Infinite-range Ising spin glass with a transverse field under the static approximation

Do-Hyun Kim and Jong-Jean Kim*
Department of Physics, Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea
(Received 21 March 2002; published 20 August 2002)

In this paper we investigate for the infinite-range Ising spin-glass model [i.e., the Sherrington-Kirkpatrick (SK) model] with a transverse field under the static approximation by using the imaginary-time replica formalism. From the investigations we show three important results: First, we show that a replica-symmetric quantum spin-glass phase is stable in most of the area of the spin-glass phase in the temperature-transverse field phase diagram. This confirms the existence of a stable replica-symmetric spin glass phase under the static approximation, which is contrary to some previous results derived without the static approximation where the replica-symmetric solution is always unstable in the whole spin-glass phase. Second, we show our theoretical result for the nonlinear susceptibility χ_{nl} which conforms to the experimental result of nonlinear susceptibility measurement by Wu *et al.* [Phys. Rev. Lett. **71**, 1919 (1993)] in a quantum spin glass $LiHo_xY_{1-x}F_4$. Third, in a classical (SK) spin-glass system, we confirm the anomaly in the second temperature derivative of C_H/T near the glass transition temperature T_g, associated possibly with the field-dependent variation of entropy in the spin glass transition, which agrees with the previous experimental observation in a classical spin glass system *Cu*Mn by Fogle *et al.* [Phys. Rev. Lett. **50**, 1815 (1983)]. We also show that this anomaly is suppressed by the nonzero transverse field of the quantum spin glass system, by which we can check for the quantum tunneling competing against spin freezing.

DOI: 10.1103/PhysRevB.66.054432 PACS number(s): 75.10.Jm, 75.10.Nr, 75.40.Cx

I. INTRODUCTION

A spin glass is a complex system characterized by quenched randomness and frustration leading to an irreversible freezing of spins to metastable states without long-range spatial order below the glass transition temperature (T_g).[1] A great many theoretical models for spin-glass systems have been studied extensively using analytic solutions or computer simulation techniques.[1] But the theory of spin glasses has been concerned mostly with mean-field calculations based on infinite-range models whose prototype is the Sherrington-Kirkpatrick (SK) model.[2] The de Almeida-Thouless (AT) line[3] of the SK model, in the presence of a mean interaction strength or an external field, separates a high-temperature paramagnetic or ferromagnetic phase where the order parameter can be determined as unique from a low-temperature spin-glass phase which is defined in terms of an infinite number of order parameters, i.e., an order parameter distribution function. Below the AT line the spin-glass phase is under the replica symmetry breaking (RSB), and in this phase there exists no stable-free energy minimum with single-valued magnetization and a spin-glass order parameter, but there may be metastable free-energy valleys with varying local magnetizations.[4]

The SK model of a spin glass predicts important qualitative distinctions such as a cusp of linear susceptibility χ. There have been variations of the SK model by the addition of extra terms to the SK Hamiltonian applicable for systems analogous to a spin glass. A well-known extension of the SK model is the SK model with a transverse field, or the Ising spin glass model with tunneling[5] for proton glass, a dielectric analogous system of a spin glass formed from a mixed crystal between ferroelectrics and antiferroelectrics such as $Rb_{1-x}(NH_4)_xH_2PO_4$ (RADP-x).[6] This SK model with a transverse field has been a useful model for the quantum spin glass.[7]

A quantum spin glass, which has received great concern since a seminal paper by Bray and Moore,[8] has an interesting feature that the glass transition in the system may be driven by not only thermal but also quantum fluctuations. In the case of the SK model with a transverse field, the transverse field (Γ) introduces channels of quantum relaxation to bypass the activation barriers of classical spin glass suppressing the glass transition. As the quantum fluctuations tuned by the transverse field (Γ) carry spin-flips detrimental to the spin-glass phase, under a sufficiently large transverse field a phase boundary can be introduced in a spin glass even at zero temperature.[7]

The SK model with a transverse field may thus be applied to a quantum spin-glass system of $LiHo_xY_{1-x}F_4$,[9–11] a site-diluted and isostructural derivative of the dipolar-coupled Ising ferromagnet $LiHoF_4$ ($T_c=1.53$ K). In the absence of a magnetic field $LiHo_xY_{1-x}F_4$ is a conventional spin glass with a glass transition temperature $T_g(x)$. But an externally tunnable magnetic field H_t transverse to the magnetic easy axis, which is parallel to the c axis in this tetragonal system, induces quantum tunneling through the barrier separating the two degenerate ground states of the Ho^{3+} ions, and thus yields a splitting of the ground-state doublet. It is this splitting, proportional to H_t^2 in the lowest order, which plays the role of transverse field (Γ) in the present model Hamiltonian. As quantum tunneling competes against spin freezing, the spin-glass ground state is expected to be suppressed at any temperature if the tunneling splitting is sufficiently high. It was experimentally observed that $LiHo_{0.167}Y_{0.833}F_4$ is paramagnetic at all temperatures above $H_{tc}(0)\simeq 12$ kOe, and below this critical field a line of temperature dependent critical field $H_{tc}(T)$ separates between paramagnetic and spin glass phases.[9,10] The SK model with a transverse field has thus been a useful model for the quantum spin glass $LiHo_xY_{1-x}F_4$, especially for the phase diagram[7] and the dynamic linear susceptibility,[12] which were in qualitative agreement with experimental measurements. Although the SK model with a transverse field has been the most realistic model for quantum-spin glass $LiHo_xY_{1-x}F_4$, there are three unsolved problems in this model.

(a) Whether the replica-symmetric solution in the quantum spin glass phase of the present model is stable, or not, as in the classical case. One of the interesting questions in quantum spin-glass systems concerns the possibility of tunneling through the barriers of the free-energy landscape in the classical spin glass of the SK model due to quantum fluctuations by the transverse field in the transverse field SK model. In the classical case, the barriers separating the valleys increase in height with the macroscopic size of the system. In the thermodynamic limit it thus becomes unable to cross the barrier by thermal fluctuations, thereby causing nonergodicity with multidegenerate thermodynamic states. Quantum spin, however, should not necessarily yield to the barrier height, and since the barrier width in configuration space decreases with increasing system size, it may tunnel through such barriers by quantum fluctuations. If the quantum fluctuations are strong enough to cause tunneling between energy barriers separating degenerate local minimum thermodynamic states, then an ergodic replica-symmetric solution in the spin glass state may be stable. There has been controversy about the nature of the spin glass phase of the SK model with a transverse field : Thirumalai *et al.*,[13] using the static approximation proposed by Bray and Moore,[8] showed that there was a small intermediate region in the spin-glass phase where a replica-symmetric solution was stable, unlike the classical SK model without transverse field. Ray *et al.* performed Monte Carlo simulations which supported the stability of the replica-symmetric solution in the whole spin-glass phase.[14] On the other hand, Büttner and Usadel[15] predicted, without assuming the spin self-interaction term to be static, that the replica-symmetric solution was always unstable in the whole spin glass phase. Goldschmidt and Lai[16] and, independently, Büttner and Usadel[17] obtained the one-step replica-symmetry-breaking solution also without assuming the spin self-interaction term to be static, and found no evidence to support an intermediate spin glass phase with replica symmetry.

(b) Can the present model show existence of the first-order spin glass transition? This feature was suggested from nonlinear susceptibility measurements by Wu *et al.*:[10] Above 25 mK, the phase transition was of the second order, as indicated by a divergence of the nonlinear susceptibility χ_{nl}. Below 25 mK, however, the divergence of the nonlinear susceptibility was changed to a flat maximum, when the imaginary part of the low-frequency linear susceptibility showed a sharp peak. Wu *et al.* concluded these features to suggest that at low temperatures the transverse-field-induced spin glass transition turned to the first-order. But there has been no evidence for this first-order transition in the quantum spin-glass models including the SK model with a transverse field. Very recently, Cugliandolo *et al.*[18] showed, from a quantum *p*-spin spherical spin glass model, a tricritical point that divides the critical transverse field line $\Gamma_c(T)$ by a second-order glass transition and a first-order transition.

(c) Can the present model show the anomaly in the temperature derivatives of the specific heat C_H? In spin-glass systems the magnetic contribution to specific heat C_H shows a broad maximum and no anomaly at the glass transition temperature T_g,[1,19] in contrast to the cusp anomalies in χ and χ_{nl}. The absence of a drastic change in C_H at the glass transition temperature implicates that the change of internal energy may be too small to be observable and magnetic anomalies such as the cusp in susceptibility may be produced by a comparatively few degrees of freedom. But the second temperature derivative of the specific heat divided by temperature C_H/T has been found to depict a weak anomaly near the glass transition temperature and was attributed to the entropy variation in the spin glass transition, which was observed in the classical spin glass CuMn by Fogle *et al.*[20] A question arises: Is there also such a thermodynamic anomaly in a quantum spin glass ($\Gamma\neq 0$ in the present model)? If so, is there any change in the anomaly with an increase of the Γ value? It may be impossible to check for this anomaly from experiment because a very low temperature (below 1K as in $LiHo_xY_{1-x}F_4$) is required for a quantum spin-glass transition. At $T\neq 0$ the quantum-mechanical effects may not be dominant in comparison with thermal effects; then there may exist such a thermodynamic anomaly even in a quantum spin-glass transition because thermal fluctuations at nonzero temperature may have a larger thermodynamic effect in a spin glass than the quantum fluctuation induced by nonzero Γ. As quantum tunneling competes against thermodynamic spin freezing, we can also expect that the anomaly will be gradually suppressed by an increase of the Γ value.

Our goals are thus rewritten as follows:

(1) We are to investigate whether the replica-symmetric solution of the SK model with a transverse field is unstable in the whole spin-glass phase or not. For this, we will use the static approximation for all order parameters and calculate the AT line analytically, which will be compared with phase boundary line. If our conclusion is same as the one of Thirumalai *et al.*,[13] then we can accept that the quantum spin-glass phase may be replica symmetric under the condition of the static approximation. But if not, then we must accept that spin glass phase always shows replica-symmetry-breaking, irrespective of static or not, classical or quantum.

(2) Though no evidence has been found that the transverse-field-induced spin-glass transition becomes of first order, we calculate the nonlinear susceptibility, which is found to conform with the nonlinear susceptibility measurement by Wu *et al.*[10] This suggests that the SK model with a transverse field may be a good realistic model for the quantum spin glass $LiHo_xY_{1-x}F_4$.

(3) We examine whether there exists anomaly in the second temperature derivative of the specific heat divided by temperature C_H/T near the spin-glass transition temperature of the SK model with a transverse field or not. We will show that the classical SK model ($\Gamma=0$) depicts the anomaly near the glass transition temperature, and check whether the quantum spin glass transition ($\Gamma\neq 0$) has any (Γ dependent) anomaly or not.

In our paper we use the imaginary-time replica formalism which was introduced to quantum Heisenberg spin-glass model by Bray and Moore,[8] and applied to the SK model with a transverse field[21] and the quantum p-spin spherical spin-glass model.[18] This formalism is widely known for a quantum spin glass together with the Trotter-Suzuki formalism,[22] which has also been used in many works of quantum spin glass.[13,16,17] We take the static approximation in the imaginary-time replica formalism to see whether the replica-symmetric solution of the present model is always unstable in the whole spin-glass phase under the static approximation or not, and also because the static approximation is the only practical choice to obtain analytic solutions instead of solutions of numerical simulations for the free energy, order parameters, resulting phase diagrams, and other physical quantities such as nonlinear susceptibility χ_{nl} and specific heat C_H.

We investigate for the analytic solution of our concerning model to obtain the free energy, order parameters and physical quantities under the static approximation. First we will present the procedure of imaginary-time replica formalism to obtain a general solution in Sec. II. We obtain the replica-symmetric solution under static approximation in Sec. III. We derive the AT stability condition of the present model in Sec. IV. We obtain the replica-symmetry-breaking solution under the static approximation in Sec. V. From the solutions we will determine various phase diagrams for the system in Sec. VI. We present an analytic solution for nonlinear susceptibility χ_{nl} to be compared with experimental data[10] of a quantum spin glass $\mathrm{LiHo}_x\mathrm{Y}_{1-x}\mathrm{F}_4$ in Sec. VII. We will also show some results associated with the specific heat C_H to be compared with experimental data of classical spin glass in Sec. VIII. In Sec. IX we will give our conclusion of the present studies.

II. MODEL

The Hamiltonian of the infinite-range Ising model spin glass with a transverse field is given by[7]

$$\mathcal{H}=-\sum_{(i,j)} J_{ij}\sigma_i^z\sigma_j^z-\Gamma\sum_i \sigma_i^x-H\sum_i \sigma_i^z, \tag{1}$$

where σ^z and σ^x denote the Pauli matrices, i.e.,

$$\sigma^z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad\text{and}\quad \sigma^x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{2}$$

(i,j) are distinct pairs of spins with the total number N, J_{ij} are the quenched random exchange interaction variables, Γ is the transverse field, and H is the external longitudinal field. The distribution of J_{ij} is taken to be Gaussian[2] with a mean of J_0/N and a variance of J^2/N, i.e.,

$$P(J_{ij})=\sqrt{N/2\pi J^2}\,\exp\{-N(J_{ij}-J_0/N)^2/2J^2\}, \tag{3}$$

where the factor $1/N$ makes the thermodynamic quantities finite for $N\to\infty$.

By the imaginary-time formalism[8,21], the partition function of the system can be written as

$$Z=\mathrm{Tr}\exp\left[\bar{\beta}\Gamma\sum_i \sigma_i^x\right]\mathcal{T}\times\exp\left[\int_0^{\bar{\beta}}d\tau\left\{\sum_{ij} J_{ij}\sigma_i^z(\tau)\sigma_j^z(\tau)+H\sum_i \sigma_i^z(\tau)\right\}\right], \tag{4}$$

where τ is the imaginary time, $\mathcal{T}$ is the time ordering operator, $\sigma^z(\tau)$ are the operators in the interaction representation, [i.e., $\sigma^z(\tau)=\exp(\mathcal{H}_0\tau)\sigma^z\exp(-\mathcal{H}_0\tau)$, where $\mathcal{H}_0=-\Gamma\Sigma_i\sigma_i^x$] and $\bar{\beta}=1/T$ (where $k_B\equiv 1$ for simplicity). For this model the free energy can be evaluated using the replica method[23] : $-\bar{\beta}F=[\ln Z]_J=\lim_{n\to 0}(1/n)([Z^n]_J-1)$, where $[\ ]_J$ indicates an average over the quenched disorder of J_{ij}.

The n-replicated partition function of the system can be written as

$$Z^n=\mathrm{Tr}\exp\left[\bar{\beta}\Gamma\sum_i\sum_{\alpha=1}^{n} \sigma_{i\alpha}^x\right]\mathcal{T}\times\exp\left[\int_0^{\bar{\beta}}d\tau\left\{\sum_{ij}\sum_{\alpha=1}^{n} J_{ij}\sigma_{i\alpha}^z(\tau)\sigma_{j\alpha}^z(\tau)+H\sum_i\sum_{\alpha=1}^{n}\sigma_{i\alpha}^z(\tau)\right\}\right], \tag{5}$$

where α denotes the replica index. Performing the averaging Z^n by $P(J_{ij})$ and rearranging terms, one obtains

$$\begin{aligned}[Z^n]_J=&\mathrm{Tr}\exp\left[\bar{\beta}\Gamma\sum_{i\alpha}\sigma_{i\alpha}^x\right]\mathcal{T}\\ &\times\exp\left[\frac{J^2}{2N}\int_0^{\bar{\beta}}d\tau\int_0^{\bar{\beta}}d\tau'\left\{\sum_{(\alpha\beta)}\left(\sum_i \sigma_{i\alpha}^z(\tau)\sigma_{i\beta}^z(\tau')\right)^2\right.\right.\\ &\left.+\sum_\alpha\left(\sum_i \sigma_{i\alpha}^z(\tau)\sigma_{i\alpha}^z(\tau')\right)^2\right\}\\ &+\frac{J_0}{N}\int_0^{\bar{\beta}}d\tau\sum_\alpha\left(\sum_i \sigma_{i\alpha}^z(\tau)\right)^2\\ &\left.+H\int_0^{\bar{\beta}}d\tau\sum_\alpha\left(\sum_i\sigma_{i\alpha}^z(\tau)\right)\right],\end{aligned} \tag{6}$$

where $(\alpha\beta)$ denotes a summation over α and $\beta(\neq\alpha)$. The squares $[\Sigma_i\sigma_{i\alpha}^z(\tau)\sigma_{i\beta(\alpha)}^z(\tau')]^2$ and $[\Sigma_i\sigma_{i\alpha}^z(\tau)]^2$ can be simplified using the Hubbard-Stratonovitch transformation[2]

$$\exp\left\{\frac{1}{2}\lambda a^2\right\}=\sqrt{\frac{\lambda}{2\pi}}\int_{-\infty}^{\infty}dx\,\exp\left\{-\frac{1}{2}\lambda x^2+\lambda a x\right\}. \tag{7}$$

The resulting free energy can thus be obtained as[2,8,13]

$$\bar{\beta}F = -\lim_{n\to 0}\frac{1}{n}\left[\int\prod_{(\alpha\beta)}\sqrt{\frac{NJ^2}{2\pi}}\,dy^{\alpha\beta}(\tau,\tau')\times\prod_{\alpha}\sqrt{\frac{NJ^2}{2\pi}}\,dw^{\alpha\alpha}(\tau,\tau')\times\prod_{\alpha}\sqrt{\frac{NJ_0}{2\pi}}\,dx^{\alpha}(\tau)\exp\{-NG(y^{\alpha\beta},w^{\alpha\alpha},x^{\alpha})\}-1\right],$$

where

$$G(y^{\alpha\beta},w^{\alpha\alpha},x^{\alpha}) \equiv \frac{1}{4}J^2\int_0^{\bar{\beta}}d\tau\int_0^{\bar{\beta}}d\tau'\left[\sum_{(\alpha\beta)}(y^{\alpha\beta}(\tau,\tau'))^2+\sum_{\alpha}(w^{\alpha\alpha}(\tau,\tau'))^2\right]+\frac{1}{2}J_0\int_0^{\bar{\beta}}d\tau\sum_{\alpha}(x^{\alpha}(\tau))^2-\ln\mathrm{Tr}\exp(\tilde{\mathcal{H}}),$$

with

$$\exp(\tilde{\mathcal{H}})\equiv\exp\left[\bar{\beta}\Gamma\sum_{\alpha}\sigma_{\alpha}^{x}\right]\mathcal{T}\times\exp\left\{\frac{1}{2}J^2\int_0^{\bar{\beta}}d\tau\int_0^{\bar{\beta}}d\tau'\times\left[\sum_{(\alpha\beta)}y^{\alpha\beta}(\tau,\tau')\sigma_{\alpha}^{z}(\tau)\sigma_{\beta}^{z}(\tau')+\sum_{\alpha}w^{\alpha\alpha}(\tau,\tau')\sigma_{\alpha}^{z}(\tau)\sigma_{\alpha}^{z}(\tau')\right]+\int_0^{\bar{\beta}}d\tau\sum_{\alpha}(J_0x^{\alpha}(\tau)+H)\sigma_{\alpha}^{z}(\tau)\right\}.$$

Here the trace Tr is over n replicas at a single spin site.

In the thermodynamic limit ($N\to\infty$) the integrals can be performed by the method of steepest descent,[2,8,13]

$$\int dy\exp\{-NG(y)\}\approx\int dy\exp\left\{-NG(y_0)-\frac{1}{2}NG''(y_0)(y-y_0)^2+\cdots\right\},\tag{8}$$

where $G'(y_0)=0$ defines a saddle point y_0. The Gaussian term can be ignored for $N\to\infty$ with $G''(y_0)\geqslant 0$. Otherwise, the resulting integral diverges and the saddle-point procedure fails. If we assume $G''(y^{\alpha\beta},w^{\alpha\alpha},x^{\alpha})\geqslant 0$, then this assumption enables us to replace $y^{\alpha\beta}(\tau,\tau')$, $w^{\alpha\alpha}(\tau,\tau')$, and $x^{\alpha}(\tau)$ by their stationary values $Q^{\alpha\beta}(\tau,\tau')$, $R^{\alpha\alpha}(\tau,\tau')$, and $M^{\alpha}(\tau)$, respectively:

$$Q^{\alpha\beta}(\tau,\tau')\equiv\langle\mathcal{T}\sigma_{\alpha}^{z}(\tau)\sigma_{\beta}^{z}(\tau')\rangle,$$

$$R^{\alpha\alpha}(\tau,\tau')\equiv\langle\mathcal{T}\sigma_{\alpha}^{z}(\tau)\sigma_{\alpha}^{z}(\tau')\rangle,\quad M^{\alpha}(\tau)\equiv\langle\sigma_{\alpha}^{z}(\tau)\rangle,\tag{9}$$

where $Q^{\alpha\beta}(\tau,\tau')$, $R^{\alpha\alpha}(\tau,\tau')$, and $M^{\alpha}(\tau)$ represent the spin-glass order parameter, the spin self-interaction, and the magnetization, respectively, and the $\langle\ \rangle$ averages are taken with respect to the effective Hamiltonian $\tilde{\mathcal{H}}=\tilde{\mathcal{H}}[Q^{\alpha\beta}(\tau,\tau'),R^{\alpha\alpha}(\tau,\tau'),M^{\alpha}(\tau)]$.

Now we make the static approximation[8,13,24] by $Q^{\alpha\beta}(\tau,\tau')=Q^{\alpha\beta}$, $R^{\alpha\alpha}(\tau,\tau')=R^{\alpha\alpha}$ and $M^{\alpha}(\tau)=M^{\alpha}$. Then the intensive free energy $f\equiv\lim_{N\to\infty}F/N$ can be given as

$$\bar{\beta}f=\lim_{n\to 0}\frac{1}{n}\left[\frac{1}{4}(\bar{\beta}J)^2\left\{\sum_{(\alpha\beta)}(Q^{\alpha\beta})^2+\sum_{\alpha}(R^{\alpha\alpha})^2\right\}+\frac{1}{2}\bar{\beta}J_0\sum_{\alpha}(M^{\alpha})^2-\ln\mathrm{Tr}\exp(\tilde{\mathcal{H}}')\right],\tag{10}$$

with the effective Hamiltonian

$$\tilde{\mathcal{H}}'\equiv\frac{1}{2}(\bar{\beta}J)^2\left[\sum_{(\alpha\beta)}Q^{\alpha\beta}\sigma_{\alpha}^{z}\sigma_{\beta}^{z}+\sum_{\alpha}R^{\alpha\alpha}(\sigma_{\alpha}^{z})^2\right]+\bar{\beta}\sum_{\alpha}(J_0M^{\alpha}\sigma_{\alpha}^{z}+H\sigma_{\alpha}^{z}+\Gamma\sigma_{\alpha}^{x}).$$

III. REPLICA-SYMMETRIC SOLUTION

Here we take the replica-symmetric assumption[2,13] to set $Q^{\alpha\beta}=Q$ for all $\alpha\neq\beta$, and $R^{\alpha\alpha}=R$ and $M^{\alpha}=M$ for all α. By applying the Hubbard-Stratonovitch transform [Eq. (7)] to $(\Sigma_{\alpha}\sigma_{\alpha}^{z})^2$, the free energy can be written as follows:

$$\bar{\beta}f=\frac{1}{4}(\bar{\beta}J)^2(R^2-Q^2)+\frac{1}{2}\bar{\beta}J_0M^2-\lim_{n\to 0}\frac{1}{n}\times\ln\left[\int\mathcal{D}z\,\mathrm{Tr}\exp\left\{\bar{\beta}\sum_{\alpha}^{n}\left(\frac{1}{2}\bar{\beta}J^2(R-Q)(\sigma_{\alpha}^{z})^2+H_z\sigma_{\alpha}^{z}+\Gamma\sigma_{\alpha}^{x}\right)\right\}\right],\tag{11}$$

where $\int\mathcal{D}z\cdots\equiv(1/\sqrt{2\pi})\int_{-\infty}^{\infty}dz\exp\{-\frac{1}{2}z^2\}\cdots$, and $H_z\equiv J\sqrt{Q}z+J_0M+H$. By using $X^n=\exp\{n\ln X\}\approx 1+n\ln X$ for $n\to 0$ with $X\equiv\mathrm{Tr}_{\alpha}\exp\{\bar{\beta}(\frac{1}{2}\bar{\beta}J^2(R-Q)(\sigma_{\alpha}^{z})^2+H_z\sigma_{\alpha}^{z}+\Gamma\sigma_{\alpha}^{x})\}$, where the trace Tr_{α} is taken over a specific replica at a single spin site, and $\ln(1+nA)\approx nA$ for $n\to 0$, we finally obtain

$$\bar{\beta}f=-\frac{1}{4}(\bar{\beta}J)^2(R-Q)(2-R-Q)+\frac{1}{2}\bar{\beta}J_0M^2-\int\mathcal{D}z\ln[2\cosh(\bar{\beta}\sqrt{H_z^2+\Gamma^2})].\tag{12}$$

We can determine M, Q and R by the condition that f resumes the stable extrema when they are the replica-symmetric solutions.[2,13] From this extremal condition we can obtain the self-consistent equations of M, Q, and R:

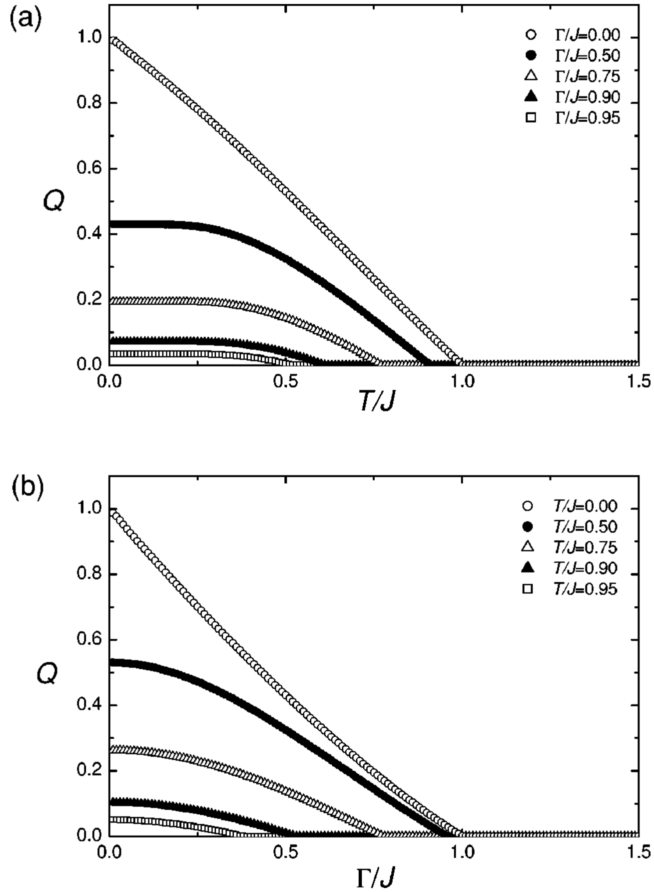

FIG. 1. (a) Spin-glass order parameter Q vs temperature T/J at various transverse fields Γ/J with $H/J=0.0$ and $J_0/J=0.0$. (b) Spin-glass order parameter Q vs transverse field Γ/J at various temperatures T/J with $H/J=0.0$ and $J_0/J=0.0$.

$$M=\int \mathcal{D}z\ \frac{H_z}{\sqrt{H_z^2+\Gamma^2}}\tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2}), \tag{13}$$

$$Q=\int \mathcal{D}z\ \frac{H_z^2}{H_z^2+\Gamma^2}\tanh^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}), \tag{14}$$

$$R=\int \mathcal{D}z\ \frac{H_z^2}{H_z^2+\Gamma^2} \tag{15}$$

We have $R=1.0$ at $\Gamma=0.0$, when the free energy f, magnetization M, and spin-glass order parameter Q are reduced to those of the SK model.[2] This criterion was not satisfied in the earlier work of Thirumalai *et al.*[13,25] From the above solutions we can derive the replica-symmetric linear susceptibility χ_0 for $J_0=H=0.0$ as

$$\chi_0=\bar{\beta}(R-Q), \tag{16}$$

and at $\Gamma=0.0$ χ_0 can be seen to satisfy the so-called Fischer relation.[26]

In Fig. 1(a) we show the spin-glass order parameter Q vs temperature T/J at various transverse fields Γ/J with a zero external longitudinal field $H/J=0.0$ and zero average interaction strength $J_0/J=0.0$. As Γ/J increases, Q values are seen to decrease in the whole temperature range and the spin-glass transition temperature is lowered. At $\Gamma/J=0.0$, Q increases to 1 with decreasing temperature to zero. At $\Gamma/J \neq 0.0$, however, Q increases with decreasing temperature below the transition temperature but to a saturation value less than 1, which decreases with increasing Γ/J. In Fig. 1(b) we show the spin-glass order parameter Q vs transverse field Γ/J at various temperatures T/J with $H/J=0.0$ and $J_0/J=0.0$. As T/J increases, Q values are seen to decrease in the whole transverse field range and the critical transverse field Γ_c of the spin-glass transition is lowered. At $T/J=0.0$, Q increases to 1 with decreasing transverse field to zero. At $T/J\neq 0.0$, however, Q increases with decreasing transverse field below a critical field but to a maximum value less than 1, which depends on T/J.

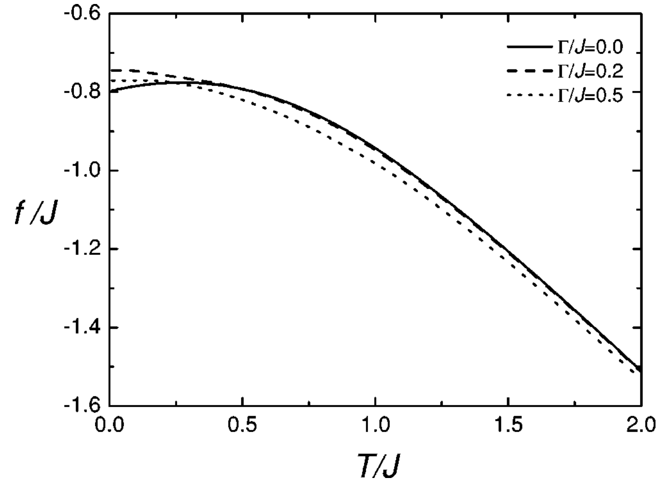

FIG. 2. Free energy f vs temperature T/J at three selected transverse fields ($\Gamma/J=0.0$, 0.2, and 0.5) with $H/J=0.0$ and $J_0/J=0.0$.

In Fig. 2 we show the free energy f vs temperature T/J at three selected transverse fields Γ/J with $H/J=0.0$ and $J_0/J=0.0$. The free energy f of the $\Gamma/J=0.0$ (SK) case shows a maximum at some specific temperature T_0, and decreases as the temperature is lowered below T_0. The entropy S ($\equiv -\partial f/\partial T$) then has negative values in the low-temperature region below T_0. This fact violates the third law of thermodynamics, by which such concepts as AT stability condition and replica symmetry breaking have been introduced in spin-glass theories.[1] But free energy f for $\Gamma/J=0.2$ and 0.5 shows no maximum at finite temperature and increases as the temperature is lowered to zero. The entropy S thus approaches zero as temperature goes to zero, which is different from the $\Gamma/J=0$ (SK) case. Thus we may ask whether a replica-symmetry-breaking spin glass (RSB-SG) phase exists in the case of a nonzero transverse field ($\Gamma/J=0.2$ or 0.5). This question will be examined in Sec. VI.

IV. DE ALMEIDA–THOULESS STABILITY CONDITION

We want to show that the assumption $G''(y^{\alpha\beta},w^{\alpha\alpha},x^{\alpha})\geqslant 0$ in Eq. (8) gives the stability condition for our replica-symmetric solutions. We start with small variable expansions of $y^{\alpha\beta}$, $w^{\alpha\alpha}$, and x^{α} as

$$\begin{aligned} y^{\alpha\beta}(\tau,\tau') &= Q^{\alpha\beta}(\tau,\tau') + \eta^{\alpha\beta}(\tau,\tau'), \\ w^{\alpha\alpha}(\tau,\tau') &= R^{\alpha\alpha}(\tau,\tau') + \xi^{\alpha\alpha}(\tau,\tau'), \\ x^{\alpha}(\tau) &= M^{\alpha}(\tau) + \epsilon^{\alpha}(\tau), \end{aligned} \tag{17}$$

and expand the free energy $\bar{\beta}F/N$ to second order in fluctuations $\eta^{\alpha\beta}(\tau,\tau')$, $\xi^{\alpha\alpha}(\tau,\tau')$, and $\epsilon^{\alpha}(\tau)$. The second derivative of G generates three- and four-spin couplings as follows:

$$O_{\alpha\beta\mu}(\tau,\tau',\tau'') \equiv \langle \mathcal{T}\sigma^z_\alpha(\tau)\sigma^z_\beta(\tau')\sigma^z_\mu(\tau'')\rangle,$$

$$P_{\alpha\beta\mu\nu}(\tau,\tau',\tau'',\tau''') \equiv \langle \mathcal{T}\sigma^z_\alpha(\tau)\sigma^z_\beta(\tau')\sigma^z_\mu(\tau'')\sigma^z_\nu(\tau''')\rangle. \tag{18}$$

Here the $\langle\,\rangle$ averages are taken with respect to the effective Hamiltonian $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}[Q^{\alpha\beta}(\tau,\tau'), R^{\alpha\alpha}(\tau,\tau'), M^{\alpha}(\tau)]$. The deviation of the free energy $\bar{\beta}F/N$ from its stationary value is then given by

$$\bar{\beta}[F(y^{\alpha\beta},w^{\alpha\alpha},x^{\alpha}) - F(Q^{\alpha\beta},R^{\alpha\alpha},M^{\alpha})]/N = -\Delta/2, \tag{19}$$

where

$$\begin{aligned} \Delta = {} & \sum_{(\alpha\beta)(\mu\nu)} \{(\bar{\beta}J)^2\delta_{(\alpha\beta)(\mu\nu)} - (\bar{\beta}J)^4[P_{\alpha\beta\mu\nu}(\tau,\tau',\tau'',\tau''') - Q_{\alpha\beta}(\tau,\tau')Q_{\mu\nu}(\tau'',\tau''')]\}\,\eta^{\alpha\beta}(\tau,\tau')\,\eta^{\mu\nu}(\tau'',\tau''') \\ & + \sum_{\alpha\beta} \{(\bar{\beta}J)^2\delta_{\alpha\beta} - (\bar{\beta}J)^4[P_{\alpha\alpha\beta\beta}(\tau,\tau',\tau'',\tau''') - R_{\alpha\alpha}(\tau,\tau')R_{\beta\beta}(\tau,\tau')]\}\,\xi^{\alpha\alpha}(\tau,\tau')\,\xi^{\beta\beta}(\tau'',\tau''') \\ & + \sum_{\alpha\beta} \{\bar{\beta}J_0\delta_{\alpha\beta} - (\bar{\beta}J_0)^2[Q_{\alpha\beta}(\tau,\tau') - M_\alpha(\tau)M_\beta(\tau')]\}\,\epsilon^{\alpha}(\tau)\,\epsilon^{\beta}(\tau') - \sum_{(\alpha\beta)\mu} (\bar{\beta}J)^4[P_{\alpha\beta\mu\mu}(\tau,\tau',\tau'',\tau''') \\ & - Q_{\alpha\beta}(\tau,\tau')R_{\mu\mu}(\tau'',\tau''')]\,\eta^{\alpha\beta}(\tau,\tau')\,\xi^{\mu\mu}(\tau'',\tau''') - \sum_{(\alpha\beta)\mu} \bar{\beta}^3J^2J_0[O_{\alpha\beta\mu}(\tau,\tau',\tau'') \\ & - Q_{\alpha\beta}(\tau,\tau')M_\mu(\tau'')]\,\eta^{\alpha\beta}(\tau,\tau')\,\epsilon^{\mu}(\tau'') - \sum_{\alpha\beta} \bar{\beta}^3J^2J_0[O_{\alpha\alpha\beta}(\tau,\tau',\tau'') - R_{\alpha\alpha}(\tau,\tau')M_\beta(\tau'')]\,\xi^{\alpha\alpha}(\tau,\tau')\,\epsilon^{\beta}(\tau''). \end{aligned} \tag{20}$$

Here Δ should be positive definite for the solutions $Q^{\alpha\beta}(\tau,\tau')$, $R^{\alpha\alpha}(\tau,\tau')$, and $M^{\alpha}(\tau)$ to be stable.

We analyze de Almeida-Thouless (AT) stability condition of the replica-symmetric solution in the concerned model under static approximation. This condition should verify whether there exists any replica-symmetric region in the spin-glass state under the static approximation, as suggested by Thirumalai *et al.*,[13] or not.

As shown in the Appendix, the stability condition of the replica-symmetric solution under the static approximation is given by

$$(\bar{\beta}J)^{-2} \geqslant \int \mathcal{D}z \frac{H_z^4}{(H_z^2+\Gamma^2)^2}\,\mathrm{sech}^4(\bar{\beta}\sqrt{H_z^2+\Gamma^2}), \tag{21}$$

which is reduced to the result of de Almeida and Thouless[3] at $\Gamma = 0.0$. The above result represents the AT stability condition extended to the case of a nonzero transverse field, and when this condition is not satisfied, the phase corresponds to the replica-symmetry-breaking one. In this phase we have no stable free-energy minimum with a single-valued magnetization M and spin-glass order parameter Q but there may be metastable free-energy valleys with varying local magnetizations.[4]

In Fig. 3 we show the temperature dependence of the AT stability region at selected transverse fields Γ/J with $H/J = 0.0$ and $J_0/J = 0.0$. When $\Gamma/J = 0.0$ (SK model), the AT line of Eq. (21) locates the stability limit at $T/J = 1.0$. But the AT line extends the stable region to zero temperature when Γ/J increases to a very small value 0.06. It thus seems

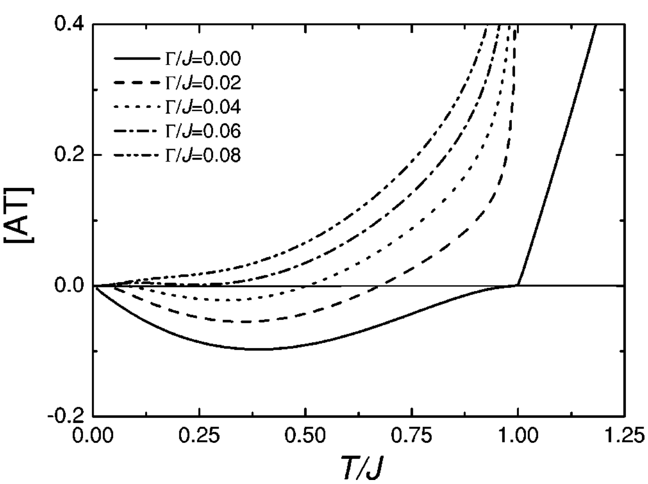

FIG. 3. [AT] vs temperature T/J at selected transverse fields Γ/J with $H/J = 0.0$ and $J_0/J = 0.0$. Here [AT] in the vertical axis represents the integral $(T/J)^2 - \int \mathcal{D}z(H_z^4/(H_z^2+\Gamma^2)^2) \times \mathrm{sech}^4(\bar{\beta}\sqrt{H_z^2+\Gamma^2})$ of the AT stability condition.

that the replica-symmetry-breaking phase will be in a very narrow region in the temperature (T/J)–transverse field (Γ/J) plane, which will be shown in Sec. VI.

V. REPLICA-SYMMETRY-BREAKING SOLUTION

We can obtain the one-step replica-symmetry-breaking (1RSB) solution of the present model under the static approximation. We use Parisi's parametrization scheme of replica symmetry breaking as in the case of the SK model[4] : for an $n\times n$ matrix $\{Q^{\alpha\beta}\}$ in the replica spin space, the n replicas of $\{Q^{\alpha\beta}\}$ are divided into n/m groups of m replicas, assuming that n must be a multiple of m, so that $\{Q^{\alpha\beta}\}$ consists of $m\times m$ diagonal matrices (in which all the diagonal elements are zero and off-diagonal elements are Q_1) and $m\times m$ off-diagonal matrices (in which all the elements are Q_0). For this 1RSB scheme in the $n\to 0$ limit, the free energy f_{1RSB} is given by

$$\bar{\beta} f_{1RSB}=-\frac{1}{4}(\bar{\beta}J)^2\{(R-Q_1)(2-R-Q_1)-m(Q_1^2-Q_0^2)\}+\frac{1}{2}\bar{\beta}J_0M^2-\frac{1}{m}\int \mathcal{D}z \times\ln\left[\int \mathcal{D}y \cosh^m\left(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2}\right)\right], \tag{22}$$

where $H_z^\prime\equiv J(\sqrt{Q_0}z+\sqrt{Q_1-Q_0}y)+J_0M+H$. From the extremal condition of f_{1RSB} we can obtain the self-consistent equations for m, M, Q_0, Q_1, and R:

$$\frac{1}{4}(\bar{\beta}J)^2m^2(Q_1^2-Q_0^2)=-\int \mathcal{D}z\ln\left[\int \mathcal{D}y \cosh^m\left(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2}\right)\right]+m\int \mathcal{D}z\frac{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})\ln[\cosh(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})]}{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}, \tag{23}$$

$$M=\int \mathcal{D}z\frac{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})(H_z^\prime/\sqrt{H_z^{\prime 2}+\Gamma^2})\tanh(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}, \tag{24}$$

$$Q_0=\int \mathcal{D}z\left[\frac{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})(H_z^\prime/\sqrt{H_z^{\prime 2}+\Gamma^2})\tanh(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}\right]^2, \tag{25}$$

$$Q_1=\int \mathcal{D}z\frac{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})(H_z^{\prime 2}/(H_z^{\prime 2}+\Gamma^2))\tanh^2(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}, \tag{26}$$

$$R=\int \mathcal{D}z\frac{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})(H_z^{\prime 2}/(H_z^{\prime 2}+\Gamma^2))}{\int \mathcal{D}y \cosh^m(\bar{\beta}\sqrt{H_z^{\prime 2}+\Gamma^2})}. \tag{27}$$

We have $R=1.0$ at $\Gamma=0.0$, when the free energy f_{1RSB}, the magnetization M, and the spin-glass order parameters Q_0 and Q_1 are reduced exactly to the Parisi solution of the SK model.[4,27] From the above solutions we can obtain the linear susceptibility χ_{1RSB} for $J_0=H=0.0$ as

$$\chi_{1RSB}=\bar{\beta}[R-Q_1+m(Q_1-Q_0)] \tag{28}$$

The replica-symmetric results of Eqs. (12)–(16) are recovered when we set $Q_1=Q_0$.

VI. PHASE DIAGRAMS

Now we obtain phase diagrams not only in the temperature (T)–transverse field (Γ) plane but also in the temperature (T)–average interaction strength (J_0), the transverse field (Γ)–average interaction strength (J_0), the temperature (T)–external longitudinal field (H), and the transverse field

(Γ)–external longitudinal field (H) planes from the replica-symmetric solutions of Sec. III. For the case of $H=0$ the paramagnetic phase can be defined from $M=0$ and $Q=0$, the spin-glass phase from $M=0$ but $Q\neq 0$, and the ferromagnetic phase from $M\neq 0$ and $Q\neq 0$. (The cases $H\neq 0$ are trivial since both magnetization M and the spin-glass order parameter Q become nonzero.) The vanishing of M to zero thus gives a phase boundary between the ferromagnetic phase and others,

$$J_0^{-1}=\int \mathcal{D}z\left[\frac{\Gamma^2}{(H_z^2+\Gamma^2)^{3/2}}\tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2})+\bar{\beta}\frac{H_z^2}{H_z^2+\Gamma^2}\,\mathrm{sech}^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2})\right], \quad (29)$$

where we have $H_z\equiv J\sqrt{Q}z$ with $M=H=0$. In addition, the condition of Q vanishing to zero gives the phase boundary between the paramagnetic phase and others,

$$J^{-2}=\int \mathcal{D}z z^2\left[\frac{\Gamma^2}{(H_z^2+\Gamma^2)^2}\tanh^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2})+\bar{\beta}\frac{H_z^2}{(H_z^2+\Gamma^2)^{3/2}}\tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2})\times\mathrm{sech}^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2})\right], \quad (30)$$

where we have $H_z\equiv J_0M$ with $Q=H=0$. A further restriction of $M=0$, leading to $H_z=0$, simplifies this phase boundary to

$$\Gamma/J=\tanh(\bar{\beta}\Gamma), \quad (31)$$

which is the same result as for the SK model with a transverse field.[28]

The de Almeida-Thouless instability line (the AT line) separates the spin-glass phase from the paramagnetic (or ferromagnetic) phase,

$$(\bar{\beta}J)^{-2}=\int \mathcal{D}z\frac{H_z^4}{(H_z^2+\Gamma^2)^2}\mathrm{sech}^4(\bar{\beta}\sqrt{H_z^2+\Gamma^2}), \quad (32)$$

where $H_z\equiv J\sqrt{Q}z+J_0M+H$. From the above equations all together we can complete the phase diagrams of the infinite-range Ising model spin glass with a transverse field.

Figure 4 is a phase diagram in the temperature (T/J)–transverse field (Γ/J) plane presenting the paramagnetic–spin-glass phase boundary of Eq. (31) and the AT line of Eq. (32) for $H/J=0.0$ and $0.0\leq J_0/J\leq 1.0$. The AT line reaches zero temperature as Γ/J increases to 0.06. The critical Γ/J value of the zero-temperature glass transition, $\Gamma_c(0)/J$ is 1.0 but the zero temperature crossing of the AT line is at 0.06. The range of Γ/J for the RSB-SG phase thus becomes very narrow.

Thirumalai *et al.*[13] studied only the region close to the paramagnetic–spin-glass phase boundary to check for the existence of the replica symmetric spin-glass (RS-SG) phase by using perturbation expansion of R and Q. But we have considered the whole region of temperature (T/J)–transverse field (Γ/J) plane by analyzing the AT stability condition to confirm that RS-SG phase is stable over a wider region of the spin-glass phase. All the cases of $\Gamma/J>0.06$ thus do not have an effect of RSB, and the RS solution of Sec. III should be valid in the whole temperature region.

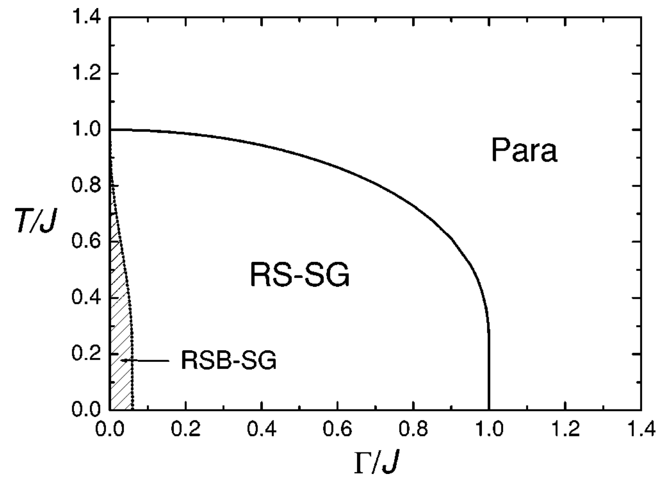

FIG. 4. Phase diagram in the temperature (T/J)–transverse field (Γ/J) plane depicting the paramagnetic–spin glass phase boundary for $H/J=0.0$ and $0.0\leq J_0/J\leq 1.0$. The solid line denotes the paramagnetic—RS-SG phase boundary, and the shaded region under the dotted line indicates the RSB-SG phase under the AT line. Equations (31)–(34) are used.

In Figs. 5 and 6, we show the phase diagrams under a zero external longitudinal field ($H/J=0.0$). The phase boundaries among paramagnetic, spin glass, ferromagnetic, and mixed (spin glass + ferromagnetic) phases are determined by Eqs. (29), (30), and (32). The paramagnetic-ferromagnetic phase boundary is given by the overlap between Eqs. (29) and (30), the paramagnetic and RS-SG phase boundary by Eq. (30), and the boundary between RS-SG and RSB-SG phase, and the ferromagnetic-mixed phase boundary by the AT line of Eq. (32). The spin-glass–mixed phase boundary for $T/J<1.0$ is given by a vertical straight line.[29]

In Figs. 5(a) and 5(b) we show the phase diagrams in the temperature (T/J)–average interaction bias (J_0/J) plane at transverse fields of $\Gamma/J=0.04$ and 0.50. From the AT line at $\Gamma/J=0.04$ we can confirm the existence of both the RSB-SG phase and a mixed phase characterized by nonzero M and the AT instability. The area of RSB-SG and mixed phase regions diminishes with a rapid fall off of the AT line as Γ/J increases as can be seen from Fig. 5(b).

In Fig. 6 we show the phase diagram in the transverse field (Γ/J)–average interaction bias (J_0/J) plane at temperature $T/J=0.50$. We can see that the RSB-SG phase has a very small area near the zero transverse field.

In Figs. 7 and 8, we show the phase diagrams under a zero average interaction bias ($J_0/J=0.0$). The phase boundary between paramagnetic and RS-SG phases is determined from

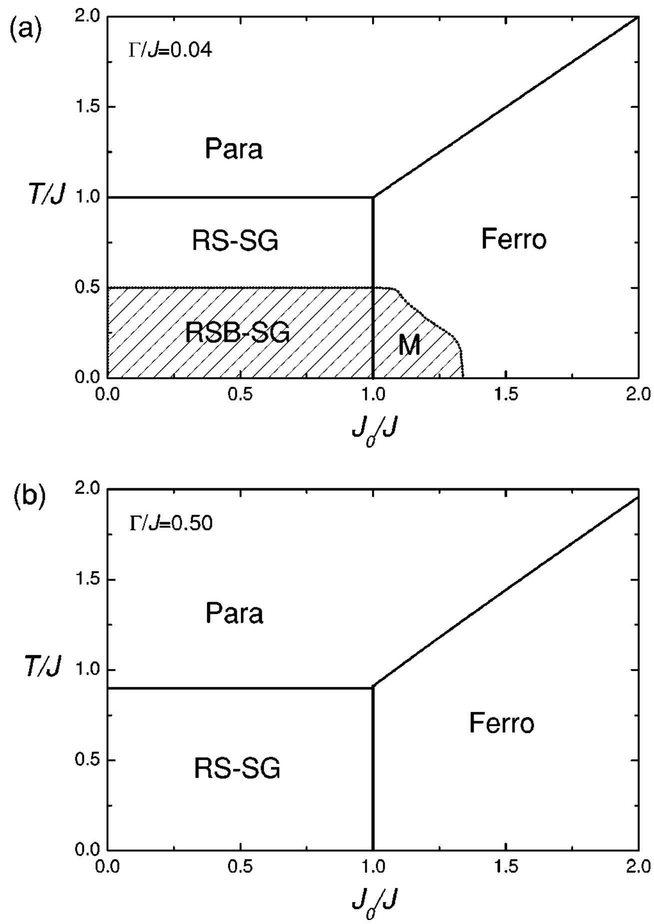

FIG. 5. Phase diagrams in the temperature (T/J)–average interaction bias (J_0/J) plane at transverse fields (a) $\Gamma/J=0.04$ and (b) $\Gamma/J=0.50$. The solid line denotes the paramagnetic–RS-SG/ferromagnetic phase boundary, and the shaded region under the dotted line in (a) indicates the RSB phase [RSB-SG or mixed (M) phase] under the AT line. Equations (30) and (32) are used.

the peaks of the linear susceptibility χ (or the nonlinear susceptibility χ_{nl}), and the boundary between RS-SG and RSB-SG phases is shown by AT line.

In Figs. 7(a) and 7(b) we show the phase diagrams in the temperature (T/J)–external longitudinal field (H/J) plane at transverse fields of $\Gamma/J=0.04$ and 0.50. The $T-H$ AT line at $\Gamma/J=0.04$ separates the RS-SG phase from the RSB-SG phase, where the AT line at a high value of H/J shows a rapid drop in contrast to the case of the SK model. The area of the RSB-SG phase region can be seen to diminish with a rapid falloff of the AT line as Γ/J increases.

In Fig. 8 we show the phase diagram in the transverse field (Γ/J)—external longitudinal field (H/J) plane at a temperature of $T/J=0.50$. The RSB-SG phase is restricted to a very small region near the zero transverse field.

Our result is different from the results of Büttner and Usadel[15,17] and Goldschmidt and Lai,[16] where there is no stable RS-SG phase in the whole spin glass phase. The only difference between our case considering the static approximation and the other cases lies in the spin self-interaction $R^{\alpha\alpha}(\tau,\tau')=\langle \mathcal{T}\sigma_\alpha^z(\tau)\sigma_\alpha^z(\tau')\rangle$, whether it is static[24] (i.e., independent of $\tau-\tau'$) or not.

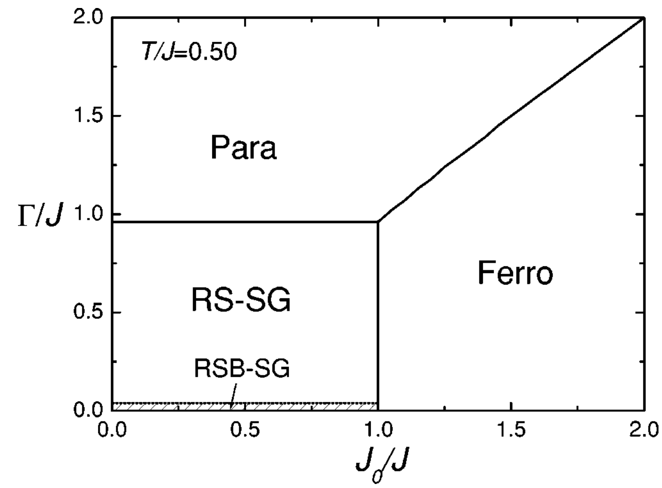

FIG. 6. Phase diagram in the transverse field (Γ/J)—average interaction bias (J_0/J) plane at temperature $T/J=0.50$. The solid line denotes the paramagnetic—RS-SG/ferromagnetic phase boundary, and the shaded region under the dotted line indicates the RSB-SG phase under the AT line. Equations (30) and (32) are used.

The result of Goldschmidt and Lai[16] following the Trotter-Suzuki formalism can be translated to the imaginary-time formalism. The relation between $R^{\alpha\alpha}(\tau,\tau')$ and $|\tau-\tau'|$ then shows that when the static approximation is not used, $R^{\alpha\alpha}(\tau,\tau')$ increases monotonically as $|\tau-\tau'|$ is reduced to zero (see Fig. 4 of Ref. 16). The relation between $R^{\alpha\alpha}(\tau,\tau')$ and $|\tau-\tau'|$ has a direct effect on the equation of AT line and thus on the integral term of the AT line equation. This integral, proportional to the square of $R^{\alpha\alpha}(\tau,\tau')$, increases as $|\tau-\tau'|$ decreases. When the static approximation is not enforced the RS-SG phase may thus diminish, as predicted from the other results.

VII. NONLINEAR SUSCEPTIBILITY

Nonlinear susceptibility χ_{nl} has drawn a great concern due to a possibility of divergence at the glass transition temperature T_g in canonical[1,30] and quantum spin-glass systems.[10] As the linear susceptibility χ shows a cusp instead of a divergence at T_g in spin-glass systems, many research workers searched for a quantity of a divergence anomaly at T_g,[1,31] and defined the spin-glass susceptibility $\chi_{SG} \equiv(\overline{\beta}^2/N)\Sigma_{i,j}^N[(\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle)^2]_J$. This spin-glass susceptibility χ_{SG} can be determined by a measurement of the third-order nonlinear susceptibility[32] $\chi_{nl}\equiv-(1/3!)\times(\partial^3M/\partial H^3)$ from a simple relation[1,33] at $T>T_g$ and $H=0$:

$$\chi_{nl}=\overline{\beta}(\chi_{SG}-2\overline{\beta}^2/3). \qquad (33)$$

Measurements of χ_{nl} can therefore give important information about the spin-glass transition, and the nonlinear susceptibility χ_{nl} of spin glasses has been studied by many experimental measurements.[10,30]

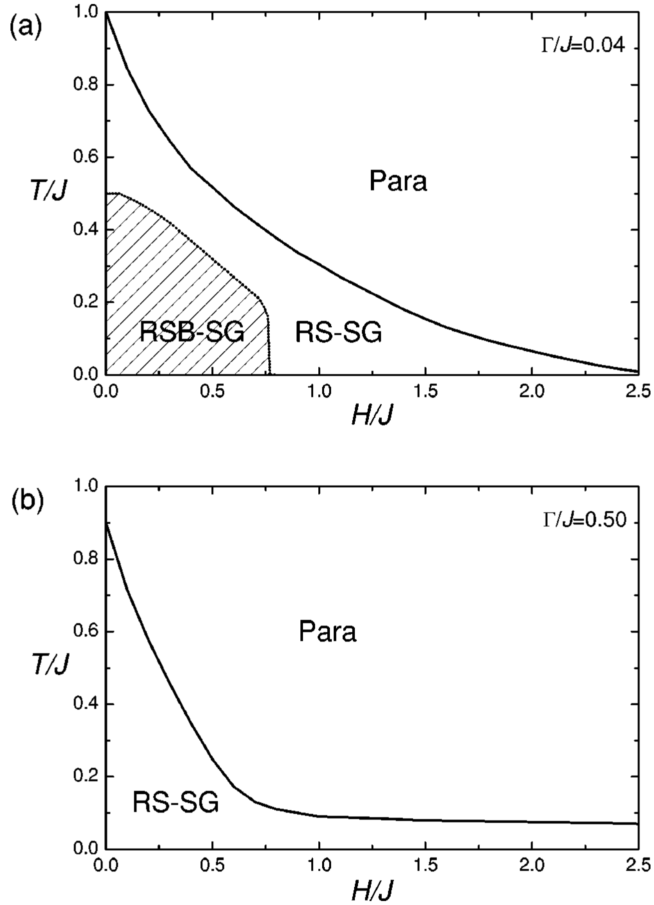

FIG. 7. Phase diagrams in the temperature (T/J)—longitudinal field (H/J) plane at transverse fields (a) $\Gamma/J=0.04$ and (b) $\Gamma/J=0.50$. The solid line denotes the paramagnetic–RS-SG phase boundary as determined by peaks of the linear susceptibility χ, and the shaded region under the dotted line in (a) indicates the RSB-SG phase under the AT line.

Before a consensus was drawn from experimental measurements of χ_{nl} in a spin glass, theoretical opinions prevailed that χ_{nl} should diverge in a power law of

$$\chi_{nl}(T)\propto|(T-T_g)/T_g|^{-\gamma'}, \tag{34}$$

with γ' corresponding to some critical exponent of the spin-glass transition as the temperature approaches T_g.[33–36] In particular $\gamma'=1.0$ was obtained from the SK theory[34] and by Landau-type phenomenological theory[35,36] for $T>T_g$. Most experimental results then tended to support the glass transition divergence by fitting with a sample-dependent γ' value.

On the other hand, there have been many theoretical attempts to calculate the nonlinear susceptibility $\chi_{nl}(\Gamma)$ in the zero-temperature quantum transition of an Ising model spin glass with a transverse field.[37–40] Analytic works[37,38] and two-dimensional (2D) and 3D Monte Carlo simulations[39,40] all made the conclusion that $\chi_{nl}(\Gamma)$ should diverge as transverse field Γ approaches a critical value Γ_c at zero temperature. Similarly to the power law of Eq. (34), Read *et al.*[37] showed that $\chi_{nl}(\Gamma)$ should diverge as

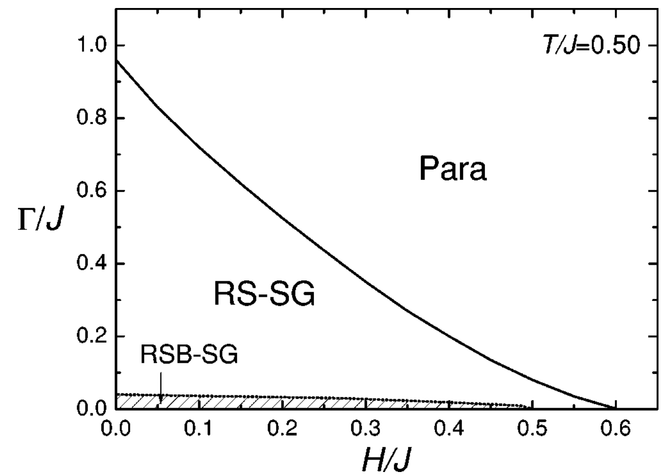

FIG. 8. Phase diagram in the transverse field (Γ/J)–longitudinal field (H/J) plane at temperature $T/J=0.50$. The solid line denotes the paramagnetic–RS-SG phase boundary as determined by peaks of the linear susceptibility χ, and the shaded region under the dotted line indicates the RSB-SG phase under the AT line.

$$\chi_{nl}(\Gamma)\propto|(\Gamma-\Gamma_c)/\Gamma_c|^{-\delta'}, \tag{35}$$

with a power-law exponent $\delta'=1.0$ as the transverse field approaches Γ_c for $\Gamma>\Gamma_c$. The experimental observation in the dipolar-coupled Ising spin glass $LiHo_xY_{1-x}F_4$ showed that its nonlinear susceptibility $\chi_{nl}(\Gamma)$, measured at $T=98$ mK, had a divergence as the transverse field Γ approached a critical value Γ_c, but was best fitted by $\delta'\simeq 0.20$.[10]

The replica-symmetric nonlinear susceptibility χ_{nl} in the third order is defined by the coefficient of the third-order term in the expansion of M [Eq. (13)],

$$\chi_{nl}\equiv-\frac{1}{3!}\frac{\partial^3 M}{\partial H^3}$$
$$=-\frac{1}{6}\left[\frac{1}{(1-J_0\chi_0)^2}\frac{\partial^2\chi_0}{\partial H^2}+\frac{2J_0}{(1-J_0\chi_0)^3}\left(\frac{\partial\chi_0}{\partial H}\right)^2\right], \tag{36}$$

where χ_0 is the replica-symmetric linear susceptibility for $J_0=0.0$, and is given by

$$\chi_0=B_0+\bar{\beta}C_0+A(B_1+\bar{\beta}C_1),$$

with

$$A\equiv\frac{D_0+\bar{\beta}E_0}{\sqrt{Q}-(D_1+\bar{\beta}E_1)},$$

$$B_n\equiv\int\mathcal{D}z z^n\frac{\Gamma^2}{(H_z^2+\Gamma^2)^{3/2}}\tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2}),$$

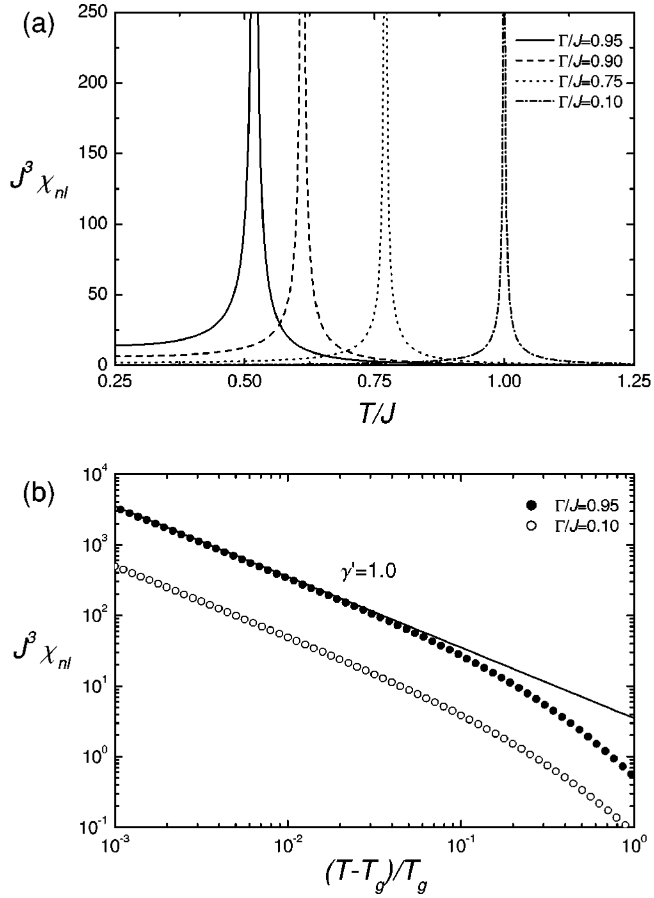

FIG. 9. (a) Nonlinear susceptibility χ_{nl} vs temperature T/J at four selected transverse fields ($\Gamma/J=0.10$, 0.75, 0.90, and 0.95) with $H/J=0.0$ and $J_0/J=0.0$. (b) A log-log plot of the nonlinear susceptibility χ_{nl} vs reduced temperature $(T-T_g)/T_g$ [in the temperature region $(T-T_g)/T_g<1$ of general experimental concerns (Ref. 30)] at two selected transverse fields ($\Gamma/J=0.10$ and 0.95) with $H/J=0.0$ and $J_0/J=0.0$.

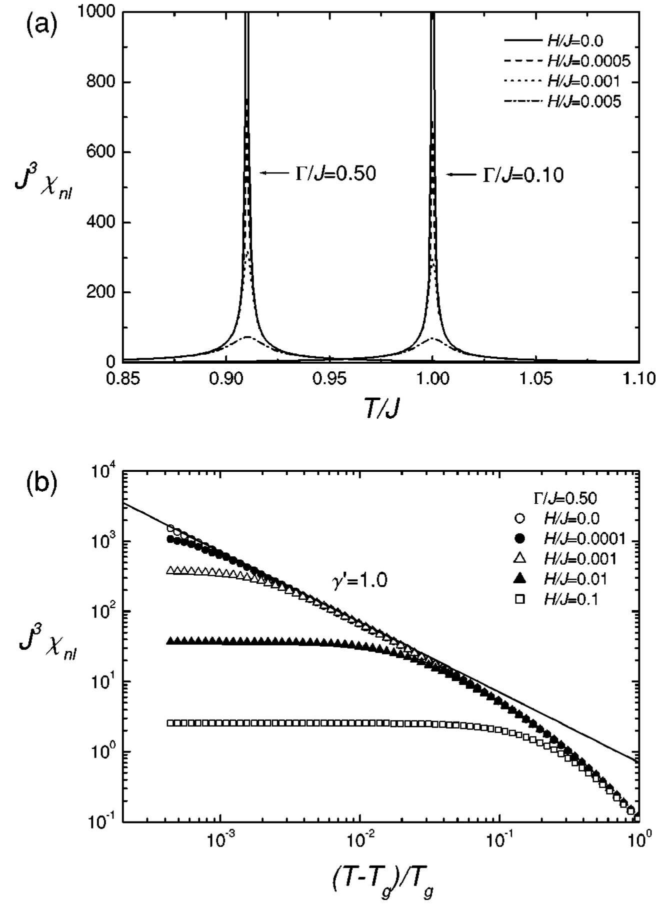

FIG. 10. (a) Nonlinear susceptibility χ_{nl} vs temperature T/J at selected transverse fields ($\Gamma/J=0.10$ and 0.50) and longitudinal fields (H/J) with $J_0/J=0.0$. (b) A log-log plot of the nonlinear susceptibility χ_{nl} vs reduced temperature $(T-T_g)/T_g$ (in the temperature region $(T-T_g)/T_g<1$) at a fixed transverse field $\Gamma/J=0.50$, and selected longitudinal fields (H/J) for $J_0/J=0.0$.

$$C_n \equiv \int \mathcal{D}z z^n \frac{H_z^2}{H_z^2+\Gamma^2} \operatorname{sech}^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}),$$

$$D_n \equiv \int \mathcal{D}z z^n \frac{H_z\Gamma^2}{(H_z^2+\Gamma^2)^2} \tanh^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}),$$

$$E_n \equiv \int \mathcal{D}z z^n \frac{H_z^3}{(H_z^2+\Gamma^2)^{3/2}} \tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \times \operatorname{sech}^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}).$$

For $H=0.0$ we can see B_0, B_1, and C_1 vanishing to obtain $\chi_0=\bar{\beta}C_0=\bar{\beta}(R-Q)$ [Eq. (16)].

In Fig. 9(a) we present the nonlinear susceptibility χ_{nl} vs temperature T/J at four selected transverse fields ($\Gamma/J=0.10$, 0.75, 0.90, and 0.95) with $H/J=0.0$ and $J_0/J=0.0$. From this figure we can see that χ_{nl} diverges to infinity at $T=T_g$. As Γ/J is increased, T_g is shifted to lower temperatures. In Fig. 9(b) we redraw our results for a log-log plot of the nonlinear susceptibility χ_{nl} vs reduced temperature $(T-T_g)/T_g$ in the temperature region $0.001<(T-T_g)/T_g<1$ of experimental concerns[30] at two selected transverse fields $\Gamma/J=0.10$ and 0.95 with $H/J=0.0$ and $J_0/J=0.0$ to check for a power-law divergence at $T>T_g$. When we attempt to fit this curve by Eq. (34), we can see that χ_{nl} shows a divergence to infinity with a power-law exponent $\gamma'=1.0$ as the temperature approaches the glass transition temperature T_g closely, which agrees with the mean-field theories.[34–36] But, if we fit in the region $0.1<(T-T_g)/T_g<1$, we obtain γ' larger than 1.0, which agrees with the previous experimental works.[30] We can thus see that the prediction that χ_{nl} diverges with $\gamma'=1.0$ may be valid only in the temperature region close to T_g, i.e., $(T-T_g)/T_g<0.1$.

In Fig. 10(a) we present the nonlinear susceptibility χ_{nl} vs temperature T/J at two selected transverse fields $\Gamma/J=0.10$ and 0.50 with various longitudinal fields (H/J) and $J_0/J=0.0$. At zero longitudinal field we can see that χ_{nl} diverges to infinity at $T=T_g$. But the nonzero external field smeared out the divergence of χ_{nl}. We can thus conclude that the longitudinal field may smear out the transition anomalies not only in the linear susceptibility χ (see Ref. 2) but also in the nonlinear susceptibility χ_{nl}. In particular, a small field such as $H/J=0.005$, which has a negligible effect on the cusp in linear susceptibility χ, can make the divergence of χ_{nl} at T_g smear out. In Fig. 10(b) we redraw our results in a log-log plot of the nonlinear susceptibility χ_{nl} vs reduced temperature $(T-T_g)/T_g$ at a fixed transverse field $\Gamma/J=0.50$ with various external longitudinal fields (H/J) and $J_0/J=0.0$ to check for a power-law divergence at $T>T_g$. This figure shows a strong dependence of the nonlinear susceptibility χ_{nl} on the external field in the temperature region of $(T-T_g)/T_g<0.01$, where most experimental data are reported.[30] At $H/J=0.0$ we can see that χ_{nl} shows a divergence to infinity with a power-law exponent $\gamma'=1.0$ as the temperature approaches the glass transition temperature T_g, which agrees with the mean-field theories.[34–36] But the nonzero longitudinal field can be seen to prevent χ_{nl} from divergence, and make it flatten out as temperature approaches T_g. Most experimental results[30] indeed show this flattening of χ_{nl} near T_g, which may well result from the applied field effect in the χ_{nl} measurements. In the temperature region of $0.1<(T-T_g)/T_g<1$, we can see that χ_{nl} becomes independent of the applied field with a fixed slope larger than $\gamma'=1.0$.

In Fig. 11(a) we present the nonlinear susceptibility χ_{nl} vs transverse field Γ/J at four selected temperatures $(T/J=0.01,\ 0.75,\ 0.90,\ \text{and}\ 0.95)$ with $H/J=0.0$ and $J_0/J=0.0$. From the figure we can see that χ_{nl} diverges to infinity as the transverse field approaches a critical value Γ_c. As T/J is increased, Γ_c is shifted to lower values. In Fig. 11(b) we redraw our results in a log-log plot of the nonlinear susceptibility χ_{nl} vs reduced transverse field $(\Gamma-\Gamma_c)/\Gamma_c$ in the field region $0.001<(\Gamma-\Gamma_c)/\Gamma_c<10$ corresponding to the experimental concerns of Wu *et al.*[10] at two selected temperatures $T/J=0.01$ and 0.95 with $H/J=0.0$ and $J_0/J=0.0$ to check for a power-law divergence at $T>T_g$. When we fit this result by Eq. (35), we can see that χ_{nl} diverges to infinity with a power-law exponent $\delta'=1.0$ as the transverse field approaches a critical value Γ_c not only at zero temperature but also finite temperatures. But, if we try to fit in the region $0.1<(\Gamma-\Gamma_c)/\Gamma_c<10$, we obtain δ' larger than 1.0. Wu *et al.*[10] measured $\chi_{nl}(\Gamma)$ at $T=98$ mK, observing a power-law divergence with an exponent $\delta'\simeq0.20$. We can thus confirm that the prediction of divergence in $\chi_{nl}(\Gamma)$ with $\delta'=1.0$ may be valid only in the region $(\Gamma-\Gamma_c)/\Gamma_c<0.1$.

In Fig. 12(a) we present the nonlinear susceptibility χ_{nl} vs transverse field Γ/J at two selected temperatures $T/J=0.01$ and 0.50 with various external longitudinal fields (H/J) and $J_0/J=0.0$. We can see that zero field χ_{nl} diverges to infinity at $\Gamma=\Gamma_c$ but at finite longitudinal field the divergence of χ_{nl} smears out. In Fig. 12(b) we redraw our results in a log-log plot of the nonlinear susceptibility χ_{nl} vs reduced transverse field $(\Gamma-\Gamma_c)/\Gamma_c$ at a selected temperature $T/J=0.50$ for various longitudinal fields (H/J) and $J_0/J=0.0$ to check for a power-law divergence at $T>T_g$. Varying characteristics of the power-law divergence and a flattening out were observed in $\chi_{nl}(\Gamma,H)$ of the dipolar-coupled Ising spin glass $LiHo_xY_{1-x}F_4$ under external Γ and H fields.[10]

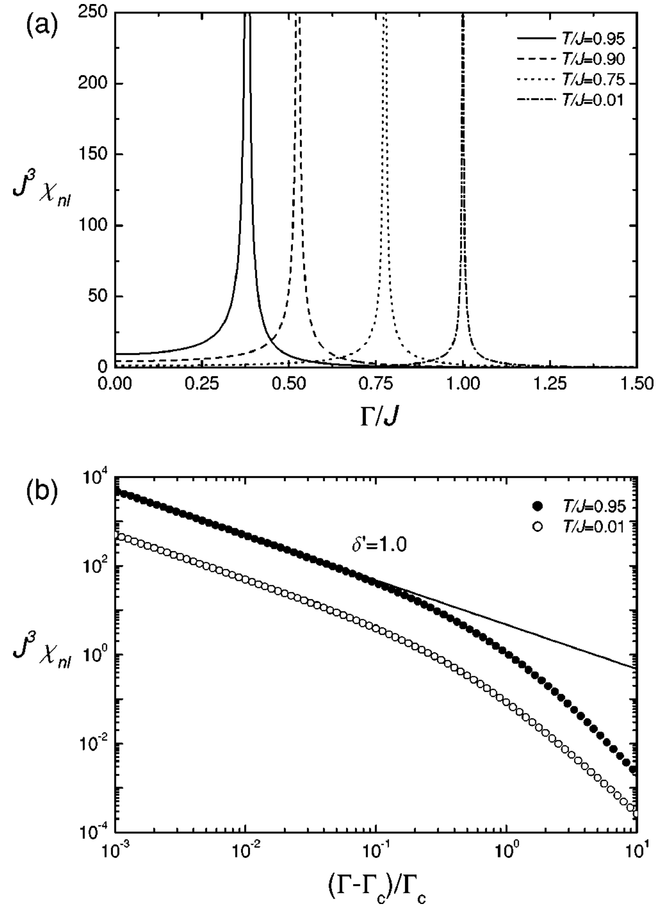

FIG. 11. (a) Nonlinear susceptibility χ_{nl} vs transverse field Γ/J at four selected temperatures $(T/J=0.01,\ 0.75,\ 0.90,\ \text{and}\ 0.95)$ with $H/J=0.0$ and $J_0/J=0.0$. (b) A log-log plot of the nonlinear susceptibility χ_{nl} vs reduced transverse field $(\Gamma-\Gamma_c)/\Gamma_c$ [in the region of experimental concerns of Wu *et al.* (Ref. 10)] at two selected temperatures $(T/J=0.01$ and $0.95)$ with $H/J=0.0$ and $J_0/J=0.0$.

In the laboratory experiment, it is very difficult to apply a magnetic field $H_t\propto\sqrt{\Gamma}$ exactly transverse to the magnetic easy axis of the $LiHo_xY_{1-x}F_4$ sample, parallel to the c axis in this tetragonal system. A tilt of only 1°, bringing about a very small longitudinal field of $H_i\equiv H_t\sin(1°)$, can influence the nonlinear susceptibility χ_{nl}, which is very sensitive to longitudinal field, as can be seen from Figs. 10 and 12. The longitudinal field $H_i\equiv H_t\sin(\theta)$ induced by a tilt θ will increase in proportion to H_t. In an experiment where the transverse magnetic field H_t is externally tunable, this tilt-induced effect cannot be neglected. Figure 13 shows this tilt effect, which resembles the observations of Wu *et al.*[10]

Figure 13(a) shows the nonlinear susceptibility χ_{nl} as a function of transverse magnetic field $H_t\equiv\sqrt{\Gamma/J}$ at four selected temperatures $T/J=0.01$, 0.75, 0.90, and 0.95 for an induced longitudinal field of $H_i/J\equiv H_t\sin(\theta)$ with $\theta=1°$ and $J_0/J=0.0$. At a very small tilt angle $\theta=1°$, the divergence of the nonlinear susceptibility can be seen to be suppressed with only the external transverse field H_t applied and the temperature T lowered, which shows a resemblance to the results of Wu *et al.*[10] Figure 13(b) shows the linear susceptibility χ obtained under the same conditions as Fig. 13(a). Though

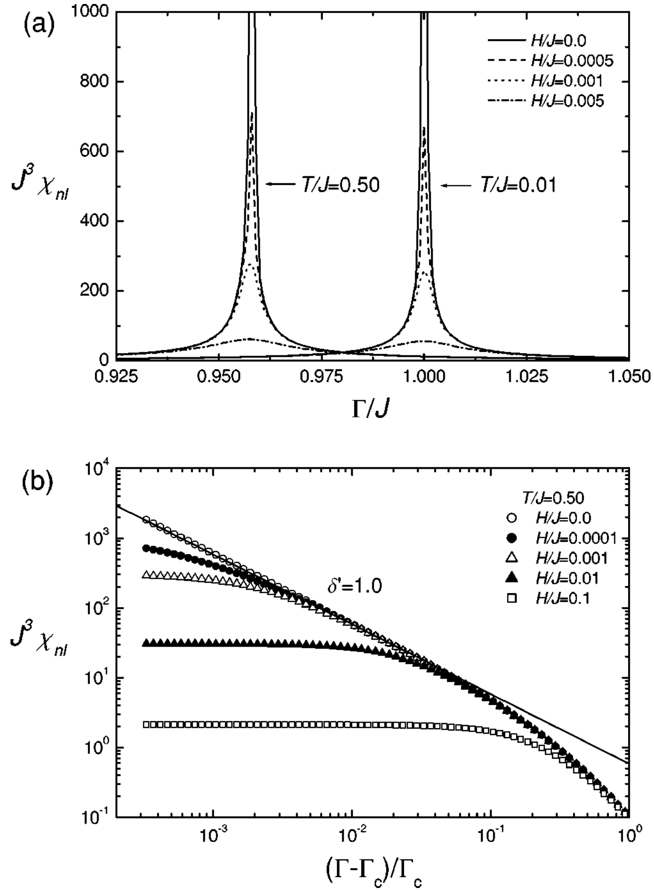

FIG. 12. (a) Nonlinear susceptibility χ_{nl} vs transverse field Γ/J at two fixed temperatures ($T/J=0.01$ and 0.50) and selected longitudinal fields (H/J) with $J_0/J=0.0$. (b) A log-log plot of the nonlinear susceptibility χ_{nl} vs reduced transverse field $(\Gamma-\Gamma_c)/\Gamma_c$ (in the region $(\Gamma-\Gamma_c)/\Gamma_c<1$) at a fixed temperature $T/J=0.50$ and selected longitudinal fields (H/J) with $J_0/J=0.0$.

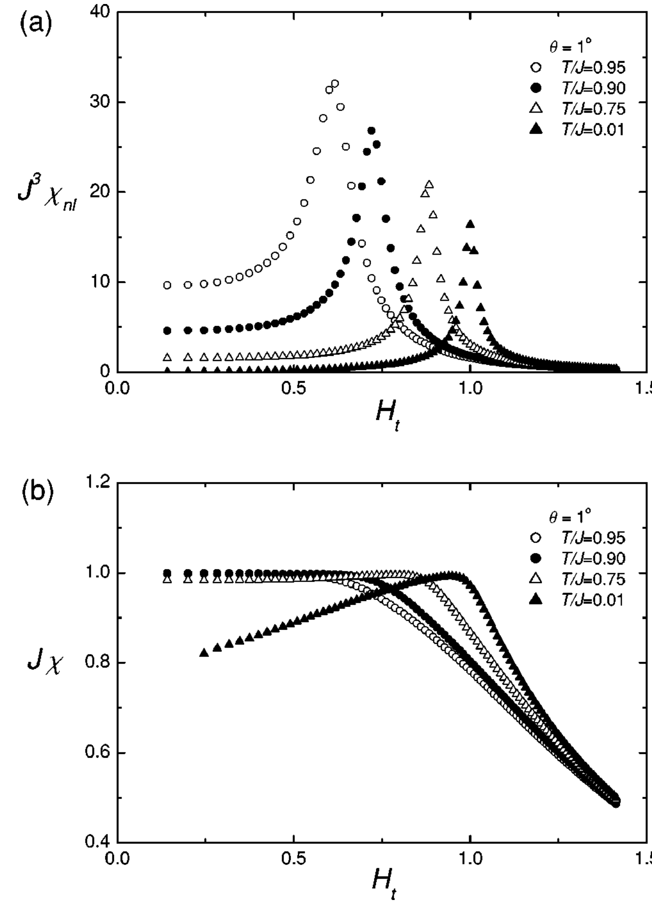

FIG. 13. (a) Nonlinear susceptibility χ_{nl} as a function of the transverse magnetic field $H_t \equiv \sqrt{\Gamma/J}$ at four selected temperatures ($T/J=0.01$, 0.75, 0.90, and 0.95) with $\theta=1°$ and $J_0/J=0.0$. The misfit angle θ can induce a longitudinal field of $H_i/J \equiv H_t \sin(\theta)$. (b) Linear susceptibility χ as a function of H_t, obtained under the same condition as (a).

our static linear susceptibility cannot be compared directly with the frequency-dependent linear susceptibility of Wu *et al.*, [10] our linear susceptibility also shows a stronger cusp at higher H_t with lowering T in conformity with the results of Wu *et al.*[10] and Rozenberg and Grempel.[41] Our results, however, do not show any evidence of a first-order transition. Our result for the SK model with a transverse field can thus explain the experimental observations in the quantum spin glass $LiHo_xY_{1-x}F_4$, in which the divergence of the nonlinear susceptibility becomes suppressed whereas the linear susceptibility continues to show a sharp peak, although it does not confirm the first-order transition.

VIII. SPECIFIC HEAT

In our theoretical calculations the constant-field magnetic specific heat C_H can be obtained from the second-order temperature derivative of the free energy [Eq. (12)],

$$C_H \equiv -T\frac{\partial^2 f}{\partial T^2}\bigg|_H \equiv T\frac{\partial S}{\partial T}\bigg|_H \tag{37}$$

where $S \equiv \partial f/\partial T$ is entropy.

In Fig. 14(a) and 14(b) we present the specific heat C_H vs temperature T/J at various external longitudinal fields (H/J) and transverse fields $\Gamma/J=0.00$ and 0.50 with $J_0/J=0.0$. As Fig. 14(a) shows, at $\Gamma/J=0.00$ (SK system) and $H/J=0.0$, C_H is proportional to T^{-2} in the paramagnetic phase ($T>T_g$ and $0\leq J_0/J\leq 1$), but linear on the low-temperature side.[2] As Figs. 14(a) and 14(b) show, a cusp appears at the glass transition temperature for all values of Γ/J when H/J is zero as in the SK model.[2] When H/J is given a nonzero finite value, however, the cusp becomes smeared, to give a broad maximum shifting to high-temperature side as H/J increases, which agrees with experimental observations by Brodale *et al.*[42]

In Figs. 15(a) and 15(b) we present the specific heat divided by temperature ($C_H/T=\partial S/\partial T|_H$) and its second derivative by temperature ($\partial^2(C_H/T)/\partial T^2=\partial^3 S/\partial T^3|_H$) vs temperature ($T/J$) at $\Gamma/J=0.00$ (SK system) and various external longitudinal fields (H/J) for $J_0/J=0.0$. Figure 15(a) shows the well-known broad maximum without any distinction at the glass transition temperature T_g under a finite external longitudinal field H/J. With increasing H/J the maximum peaks shift to lower temperatures and decrease

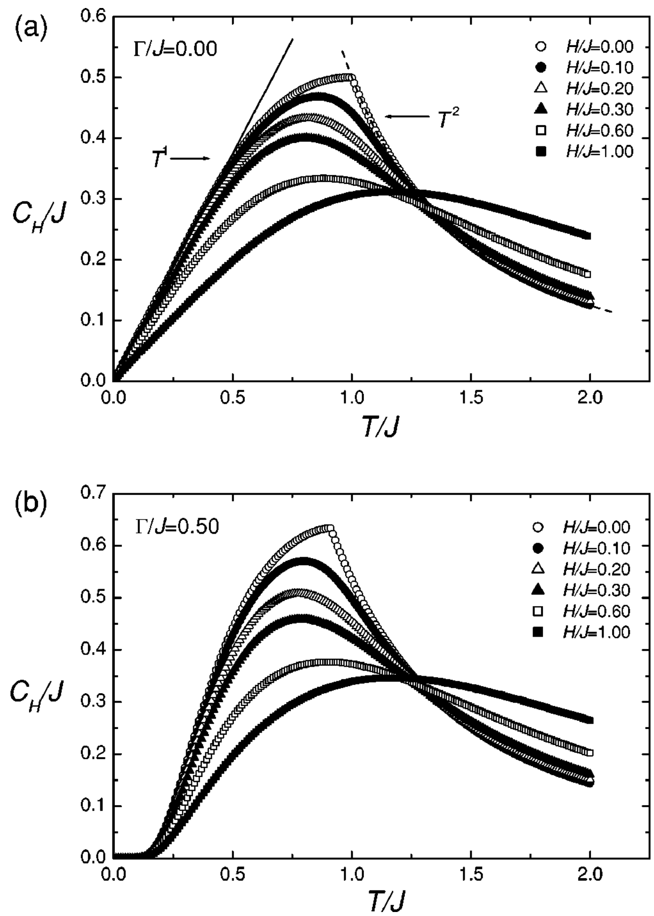

FIG. 14. Specific heat C_H vs temperature T/J at various longitudinal fields (H/J) with $J_0/J=0.0$ and the transverse field fixed at (a) $\Gamma/J=0.00$ (SK system) and (b) $\Gamma/J=0.50$. The difference is distinctive at very low temperatures near zero.

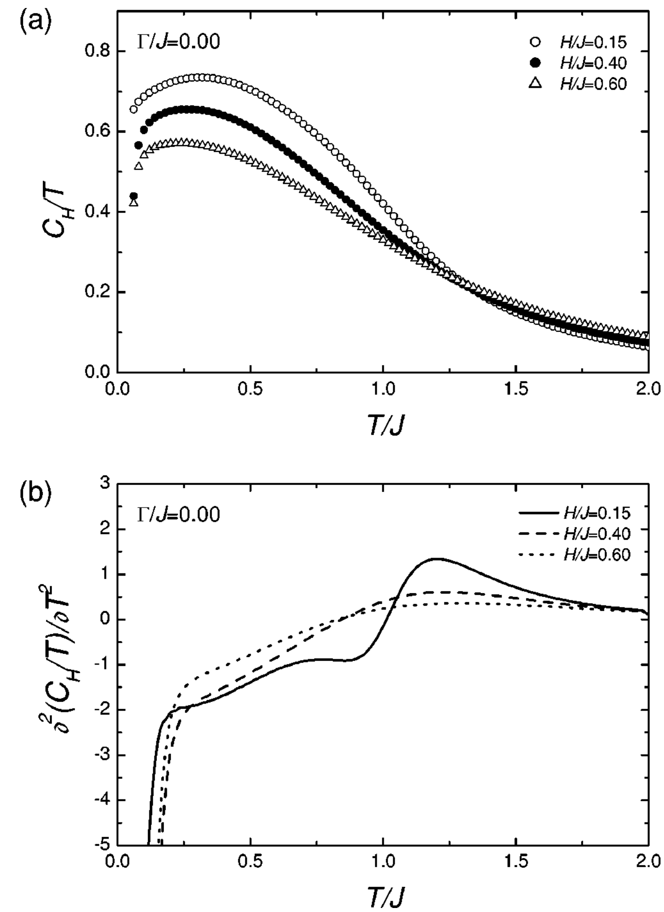

FIG. 15. (a) Specific heat divided by temperature (C_H/T) and (b) its second temperature derivative $[\partial^2(C_H/T)/\partial T^2]$ vs temperature T/J at $\Gamma/J=0.00$ (SK system) and fixed longitudinal fields (H/J) for $J_0/J=0.0$.

in magnitude, in agreement with the experimental observations.[20,43] Figure 15(b) shows an anomaly in $\partial^2(C_H/T)/\partial T^2$ near the glass transition temperature T_g, which disappears with peak shifting to lower temperatures as the external field H/J is increased. This may be evidence of the field-dependent entropy variation in the spin glass transition, which was observed by Fogle *et al.*[20]

In Figs. 16(a) and 16(b) we present $\partial^2(C_H/T)/\partial T^2$ vs T/J at $\Gamma/J=0.50$ and 0.90, respectively, in the same condition with Fig. 15. Figure 16(a) shows a weaker anomaly as compared with the $\Gamma/J=0.00$ case near the glass transition temperature T_g. We can thus see that there exists a thermodynamic anomaly even in the quantum spin-glass transition with nonzero Γ value, because thermal fluctuations at nonzero temperature may have a larger thermodynamic effect in a spin glass than the quantum fluctuation induced by nonzero Γ. Figure 16(b) shows an even weaker anomaly at a larger Γ value. We can thus see that the anomaly is gradually suppressed with an increase of the Γ value, which checks for the quantum tunneling competing against spin freezing. This anomaly disappears with peak shifting to lower temperatures as the external field H/J is increased, which gives, as shown in Fig. 15(b), evidence of the field-dependent entropy variation in the spin glass transition.

IX. CONCLUSION

We have obtained replica-symmetric and replica-symmetry-breaking solutions of the SK model with a transverse field under a static approximation by using the imaginary-time replica formalism. The de Almeida–Thouless stability line in the model has been determined. From these we could obtain various phase diagrams and theoretical results of measurable quantities such as the nonlinear susceptibility χ_{nl} and specific heat C_H. From the above investigations we have shown three important results as follows:

First we have shown that a replica-symmetric quantum spin-glass phase is stable in most parts of the spin-glass phase in the temperature-transverse field phase diagram. This confirms the existence of a stable replica-symmetric spin glass phase under the static approximation, which agrees with Thirumalai *et al.*[13] But our result is in contrast with other results[15–17] avoiding the static approximation, where the replica-symmetric solution is always unstable in the whole spin-glass phase. The difference originates from the static approximation applied to the spin self-interaction $R^{\alpha\alpha}(\tau,\tau')$. When the static approximation is not employed, the RS-SG phase may diminish.

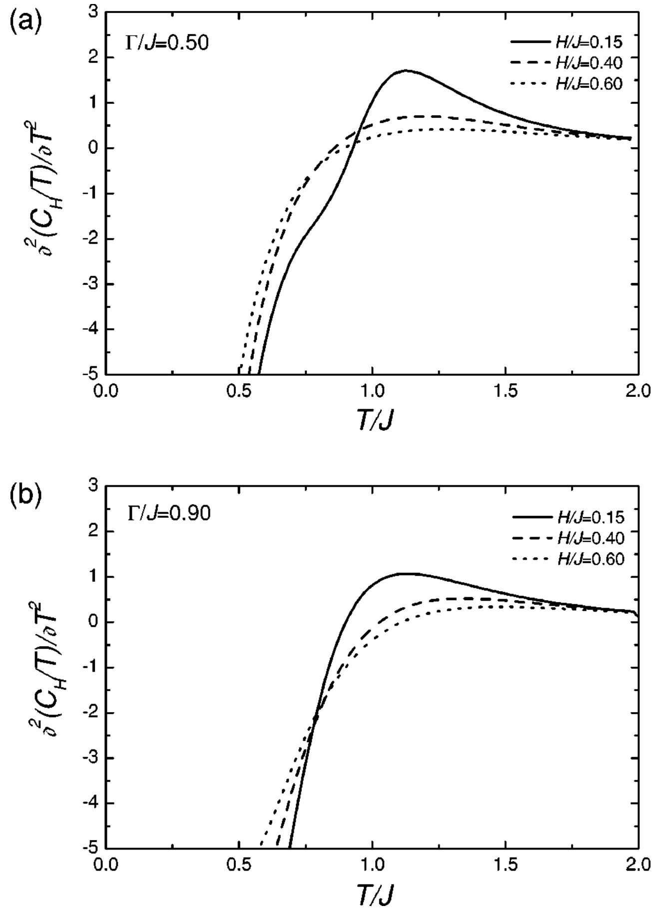

FIG. 16. $\partial^2(C_H/T)/\partial T^2$ vs temperature T/J at (a) $\Gamma/J=0.50$ and (b) $\Gamma/J=0.90$, with the other conditions the same as in Fig. 15.

Second we have shown a theoretical result of the nonlinear susceptibility χ_{nl} conforming with the experimental result of Wu *et al.*[10] This suggests that the SK model with a transverse field may be a realistic model for the quantum spin glass $\mathrm{LiHo}_x\mathrm{Y}_{1-x}\mathrm{F}_4$, though it does not confirm the first-order transition.

Third, in a classical (SK) spin-glass system with a zero transverse field, we have confirmed the anomaly in the second temperature derivative of C_H/T near the glass transition temperature T_g, possibly associated with the field-dependent variation of the entropy in the spin glass transition, in agreement with the earlier experimental observation by Fogle *et al.*[20] We have also shown that this anomaly is suppressed by the nonzero transverse field in a quantum spin-glass system, confirming that quantum tunneling competes against spin freezing. It would be most welcome if a theory is discovered that goes beyond the static approximation used in this paper to obtain analytic solutions as for nonlinear susceptibility and specific heat, and to understand more results of numerical simulations and experimental measurements of quantum spin glasses.

ACKNOWLEDGMENT

This work was supported by the Korea Research Foundation Grant (KRF-2001).

APPENDIX: DE ALMEIDA–THOULESS STABILITY CONDITION UNDER THE STATIC APPROXIMATION

We give detalis of derivation for the de Almeida–Thouless stability condition of the replica-symmetric solution under the static approximation. Under the static approximation, the matrix Δ of Eq. (20) can be seen to take 11 different types of elements.

(1) The coefficients of $\eta^{\alpha\beta}\eta^{\mu\nu}$ terms have three types of elements:

$$(\bar{\beta}J)^2\delta_{(\alpha\beta)(\mu\nu)}-(\bar{\beta}J)^4(P_{\alpha\beta\mu\nu}-Q_{\alpha\beta}Q_{\mu\nu}) = \begin{cases} (\bar{\beta}J)^2-(\bar{\beta}J)^4(P_{22}-Q^2)\equiv\bar{P} & (\alpha=\mu,\beta=\nu)\\ -(\bar{\beta}J)^4(P_{211}-Q^2)\equiv\bar{Q} & (\alpha=\mu,\beta\neq\nu)\\ -(\bar{\beta}J)^4(P_{1111}-Q^2)\equiv\bar{R} & (\alpha\neq\mu,\beta\neq\nu). \end{cases} \tag{A1}$$

(2) The coefficients of $\xi^{\alpha\alpha}\xi^{\beta\beta}$ terms have two types of elements:

$$(\bar{\beta}J)^2\delta_{\alpha\beta}-(\bar{\beta}J)^4(P_{\alpha\alpha\beta\beta}-R_{\alpha\alpha}R_{\beta\beta}) = \begin{cases} (\bar{\beta}J)^2-(\bar{\beta}J)^4(P_{22}-R^2)\equiv\bar{E} & (\alpha=\beta)\\ -(\bar{\beta}J)^4(P_{22}-R^2)\equiv\bar{F} & (\alpha\neq\beta). \end{cases} \tag{A2}$$

(3) The coefficients of $\epsilon^\alpha\epsilon^\beta$ terms have two types of elements:

$$\bar{\beta}J_0\delta_{\alpha\beta}-(\bar{\beta}J_0)^2(Q_{\alpha\beta}-M_\alpha M_\beta) = \begin{cases} \bar{\beta}J_0-(\bar{\beta}J_0)^2(R-M^2)\equiv\bar{A} & (\alpha=\beta)\\ -(\bar{\beta}J_0)^2(Q-M^2)\equiv\bar{B} & (\alpha\neq\beta). \end{cases} \tag{A3}$$

(4) For the coefficients of the $\eta^{\alpha\beta}\xi^{\mu\mu}$ terms we have

$$-(\bar{\beta}J)^4(P_{\alpha\beta\mu\mu}-Q_{\alpha\beta}R_{\mu\mu})=-(\bar{\beta}J)^4(P_{211}-QR)\equiv\bar{G}. \tag{A4}$$

(5) The coefficients of $\eta^{\alpha\beta}\epsilon^{\mu}$ terms have two types of elements:

$$-\bar{\beta}^3J^2J_0(O_{\alpha\beta\mu}-Q_{\alpha\beta}M_\mu) = \begin{cases} -\bar{\beta}^3J^2J_0(O_{21}-QM)\equiv\bar{C} & (\mu=\alpha)\\ -\bar{\beta}^3J^2J_0(O_{111}-QM)\equiv\bar{D} & (\mu\neq\alpha). \end{cases} \tag{A5}$$

(6) For the coefficients of $\xi^{\alpha\alpha}\epsilon^{\beta}$ terms we have

$$-\bar{\beta}^3J^2J_0(O_{\alpha\alpha\beta}-R_{\alpha\alpha}M_\beta)=-\bar{\beta}^3J^2J_0(O_{21}-RM)\equiv\bar{H}. \tag{A6}$$

Variables introduced in the Eqs. (A1)–(A6) are defined as follows:

$$M=\int \mathcal{D}z\,\frac{H_z}{\sqrt{H_z^2+\Gamma^2}}\tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2}),$$

$$Q=\int \mathcal{D}z \frac{H_z^2}{H_z^2+\Gamma^2} \tanh^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \quad (\leftarrow \alpha\neq\beta),$$

$$R=\int \mathcal{D}z \frac{H_z^2}{H_z^2+\Gamma^2} (\leftarrow \alpha=\beta),$$

$$O_{111}=\int \mathcal{D}z \frac{H_z^3}{(H_z^2+\Gamma^2)^{3/2}} \tanh^3(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \ (\leftarrow \alpha\neq\mu),$$

$$O_{21}=\int \mathcal{D}z \frac{H_z^3}{(H_z^2+\Gamma^2)^{3/2}} \tanh(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \ (\leftarrow \alpha=\mu),$$

$$P_{1111}=\int \mathcal{D}z \frac{H_z^4}{(H_z^2+\Gamma^2)^2} \times\tanh^4(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \ (\leftarrow \alpha\neq\mu,\beta\neq\nu),$$

$$P_{211}=\int \mathcal{D}z \frac{H_z^4}{(H_z^2+\Gamma^2)^2} \times\tanh^2(\bar{\beta}\sqrt{H_z^2+\Gamma^2}) \ (\leftarrow \alpha=\mu,\beta\neq\nu),$$

$$P_{22}=\int \mathcal{D}z \frac{H_z^4}{(H_z^2+\Gamma^2)^2} \ (\leftarrow \alpha=\mu,\beta=\nu).$$

We obtain the stability condition for the replica-symmetric solution in accordance with de Almeida and Thouless.[3] In the paramagnetic phase we have $M=Q=\cdots=0$, and the matrix $[\Delta]$ becomes diagonal. The stability condition against ferromagnetic ordering and spin-glass formation is given by $\bar{\beta}J_0\geqslant 0$ and $(\bar{\beta}J)^2\geqslant 0$, respectively. The paramagnetic phase is thus stable under the replica-symmetric assumption.

Similarly, for generalizing to other phases, we need to find all the eigenvalues of the matrix $[\Delta]$. The condition that all these eigenvalues should be positive then gives the stability of the replica-symmetric solutions. To find the eigenvalues of the matrix $[\Delta]$, it is necessary to exploit the permutation symmetry of the matrix elements. Since the matrix is real and symmetric, the matrix order is equal to the number of linearly independent eigenvectors to be found. Hence the matrix $[\Delta]$ will have the order of $n+\frac{1}{2}n(n-1)+n=\frac{1}{2}n(n+3)$, with the eigenvectors $\{\mu\}$ given in the form

$$\mu=\begin{pmatrix}\{\epsilon^\alpha\}\\ \{\eta^{\alpha\beta}\}\\ \{\xi^{\alpha\alpha}\}\end{pmatrix} \quad (\beta<\alpha=1,2,\ldots,n) \tag{A7}$$

where $\{\epsilon^\alpha\}$ and $\{\xi^{\alpha\alpha}\}$ are column vectors with n elements and $\{\eta^{\alpha\beta}\}$ a column vector with $\frac{1}{2}n(n-1)$ elements. It turns out that we can divide the complete set of eigenvectors into three symmetry species by considering distinct eigenvalues in the limit $n\to 0$.

(1) Consider an eigenvector μ_1 of the form with elements $\epsilon^\alpha=a$ for all α, $\eta^{\alpha\beta}=b$ for all $(\alpha\beta)$ and $\xi^{\alpha\alpha}=c$ for all α. Then the eigenvalue equations become

$$\{\bar{A}+(n-1)\bar{B}-\lambda\}a+\left\{(n-1)\bar{C}+\frac{1}{2}(n-1)(n-2)\bar{D}\right\}b+n\bar{H}c=0,$$

$$\{2\bar{C}+(n-2)\bar{D}\}a+\left\{\bar{P}+2(n-2)\bar{Q}+\frac{1}{2}(n-2)(n-3)\bar{R}-\lambda\right\}b+n\bar{G}c=0,$$

$$n\bar{H}a+\frac{1}{2}n(n-1)\bar{G}b+\{\bar{E}+(n-1)\bar{F}-\lambda\}c=0, \tag{A8}$$

which can be solved for three nondegenerate eigenvalues $\{\lambda_1\}$ in the limit $n\to 0$, as given by

$$\lambda^2-(\bar{A}-\bar{B}+\bar{P}-4\bar{Q}+3\bar{R})\lambda+(\bar{A}-\bar{B})(\bar{P}-4\bar{Q}+3\bar{R})+2(\bar{C}-\bar{D})^2=0,$$

$$\lambda=\bar{E}-\bar{F}. \tag{A9}$$

(2) Consider another eigenvector μ_2 of the form with elements given by $\epsilon^\alpha=a$ for $\alpha=\theta$, $\epsilon^\alpha=b$ for $\alpha\neq\theta$, $\eta^{\alpha\beta}=c$ for α or $\beta=\theta$, $\eta^{\alpha\beta}=d$ for α, $\beta\neq\theta$, $\xi^{\alpha\alpha}=e$ for $\alpha=\theta$, and $\xi^{\alpha\alpha}=f$ for $\alpha\neq\theta$ so that each eigenvalue has $(n-1)$-fold degeneracy. By orthogonality to the eigenvector μ_1, we obtain $a=-(n-1)b$, $c=-\frac{1}{2}(n-2)d$ and $e=-(n-1)f$. Then the eigenvalue equations become

$$(\bar{A}-\bar{B}-\lambda)a+(n-1)(\bar{C}-\bar{D})c=0,$$

$$\{(n-2)/(n-1)\}(\bar{C}-\bar{D})a+\{\bar{P}+(n-4)\bar{Q}-(n-3)\bar{R}-\lambda\}c=0,$$

$$(\bar{E}+\bar{F}-\lambda)e=0, \tag{A10}$$

from which we can obtain three $(n-1)$-fold degenerate eigenvalues $\{\lambda_2\}$ in the limit $n\to 0$ as given by

$$\lambda^2-(\bar{A}-\bar{B}+\bar{P}-4\bar{Q}+3\bar{R})\lambda+(\bar{A}-\bar{B})(\bar{P}-4\bar{Q}+3\bar{R})+2(\bar{C}-\bar{D})^2=0,$$

$$\lambda=\bar{E}-\bar{F}, \tag{A11}$$

which are the same as for the case of the eigenvector μ_1.

(3) Consider an eigenvector μ_3 of the form with elements given by $\epsilon^\alpha=a$ for $\alpha=(\theta,\varphi)$, $\epsilon^\alpha=b$ for $\alpha\neq(\theta,\varphi)$, $\eta^{\alpha\beta}=c$ for $\alpha=\theta$, $\beta=\varphi$, $\eta^{\alpha\beta}=d$ for $\alpha=(\theta,\varphi)$ and $\beta\neq(\theta,\varphi)$, $\eta^{\alpha\beta}=e$ for $(\alpha,\beta)\neq(\theta,\varphi)$, $\xi^{\alpha\alpha}=f$ for $\alpha=(\theta,\varphi)$, and $\xi^{\alpha\alpha}=g$ for $\alpha\neq(\theta,\varphi)$. By orthogonality to the eigenvectors μ_1 and μ_2, we obtain the condition $a=b=f=g=0$, so that each eigenvalue has $\frac{1}{2}n(n-3)$-fold degeneracy. The orthogonality to the eigen-

vectors μ_1 and μ_2 also imposes conditions $c=-(n-2)d$ and $e=-\frac{1}{2}(n-3)d$. Eigenvalue equations are then all reduced to give

$$\lambda=\overline{P}-2\overline{Q}+\overline{R}, \tag{A12}$$

leading to $\frac{1}{2}n(n-3)$-fold degeneracy.

Thus we have a trivial solution $\lambda_{1,2}=\overline{E}-\overline{F}=(\overline{\beta}J)^2\geqslant 0$ and three nontrivial solutions in the limit $n\to 0$, given by

$$\lambda_{1,2}^2-(\overline{A}-\overline{B}+\overline{P}-4\overline{Q}+3\overline{R})\lambda_{1,2}+(\overline{A}-\overline{B})(\overline{P}-4\overline{Q}+3\overline{R})+2(\overline{C}-\overline{D})^2=0,$$

$$\lambda_3=\overline{P}-2\overline{Q}+\overline{R}. \tag{A13}$$

Numerical calculations show that $\overline{A}-\overline{B}+\overline{P}-4\overline{Q}+3\overline{R}$ and $(\overline{A}-\overline{B})(\overline{P}-4\overline{Q}+3\overline{R})+2(\overline{C}-\overline{D})^2$ are always positive, and we thus have only the eigenvalue λ_3 to be checked for the stability condition of replica-symmetric solutions:

$$\lambda_3=\overline{P}-2\overline{Q}+\overline{R}=(\overline{\beta}J)^2-(\overline{\beta}J)^4(P_{22}-2P_{211}+P_{1111})$$
$$=(\overline{\beta}J)^2-(\overline{\beta}J)^4\int \mathcal{D}z\frac{H_z^4}{(H_z^2+\Gamma^2)^2}\mathrm{sech}^4(\overline{\beta}\sqrt{H_z^2+\Gamma^2}). \tag{A14}$$

The stability condition of the replica-symmetric solution under static approximation is thus given by

$$(\overline{\beta}J)^{-2}\geqslant\int \mathcal{D}z\frac{H_z^4}{(H_z^2+\Gamma^2)^2}\mathrm{sech}^4(\overline{\beta}\sqrt{H_z^2+\Gamma^2}). \tag{A15}$$

*Corresponding author. Email address: jjkim@kaist.ac.kr

[1] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986); K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1991); J. A. Mydosh, *Spin Glasses: An Experimental Introduction* (Taylor and Francis, London, 1993); D. Sherrington, in *Physics of Novel Materials*, edited by M. P. Das (World Scientific, Singapore, 1999).

[2] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975); S. Kirkpatrick and D. Sherrington, Phys. Rev. B **17**, 4384 (1978).

[3] J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).

[4] M. Mezard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987), and references therein.

[5] R. Pirc, B. Tadić, and R. Blinc, Z. Phys. B: Condens. Matter **61**, 69 (1985); Phys. Rev. B **36**, 8607 (1987).

[6] E. Courtens, J. Phys. (France) Lett. **43**, L-199 (1982).

[7] B. K. Chakrabarti, A. Dutta, and P. Sen, *Quantum Ising Phases and Transitions in Transverse Ising Models* (Springer, Berlin, 1996) and references therein.

[8] A. J. Bray and M. A. Moore, J. Phys. C **13**, L655 (1980).

[9] W. Wu, B. Ellmann, T. F. Rosenbaum, G. Appeli, and D. H. Reich, Phys. Rev. Lett. **67**, 2076 (1991).

[10] W. Wu, D. Bitko, T. F. Rosenbaum, and G. Appeli, Phys. Rev. Lett. **71**, 1919 (1993).

[11] J. Brooke, D. Bitko, T. F. Rosenbaum, and G. Aeppli, Science **284**, 779 (1999).

[12] V. Banerjee and S. Dattagupta, J. Phys.: Condens. Matter **10**, 8351 (1998).

[13] D. Thirumalai, Q. Li, and T. R. Kirkpatrick, J. Phys. A **22**, 3339 (1989).

[14] P. Ray, B. K. Chakrabarti, and A. Chakrabarti, Phys. Rev. B **39**, 11 828 (1989).

[15] G. Büttner and K. D. Usadel, Phys. Rev. B **41**, 428 (1990).

[16] Y. Y. Goldschmidt and P.-Y. Lai, Phys. Rev. Lett. **64**, 2467 (1990).

[17] G. Büttner and K. D. Usadel, Phys. Rev. B **42**, 6385 (1990).

[18] L. F. Cugliandolo, D. R. Grempel, and C. A. da Silva Santos, Phys. Rev. Lett. **85**, 2589 (2000); Phys. Rev. B **64**, 014403 (2001).

[19] L. E. Wenger and P. H. Keesom, Phys. Rev. B **13**, 4053 (1976).

[20] W. E. Fogle, J. D. Boyer, R. A. Fisher, and N. E. Philips, Phys. Rev. Lett. **50**, 1815 (1983).

[21] Ya. V. Federov and E. F. Shender, Pis'ma Zh. Éksp. Teor. Fiz. **43**, 526 (1986) [JETP Lett. **43**, 681 (1986)].

[22] H. F. Trotter, Proc. Am. Math. Soc. **10**, 545 (1959); M. Suzuki, Prog. Theor. Phys. **56**, 1454 (1976).

[23] S. F. Edwards and P. W. Anderson, J. Phys. F: Met. Phys. **5**, 965 (1975).

[24] One may now expand, for example, $Q^{\alpha\beta}(\tau,\tau')$ in a Fourier series as $Q^{\alpha\beta}(\tau,\tau')=\Sigma_j Q_j^{\alpha\beta}\exp[i\omega_j(\tau-\tau')]=Q^{\alpha\beta}+\delta Q^{\alpha\beta}(\tau,\tau')$, where $Q^{\alpha\beta}$ and $\delta Q^{\alpha\beta}(\tau,\tau')$ denote the zero frequency mode and the contribution of nonzero frequency mode, respectively, and $\omega_j\equiv 2\pi j/\overline{\beta}(j=0,1,2,\ldots)$. By the static approximation, $Q^{\alpha\beta}(\tau,\tau')$ becomes independent of $\tau-\tau'$.

[25] The reason why the free energy *f*, magnetization *M*, and spin-glass order parameter *Q* cannot be reduced exactly to those of the SK model at $\Gamma=0.0$ in Ref. 13 (and also Ref. 5) is that the Hubbard-Stratonovitch transform [Eq. (7)] is applied to not only $(\Sigma_\alpha\sigma_\alpha^z)^2$ but also $\Sigma_\alpha(\sigma_\alpha^z)^2$ so that the integral remains inside $\int\mathcal{D}z$. [See also K. Katayama and T. Horiguchi, Physica A **267**, 271 (1999).]

[26] K. H. Fischer, Phys. Rev. Lett. **34**, 1438 (1975).

[27] G. Parisi, Phys. Lett. **73A**, 203 (1979); Phys. Rev. Lett. **43**, 1754 (1979).

[28] K. Walasek and K. Lukierska-Walasek, Phys. Rev. B **38**, 725 (1988); T. K. Kopec, J. Phys. C **21**, 6053 (1988); Y. Q. Ma and Z. Y. Li, Phys. Lett. A **148**, 134 (1990).

[29] G. Toulouse, J. Phys. (France) Lett. **41**, L-447 (1980).

[30] For example, S. Chikazawa, S. Taniguchi, H. Matsuyama, and Y. Miyako, J. Magn. Magn. Mater. **31-34**, 1355 (1983); S. Taniguchi, H. Matsuyama, S. Chikazawa, and Y. Miyako, J. Phys. Soc. Jpn. **52**, 4323 (1983); E. Zastre, R. M. Roshko, and G. Williams, Phys. Rev. B **32**, 7597 (1985); L. P. Levy and A. T. Ogielski, Phys. Rev. Lett. **57**, 3288 (1986); L. P. Levy, Phys. Rev. B **38**, 4963 (1988); B. Özçelik, K. Kiymaç, J. C. Verstelle, A. J. van Duyneveld, and J. A. Mydosh, J. Phys.: Condens. Matter **4**, 6639 (1992).

[31] R. Fisch and A. B. Harris, Phys. Rev. Lett. **38**, 785 (1977).

[32] In most experiments, the nonlinear susceptibility is usually defined as $\chi_{nl} \equiv 1 - m/(\chi H)$, and $-(1/3!)(\partial^3 m/\partial H^3)$ is denoted as χ_2 or χ_3. But we choose to define the nonlinear susceptibility χ_{nl} as $-(1/3!)(\partial^3 m/\partial H^3)$ because this is directly related to the spin-glass susceptibility χ_{SG}.
[33] J. Chalupa, Solid State Commun. **22**, 315 (1977).
[34] From the M and Q solutions of the SK model, one can obtain $M = (H/T)[1 - (H^2/3T^2)(T^2 + 2T_g^2)/(T^2 - T_g^2) + \cdots]$ for $T > T_g$. The nonlinear susceptibility $\chi_{nl} \equiv -(1/3!)(\partial^3 M/\partial H^3)$ then diverges as $T \rightarrow T_g$ in proportion to $(T - T_g)^{-1}$ (see Ref. 1).
[35] M. Suzuki, Prog. Theor. Phys. **58**, 1151 (1977).
[36] See, for example, K. Wada and H. Takayama, Prog. Theor. Phys. **64**, 327 (1980); S. Fujiki and S. Katsura, *ibid.* **65**, 1130 (1981).
[37] N. Read, S. Sachdev, and J. Ye, Phys. Rev. B **52**, 384 (1995).
[38] J. Miller and D. A. Huse, Phys. Rev. Lett. **70**, 3147 (1993).
[39] H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994); Phys. Rev. B **54**, 3328 (1996).
[40] M. Y. Guo, R. N. Bhatt, and D. A. Huse, Phys. Rev. Lett. **72**, 4137 (1994); Phys. Rev. B **54**, 3336 (1996).
[41] M. J. Rozenberg and D. R. Grempel, Phys. Rev. Lett. **81**, 2550 (1998).
[42] G. E. Brodale, R. A. Fisher, W. E. Fogle, N. E. Philips, and J. van Curen, J. Magn. Magn. Mater. **31-34**, 1331 (1983).
[43] D. L. Martin, Phys. Rev. B **21**, 1902 (1980).