Level statistics in a Heisenberg chain with random magnetic field

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(Received 22 February 2002; revised manuscript received 31 May 2002; published 19 August 2002)

Level statistics is calculated for a Heisenberg spin chain with a random magnetic field, and shown to follow that of random-matrix theory pertaining to Gaussian orthogonal ensemble, due to nonconventional time-reversal invariance. However, when a *bona fide* time reversal violation term is added, such as three spin interaction, the statistics follows that pertaining to Gaussian unitary ensemble.

DOI: 10.1103/PhysRevB.66.052416 PACS number(s): 75.10.−b

The relevance of random-matrix theory (RMT) to the investigation and analysis of the spectrum of numerous physical systems started in nuclear physics[1] (a strongly correlated particle system), continued in mesoscopic physics[2] and chaotic dynamics[3] (single particle systems), and focused recently on strongly correlated electron and spin systems.[4,5] In Ref. 4 attention was directed to the difference between integrable and nonintegrable spin and strongly correlated electron systems as far as the nearest level distribution $p(s)$ is concerned. Level statistics in a two-dimensional interacting electron system was studied in Ref. 5, with special attention to the influence of the electron-electron interaction strength U on the behavior of $p(s)$. It was shown that the distribution $p(s)$ undergoes changes between Poisson and Gaussian orthogonal ensemble (GOE) behavior as U increases.

In the present work we want to illuminate the role of RMT in *random* spin systems. The motivation is at least three-fold (beyond curiosity). First, recalling that the level statistics is intimately related to the symmetry of the Hamiltonian, the system studied below has a lower rotational symmetry than the one studied in Ref. 4. More precisely, the total spin S of the system is not conserved, but its projection S_z remains a good quantum number. Second, the randomness is introduced here by an application of a random magnetic field along the z direction. This is a time reversal breaking term which, naively speaking, is expected to lead to level statistics pertaining to Gaussian unitary ensemble (GUE) instead of the GOE statistics. We shall see below that this is not the case, the reason being that the Hamiltonian is still invariant with respect to the action of some antiunitary operator. Hence it can be represented by a real matrix. Finally, it is interesting to check the level distribution when a *bona fide* time-reversal violating interaction is added to the Hamiltonian. Such a term can be a random field acting in all three space directions but in that case the S_z invariance is broken. Instead, we suggest a non-random three body force. The corresponding level statistics turns out to be consistent with that of the GUE.

To begin, let us consider the one-dimensional Heisenberg spin-1/2 chain containing N spins with random (on-site) magnetic field in the z direction. The Hamiltonian is

$$H=\sum_{n=1}^{N}[J\mathbf{S}_n\cdot\mathbf{S}_{n+1}+h_nS_{nz}], \tag{1}$$

where $\mathbf{S}_n$ is the spin operator at site n, J is the nearest-neighbor exchange constant (periodic boundary conditions are assumed throughout), and h_n is a random magnetic field along the z direction at site n. It is assumed that h_n's are uncorrelated random numbers with zero mean and finite fluctuation width h, namely,

$$\langle h_n\rangle=0,$$
$$\langle h_nh_m\rangle=h^2\delta_{nm}. \tag{2}$$

If $\mathbf{S}\equiv\Sigma_{n=1}^{N}\mathbf{S}_n$ is the total spin operator then, evidently, $[S^2,H]\neq 0$ but $[S_z,H]=0$. Therefore, it is natural to consider Hamiltonian (1) within a subspace corresponding to a given value of S_z. Restricting ourselves to N even, the largest subspace which corresponds to $S_z=0$, has the dimension $M\equiv C_{N/2}^{N}$. A natural basis in this subspace is constructed from vectors $|s_{1z}s_{2z}\ldots s_{Nz}\rangle$ (here $s_{nz}=\pm 1/2$ is the eigenvalue of S_{nz}) in which $N/2$ of the s_{nz} are $+1/2$ and the other $N/2$ are $-1/2$. Evidently, in this basis the Hamiltonian matrix is real. Note that, unlike in the standard Heisenberg system, the sign of the exchange constant J is irrelevant here because we are interested in the *whole* spectrum.

The fact that the Hamiltonian H defined in Eq. (1) is representable by a real matrix is related to its symmetry properties. Let T_0 be the time-reversal operator. For a spin operator $\mathbf{S}_n$ one has $T_0\mathbf{S}_nT_0^{-1}=-\mathbf{S}_n$ and therefore,

$$T_0HT_0^{-1}=\sum_{n=1}^{N}[J\mathbf{S}_n\cdot\mathbf{S}_{n+1}-h_nS_{nz}]\neq H, \tag{3}$$

as expected when there is an external magnetic field. Consider, however, the operator $T=e^{i\pi S_x}T_0$. Application of the (unitary) operator $e^{i\pi S_x}$ reverses the signs of S_{ny} and S_{nz} (but not of S_{nx} of course). Hence T is an antiunitary operator which commutes with H. The existence of such an operator (referred to as non-conventional time reversal in Ref. 6) guarantees the reality of H in a proper basis. In Ref. 6 an example for such an operator is given within the theory of quantum chaos. Here it is given for a spin system.

It might be of some interest to compare the sparsity of the Hamiltonian matrix for the spin system and, say, a particle in a disordered potential (within the tight binding approximation). In the latter case, the number of non-zero elements in a

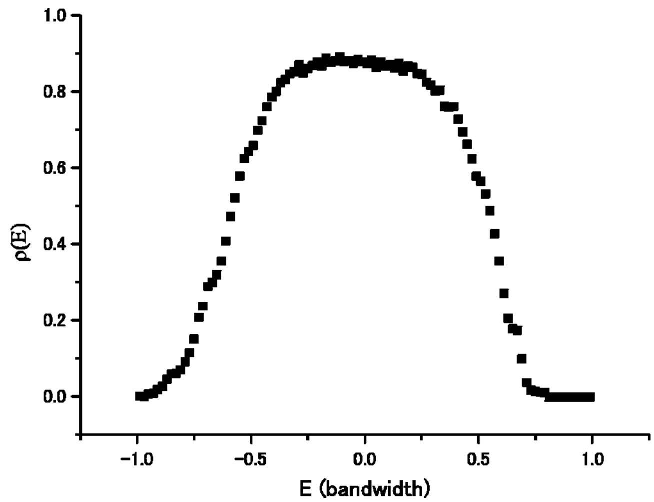

FIG. 1. Density of states (normalized by band width) for Hamiltonian (1) for a system of $N=14$ spins with $h/J=4$. The spectrum is not symmetric around $E=0$ since the ordered (Heisenberg exchange) part is not symmetric.

given row is $b=1+2d$, where d is the dimension. In the case of one dimensional spin system with $S_z=0$, the number b is not fixed, and changes between 2 and $N/2$, However, the ratio b/M (where M is the size of the matrix) decays much faster with M for the spin system.

We have carried out concrete calculations on a spin chain with $N=14$ ($M=3432$) on an ensemble of 900 matrices. By comparing results for smaller ensembles (e.g., 800 matrices) it is verified that this number of realizations is sufficient. Going beyond $N=14$ is virtually out of reach since the matrix dimension is the binomial coefficient $C^N_{N/2}$ which is too large for $N=16$. The density of states is shown in Fig. 1.

Since the density of states is not constant, a standard unfolding procedure is used to reach a spectrum $\{\lambda_n\}$ with an averaged density of states equal to unity. For the main purpose of the present study, (testing the relevance of RMT in disordered quantum spin systems) the nearest level spacing criteria might seem adequate. Nevertheless, we have gone a bit further and, for the present ensemble, computed the distribution of both nearest level spacings $p(s)=\lambda_{n+1}-\lambda_n$ and that of the next nearest level spacings $p_2(s)=\lambda_{n+2}-\lambda_n$.[8] The resulting level distributions $p(s)$ and $p_2(s)$ were tested to satisfy the normalization conditions $\int_0^\infty p(s)ds=\int_0^\infty sp(s)ds=1$, and $\int_0^\infty p_2(s)ds=\frac{1}{2}\int_0^\infty sp_2(s)ds=1$. They are displayed in Fig. 2. The distribution $p(s)$ is compared with the GOE Wigner surmise distribution $p_{WS}(s)$, while the distribution $p_2(s)$ is compared with the corresponding GOE prediction.[7] The quality of agreement with the GOE statistics is quite satisfactory. To corroborate this statement we also compare (see the inset of Fig. 2) the difference $p(s)-p_{WS}(s)$ with $p_{RMT}(s)-p_{WS}(s)$ where $p_{RMT}(s)$ is the large N prediction for $p(s)$ (see Ref. 6, Fig. 4.2, and Ref. 9). A glance at the figure suggests that the absolute value of the area pertaining to the data point curve is smaller than that pertaining to the solid curve. In other words, our data fall between $p_{WS}(s)$ and $p_{RMT}(s)$.

These calculations indicate that level statistics of Heisenberg spin chain with random magnetic field is encoded by RMT with symmetry corresponding to the GOE.

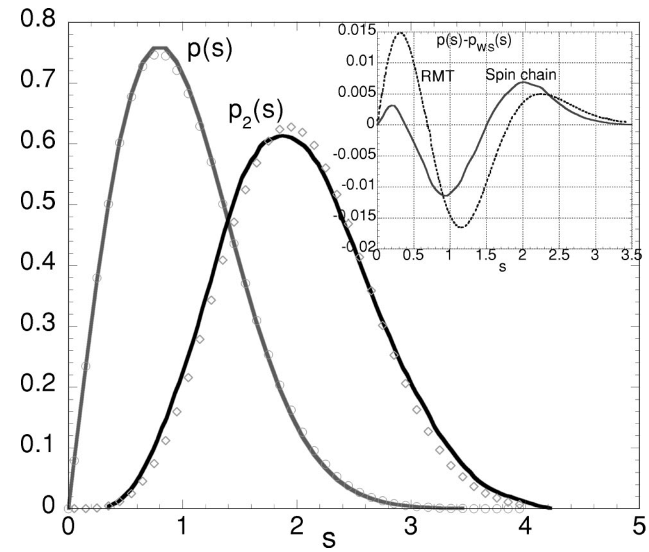

FIG. 2. Data points depict nearest (left) and next to nearest (right) level spacing distributions ($p(s)$ and $p_2(s)$ respectively) for the Hamiltonian (1) for a system of $N=14$ spins with $h/J=4$. The continuous lines correspond to Wigner surmise $p_{WS}(s)$ and the RMT prediction for $p_2(s)$ pertaining to GOE ensemble. The agreement of $p(s)$ with the Wigner surmise seems evident. A more stringent test (inset) compares $p(s)-p_{WS}(s)$ with $p_{RMT}(s)-p_{WS}(s)$.

Having demonstrated the consistence of GOE statistics with the spectrum of Hamiltonian (1), we now ask the question what model of spin chain can exhibit a GUE statistics? We have already seen that a random magnetic field along the z direction is not enough. Naturally, one might suggest a random magnetic field in arbitrary direction at each site, i.e., a term $\mathbf{h}_n\cdot\mathbf{S}_n$ replacing h_nS_{nz}. This choice, however, has a couple of drawbacks. First, S_z is no longer a good quantum number, which means that the matrix to be diagonalized is now much larger (for a given number N of spins). Second, there are now nondiagonal random-matrix elements, which turn the analogy with Hamiltonian for particle (within the tight-binding approximation) to be less transparent.

Instead, we introduce a three-site interaction, so that the Hamiltonian is

$$H_T=\sum_{n=1}^{N}[J\mathbf{S}_n\cdot\mathbf{S}_{n+1}+h_nS_{nz}+J_T\mathbf{S}_n\cdot[\mathbf{S}_{n+1}\times\mathbf{S}_{n+2}], \quad (4)$$

which conserves S_z. This Hamiltonian violates time-reversal invariance, but, unlike the former case, this time-reversal violation cannot be compensated for by the spin-reversal operator $e^{i\pi S_z}$. Hence it is represented by a complex Hermitian matrix.[9] The three-spin interaction in Eq. (4) is not an *ad hoc* term. It appears in earlier essays on magnetism as one goes to higher-order terms associated with interchanges of electrons belonging to different atoms. An example of an interaction term involving three spin operators is discussed in Ref. 10. It turns out that a very small mixture of three-site interaction terms ($J_T/J=1/140$) is sufficient to drive the sys-

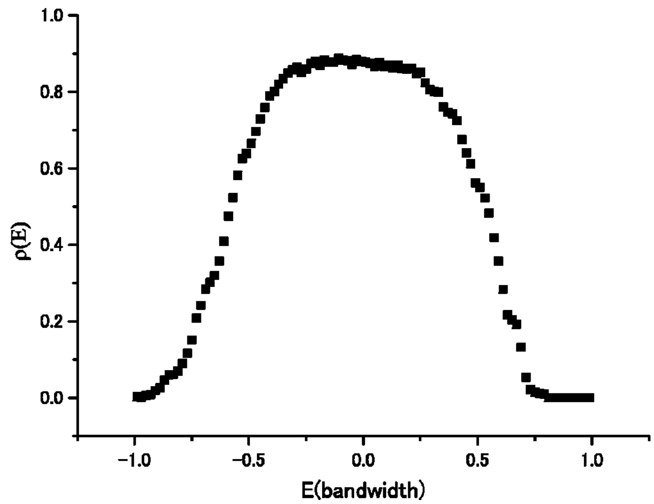

FIG. 3. Density of states (normalized by the bandwidth) for Hamiltonian (4) for a system of $N=14$ spins with $h/J=4$ and $J_T/J=1/140$. The addition of the weak time breaking term does not affect the density of states (compared with Fig. 1).

tem statistics from a GOE to a GUE, while the density of states (Fig. 3) is virtually unchanged.

The corresponding level spacing distribution is shown in Fig. 4. This time, the consistency with GUE statistics is excellent.

There is a considerable amount of work on the problem of mixture of ensembles with two different symmetries. The one relevant for our case is that represented by an ensemble of matrices $H_0+\lambda V$ where H_0 and V are *both random* matrices such that H_0 is time-reversal invariant while V is not.[6] Here λ is a real parameter, and the two matrices are assumed to have the same mean level spacing. The main goal is to determine the distribution $P(H,\lambda)\equiv\langle\delta(H-H_0-\lambda V\rangle$ (averaging over the GOE for H_0 and the GUE for V). In a less rigorous sense, the question is at which value of λ the level statistics crosses from the GOE (which is exact for $\lambda=0$) to that of the GUE (valid at $\lambda=\infty$). The answer to these questions is provided by Dyson's Brownian-motion model[1] which shows that $P(H,\lambda)$ satisfies a certain Fokker-Planck equation. The solution of this equation (see Ref. 6, equation 6.15.12) shows that the crossover occurs at $\lambda^2\gg 1$. Comparing the separate spectra of the GOE and GUE parts of Hamiltonian (4), the parameter λ in our case is $\lambda\approx 0.8$, which evidently does not satisfy the condition above. It should be kept in mind, however, that in our case the time-reversal violating matrix is *deterministic*. Combined with the fact that Hamiltonian (4) is a sparse matrix and not a fully random matrix in the sense of RMT, the application of Dyson's Brownian-motion model[1] in the present case should be examined with care. Investigation of ensembles containing random and deterministic parts has been carried out by Brezin and Hikami[11], but it does not cover the present case specified by Hamiltonian (4).

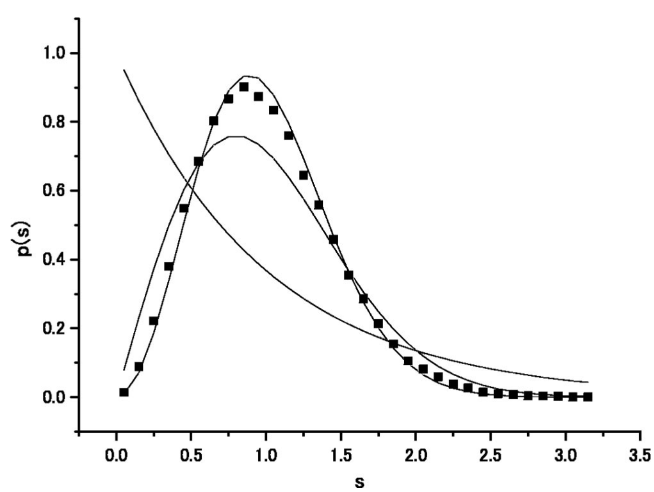

FIG. 4. Nearest level spacing distribution for Hamiltonian (4) for a system of $N=14$ spins with $h/J=4$ and $J_T/J=1/140$. The continuous lines correspond to Poisson, GOE, and GUE statistics. The agreement with the GUE statistics is evident.

Let us then present some heuristic arguments about the minimum value of J_T for which the GUE statistics emerges. Comparing the forms of spin and tight-binding models (not in the rigorous Holstein Primakoff sense) the Heisenberg interaction can be thought of as a hopping term while the random field term can be considered as a site energy contribution. Of course, the spin operators are not particle operators but still, a term like $S_n^+S_{n+1}^-$ can be interpreted as "destroying a particle" in site $n+1$ and "creating a particle" at site n. The three spin term can be written as

$$J_T\mathbf{S}_n\cdot[\mathbf{S}_{n+1}\times\mathbf{S}_{n+2}]=iJ_TS_{jz}\varepsilon_{jkl}S_k^+S_l^-, \tag{5}$$

where jkl run on $n,n+1,n+2$ and the summation convention applies. Together with the Heisenberg term $J\mathbf{S}_n\cdot\mathbf{S}_{n+1}$ we can think of it as a deterministic hopping with coefficient $J+iaJ_T$ where a takes into account the number of hopping terms in equation 5 and their range (nearest and next nearest ones), as well as some mean-field values of S_{jz}. There are three terms in the scalar product; each one multiplies two terms in the vector product. Such very rough estimates suggests $5\leqslant a\leqslant 10$. Thus, each hopping carries a phase ϕ which, for small J_T/J is equal to aJ_T/J. It is then expected that the crossover region from the GOE to the GUE statistics occurs when the overall phase $\Phi=NaJ_T/J\approx 2\pi$.

In conclusion, we have demonstrated the relevance of random-matrix theory for disordered quantum spin systems, manifested through the distribution of nearest level spacings. For the Heisenberg spin chain with random magnetic field along the z direction Hamiltonian (1) breaks time reversal but there is still an antiunitary operator with which it commutes, leading to a real matrix representation and GOE statistics. Adding a small three site term to Hamiltonian (1) leads to the Hamiltonian (4) which generically breaks the time-reversal invariance. Whereas the density of states remains virtually intact, the nearest level spacing distribution shifts from the GOE to the GUE.

Discussions with B. L. Altshuler and D. Cohen are highly acknowledged. This work was partially supported by ISF, BSF, DIP and Minerva grants. One of us (Y.A.) acknowledges a visitor grant from the CNRS.

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