[28] though the error may be larger since the quality of data collapse is perceived by eye. The value found in this way is 0.067 and 0.12 for $T=1.8$ and 1.5, respectively. It is close to that found from extrapolation, corroborating each other. Subtracting H_0 affects the values of z and z/n, which have also been given in Table I in the second line of the relevant rows. The standard deviations in this case have not included those arising from H_0. Note that consistently, $1-z/n \approx \beta$ within statistical errors.

Having considered the static hysteresis, we now show data collapse and its consequences as a result of scaling. Figures 2 and 3 display the averaged correlation function and magnetization M vs the rescaled field $(H-H_0)R^{-1+z/n}$ with $z/n=0.62$ at $T=1.8$, respectively. It is seen that all original curves of various rates shown in the insets approach a scaling form almost independent of the sweep rates as the levels of renormalization increase. Size independence is also clearly seen from the insets. In particular, the large rates have reached a single valley for $m=4$ in Fig. 2, though the small rates have not. However, for $m=5$ and 6, the latter have almost joined the others, though as pointed out above, fluctuations become relatively large and so the feature becomes diffused beyond the minima. In Fig. 3, although the curves of $R=0.0001$ appear not to collapse as well as others, for $m \geqslant 3$, it can be seen that all the other rates coincide impressively in a single curve. This strongly supports our theory. Moreover, this also implies the scaling of the area, $A = \oint M dH \propto R^{1-z/n}$, or a scaling law $\alpha \approx 1-z/n$ from Eq. (2). We found α to be 0.381(9) for $T=1.8$ and 0.326(9) for $T=1.5$ on $L=256$ lattices (this L is several times bigger than those previous studied), both of which satisfy this law within statistical errors and agree with previous studies.[6,15,17] Therefore, the origin of the dynamic hysteresis arises from the rate exponent n, which characterizes the response of the system to the driving rate. If this response keeps pace with the dynamical evolution of the system itself, no dynamic hysteresis would appear.

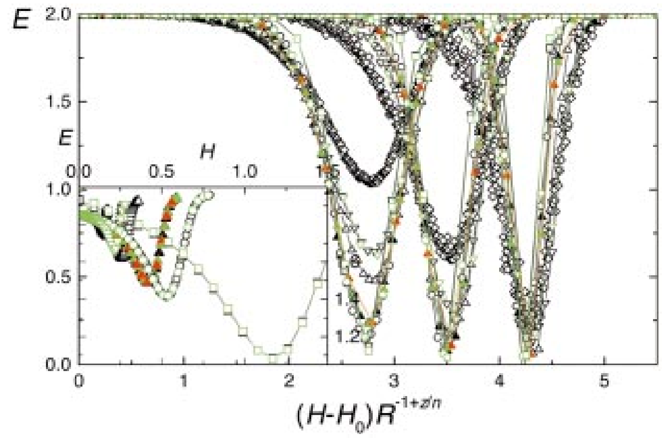

FIG. 2. (Color) Nearest-neighbor correlation function E vs the rescaled field $(H-H_0)R^{-1+z/n}$ at $T=1.8$. The three main valleys correspond, respectively, to $m=3$, 4, and 5, the latter two having been shifted by $+0.75$ and $+1.5$ relative to the first one for clarity. The inset shows the original curves. All curves have been averaged over different realizations of random numbers. Opened diamonds denote $R=0.0001$, down-triangles 0.0005, up-triangles 0.001, circles 0.01, squares 0.09, and filled up-triangles 0.005. Colors are the same as in Fig. 1. Lines are only a guide to the eye.

Finally we discuss the effect of the temperature. We used the same temperature for the large and small lattices in the matching above. This is supported by the results, especially

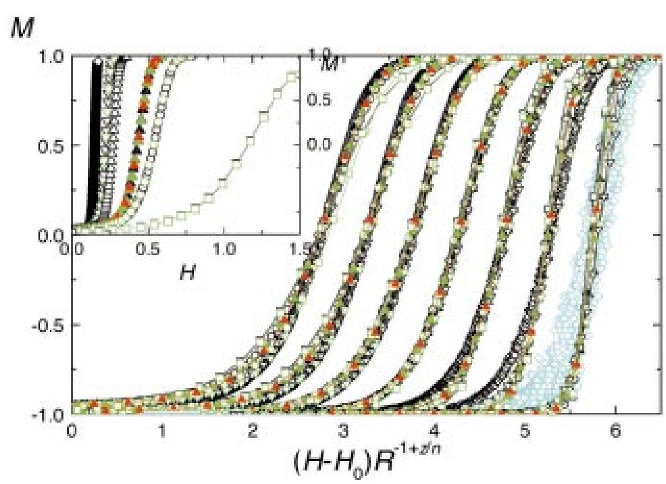

FIG. 3. (Color) Magnetization M vs $(H-H_0)R^{-1+z/n}$ at $T=1.8$. The curves correspond respectively to $m=0$ to 6, each of which has been shifted by $+0.5$ relative to its preceding one for clarity. The inset shows the original curves. All curves have also been averaged except the $m=6$, $R=0.0001$ one showing the possible diffusiveness (in cyan for contrast). Symbols and colors have the same meaning as in Fig. 2.

the striking coincidence in Fig. 3 for $m \geqslant 3$. A common expectation in application of the MCRG method to undercritical systems is that temperature is an irrelevant parameter; as the number of RG increases, the system is driven to the zero-temperature fixed point, so universal behavior reveals. However, a direct attempt to match quantities at $T=1.8$ to those at $T=1.5$ by the same method failed to yield an exponent that could make the curves overlapped. There are several possible reasons for this. The temperature may be a variable that renormalizes slowly or does not renormalize similar to the large-N model in 2D [16], or the present available levels of coarse graining are not sufficient to reach the fixed point for the different-temperature matching, or more subtly, each rate has to be run at its particular temperature if the temperature has to be renormalized like the 3D case of the large-N model.[16] Our present numerical results could not yet give a conclusive answer because of the fluctuations in the data. Further work is needed and is underway.

Summarizing, we have extended the MCRG technique to the 2D Ising model, and found from first principles that the dynamic scaling behavior of hysteresis originates from a rate exponent that characterizes the response of the system to the sweep rate of the field driving the transition. The static hysteresis has been determined which is important for the scaling behavior to occur. A scaling law relating the rate exponent to the area exponent has also been attained.

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