Minigap in a long disordered SNS junction: Analytical results

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We review and refine analytical results on the density of states in a long disordered superconductor–normal-metal–superconductor junction with transparent interfaces. Our analysis includes the behavior of the minigap near phase differences zero and π across the junction, as well as the density of states at energies much larger than the minigap but much smaller than the superconducting gap.

A superconductor in contact with a normal metal induces pairing correlations in the metal, a phenomenon known as the proximity effect. One of the most remarkable consequences of such induced correlations is the appearance of a gap, usually referred to as the "minigap," in the electronic excitation spectrum of the normal metal; see Ref. 1 for an early introduction and Refs. 2 and 3 for a detailed derivation. A very common setup exhibiting a minigap is the superconductor–normal-metal–superconductor (SNS) junction made of two superconducting leads connected via a disordered normal layer. The gap in this junction is determined by the diffusion time across the normal layer and is sensitive to the phase difference across the junction, reaching a maximum at zero phase difference and vanishing at the phase difference π.[4]

The appearance of a minigap and its phase dependence is well understood in the quasiclassical description. Provided the scattering length is much larger than the Fermi wave length but much smaller than the coherence length in the normal metal, the motion of the electrons in the normal layer is diffusive and the proximity effect may be described by the Usadel equations.[5] These equations are nonlinear, which complicates their analytical treatment except for several simple limits. One of the cases most accessible to an analytical treatment is the limit of a long disordered SNS junction (with the minigap energy scale much smaller than the superconducting gap) with transparent normal-metal–superconductor interfaces. The spectral properties of such a junction have been previously studied in Refs. 4 and 6, and we find it possible to further improve on those results. In this note we revisit this problem, refining some of the existing results and replacing numerical answers with analytical ones. We design this note as a quick reference on the structure of the minigap in a long disordered SNS junction which may be useful in view of renewed interest in such systems in connection with problems related to π-junctions[7,8] and mesoscopic fluctuations.[9,10] As a byproduct, we derive two useful identities for solutions to Usadel equations which simplify our analytical calculations.

Assuming a quasi-one-dimensional geometry of the contact, the proximity effect in the normal layer may be described via the Usadel equations (in our paper we conform to the definitions of Ref. 4)

$$\frac{1}{2}\partial_x^2\theta + i\varepsilon\sin\theta - \frac{1}{4}(\partial_x\chi)^2\sin 2\theta = 0,$$

$$\partial_x(\partial_x\chi\sin^2\theta) = 0, \tag{1}$$

where $\theta(x,\varepsilon)$ and $\chi(x,\varepsilon)$ are the variables parameterizing the zero angular momentum component of the Green's functions, $g = \cos\theta$ and $f = \sin\theta\exp(i\chi)$; here χ is the phase of the superconducting correlations, and the local density of states $\rho(x,\varepsilon)$ (in units of the normal electron density in the bulk) is given by

$$\rho(x,\varepsilon) = \mathrm{Re}\cos\theta. \tag{2}$$

Note that we measure lengths in units of the junction width $L = 1$ and the unit of energy ε is the Thouless energy $E_c = D/L^2$, with D the diffusion constant in the normal metal. We further assume that the Thouless energy E_c, as well as all other energy scales in the problem, are much smaller than the superconducting gap Δ. In this limit, the energy scale Δ is (to leading order) excluded from the Usadel equations. At energies of order E_c, the relevant normal-metal coherence length determining the applicability of Usadel equations equals the junction width L.

Within the superconducting leads, $\theta = \pi/2$ (at $\varepsilon \ll \Delta$) and χ equals the phase of the superconducting order parameter. Inside the junction, both θ and χ turn into complex functions. For simplicity, we choose ideally transparent interfaces and assume the normal metal to be much more disordered than the superconductor; in this case the boundary conditions become "rigid"[9,11]:

$$\theta(x=0) = \theta(x=1) = \frac{\pi}{2},$$

$$\chi(x=0) = 0, \quad \chi(x=1) = \chi_0. \tag{3}$$

Equations (1)–(3) form a closed set determining the density of states (with χ_0 and ε as input parameters). They have been analyzed in Refs. 4 and 6 and in this paper we extend their results; we list them first and sketch their derivation afterwards.

(i) The complex integral "density of states" defined as

$$\hat{\rho}(\varepsilon) = \int_0^1 dx\cos\theta(x,\varepsilon) \tag{4}$$

obeys the relations

$$\hat{\rho} + 2\varepsilon \frac{\partial \hat{\rho}}{\partial \varepsilon} = i \frac{\partial B}{\partial \varepsilon}, \tag{5}$$

$$2 \frac{\partial \hat{\rho}}{\partial \chi_0} = -i \frac{\partial C}{\partial \varepsilon}, \tag{6}$$

where $B(\varepsilon, \chi_0)$ and $C(\varepsilon, \chi_0)$ are integrals of Eq. (1);

$$B = \frac{1}{4}(\partial_x \theta)^2 - i\varepsilon \cos \theta + \frac{C^2}{4 \sin^2 \theta}, \tag{7}$$

$$C = (\partial_x \chi) \sin^2 \theta. \tag{8}$$

(ii) Near $\chi_0 \to 0$ and $\chi_0 \to \pi$, the phase dependence of the minigap $E_g(\chi_0)$ (in units of E_c) involves the leading terms

$$E_g(\chi_0) = C_2(1 - C_1 \chi_0^2), \quad \chi_0 \ll \pi, \tag{9}$$

$$E_g(\chi_0) = C_3(\pi - \chi_0), \quad \pi - \chi_0 \ll \pi. \tag{10}$$

The value of C_2 was derived in Ref. 4,

$$C_2 = \left[\max_{\vartheta_0} \int_0^{\vartheta_0} \frac{d\vartheta}{(\sinh \vartheta_0 - \sinh \vartheta)^{1/2}} \right]^2 \approx 3.122, \tag{11}$$

with the maximum attained at $\hat{\vartheta}_0 \approx 1.421$. The value of C_1 reported in Ref. 4 is incorrect by an order of magnitude and we find the correct value

$$C_1 = \frac{\int_0^{\hat{\vartheta}_0} \dfrac{d\vartheta (\sinh \hat{\vartheta}_0 + \sinh \vartheta)}{(\sinh \hat{\vartheta}_0 - \sinh \vartheta)^{1/2} \cosh^2 \vartheta \cosh^2 \hat{\vartheta}_0}}{4 \sqrt{C_2} \left[\int_0^{\hat{\vartheta}_0} \dfrac{d\vartheta}{(\sinh \hat{\vartheta}_0 - \sinh \vartheta)^{1/2} \cosh^2 \vartheta} \right]^2} \approx 0.0921. \tag{12}$$

For C_3 we find the analytic result close to the numerical value reported in Ref. 4,

$$C_3 = \frac{\pi^2}{4} \approx 2.467. \tag{13}$$

(iii) The integral density of states $\rho(\varepsilon) = \mathrm{Re}\, \hat{\rho}(\varepsilon)$ has a square-root singularity at $\varepsilon = E_g$,[4]

$$\rho(\varepsilon) \sim C_4(\chi_0) \left(\frac{\varepsilon - E_g}{E_g} \right)^{1/2}, \tag{14}$$

with the coefficient C_4 diverging at $\chi_0 \to \pi$ as

$$C_4(\chi_0) = \alpha (\pi - \chi_0)^{-2/3}. \tag{15}$$

This asymptotic form has been found numerically in Refs. 4 and 6 and we confirm it here analytically, together with the value for α:

$$\alpha = \frac{1}{\pi \sqrt{6}} \left(\frac{1}{2\pi} - \frac{3\pi}{64} \right)^{-2/3} \approx 2.494. \tag{16}$$

(iv) At energies ε much higher than E_c but much lower than the superconducting gap, the leading corrections to the (integral) density of states are

$$\rho(\varepsilon, \chi_0) \approx 1 - \frac{C_5}{\sqrt{\varepsilon}} + e^{-\sqrt{\varepsilon}} C_6(\varepsilon) \cos(\chi_0), \tag{17}$$

where

$$C_5 = 2 - \sqrt{2} \approx 0.586, \tag{18}$$

$$C_6(\varepsilon) = 16 \tan^2 \frac{\pi}{8} \left[-\cos \sqrt{\varepsilon} + \frac{\cos \sqrt{\varepsilon} - \sin \sqrt{\varepsilon}}{2\sqrt{\varepsilon}} \right]. \tag{19}$$

The verification of (i) is straightforward: We integrate the Usadel equations (1) once and denote the coordinate-independent integrals by B and C as in Eqs. (7) and (8). Relations (5) and (6) are then obtained via integration by parts in x and repeated use of the Usadel equations; see the Appendix for details. Relation (5) expresses the conservation of the total number of states as a function of χ_0; indeed, Eq. (5) implies that $\hat{\rho}(\varepsilon)$ is a total derivative in energy of the expression $(2\varepsilon\hat{\rho} - iB)$ which is independent of χ_0 at large energies [this follows from our discussion of the result (iv) below]; therefore, the integral of $\hat{\rho}(\varepsilon)$ over energies is independent of χ_0. Identity (6) follows from the fact that the supercurrent (which equals $\mathrm{Im}\,C$ in appropriate units[5]) can be expressed as the derivative of the free energy with respect to χ_0; we verify Eq. (8) directly on the level of Usadel equation in the Appendix.

To obtain the results of paragraph (ii) we follow the usual procedure in integrating the Usadel equations: below the gap, $\theta(x)$ takes the form $\theta = \pi/2 + i\vartheta$, where $\vartheta(x)$ is real and varies from 0 at the N–S interface to its maximal value ϑ_0 in the middle of the normal layer. From Eqs. (7) and (8) we obtain the differential equations

$$\partial_x \vartheta = \sqrt{f(\vartheta_0) - f(\vartheta)} / \sqrt{\varepsilon}, \quad \partial_x \chi = C \cosh^{-2} \vartheta, \tag{20}$$

where $f(\vartheta) = \sinh(\vartheta) - (C^2/4\varepsilon) \cosh^{-2} \vartheta$. Their integration over half the junction provides us with the solutions of the Usadel equations in the forms

$$\sqrt{\varepsilon} = \int_0^{\vartheta_0} \frac{d\vartheta}{[f(\vartheta_0) - f(\vartheta)]^{1/2}},$$

$$\chi_0 = \frac{C}{\sqrt{\varepsilon}} \int_0^{\vartheta_0} \frac{d\vartheta}{\cosh^2 \vartheta [f(\vartheta_0) - f(\vartheta)]^{1/2}}, \tag{21}$$

expressing ε and χ_0 in terms of the new parameters $C/\sqrt{\varepsilon}$ and ϑ_0. The minigap $E_g(\chi_0)$ is defined as the maximal energy ε compatible with this solution [with a real $\vartheta(x)$]. Equation (12) is now easily obtained by expanding in C, which is the small parameter near $\chi_0 = 0$, while result (13) is obtained from an expansion in ε, the small parameter close to $\chi_0 = \pi$ where the minigap vanishes (also see the derivation below).

A more accurate expansion in small ε is necessary to obtain the results of paragraph (iii). The density of states near the gap edge [Eq. (14)] is derived with the help of relation (5). The key idea of the calculation is that most quantities are regular functions of the parameters ϑ_0 and ε/C^2 (except for the point $\chi_0 = \pi$, where the minigap vanishes). The square-root singularity (14) appears when inverting regular functions at the extremal point. Remarkably, in this way we can compute the density of states at the gap edge using the solutions to the Usadel equations *below* the gap and analytically continuing them to energies above the gap.

Close to $\chi_0 = \pi$ the energy ε is small and we can expand Eq. (21) in the small parameter $\delta = \varepsilon/C^2$:

$$\varepsilon = (\pi^2 \cosh^2 \vartheta_0)\delta - \pi\left(\frac{1}{2} - \frac{\pi}{8}\right)e^{5\vartheta_0}\delta^2 + \cdots, \quad (22)$$

$$\chi_0 = \pi - \frac{4\cosh^3 \vartheta_0}{\sinh \vartheta_0}\delta + \left(\frac{3\pi}{16} - \frac{1}{2}\right)e^{5\vartheta_0}\delta^2 + \cdots. \quad (23)$$

We will show below that the solutions at the gap edge near $\chi_0 = \pi$ involve a large parameter $\vartheta_0 \gg 1$, and hence we may keep only the leading terms in e^{ϑ_0} in the coefficients of δ^2. Next we invert Eq. (23) to find

$$\delta \approx \frac{\sinh \vartheta_0}{4\cosh^3 \vartheta_0}(\pi - \chi_0) + \left[\frac{3\pi}{16} - \frac{1}{2}\right]e^{-\vartheta_0}(\pi - \chi_0)^2, \quad (24)$$

and substitute the result into Eq. (22) to express the energy ε as a power series in $(\pi - \chi_0)$ and as a function of ϑ_0:

$$\varepsilon \approx \frac{\pi^2}{4}(\pi - \chi_0)\tanh \vartheta_0 - \left[\frac{\pi}{2} - \frac{3\pi^3}{64}\right]e^{\vartheta_0}(\pi - \chi_0)^2. \quad (25)$$

The gap edge is given by maximizing ε in Eq. (25) as a function of ϑ_0; the corresponding value $\bar{\vartheta}_0$ maximizing ε is

$$\bar{\vartheta}_0 = \frac{1}{3}\ln\frac{\chi_c}{\pi - \chi_0}, \quad \chi_c = \left(\frac{1}{2\pi} - \frac{3\pi}{64}\right)^{-1} \approx 84.08; \quad (26)$$

we see that $\bar{\vartheta}_0$ is indeed large at the gap edge (for small $\pi - \chi_0$), albeit only logarithmically and with a small prefactor $1/3$. Upon substitution into Eq. (25), this result also gives us the next-order correction to Eq. (10):

$$E_g(\chi_0) = \frac{\pi^2}{4}(\pi - \chi_0)\left[1 - 6\left(\frac{\pi - \chi_0}{\chi_c}\right)^{2/3}\right]. \quad (27)$$

We are now prepared to derive the density-of-states singularity at the gap edge [Eqs. (14)–(16)]. The maximum of the function $\varepsilon(\vartheta_0)$ at $\bar{\vartheta}_0$ is expressed in the relation $\varepsilon(\vartheta_0) \approx E_g - |\partial_{\vartheta_0}^2 \varepsilon|_{\bar{\vartheta}_0}[\bar{\vartheta}_0 - \vartheta_0]^2/2$; its inversion produces a square-root singularity at the gap edge in the function $\vartheta_0(\varepsilon)$,

$$\vartheta_0(\varepsilon) = \bar{\vartheta}_0 \pm i \sqrt{\frac{2(\varepsilon - E_g)}{|\partial_{\vartheta_0}^2 \varepsilon|_{\bar{\vartheta}_0}}}, \quad (28)$$

and hence ϑ_0 develops an imaginary part at energies $\varepsilon > E_g$. The singularity in $\vartheta_0(\varepsilon)$ translates into a square-root singularity in $B(\varepsilon)$; evaluating Eq. (7) in the junction middle and expressing C with the help of Eq. (24), we obtain

$$B = \varepsilon\left[\frac{\coth \vartheta_0}{\pi - \chi_0} - \frac{3\pi}{16}e^{\vartheta_0}\right]$$

$$\approx \text{Re}B \pm i\varepsilon \sqrt{\frac{2(\varepsilon - E_g)}{|\partial_{\vartheta_0}^2 \varepsilon|_{\bar{\vartheta}_0}}}[(\pi - \chi_0)^{-1}\sinh^{-2}\bar{\vartheta}_0$$

$$+ 3\pi e^{\bar{\vartheta}_0}/16]. \quad (29)$$

This singularity in B further translates, via Eq. (5), into a square-root singularity in the density of states $\rho(\varepsilon)$. Evaluating the coefficient in Eq. (29) with the use of Eq. (25) for $\varepsilon(\vartheta_0)$ and of Eq. (26) for $\bar{\vartheta}_0$, we arrive at the final results [Eqs. (14)–(16)]. The coefficient (16) agrees with the numerical findings in Ref. 4.

Another regime where the density of states is amenable to a simple analytic solution is at energies much larger than E_c but much smaller than Δ. In this limit, the coupling between the superconducting leads is weak, which allows us to derive the results of paragraph (iv). In Eq. (17), the term proportional to $\varepsilon^{-1/2}$ is due to the suppression of the density of states near the interfaces, and the exponentially small term proportional to $\cos \chi_0$ results from the Josephson coupling.

At $\varepsilon \gg 1$, the Usadel equations may be solved by matching the solutions for the two semi-infinite N–S systems; such a solution takes the form[9]

$$\theta_{NS}(x,\varepsilon) = 4\arctan\left(e^{-\kappa x}\tan\frac{\pi}{8}\right), \quad (30)$$

where $\kappa = \sqrt{-2i\varepsilon}$ (Re $\kappa > 0$) and the normal layer is at $x > 0$. Matching of such solutions was performed in Ref. 12 by observing that for $|\theta| \ll 1$ (i.e., everywhere in the junction, except for the very thin layers at the interfaces) the Usadel equations become linear in the variables $f = \theta e^{i\chi}$ and $\bar{f} = \theta e^{-i\chi}$. Therefore the solution near the middle of the junction is given through the simple sums

$$f(x) = \theta_{NS}(x) + e^{i\chi_0}\theta_{NS}(1-x),$$

$$\bar{f}(x) = \theta_{NS}(x) + e^{-i\chi_0}\theta_{NS}(1-x). \quad (31)$$

From this solution we easily find, using Eqs. (7) and (8),

$$C(\varepsilon) = 32\tan^2\frac{\pi}{8}\kappa e^{-\kappa}\sin \chi_0, \quad (32)$$

$$B(\varepsilon) = -i\varepsilon\left[1 - 32\tan^2\frac{\pi}{8}e^{-\kappa}\cos \chi_0\right]. \quad (33)$$

With the help of identities (5) and (6) this immediately implies results (17) and (19). The value of C_5 is left undetermined by this method but may easily be obtained from di-

rectly integrating the local density of states (2) corresponding to the $N-S$ solutions (30), which leads to result (18).

APPENDIX

In order to verify Eq. (5) we use definition (7) of B to re-express

$$\frac{\partial}{\partial\varepsilon}(iB-2\varepsilon\hat{\rho})=\frac{\partial}{\partial\varepsilon}\int_0^1 dx\left[\frac{i}{4}(\partial_x\theta)^2-\varepsilon\cos\theta\right.$$

$$\left.+\frac{i}{4}(\partial_x\chi)^2\sin^2\theta\right]$$

$$=\int_0^1 dx\left[\frac{i}{2}(\partial_x\theta)\partial_x\left(\frac{\partial\theta}{\partial\varepsilon}\right)-\cos\theta+\varepsilon\sin\theta\frac{\partial\theta}{\partial\varepsilon}\right.$$

$$+\frac{i}{2}(\partial_x\chi)\partial_x\left(\frac{\partial\chi}{\partial\varepsilon}\right)\sin^2\theta$$

$$\left.+\frac{i}{2}(\partial_x\chi)^2\sin\theta\cos\theta\frac{\partial\theta}{\partial\varepsilon}\right]. \qquad (A1)$$

The last integral contains five terms. After integrating the first term by parts in x (separating $\partial\theta/\partial\varepsilon$), it vanishes against the third and fifth terms by virtue of the first Usadel equation (1). Integrating the fourth term in Eq. (A1) by parts in x annihilates it due to the second Usadel equation; we thus arrive at

$$\frac{\partial}{\partial\varepsilon}(iB-2\varepsilon\hat{\rho})=-\hat{\rho}, \qquad (A2)$$

which is equivalent to Eq. (5).

In order to verify Eq. (6) we use the identity $\partial_{\chi_0}\int_0^1 dx(\partial_x\chi)^2\sin^2\theta=2C+\int_0^1 dx(\partial_x\chi)^2\partial_{\chi_0}\sin^2\theta$ to re-express $\partial C/\partial\varepsilon$ as

$$-i\frac{\partial C}{\partial\varepsilon}=-\frac{i}{2}\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\chi_0}\int_0^1 dx(\partial_x\chi)^2\sin^2\theta$$

$$+\frac{i}{2}\frac{\partial}{\partial\varepsilon}\int_0^1 dx(\partial_x\chi)^2\sin 2\theta\frac{\partial\theta}{\partial\chi_0}. \qquad (A3)$$

We use the second Usadel equation to differentiate the first of the two terms in ε;

$$\frac{\partial}{\partial\varepsilon}\int_0^1 dx(\partial_x\chi)^2\sin^2\theta=2\int_0^1 dx\partial_x\left(\frac{\partial\chi}{\partial\varepsilon}\right)(\partial_x\chi)\sin^2\theta$$

$$+\int_0^1 dx(\partial_x\chi)^2\sin 2\theta\frac{\partial\theta}{\partial\varepsilon}. \qquad (A4)$$

On the right-hand side of Eq. (A4), the first term is annihilated after integration by parts, and we finally arrive at

$$-i\frac{\partial C}{\partial\varepsilon}=-\frac{i}{2}\frac{\partial}{\partial\chi_0}\int_0^1 dx(\partial_x\chi)^2\sin 2\theta\frac{\partial\theta}{\partial\varepsilon}$$

$$+\frac{i}{2}\frac{\partial}{\partial\varepsilon}\int_0^1 dx(\partial_x\chi)^2\sin 2\theta\frac{\partial\theta}{\partial\chi_0}$$

$$=-i\frac{\partial}{\partial\chi_0}\int_0^1 dx(\partial_x^2\theta+2i\varepsilon\sin\theta)\frac{\partial\theta}{\partial\varepsilon}$$

$$+i\frac{\partial}{\partial\varepsilon}\int_0^1 dx(\partial_x^2\theta+2i\varepsilon\sin\theta)\frac{\partial\theta}{\partial\chi_0}$$

$$=-2\int_0^1 dx\sin\theta\frac{\partial\theta}{\partial\chi_0}=2\frac{\partial}{\partial\chi_0}\hat{\rho}, \qquad (A5)$$

where we have used the first Usadel equation.

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