

Quantum criticalities in a two-leg antiferromagnetic $S = \frac{1}{2}$ ladder induced by a staggered magnetic field

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We study a two-leg antiferromagnetic spin-1/2 ladder in the presence of a staggered magnetic field. We consider two parameter regimes: strong (weak) coupling along the legs and weak (strong) coupling along the rungs. In both cases, the staggered field drives the Haldane spin-liquid phase of the ladder towards a Gaussian quantum criticality. In a generalized spin ladder with a non-Haldane, spontaneously dimerized phase, the staggered magnetic field induces an Ising quantum critical regime. In the vicinity of the critical lines, we derive low-energy effective field theories and use these descriptions to determine the dynamical response functions, the staggered spin susceptibility, and the string order parameter.

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I. INTRODUCTION

The problem of quantum critical points (QCP's) is one of the most important issues in the physics of strongly correlated electron systems, in particular in the context of high- T_c superconductivity¹ and heavy-fermion compounds.² Recently this problem attracted much interest also in the context of one-dimensional (1D) quantum systems, such as 1D interacting electrons and antiferromagnetic spin chains and ladders, where a detailed description of QCP's is available due to the powerful nonperturbative techniques based on conformal field theory, bosonization, and integrability. It is well known that universal properties of 1D quantum systems can be described on the basis of a properly chosen conformally invariant theory deformed by a number of perturbations consistent with the structure and symmetry of the underlying microscopic model. Quantum criticalities can then emerge due to the competition between two (or more) relevant perturbations which, when acting alone, would drive the system to qualitatively different strong-coupling massive phases that cannot be smoothly connected by a continuous path in the parameter space of the model. An example of a theory of this kind, displaying an Ising quantum criticality, is the so-called double-frequency sine-Gordon model.³⁻⁵

In connection with 1D quantum antiferromagnets, a plausible QCP scenario was anticipated some time ago by Affleck and Haldane.⁶ They argued that a massive phase of a translationally invariant spin-chain Hamiltonian can be pushed towards quantum criticality by an external, parity-breaking perturbation. Typical examples of such perturbations are an explicit dimerization, whose role in the formation of QCP's has already been analyzed in several spin-chain and spin-ladder models,⁷⁻¹⁰ and a staggered magnetic field. While dimerization of quantum spin chains and ladders is quite realistic because it can originate, for instance, from the spin-phonon coupling, the case of a static magnetic field whose sign alternates on a microscopic scale used to be regarded as not achievable in experimental conditions.

However, recently two beautiful experimental realizations of staggered magnetic fields in quasi-1D magnetic insulators have been discovered. The first concerns the spin-1/2 antiferromagnetic chain compound copper benzoate.¹¹ Due to the low crystalline symmetry, the magnetic field H couples to the effective spins 1/2 through a gyromagnetic *tensor*¹²

$$H_{\text{magn}} = \sum_n \sum_{\alpha, \beta} [g_{\alpha\beta}^u + (-1)^n g_{\alpha\beta}^{\text{st}}] H_\alpha S_n^\beta.$$

Application of a uniform magnetic field \mathbf{H} thus induces a staggered field \mathbf{h} in a direction perpendicular to \mathbf{H} . In copper benzoate there is a second mechanism that gives rise to a staggered internal magnetic field. It derives from the staggered Dzyaloshinskii-Moriya (DM) interaction along the chain direction,

$$\mathcal{H}_{\text{DM}} = \sum_j (-1)^j \mathbf{D} \cdot (\mathbf{S}_j \times \mathbf{S}_{j+1}),$$

which, when a uniform field \mathbf{H} is applied, induces a staggered component proportional to $\mathbf{H} \times \mathbf{D}$.¹³ The presence of a staggered field has been shown to lead to a variety of very interesting consequences.¹³⁻¹⁶ The staggered field scenario described above is by no means specific to copper benzoate. There are at least two other materials, Yb_4As_3 (Ref. 17) and $[\text{PM} \cdot \text{Cu}(\text{NO}_3)_2 \cdot (\text{H}_2\text{O})_2]_n$ (Ref. 18), whose properties in a magnetic field are controlled by the same mechanism.

A completely different mechanism that leads to the generation of a staggered field has recently been discovered for spin-1 Haldane gap compounds of the type $R_2\text{BaNiO}_3$,^{19,20} where R is a magnetic rare earth. In these materials, the rare-earth ions are only weakly coupled to the Ni chains, but interact strongly with one another. They may be considered to reside on a separate sublattice that undergoes a Néel transition at a rather high temperature T_N . The effect of the resulting antiferromagnetic order is to induce an effective staggered magnetic field along the Ni (spin-1 Heisenberg) chains below T_N .

In this paper, we study the effect of an external staggered magnetic field on the low-energy properties of the spin-1/2 antiferromagnetic two-chain Heisenberg ladder. Although at present no analogous mechanism for the generation of a staggered field has been found for Heisenberg ladders compounds, in principle either of the two scenarios mentioned above is possible. One may well expect that a staggered field will be realized in a ladder compound before long, so that addressing this problem is not only of academic interest.

The Hamiltonian of the “standard” ladder is

$$H_{\text{stand}} = J \sum_{j=1,2} \sum_n S_{j,n} \cdot S_{j,n+1} + J_{\perp} \sum_n S_{1,n} \cdot S_{2,n} - h \sum_{a=1,2} \sum_n (-1)^n S_{a,n}^z, \quad (1)$$

where J and J_{\perp} are antiferromagnetic exchange coupling constants in the “leg” and “rung” directions, respectively. We employ weak-coupling ($J_{\perp} \ll J$) and strong-coupling ($J_{\perp} \gg J$) approaches to show that there exists a critical value of the staggered magnetic field, $h = h_c(J_{\perp}, J)$, for which the system displays a Gaussian U(1) criticality with central charge $C = 1$, characterized by nonuniversal critical exponents. Both at $h < h_c$ and $h > h_c$ the spectrum is gapped, and the spin correlations are commensurate with the underlying lattice. This is different from the spin ladder in a uniform magnetic field²¹ which induces a transition from the gapped commensurate phase ($h < h_c$) to a gapless incommensurate phase ($h > h_c$). Comparing the results of the weak-coupling and strong-coupling approaches, we find that near the critical point the low-energy properties of the spin ladder are adequately described in terms of a XXZ spin-1/2 chain with a J - and J_{\perp} -dependent exchange anisotropy and an effective staggered magnetic field proportional to $h - h_c$. Hence we expect that the existence of the U(1) QCP is a universal property of the standard Heisenberg spin-1/2 ladder in a staggered field.

The critical surface $h_c(J_{\perp}, J)$ separates two massive phases: an anisotropic Haldane spin-liquid phase at $h < h_c$ with coherent $S^z = \pm 1$ and $S^z = 0$ magnon excitations having different, field-dependent, mass gaps, and another massive phase at $h > h_c$ in which the spin excitation spectrum at $q \sim \pi$ still includes coherent transverse ($S^z = \pm 1$) magnons, whereas the $S^z = 0$ modes transform to an incoherent background. The transition is associated with softening of the $S^z = \pm 1$ spin-doublet modes and is characterized by a divergent staggered magnetic susceptibility.

We also discuss the properties of a generalized ladder,

$$H_{\text{gen}} = H_{\text{stand}} + V \sum_n (S_{1,n} \cdot S_{1,n+1})(S_{2,n} \cdot S_{2,n+1}), \quad (2)$$

which, apart from the on-rung interchain exchange J_{\perp} , also includes a four-spin interaction V . This model is interesting because it can display non-Haldane, spontaneously dimerized massive phases if the biquadratic interaction V is sufficiently large.²² The existence of such interactions in ladder compounds is supported by recent neutron scattering

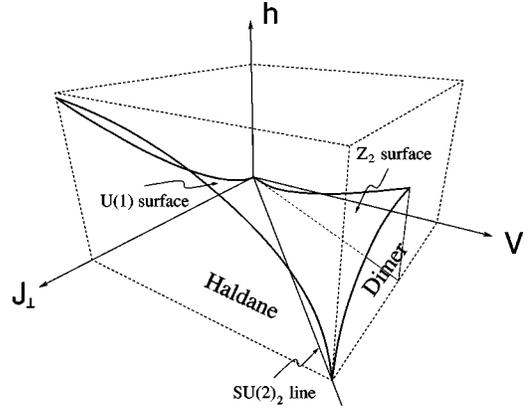


FIG. 1. Phase diagram of the generalized spin ladder in a staggered magnetic field in the limit $J_{\perp}, V \ll J$. The $SU(2)_2$ critical line, which at $h = 0$ separates the Haldane spin-liquid phase from the spontaneously dimerized phase, is split by the field into U(1) and Z_2 critical surfaces.

experiments.²³ We show that, in the weak-coupling regime, the staggered magnetic field can drive the non-Haldane phase to an Ising quantum criticality (with central charge $C = 1/2$), where the spontaneous dimerization vanishes and the staggered magnetic susceptibility is logarithmically divergent. In the dimerized phase ($h < h_c$) the spin excitation spectrum is entirely incoherent, whereas at $h > h_c$, as in the large- h phase of the standard ladder, the coherent $S^z = \pm 1$ magnons are recovered.

The qualitative phase diagram in the (J_{\perp}, h, V) space is shown in Fig. 1. We note that the U(1) (Gaussian, $C = 1$) and Z_2 (Ising, $C = 1/2$) critical surfaces merge at $h = 0$ into a critical line, which has been shown earlier²² to belong to the universality class of the $SU(2)_2$ Wess-Zumino-Novikov-Witten (WZNW) model with central charge $C = 3/2$.

The paper is organized as follows. In Sec. II, we study the phase diagram of the standard spin-ladder model (1) in the weak-coupling limit ($J_{\perp}, h \ll J$). Here we employ a field-theoretical approach which represents the low-energy sector of the spin ladder as an $SO(3) \times Z_2$ -symmetric model of four noncritical 2D Ising systems.^{24,22} Integrating out the fast degrees of freedom associated with collective singlet excitations, we derive an effective action which describes the triplet sector with anisotropy induced by the staggered field. We demonstrate the existence of a Gaussian criticality and show that, close to the critical point, the model is described in terms of a spin-1/2 XXZ chain with a small anisotropy parameter and an effective staggered magnetic field $h^* \propto h - h_c$. In this section we also derive projections of all physical fields onto the low-energy triplet sector.

In Sec. III we discuss general properties of the dynamical structure factor, measured in neutron scattering experiments, for quantum spin chains and ladders in a staggered magnetic field. In Sec. IV we determine the dynamical structure factor of a weakly coupled spin ladder for momentum transfers along the “leg” direction close to π and 0 in both massive phases and at criticality. In Sec. V we study the induced staggered magnetization and show that at the U(1) transition the staggered susceptibility is divergent with a nonuniversal critical exponent.

In Sec. VI we consider the model (1) in the strong-coupling limit ($J_\perp, h \gg J$). Treating the exchange J as a perturbation and projecting the Hamiltonian onto the subspace of the low-energy states, we arrive at an effective strongly anisotropic spin-1/2 Heisenberg chain model in a staggered magnetic field $h^* = h - J_\perp + J/2 + \dots$. As opposed to the weak-coupling case, in the strong-coupling limit the relationship between the parameters of the original and the effective low-energy model is known with any desired degree of accuracy. In Sec. VII we exploit the exact integrability of the sine-Gordon model and apply the form factor method to achieve a quantitative description of the dynamical properties of the model in the strong-coupling limit. This section also contains a brief overview of the form factor approach.

In Sec. VIII, we turn to the generalized ladder (2) and describe the Ising transition in the non-Haldane phase, induced by the staggered magnetic field. In Sec. IX we address the topological order of the generalized ladder model in the staggered field and analyze the field dependence of the longitudinal and transverse components of the nonlocal string order parameter in various parts of the phase diagram shown in Fig. 1.

A discussion of the results obtained from the weak- and strong-coupling regimes and our final conclusions are given in Sec. X. The paper is supplemented with four Appendixes which contain some technical details used in the main text.

II. WEAK-COUPLING LIMIT

A. Ising-model description of the Heisenberg ladder

In this and next subsections we will be concerned with a weakly coupled spin ladder in a small staggered magnetic field: $J_\perp, h \ll J$. We start our discussion with a brief overview of the effective field-theoretical model describing universal properties of the spin-liquid state in the antiferromagnetic two-leg Heisenberg ladder.^{24,22}

It is well known that a single $S=1/2$ Heisenberg chain is critical and has massless $S=1/2$ spinons as elementary excitations.²⁵ When a small J_\perp is switched on, the spinons of the originally decoupled chains get confined to form gapped triplet and singlet excitations (for a review see Ref. 26). The field theory that accounts for the spinon-magnon transmutation in the two-leg spin-1/2 ladder represents an $O(3) \times Z_2$ -symmetric model of four massive real (Majorana) fermions or, equivalently, four noncritical 2D Ising models.^{24,22,27} In the continuum limit, the Hamiltonian density

$$\mathcal{H}_M = \sum_{a=1,2,3} \left[-\frac{iv_t}{2} (\xi_R^a \partial_x \xi_R^a - \xi_L^a \partial_x \xi_L^a) - im_t \xi_R^a \xi_L^a \right] - \frac{iv_s}{2} (\xi_R^4 \partial_x \xi_R^4 - \xi_L^4 \partial_x \xi_L^4) - im_s \xi_R^4 \xi_L^4 + \mathcal{H}_{\text{marg}} \quad (3)$$

describes a degenerate triplet of Majorana fields, $\xi_\nu = (\xi_\nu^1, \xi_\nu^2, \xi_\nu^3)$ ($\nu=R,L$), and a singlet Majorana field $\xi_{R,L}^4$. The velocities $v_{t,s}$ are proportional to $J\alpha$, where α is a short-distance cutoff of the theory. The triplet and singlet masses are given by

$$m_t = J_\perp \lambda, \quad m_s = -3J_\perp \lambda, \quad (4)$$

where λ is a nonuniversal constant. The corresponding correlation lengths in the triplet and singlet sectors of the model are

$$l_{t,s} \sim v_{t,s} / |m_{t,s}|, \quad (5)$$

which for a weakly coupled ladder are macroscopically large ($l_{t,s} \gg \alpha$). The last term in Eq. (3) describes a weak interaction between the Majorana fermions,

$$\mathcal{H}_{\text{marg}} = \frac{1}{2} g_1 (\xi_R \cdot \xi_L)^2 + g_2 (\xi_R \cdot \xi_L) (\xi_R^4 \xi_L^4), \quad (6)$$

where $g_1 = -g_2 = \frac{1}{2} \pi \alpha J_\perp$.

For the generalized ladder (2), the low-energy effective model is still of the form (3), (6), with the only difference that the triplet and singlet masses

$$m_t = J_\perp \lambda - V \lambda', \quad m_s = -3J_\perp \lambda - V \lambda' \quad (7)$$

(λ' is another nonuniversal constant) can be varied independently.

In the continuum description, the local spin densities of the two Heisenberg chains, $S_j(x)$ ($j=1,2$), are contributed by low-energy spin-fluctuation modes centered in the momentum space at $q=0$ and $q=\pi$. Accordingly,

$$S_j(x) = J_{jR}(x) + J_{jL}(x) + (-1)^{x/a_0} n_j(x). \quad (8)$$

The chiral components of the vector currents $J_{j;R,L}$ (i.e., the smooth parts of the spin densities) can be expressed locally in terms of the Majorana bilinears

$$I_\nu = J_{1\nu} + J_{2\nu} = -\frac{i}{2} (\xi_\nu \times \xi_\nu), \quad (9a)$$

$$K_\nu = J_{1\nu} - J_{2\nu} = i \xi_\nu^4 \xi_\nu^4 \quad (\nu=R,L). \quad (9b)$$

However, the most strongly fluctuating fields of the spin ladder, the staggered magnetizations $n_j(x)$ and dimerization fields $\epsilon_j(x) \rightarrow (-1)^n S_{j,n} \cdot S_{j,n+1}$, all with scaling dimension 1/2, are nonlocal with respect to ξ, ξ^4 . These fields, however, admit a representation in terms of the order (σ) and disorder (μ) operators of the related noncritical Ising models:^{24,22,27}

$$n^+ \sim (1/\alpha) (\mu_1 \sigma_2 \sigma_3 \mu_4, \sigma_1 \mu_2 \sigma_3 \mu_4, \sigma_1 \sigma_2 \mu_3 \mu_4), \quad (10a)$$

$$n^- \sim (1/\alpha) (\sigma_1 \mu_2 \mu_3 \sigma_4, \mu_1 \sigma_2 \mu_3 \sigma_4, \mu_1 \mu_2 \sigma_3 \sigma_4), \quad (10b)$$

$$\epsilon^+ \sim (1/\alpha) \mu_1 \mu_2 \mu_3 \mu_4, \quad \epsilon^- \sim (1/\alpha) \sigma_1 \sigma_2 \sigma_3 \sigma_4, \quad (10c)$$

where $n^\pm = n_1 \pm n_2$ and $\epsilon^\pm = \epsilon_1 \pm \epsilon_2$.

Since the correlation lengths $l_{t,s}$ are large, all Ising systems are slightly noncritical. Whether they occur in the ordered or disordered phase depends on the sign of the corresponding mass $m \propto (T - T_c)/T_c$. The crucial property of the standard ladder is that the signs of the triplet and singlet Majorana masses are always *opposite*. This fact, together

with the known asymptotics of the two-point correlation functions of a noncritical Ising model,²⁸ leads to the observation that the dynamical spin susceptibility of the antiferromagnetic Heisenberg ladder, $\chi''(q, \omega)$, obtained by Fourier transforming the correlation function $\langle \mathbf{n}^-(x, \tau) \cdot \mathbf{n}^-(0, 0) \rangle$, exhibits a coherent $S=1$ single-magnon peak at $\omega^2 = (\pi - q)^2 v_t^2 + m_t^2$ (with q close to π). Due to multiparticle processes, the dynamical spin susceptibility $\chi''(q \approx \pi, \omega)$ also displays an incoherent tail with a threshold at $\omega = 3m_t$. The singlet mode shows up only at higher energies, $\omega > 2m_t + |m_s| \sim 5m_t$. Thus, at low energies, the standard two-chain ladder represents a disordered spin liquid, similar to the Haldane phase of the spin-1 Heisenberg chain with a small triplet gap.

Let us now switch on a small staggered magnetic field $\mathbf{h} = \hat{h}\mathbf{z}$ which is assumed to be the same for the both chains of the ladder. The field couples to the total staggered magnetization n_z^+ , and the Hamiltonian density becomes

$$\mathcal{H} = \mathcal{H}_M - (\bar{h}/\alpha) \sigma_1 \sigma_2 \mu_3 \mu_4, \quad (11)$$

with $\bar{h} \sim h$. Here a comment is in order. In spite of the already mentioned similarity between the antiferromagnetic $S=1/2$ ladder and the spin-1 chain, it would be misleading to think that the role of the staggered field in these two cases will be similar. A weakly coupled two-leg $S=1/2$ ladder can be mapped onto a spin-1 chain by formally shifting the singlet excitation band to infinity. This implies the substitutions $\mu_4 \rightarrow \langle \mu_4 \rangle = 0$, $\sigma_4 \rightarrow \langle \sigma_4 \rangle \neq 0$, in which case the $q = \pi$ component of the $S=1$ spin density is determined by the relative staggered magnetization of the spin ladder, \mathbf{n}^- [see Eq. (26a)]. So for the $S=1$ chain the magnetic term has a structure different from Eq. (11):

$$\mathcal{H}_{\text{mag}}^{S=1} = h \mu_1 \mu_2 \sigma_3. \quad (12)$$

Since $m_t > 0$, in the leading order the interaction (12) gives rise to an effective magnetic field $\tilde{h} \sim h \langle \mu_1 \mu_2 \rangle$ applied to the third Ising system ($\tilde{h} \sigma_3$). The spectrum of the $S=1$ chain in a staggered field is therefore always massive (see Ref. 29).

The existence of a U(1) transition in the model (11) can be foreseen as follows. Since the triplet of Ising copies is disordered, the magnetic term in Eq. (11) can be approximately replaced by $\tilde{h} \sigma_1 \sigma_2 \mu_4$, where $\tilde{h} \sim \bar{h} \langle \mu_3 \rangle$. Making a duality transformation in the fourth (singlet) Ising copy ($m_s \rightarrow -m_s$, $\mu_4 \leftrightarrow \sigma_4$), one arrives at a system of three disordered Ising models with the underlying $U(1) \times Z_2$ symmetry, coupled by the interaction $\tilde{h} \sigma_1 \sigma_2 \sigma_4$. This is an Ising-model representation of an anisotropic spin-1 chain close to the integrable, multicritical point,³⁰ with a perturbation representing a parity-breaking, dimerization field. In the isotropic, SU(2)-symmetric case, this model is known^{7-10,31} to exhibit a QCP where it becomes equivalent to the $S=1/2$ Heisenberg chain [SU(2)₁ WZNW universality class]. A finite easy-plane anisotropy transforms this criticality to a Gaussian one.

B. Effective action in the triplet sector

Turning back to Eq. (11), we notice that the magnetic term contains the disorder operator μ_4 of the singlet Ising system which has zero expectation value and represents “fast” degrees of freedom of the system [although the ratio $|m_s|/m_t \approx 3$ may not seem large enough, it can be significantly increased in the generalized model (2) with the “spin-liquid” condition $m_s m_t < 0$ still preserved; see Eqs. (7)]. We will therefore integrate the singlet mode out to obtain an effective action in the triplet sector. The existence of a Gaussian criticality will then immediately follow from the structure of this action.

We write the total Euclidean action of the model as $S = S_t + S_s + S_{st}$, where $S_t[\xi]$ and $S_s[\xi^4]$ are the contributions of triplet and singlet sectors, respectively, and

$$S_{st}[\xi, \xi^4] = \frac{1}{v} \int d^2 r [O_h(\mathbf{r}) + O_g(\mathbf{r})], \quad (13)$$

$$O_h = -(\bar{h}/\alpha) \sigma_1 \sigma_2 \mu_3 \mu_4, \quad O_g = g_2 (\xi_R \cdot \xi_L) (\xi_R^4 \xi_L^4),$$

is treated as a perturbation. Here $\mathbf{r} = (x, v\tau)$, and for simplicity we ignore the difference between the triplet and singlet velocities. Integrating over ξ^4 in the partition function

$$Z = \int D[\xi] D[\xi^4] e^{-S[\xi, \xi^4]} = \text{const} \times \int D[\xi] e^{-S_{\text{eff}}[\xi]}$$

yields the effective action in the triplet sector in the form of a cumulant expansion:

$$S_{\text{eff}}[\xi] = S_t[\xi] + \langle S_{st} \rangle_s - \frac{1}{2} [\langle S_{st}^2 \rangle_s - \langle S_{st} \rangle_s^2] + \dots, \quad (14)$$

where $\langle \dots \rangle_s$ means averaging over the free massive fermions ξ^4 .

The first-order correction in the expansion (14) gives rise to a small renormalization of the triplet mass

$$m_t \rightarrow m_t + m_s \frac{g_2}{2\pi v} \ln \left(\frac{v}{|m_s| \alpha} \right).$$

The cross term proportional to $g_2 h$ involves the correlator $\langle \mu_4(\mathbf{r}_1) \xi_R^4(\mathbf{r}_2) \xi_L^4(\mathbf{r}_2) \rangle_s$, which vanishes due to the unbroken ($\mu \rightarrow -\mu$, $\xi_{R,L} \rightarrow -\xi_{R,L}$) symmetry of an ordered Ising model (see Appendix A). In the second order in g_2 , one obtains terms leading to renormalization of the velocity v_t and coupling constant g_1 . Assuming that all these renormalizations are already taken into account, we are left with the following expression for the effective action:

$$S_{\text{eff}}[\xi] = S_t[\xi] - \frac{\bar{h}^2}{2v^2 \alpha^2} \int d^2 r_1 d^2 r_2 O_t(\mathbf{r}_1) O_t(\mathbf{r}_2) \times \langle \mu_4(\mathbf{r}_1) \mu_4(\mathbf{r}_2) \rangle_s, \quad (15)$$

where $O_t = \sigma_1 \sigma_2 \mu_3$.

The fourth Ising copy is ordered ($m_s < 0$); so the correlation function in Eq. (15) decays exponentially at distances $r \sim l_s$ (Ref. 28):

$$\langle \mu_4(\mathbf{r}) \mu_4(\mathbf{0}) \rangle_s \approx \frac{A_1 (\alpha/l_s)^{1/4}}{\sqrt{2\pi r/l_s}} e^{-r/l_s} + O(e^{-3r/l_s}), \quad (16)$$

where A_1 is a nonuniversal parameter. Under the assumption that $|m_s|/m_t \gg 1$, the correlation length in the triplet sector, l_t , is much larger than that in the singlet sector, l_s . Therefore the product $\mathcal{O}_t(\mathbf{r}_1)\mathcal{O}_t(\mathbf{r}_2)$ in Eq. (15) can be treated by means of the short-distance operator product expansion (OPE).³² Using the fusion rules for two Ising order and disorder operators [see Appendix A, Eqs. (A7a) and (A7b)], we find that

$$\begin{aligned} S_{\text{eff}}[\xi] = & S_t[\xi] + \frac{iC_1}{\alpha} \left(\frac{\bar{h}}{m_s} \right)^2 \int d^2\mathbf{r} (\xi_R^1 \xi_L^1 + \xi_R^2 \xi_L^2 - \xi_R^3 \xi_L^3) \\ & + C_2 \left(\frac{l_s}{\alpha} \right) \left(\frac{\bar{h}}{m_s} \right)^2 \int d^2\mathbf{r} [\xi_R^1 \xi_L^1 \xi_R^2 \xi_L^2 - (\xi_R^1 \xi_L^1 \\ & + \xi_R^2 \xi_L^2) \xi_R^3 \xi_L^3], \end{aligned} \quad (17)$$

where C_1 and C_2 are positive numerical constants.

Thus, we arrive at the following effective Hamiltonian for the triplet degrees of freedom

$$\begin{aligned} \mathcal{H}_{t, \text{eff}} = & -\frac{iv}{2} (\xi_R \cdot \partial_x \xi_R - \xi_L \cdot \partial_x \xi_L) - im_a \xi_R^a \xi_L^a + g_{\parallel} I_R^3 I_L^3 \\ & + g_{\perp} (I_R^1 I_L^1 + I_R^2 I_L^2), \end{aligned} \quad (18)$$

which has the same structure as the field-theoretical model suggested by Tsvelik³³ to describe the Heisenberg spin-1 chain with a biquadratic term and a single-ion anisotropy. The staggered magnetic field introduces anisotropy in the spin ladder and effectively lowers the SO(3) symmetry of the Majorana triplet down to SO(2) \times Z₂ by splitting the masses,

$$m_d \equiv m_1 = m_2 = m_t - C_1 \left(\frac{v}{\alpha} \right) \left(\frac{\bar{h}}{m_s} \right)^2, \quad (19a)$$

$$m_3 = m_t + C_1 \left(\frac{v}{\alpha} \right) \left(\frac{\bar{h}}{m_s} \right)^2, \quad (19b)$$

and by renormalizing the coupling constants of the marginal current-current interaction

$$g_{\parallel} = g_1 + C_2 v \left(\frac{l_s}{\alpha} \right) \left(\frac{\bar{h}}{|m_s|} \right)^2, \quad (20a)$$

$$g_{\perp} = g_1 - C_2 v \left(\frac{l_s}{\alpha} \right) \left(\frac{\bar{h}}{|m_s|} \right)^2. \quad (20b)$$

From Eq. (19) it follows that increasing the staggered magnetic field increases the mass m_3 whereas the mass of the Majorana doublet, m_d , decreases and vanishes at a critical value of the field

$$h_c \propto |m_s| \left(\frac{\alpha}{l_t} \right)^{1/2} \propto J_{\perp}^{3/2} / J^{1/2}. \quad (21)$$

At this point the critical degrees of freedom are represented by a degenerate doublet of massless Majorana fermions with a marginal current-current interaction $g_{\parallel}(h_c)$. This is a typical Gaussian [U(1)] criticality with central charge $C=1$. The vicinity of the critical point where the Majorana doublet (ξ^1, ξ^2) becomes very soft is described by the off-critical Askin-Teller model or, equivalently, the quantum sine-Gordon model (SGM) for a scalar field Φ :

$$\mathcal{H}_{t, \text{eff}} \approx \frac{v}{2} [(\partial_x \Phi)^2 + (\partial_x \Theta)^2] - \frac{m_d}{\pi \alpha} \cos \sqrt{4\pi K} \Phi. \quad (22)$$

Here Θ is the field dual to Φ , the parameter

$$K = 1 - (g_{\parallel}/2\pi v) + O(g_{\parallel}^2) \quad (23)$$

determines the (coupling-dependent) compactification radius of the field Φ and

$$m_d \propto m_t (h_c - h) / h_c.$$

In the vicinity of the critical point, the spectral gap scales as the renormalized mass of the SGM (22):

$$M_d \propto \frac{v}{\alpha} \left(\frac{|m_d| \alpha}{v} \right)^{1/(2-K)} \text{sgn}(m_d). \quad (24)$$

In Sec. VI, we show [see Eq. (80)] that, in the strong-coupling limit ($J_{\perp}, h \gg J$), the effective model describing the low-energy properties of the spin ladder at $h \sim h_c$ represents an anisotropic (XXZ) spin-1/2 Heisenberg chain with the parameter Δ close to 1/2 and an effective staggered magnetic field $h^* \sim h - h_c$. In the continuum limit, this quantum lattice model transforms to the SGM (22). The only difference between the weak-coupling and strong-coupling regimes is that in the latter case the parameter K is close to 3/4.

The description of the low-energy part of the spectrum of the original model (11) in terms of the effective anisotropic spin-1 chain (18) holds if $h \ll J_{\perp}$. As follows from Eq. (21), this condition is satisfied at $h < h_c$ and also in some region above the critical field. However, if the field reaches values $h \sim |m_s|$, the singlet mode becomes as important as the triplet ones, and the effective model (18) is no longer applicable. This regime is difficult to tackle analytically. On the other hand, if the field h is further increased and occurs in the range $|m_s| \ll h \ll J$, the role of the interchain exchange J_{\perp} becomes subdominant, and the original model reduces to two decoupled Heisenberg chains in a weak staggered magnetic field. In this case exact results are available, because in the scaling limit each such chain is described by a SGM with a coupling constant $\beta^2 = 2\pi$ (see, e.g., Ref. 27, Chap. 22):

$$\mathcal{H} \rightarrow \sum_{j=1,2} \mathcal{H}_j, \quad (25)$$

$$\mathcal{H}_j = \frac{v}{2} [(\partial_x \Theta_j)^2 + (\partial_x \Phi_j)^2] - \frac{\lambda(\bar{h})}{2\pi \alpha} \sin \sqrt{2\pi} \Phi_j,$$

where $\lambda(\bar{h}) \sim \bar{h}$ [see Refs. 15 and 16 for an accurate estimation of $\lambda(\bar{h})$]. (Technically, this mapping can be achieved either directly, i.e., using the rules of Abelian bosonization of

the $S=1/2$ Heisenberg chain,^{34,6} or by establishing the correspondence between two pairs of the Majorana fields and two bosonic fields,

$$(\xi_1, \xi_2) \leftrightarrow \Phi_+, \quad (\xi_3, \xi_4) \leftrightarrow \Phi_-, \quad \Phi_{\pm} = \frac{\Phi_1 \pm \Phi_2}{\sqrt{2}},$$

and using formulas (A6a) of Appendix A to bosonize the magnetic term $\sigma_1 \sigma_2 \mu_3 \mu_4$)

C. Projecting operators onto the low-energy sector

Since the fourth (singlet) Ising system has the largest energy gap and stays ordered across the transition ($m_s < 0$), at energies $\omega \ll |m_s|$ the order operator σ_4 can be replaced by its nonzero expectation value $\langle \sigma_4 \rangle \sim (\alpha/l_s)^{1/8}$. Under this substitution the relative staggered magnetization \mathbf{n}^- and dimerization field ϵ^- , defined in Eqs. (10b) and (10c), become projected onto the low-energy, triplet sector of the model described by the effective action (17) and (18):

$$\mathbf{n}^- \rightarrow (\alpha/l_s)^{1/8} \mathbf{N}^-,$$

$$N^- \sim (1/\alpha)(\sigma_1 \mu_2 \mu_3, \mu_1 \sigma_2 \mu_3, \mu_1 \mu_2 \sigma_3); \quad (26a)$$

$$\epsilon^- \rightarrow (\alpha/l_s)^{1/8} E^-, \quad E^- \sim (1/\alpha) \sigma_1 \sigma_2 \sigma_3. \quad (26b)$$

On the other hand, the total staggered magnetization \mathbf{n}^+ and dimerization field ϵ^+ are both proportional to the disorder operator μ_4 whose correlations are exponentially decaying at short distances, $r \sim l_s$ [see Eq. (16)]. Therefore, one might conclude that these fields are short ranged and the spectral weight of their fluctuations is only nonzero in the high-energy region $\omega \sim |m_s|$. By the same argument, the smooth part of the relative magnetization, \mathbf{K} , Eq. (9b), proportional to the singlet Majorana field ξ^4 , would also appear as a ‘‘high-energy’’ field. However, this conclusion cannot be correct for the following reason.

It is true that, once the high-energy singlet modes are integrated out, the operator \mathbf{n}^+ defined in Eq. (10a) has no projection onto the lower-energy sector. However, Eq. (10a) is the *zeroth-order* definition of this operator with respect to the staggered field which couples the high- and low-energy modes. In fact, apart from the always existing short-ranged part, the operator \mathbf{n}^+ contains a strongly fluctuating piece, which originates from a finite admixture of low-energy states occurring already in the first order in h . This can be easily understood from the fact that reduction of the original action of the model to the effective one is equivalent to a unitary transformation of the quantum Hamiltonian of the system that projects it to the subspace of the low-energy states. But the same unitary transformation should be applied to physical operators to single out their low-energy projections.

The low-energy projections of seemingly high-energy operators can be extracted from the second-order perturbative corrections to the corresponding correlation functions. Equivalently (and more formally), this can be done by fusing a local operator $\mathcal{O}_0(\mathbf{r})$, originally defined as a short-ranged field with the perturbative part of the total action,

$$\begin{aligned} \mathcal{O}_0(\mathbf{r}) &\rightarrow \mathcal{O}(\mathbf{r}) \\ &= \mathcal{O}_0(\mathbf{r}) e^{-S_{st}} \\ &= \mathcal{O}_0(\mathbf{r}) + \left(\frac{h}{v}\right) \int d^2 \mathbf{r}_1 \langle \mathcal{O}_0(\mathbf{r}) n_z^+(\mathbf{r}_1) \rangle_s + \mathcal{O}(h^2), \end{aligned} \quad (27)$$

and averaging the first-order term over the fast singlet modes. This term is just the low-energy projection we are looking for (it can be easily checked that the marginal part of the perturbation, given by the operator \mathcal{O}_g in (13), yields no mapping onto the low-energy triplet sector). This is essentially an ‘‘integrating-out’’ procedure but this time applied to the correlation functions rather than the action itself.

In Appendix B we show that the projection of the total staggered magnetization onto the whole triplet sector is of the form

$$n_z^+ \rightarrow \langle n_z^+ \rangle + i A_n \left(\frac{h}{|m_s|}\right) \left(\frac{l_s}{\alpha}\right) (\xi_R^1 \xi_L^1 + \xi_R^2 \xi_L^2 - \xi_R^3 \xi_L^3) + \dots, \quad (28a)$$

$$n_x^+ \rightarrow i A_n \left(\frac{h}{|m_s|}\right) \left(\frac{l_s}{\alpha}\right) (\xi_R^1 \xi_L^3 + \xi_R^3 \xi_L^1) + \dots, \quad (28b)$$

$$n_y^+ \rightarrow i A_n \left(\frac{h}{|m_s|}\right) \left(\frac{l_s}{\alpha}\right) (\xi_R^2 \xi_L^3 + \xi_R^3 \xi_L^2) + \dots, \quad (28c)$$

where $\langle n_z^+ \rangle$ is the average staggered magnetization induced by the field (see Sec. V), A_n is a numerical constant, and the ellipses stand for the high-energy parts of the operators. In that Appendix we also derive the first-order low-energy projection of the relative smooth magnetization of the ladder:

$$K_x \rightarrow -i A_K \left(\frac{h}{|m_s|}\right) \left(\frac{\alpha}{l_s}\right)^{1/8} \left(\frac{l_s}{v}\right) \partial_\tau N_y^-, \quad (29a)$$

$$K_y \rightarrow -i A_K \left(\frac{h}{|m_s|}\right) \left(\frac{\alpha}{l_s}\right)^{1/8} \left(\frac{l_s}{v}\right) \partial_\tau N_x^-, \quad (29b)$$

$$K_z \rightarrow A_K \left(\frac{h}{|m_s|}\right) \left(\frac{\alpha}{l_s}\right)^{1/8} l_s \partial_x E^-, \quad (29c)$$

where A_K is another numerical constant and the fields N^- and E^- are defined in Eqs. (26).

III. DYNAMICAL STRUCTURE FACTOR IN THE PRESENCE OF A STAGGERED FIELD

The scattering cross section measured in neutron scattering experiments is proportional to the dynamical structure factor $S^{\alpha\beta}(\omega, Q)$. In this section we discuss some general properties of $S^{\alpha\beta}(\omega, Q)$ in the case where a quantum spin chain or ladder is subject to parity-breaking external perturbations, such as a staggered magnetic field or explicit dimerization.

Consider first the case of a single Heisenberg chain. The dynamical structure factor is defined as the Fourier transform of the spin-spin correlation function

$$S^{\alpha\beta}(\omega, Q) = \frac{1}{2\pi N} \sum_{n,m=1}^N \int_{-\infty}^{\infty} dt e^{i\omega t - iQ(n-m)} \langle S_n^\alpha(t) S_m^\beta(0) \rangle. \quad (30)$$

Here we have set the lattice spacing $a_0 = 1$. For a translationally invariant, antiferromagnetic spin chain the spin-spin correlation function has the following asymptotic structure

$$\langle S_n^\alpha(t) S_m^\beta(0) \rangle = F_1^{\alpha\beta}(t, n-m) + (-1)^{n-m} F_2^{\alpha\beta}(t, n-m), \quad (31)$$

where $F_1(t, n)$ and $F_2(t, n)$ are slowly varying functions of n and t . According to Eq. (8), in the continuum limit these reduce to the correlation functions of the smooth and staggered magnetization, $\langle J^\alpha(t, x) J^\beta(0, 0) \rangle$ and $\langle n^\alpha(t, x) n^\beta(0, 0) \rangle$, and thus determine the dynamical structure factor in the vicinity of two *different* points: $Q \approx 0$ and $Q \approx \pi$.

When a staggered magnetic field is applied to a spin chain, the situation may seem to be different. Indeed, the translational symmetry of the underlying lattice is broken and the period of the induced magnetic structure is doubled. The spin excitation spectrum is now defined in the reduced Brillouin zone $-\pi/2 < q \leq \pi/2$, with the points $q=0$ and $q=\pi$ identified. At the same time, due to the lowered translational symmetry, the asymptotical expression (31) for the spin-spin correlation function will contain extra oscillating pieces

$$\begin{aligned} \langle S_n^\alpha(t) S_m^\beta(0) \rangle &= F_1^{\alpha\beta}(t, n-m) + (-1)^{n-m} F_2^{\alpha\beta}(t, n-m) \\ &+ (-1)^n F_3^{\alpha\beta}(t, n-m) \\ &+ (-1)^m F_4^{\alpha\beta}(t, n-m), \end{aligned} \quad (32)$$

where $F_3(t, n)$ and $F_4(t, n)$ are smooth functions that transform in the continuum limit to the mixed correlators $\langle n^\alpha(t, x) J^\beta(0, 0) \rangle$ and $\langle J^\alpha(t, x) n^\beta(0, 0) \rangle$, respectively. These correlators are nonzero in the presence of the staggered field, and the question is whether they contribute to the dynamical structure factor.

The answer to this question is negative. At a formal level, this can be shown as follows. Note that due to the broken one-site translational symmetry the spin-spin correlation function should be expanded in a double Fourier series

$$\langle S_n^\alpha(t) S_m^\beta(0) \rangle = \frac{1}{N} \sum_{k, k'} e^{i(kn - k'm)} \tilde{F}_{k, k'}^{\alpha\beta}(t), \quad (33)$$

where k and k' vary within the paramagnetic Brillouin zone ($-\pi < k \leq \pi$), and the double periodicity requires that $\tilde{F}_{k, k'}^{\alpha\beta} \neq 0$ if $k = k'$ or $k = k' + \pi$. Mapping onto the reduced Brillouin zone yields

$$\begin{aligned} \langle S_n^\alpha(t) S_m^\beta(0) \rangle &= \frac{1}{N} \sum_{|q| < \pi/2} e^{iq(n-m)} [\tilde{F}_{q, q}^{\alpha\beta}(t) \\ &+ (-1)^{n-m} \tilde{F}_{q+\pi, q+\pi}^{\alpha\beta}(t) + (-1)^n \tilde{F}_{q+\pi, q}^{\alpha\beta}(t) \\ &+ (-1)^m \tilde{F}_{q, q+\pi}^{\alpha\beta}(t)]. \end{aligned} \quad (34)$$

At small q , $\tilde{F}_{q, q}$, $\tilde{F}_{q+\pi, q+\pi}$, $\tilde{F}_{q+\pi, q}$, and $\tilde{F}_{q, q+\pi}$ are the Fourier transforms of the smooth functions $F_1(n)$, $F_2(n)$, $F_3(n)$, and $F_4(n)$, respectively. Substituting Eq. (33) into Eq. (30) we find that

$$S^{\alpha\beta}(\omega, Q) = \frac{1}{2\pi} \sum_{l, l'=-\infty}^{\infty} \tilde{F}_{Q+2\pi l, Q+2\pi l'}^{\alpha\beta}(\omega). \quad (35)$$

From Eq. (35) it follows that the structure factor is 2π periodic in Q . Secondly, the right-hand side (RHS) of Eq. (35) does not contain off-diagonal matrix elements $\tilde{F}_{Q, Q+\pi}$ and $\tilde{F}_{Q+\pi, Q}$, implying that mixed correlators do not contribute to $S^{\alpha\beta}(\omega, Q)$. Therefore,

$$S^{\alpha\beta}(\omega, Q) = \frac{1}{2\pi} \begin{cases} \tilde{F}_1^{\alpha\beta}(\omega, Q) & \text{if } Q \approx 0, \\ \tilde{F}_2^{\alpha\beta}(\omega, Q - \pi) & \text{if } Q \approx \pi, \end{cases} \quad (36)$$

where, at small Q and ω , $\tilde{F}_1^{\alpha\beta}(\omega, Q)$ and $\tilde{F}_2^{\alpha\beta}(\omega, Q)$ are Fourier transformations of the correlation functions $\langle J^\alpha(t, x) J^\beta(0, 0) \rangle$ and $\langle n^\alpha(t, x) n^\beta(0, 0) \rangle$, respectively.

The same conclusion can be reached within an equivalent but somewhat more appealing picture of diatomic cells. Define a magnetic unit cell made of two sites $(2n-1, 2n)$, and denote the corresponding spin operators by $S_{2l} = S_l^\alpha$ and $S_{2l-1} = T_l^\alpha$. There altogether are four different spin-spin correlation functions

$$\begin{aligned} g_1^{\alpha\beta}(t, l-l') &= \langle S_l^\alpha(t) S_{l'}^\beta(0) \rangle, \\ g_2^{\alpha\beta}(t, l-l') &= \langle T_l^\alpha(t) T_{l'}^\beta(0) \rangle, \\ g_3^{\alpha\beta}(t, l-l') &= \langle S_l^\alpha(t) T_{l'}^\beta(0) \rangle, \\ g_4^{\alpha\beta}(t, l-l') &= \langle T_l^\alpha(t) S_{l'}^\beta(0) \rangle, \end{aligned}$$

whose Fourier transforms

$$\tilde{g}_a(\omega, q) = \sum_{l=1}^{N/2} \int_{-\infty}^{\infty} dt \exp(i\omega t - iq[2la_0]) g_a(t, l)$$

have the periodicity of the reduced Brillouin zone:

$$\tilde{g}_a(\omega, q + \pi) = \tilde{g}_a(\omega, q).$$

It then follows from the definition (30) that

$$\begin{aligned} S^{\alpha\beta}(\omega, Q) &= \frac{1}{4\pi} [\tilde{g}_1^{\alpha\beta}(\omega, Q) + \tilde{g}_2^{\alpha\beta}(\omega, Q) \\ &+ e^{-iQa_0} \tilde{g}_3^{\alpha\beta}(\omega, Q) + e^{iQa_0} \tilde{g}_4^{\alpha\beta}(\omega, Q)]. \end{aligned} \quad (37)$$

This expression shows that, although the spin correlation functions $\tilde{g}_a(\omega, q)$ have the periodicity of the reduced Brillouin zone, the dynamical structure factor does not; it rather retains the periodicity of the paramagnetic Brillouin zone. Thus, for the dynamical structure factor $S^{\alpha\beta}(\omega, Q)$, the points $Q=0$ and $Q=\pi/a_0$ are *inequivalent* even in the presence of a staggered magnetic field.

The functions $g_a(t, l)$ can be easily expressed in terms of the functions $F_a(t, n)$, defined in Eq. (32). Using then Eq. (37) one finds that

$$S^{\alpha\beta}(\omega, Q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \sum_{n=1}^N e^{i\omega t + iQn} F_1^{\alpha\beta}(t, n) + \int_{-\infty}^{\infty} dt \sum_{n=1}^N e^{i\omega t + i(Q+\pi)n} F_2^{\alpha\beta}(t, n).$$

Using the fact that $F_a(t, n)$ are slowly varying functions of n , we finally arrive at the result (36) where

$$\tilde{F}_a^{\alpha\beta}(\omega, p) = \int_{-\infty}^{\infty} dt \sum_{n=1}^N e^{i\omega t + ipn} F_a^{\alpha\beta}(t, n). \quad (38)$$

The generalization to the case of the two-leg ladder is straightforward. The structure factor is defined by

$$S^{\alpha\beta}(\omega, q, q_{\perp}) = \frac{1}{4\pi N} \sum_{a,b=1}^2 \sum_{n,m=1}^N \int_{-\infty}^{\infty} dt e^{i\omega t} \times e^{-iq(n-m) - iq_{\perp}(a-b)} \langle S_{a,n}^{\alpha}(t) S_{b,m}^{\beta}(0) \rangle, \quad (39)$$

where we have introduced a transverse momentum q_{\perp} that can take only the two values 0 and π . Information about the low-energy part of the spin fluctuation spectrum is contained in the staggered ($q \approx \pi$) and smooth ($q \approx 0$) parts of the structure factor for both values of q_{\perp} . All these four cases will be considered separately. Below we adopt the notation

$$s^2 = \omega^2 - q^2 v^2, \quad (40)$$

where q stands for a small momentum deviation either from 0 or π .

IV. DYNAMICAL STRUCTURE FACTOR IN THE WEAK-COUPLING LIMIT

In this section we determine the dynamical structure factor at low energies in the weak-coupling regime. In Secs. IV A–IV D it will be assumed that the staggered magnetic field is much smaller than the singlet gap $|m_s|$. In this case the relevant correlation functions can be estimated using the effective action in the triplet sector and the low-energy projections of the corresponding physical fields, discussed in the preceding section. In Sec. IV F we will consider another limiting case $|m_s| \ll h \ll J$, which will be treated on the basis of the model (25).

A. Structure factor at $q_{\perp} = \pi$, $q \approx \pi$

Of primary importance is the evolution of the coherent triplet peak displayed by the dynamical structure factor of the Heisenberg ladder under the action of the gradually increasing staggered magnetic field. This information is contained in the spectral properties of the relative staggered magnetization n^- whose low-energy projection is given by

Eq. (26a). We therefore start our discussion with the case $q_{\perp} = \pi$ and consider the real-space–imaginary-time correlation functions

$$D_{\alpha\alpha}^{(-)}(\mathbf{r}) = (\alpha/l_s)^{1/4} \langle N_{\alpha}^{-}(\mathbf{r}) N_{\alpha}^{-}(\mathbf{0}) \rangle \quad (\alpha = x, y, z).$$

The corresponding structure factor is obtained by Fourier transformation $D_{\alpha\alpha}^{(-)}(\mathbf{r}) \rightarrow D_{\alpha\alpha}^{(-)}(\mathbf{q})$ [$\mathbf{q} = (q, \omega_n/v)$], and subsequent analytical continuation to the upper complex ω -plane ($i\omega_n \rightarrow \omega + i\delta$).

1. $h < h_c$

For fields smaller than h_c the leading asymptotics of $D_{\alpha\alpha}^{(-)}(\mathbf{r})$ are given by

$$D_{xx}^{(-)}(\mathbf{r}) = D_{yy}^{(-)}(\mathbf{r}) \propto (1/\alpha^2) (\alpha^2/l_s l_3)^{1/4} \langle \sigma_1(\mathbf{r}) \mu_2(\mathbf{r}) \sigma_1(\mathbf{0}) \mu_2(\mathbf{0}) \rangle, \quad (41a)$$

$$D_{zz}^{(-)}(\mathbf{r}) \propto (1/\alpha^2) (\alpha/l_s)^{1/4} \langle \sigma_3(\mathbf{r}) \sigma_3(\mathbf{0}) \rangle \times \langle \mu_1(\mathbf{r}) \mu_2(\mathbf{r}) \mu_1(\mathbf{0}) \mu_2(\mathbf{0}) \rangle, \quad (41b)$$

with $l_3 \sim v/m_3$. Here and in what follows it is assumed that the third Ising system (ξ^3) decouples from the doublet (ξ^1, ξ^2). This assumption is certainly correct in the vicinity of the critical point and holds on a qualitative level everywhere in the massive phases at $h \neq h_c$.

Using the local bosonic representation of the product $\sigma_1 \mu_2$, given by Eq. (A6b) of Appendix A, and properly rescaling the dual field Θ , $\Theta \rightarrow (1/\sqrt{K})\Theta$, we find that the correlator on the RHS of Eq. (41a) reduces to that of vertex operators of the dual field in the SGM (22) which, in turn, can be estimated by means of the form factor bootstrap approach:^{35–39}

$$\left\langle \cos \sqrt{\frac{\pi}{K}} \Theta(\mathbf{r}) \cos \sqrt{\frac{\pi}{K}} \Theta(\mathbf{0}) \right\rangle \sim Z(K) \frac{e^{-M_d r/v}}{\sqrt{M_d r/v}}.$$

Inspecting the RHS of Eq. (41b), we notice that the disorder operators μ_1 and μ_2 have nonzero expectation values ($m_d > 0$). On the other hand, since $m_3 > 0$, the correlator $\langle \sigma_3(\mathbf{r}) \sigma_3(\mathbf{0}) \rangle$ has the asymptotic form (16) with l_s replaced by l_3 . The obtained asymptotics for the correlation functions $D_{xx}^{(-)}(\mathbf{r})$ and $D_{zz}^{(-)}(\mathbf{r})$ lead to the following expressions for the transverse and longitudinal components of the dynamical structure factor:

$$S^{+-}(\omega, \pi + q, \pi) = C_{\perp}(h) \delta(s^2 - M_d^2), \quad (42a)$$

$$S^{zz}(\omega, \pi + q, \pi) = C_{\parallel}(h) \delta(s^2 - m_3^2), \quad (42b)$$

where

$$C_{\perp}(h), C_{\parallel}(h) \sim (|m_s| m_3 M_d^2)^{1/4}. \quad (42c)$$

We note that the incoherent continua that contribute to the dynamical susceptibilities at higher energy can be calculated by using the results of Ref. 28 (see, e.g., Ref. 40 for a similar calculation).

Thus, the coherent triplet magnon peak of the isotropic Heisenberg spin ladder, originally located at the frequency $\omega = m_t$, is split by the field in two peaks: the doublet ($S^z = \pm 1$) peak at $\omega = M_d < m_t$ and the $S^z = 0$ peak at $\omega = m_3 > m_t$. Therefore, the phase occurring at $h < h_c$ represents an anisotropic spin liquid with coherent longitudinal and transverse magnon excitations having different, field-dependent mass gaps.

Upon increasing the field, the two peaks move in opposite directions. When the critical field is approached, the background of multiparticle states with thresholds $3M_d$, $5M_d$, ..., and the doublet peak merge, and, at criticality, the four-point Ising correlation functions in Eqs. (41) follow a power-law behavior.

2. $h = h_c$

As follows from the bosonic representation of the products $\sigma_1 \mu_2$ and $\mu_1 \mu_2$ [see Eqs. (A6a) and (A6b) of Appendix A], at $h = h_c$ the four-point Ising correlators in Eqs. (41) transform to those of vertex operators in a Gaussian model and therefore display a power-law decay. One easily finds that

$$D_{xx}^{(-)}(\mathbf{r}) \propto \left(\frac{\alpha}{r}\right)^{1/2K},$$

$$D_{zz}^{(-)}(\mathbf{r}) \propto \left(\frac{\alpha}{r}\right)^{K/2} \frac{e^{-r/l_3}}{\sqrt{r/l_3}}.$$

After the Fourier transformation and analytic continuation we obtain

$$S^{+-}(\omega, \pi + q, \pi) = C'_\perp \left(\frac{v^2}{\alpha^2 s^2}\right)^{1-1/4K} \theta(s^2), \quad (43a)$$

$$S^{zz}(\omega, \pi + q, \pi) = C'_\parallel \left[\frac{v^2}{\alpha^2(s^2 - m_3^2)}\right]^{1-K/2} \theta(s^2 - m_3^2), \quad (43b)$$

where

$$C'_\perp \sim \left(\frac{\alpha}{v}\right) \left(\frac{\alpha^2}{l_s l_3}\right)^{1/4},$$

$$C'_\parallel \sim \left(\frac{\alpha}{v}\right) \left(\frac{\alpha}{l_s}\right)^{1/4} \left(\frac{\alpha}{l_3}\right)^{1/4 - K/2}, \quad (43c)$$

and $\theta(x)$ is the Heaviside step function. Both the transverse and longitudinal susceptibilities feature incoherent scattering continua with thresholds at zero energy and $\omega = m_3$, respectively.

3. $h_c < h \ll |m_s|$

At $h > h_c$ the mass m_d becomes negative and the doublet of Ising systems occurs in the ordered phase, whereas the third Ising system stays disordered ($m_3 > 0$). The behavior of $D_{xx}^{(-)}(\mathbf{r})$ is just the same as at $h < h_c$ since the duality transformation associated with the sign reversal of m_d , i.e.,

$\mu_{1,2} \leftrightarrow \sigma_{1,2}$, keeps the correlator in the RHS of Eq. (41a) unchanged. Therefore the coherent $|S^z| = 1$ magnon peak, which exists at $h < h_c$ and disappears at the critical point, is recovered in the $h > h_c$ phase. In contrast to this, the asymptotics of $D_{zz}^{(-)}(\mathbf{r})$ is changed, and at $h > h_c$ we find

$$D_{zz}^{(-)}(\mathbf{r}) \propto \frac{1}{v \alpha^2} (|m_s| m_3 M_d^2)^{1/4} \frac{e^{-2|M_d|r/v} e^{-m_3 r/v}}{|M_d|r/v \sqrt{m_3 r/v}}.$$

As a consequence, the Haldane spin liquid loses part of its coherent spectral weight at $q \sim \pi$: the $S^z = 0$ magnon is no longer seen in the longitudinal staggered structure factor $S^{zz}(\omega, \pi + q, \pi)$ and is replaced by an incoherent continuum of states with a threshold at $\omega = 2|M_d| + m_3$:

$$S^{zz}(\omega, \pi + q, \pi) = C''_\parallel \theta[s^2 - (2|M_d| + m_3)^2], \quad (44a)$$

where

$$C''_\parallel(h) \sim \left(\frac{\alpha}{v}\right) \left(\frac{\alpha^2}{l_s l_3}\right)^{1/4} \left(\frac{v}{(2|M_d| + m_3)\alpha}\right)^{1/2}. \quad (44b)$$

We will show below that, in fact, the coherent $S^z = 0$ mode still exists at $h > h_c$ but its spectral weight is shifted towards small momenta ($q \sim 0$) and is very small.

B. Structure factor at $q_\perp = \pi$, $q \approx 0$

The structure factor $S^{\alpha\alpha}(\omega, q, \pi)$ is determined by correlations of the relative magnetization

$$\mathbf{S}_{1,n} - \mathbf{S}_{2,n} \rightarrow \mathbf{K}(x),$$

$$K^a = i(\xi_R^a \xi_L^a + \xi_L^a \xi_R^a), \quad a = 1, 2, 3. \quad (45)$$

The projections of the fields K^a onto the low-energy sector are given by Eqs. (29a)–(29c).

Consider first the longitudinal structure factor. Since the third Ising system remains disordered across the transition ($m_3 > 0$), $S^{zz}(\omega, q, \pi)$ will be nonzero only at frequencies $\omega \geq m_3$. This follows from Eqs. (29c) and (26b). Note that the operator E^- in Eq. (26b) is related to N_z^- , Eq. (26a), by a duality transformation in the doublet sector. Therefore the large-distance asymptotics of $\langle E^-(\mathbf{r}) E^-(\mathbf{0}) \rangle$ can be obtained from those of $D_{zz}^{(-)}(\mathbf{r})$, estimated in the preceding subsection, by interchanging the cases $h < h_c$ and $h > h_c$. This leads to the following results:

$h < h_c$:

$$S^{zz}(\omega, q, \pi) = C''_\parallel(h) \left(\frac{h}{m_s}\right)^2 (ql_s)^2 \theta[s^2 - (2M_d + m_3)^2]; \quad (46a)$$

$h = h_c$:

$$S^{zz}(\omega, q, \pi) = C'_\parallel \left(\frac{h}{m_s}\right)^2 (ql_s)^2 \left[\frac{v^2}{\alpha^2(s^2 - m_3^2)}\right]^{1-K/2} \times \theta(s^2 - m_3^2); \quad (46b)$$

$h > h_c$:

$$S^{zz}(\omega, q, \pi) = C_{\parallel}(h) \left(\frac{h}{m_s} \right)^2 (ql_s)^2 \delta(s^2 - m_3^2), \quad (46c)$$

where the prefactors C_{\parallel} , C'_{\parallel} , and C''_{\parallel} are given by Eqs. (42c), (43c), and (44b). We see that at $h > h_c$ the coherent $S^z=0$ magnon mode is seen in the small- q part of the dynamical structure factor. However, its spectral weight is proportional to $(ql_s)^2$ and thus small.

Consider now the transverse structure factor. Using the definitions (29a) and (29b) and the known expression for the structure factor $S^{\pm}(\omega, \pi+q, \pi)$ obtained in Sec. IV A, we find the following:

$h \neq h_c$:

$$S^{\pm}(\omega, q, \pi) = C_{\perp}(h) \left(\frac{h}{m_s} \right)^2 \left(\frac{\omega}{m_s} \right)^2 \delta(s^2 - M_d^2); \quad (47a)$$

$h = h_c$:

$$S^{\pm}(\omega, q, \pi) = C'_{\perp} \left(\frac{h}{m_s} \right)^2 \left(\frac{\omega}{m_s} \right)^2 \left(\frac{v^2}{s^2 \alpha^2} \right)^{1-1/4K} \theta(s^2), \quad (47b)$$

where C_{\perp} and C'_{\perp} are given by Eqs. (42c) and (43c). Comparing the results (42a) and (47a), we see that at any nonzero h (except for the transition point), the coherent transverse ($|S^z|=1$) magnon mode is seen both in the staggered ($q \approx \pi$) and smooth ($q \approx 0$) parts of the structure factor $S^{\pm}(\omega, q, \pi)$, with the ratio of the spectral weights at $q \approx 0$ and $q \approx \pi$ being of the order of $(h/m_s)^2 (M_d/m_s)^2$.

C. Structure factor at $q_{\perp} = 0$, $q \approx \pi$

The structure factor $S^{\alpha\beta}(\omega, \pi+q, 0)$ is determined by dynamical correlations of the total staggered magnetization, \mathbf{n}^+ , whose low-energy projection is given by expressions (28a)–(28c).

1. Longitudinal structure factor

As follows from Eq. (28a), $S^{zz}(\omega, \pi+q, 0)$ displays a broad continuum of states with the lowest-energy threshold equal to $2|M_d|$. If $|h - h_c| \ll h_c$, at frequencies much less than m_3 only the doublet modes are to be taken into account. In this case we have

$$D_{zz}^{(+)}(\mathbf{r}) = A_n^2 \left(\frac{h}{m_s} \right)^2 \left(\frac{l_s}{\alpha} \right)^2 K_d(\mathbf{r}),$$

$$K_d(\mathbf{r}) = \langle \varepsilon_d(\mathbf{r}) \varepsilon_d(\mathbf{0}) \rangle, \quad (48)$$

where $\varepsilon_d = i(\xi_R^1 \xi_L^1 + \xi_R^2 \xi_L^2)$ is the energy-density operator in the doublet sector. A simple calculation, based on the assumption that the doublet fermions are free, leads to the result (see Appendix C)

$$\text{Im } K_d(q, \omega + i\delta) = \frac{1}{2v} \frac{\sqrt{s^2 - 4m_d^2}}{s}. \quad (49)$$

In fact, the square-root behavior of the structure factor near the threshold is *universal*: the effect of the interaction be-

tween the Majorana fermions ξ^1 and ξ^2 shows up only in the mass and velocity renormalization and the interaction-dependent prefactor $Z_d(K)$:

$$S^{zz}(\omega, \pi+q, 0) \propto \frac{1}{2v} Z_d^2(K) \left(\frac{h}{m_s} \right)^2 \left(\frac{l_s}{\alpha} \right)^2 \frac{\sqrt{s^2 - 4M_d^2}}{s} \theta(s^2 - 4M_d^2). \quad (50)$$

This is confirmed by the calculation done in Sec. VII E, which proceeds from the bosonized correlator (48),

$$K_d(\mathbf{r}) \propto \langle \cos \sqrt{4\pi K} \Phi(\mathbf{r}) \cos \sqrt{4\pi K} \Phi(\mathbf{0}) \rangle, \quad (51)$$

and automatically takes into account interaction effects in the doublet sector within the form factor approach^{35–37} to the SGM (22). In that calculation, the crucial fact is that, in the range $3/4 \leq K < 1$, the spectrum of the SGM (22) consists of massive quantum solitons (s) and antisolitons (\bar{s}) with the mass M_d , and one soliton-antisoliton bound state (breather) with the mass M_1 given by Eq. (91). The latter, however, is odd under parity (charge) conjugation (A12) and as a result the form factor of the parity-symmetric operator $\cos \sqrt{4\pi} \Phi$ between the vacuum and the breather state vanishes. Therefore the main contribution to the structure factor is due to the $s\bar{s}$ scattering continuum with a threshold at $2|M_d|$, where M_d is the single-soliton mass. This explains the universality of the square-root behavior (50) of the structure factor $S^{zz}(\omega, \pi+q, 0)$ near the threshold.

The result (50) neglects the contributions of multiparticle processes with thresholds at higher energies ($4|M_d|$, $6|M_d|$, etc.). As $h \rightarrow h_c$ (i.e., $M_d \rightarrow 0$), such processes become as important as the two-particle ones. Exactly at criticality the Majorana doublet becomes massless, and the interaction in the doublet sector can no longer be ignored. In this case $K_d(r) \approx (\alpha/r)^{2K}$ and

$$S^{zz}(\omega, \pi+q, 0) \propto \frac{C(K)}{v} \left(\frac{v^2}{\alpha^2 s^2} \right)^{1-K} \theta(s^2), \quad (52)$$

where $C(K) \sim [2^{1-2K}/\Gamma^2(K)](h/m_s)^2 (l_s/\alpha)^2$.

2. Transverse structure factor

The transverse structure factor $S^{xx}(\omega, \pi+q, 0)$ can be estimated in a similar manner. The main difference is that now we have two Majorana fermions (ξ^1 and ξ^3) with unequal masses (m_d and m_3). Treating both fermions as free, simple calculations along the lines of Appendix C give

$$S^{xx}(\omega, \pi+q, 0) \propto \frac{\alpha}{v} \left(\frac{h}{m_s} \right)^2 \left(\frac{l_s}{\alpha} \right)^2 \sqrt{\frac{s^2 - m_+^2}{s^2 - m_-^2}}, \quad (53)$$

$$s^2 \geq \max\{m_+^2, m_-^2\},$$

where $m_{\pm} = m_3 \pm m_d$. For different signs of the doublet mass $m_d \sim h_c - h$, the frequency dependence of the structure factor is qualitatively different (see Fig. 2). At $h < h_c$ ($m_d > 0$), $S^{xx}(\omega, \pi+q, 0)$ follows a square-root increase above the

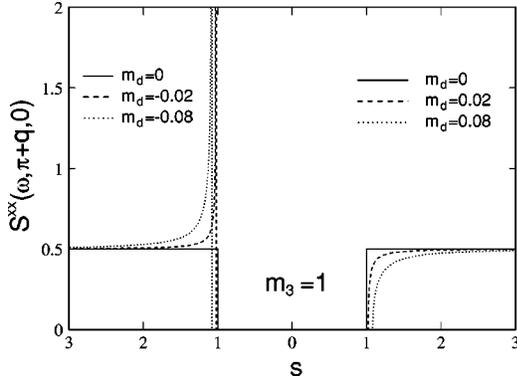


FIG. 2. Transverse structure factor at $q \approx \pi$, $q_{\perp} = 0$. For more clarity, the cases $h < h_c$ ($m_d > 0$) and $h > h_c$ ($m_d < 0$) are shown separately.

threshold m_+ which becomes steeper as $h \rightarrow h_c$. At $h > h_c$, $S^{xx}(\omega, \pi + q, 0)$ has a square-root singularity at the threshold m_- with an amplitude proportional to $|m_d|^{1/2}$.

Let us now discuss the role of the so far neglected interaction between the fermions. As we have mentioned before, near the critical point the $S^z = 0$ collective modes described by the field ξ^3 asymptotically decouple from the $S^z = \pm 1$ (doublet) ones. Hence one can always assume that the Majorana fermion ξ^3 is free and massive. Concerning the fermion ξ^2 that belongs to the interacting doublet sector, we may use the bosonized expressions for the chiral components of ξ^2 [see Eqs. (A5)] and employ a form factor expansion in the model (22) to show that

$$\begin{aligned} \langle \xi_R^2(x, \tau) \xi_R^2(0, 0) \rangle &= Z(K) K_1(|M_d z|) \left(\frac{z}{\bar{z}} \right)^{1/2}, \\ \langle \xi_L^2(x, \tau) \xi_L^2(0, 0) \rangle &= Z(K) K_1(|M_d z|) \left(\frac{\bar{z}}{z} \right)^{1/2}, \\ \langle \xi_R^2(x, \tau) \xi_L^2(0, 0) \rangle &= Z(K) K_0(|M_d z|). \end{aligned} \quad (54)$$

Here $z = \tau + ix/v$, $\bar{z} = \tau - ix/v$, and $K_n(x)$ are the MacDonald functions. In Eqs. (54) only the one-particle form factor has been taken into account (the first correction involves a contribution of three-particle processes). We see that the expressions (54) have the structure of the two-point correlators of free massive Majorana fermions. The information about the interactions in the doublet sector, i.e., the parameter K of the SGM, is contained in the renormalized mass M_d and the constant $Z(K)$. This means that, up to this prefactor, the above result (53) actually represent the first term of the exact form factor expansion and, thus, is universal close to the threshold as long as $h \neq h_c$.

According to Eq. (53), at criticality the free-fermion approximation leads to a step function $\theta(s^2 - m_3^2)$. However, this result must be revisited because at $M_d = 0$ the single-fermion propagator $\langle \xi^1(x, \tau) \xi^1(0, 0) \rangle$ transforms to that of a spinless Tomonaga-Luttinger liquid. An estimation of $S^{xx}(\omega, \pi + q, 0)$, quite similar to that done in Appendix D, leads to a power-law behavior

$$S^{xx}(\omega, \pi + q, 0)|_{\text{crit}} = \mathcal{C}(\vartheta) \left(\frac{\hbar}{|m_s|} \right)^2 \left(\frac{l_s}{\alpha} \right)^2 \left(\frac{s^2 - m_3^2}{m_3^2} \right)^{2\vartheta}, \quad (55a)$$

where

$$\mathcal{C}(\vartheta) \sim \frac{(\alpha/v)}{2^{1+2\vartheta} \Gamma(1+2\vartheta)} \left(\frac{\alpha}{l_3} \right)^{2\vartheta} \quad (55b)$$

and

$$2\vartheta = \frac{1}{2} \left(K + \frac{1}{K} \right) - 1 \quad (56)$$

is the critical exponent of the single-particle density of states in a Tomonaga-Luttinger liquid. We see that, due to an ‘‘infrared catastrophe’’ caused by the interactions in the doublet sector, the threshold discontinuity of $S^{xx}(\omega, \pi + q, 0)$ transforms to a continuous dependence. However, due to the smallness of ϑ , the power-law increase of the transverse structure factor is very steep.

D. Structure factor at $q_{\perp} = 0$, $q \approx 0$

At small q the structure factor $S^{\alpha\alpha}(\omega, q, 0)$ is determined by correlations of the smooth part of the total magnetization density. In the continuum limit, the latter can be expressed in terms of the triplet Majorana fields:

$$S_{1,n} + S_{2,n} \rightarrow \mathbf{I}(x),$$

$$\mathbf{I}^a = -(i/2) \epsilon^{abc} (\xi_R^b \xi_R^c + \xi_L^b \xi_L^c). \quad (57)$$

1. Longitudinal structure factor

The total longitudinal current $I^3 = -i \xi_{\nu}^1 \xi_{\nu}^2$ involves only the doublet Majorana modes which become critical at the transition. If the marginal interaction between these modes was absent, the structure factor at small q would display a broad continuum of states with a threshold $2|m_d|$ (Ref. 24):

$$S^{zz}(\omega, q, 0) \sim \frac{2q^2 m_d^2}{s^3 \sqrt{s^2 - 4m_d^2}}. \quad (58)$$

As opposed to the case of the energy-density correlator (51) where the free-fermion approximation correctly captures its universal features, here the interaction in the doublet sector changes the result (58) dramatically. As already mentioned, apart from soliton/antisoliton states the spectrum of the SGM (22) with $\frac{3}{4} \leq K \leq 1$ also features the first breather state. The latter is odd under charge conjugation \mathbf{C} , which inverts the sign of sine-Gordon field Φ [see Eq. (A12)]. As follows from Eq. (A3) the current operator $I^3 = \sqrt{K/\pi} \partial_x \Phi$ is also odd under \mathbf{C} and consequently has a nonzero form factor between the vacuum and the first breather state. This gives rise to a coherent δ -function peak appearing below the threshold of the two-soliton scattering continuum. Taking into account only the contributions of the first breather (with mass M_1) and the two-particle scattering states, close to the critical point we obtain

$$S^{zz}(\omega, q, 0) \propto 2 \left| \frac{\pi \xi \lambda}{\sin(\pi \xi)} \right|^2 q^2 \delta(s^2 - M_1^2) + 4\pi \frac{q^2 \sqrt{s^2 - 4M_d^2}}{s^3} \frac{1}{\cosh(2\theta_0/\xi) + \cos(\pi/\xi)} \times \exp \left(\int_0^\infty \frac{dt}{t} \frac{\sinh([1-\xi]t) \left[1 - \cosh 2t \cos\left(\frac{2\theta_0 t}{\pi}\right) \right]}{\sinh(2t) \cosh t \sinh(t\xi)} \right), \quad (59)$$

where $\theta_0 = \text{arccosh}(s/2M_d)$ and

$$\lambda = 2 \cos(\pi \xi/2) \sqrt{2 \sin(\pi \xi/2)} \exp \left(- \int_0^{\pi \xi} \frac{dt}{\sin t} \right), \quad \xi = \frac{K}{2-K}, \quad M_1 = 2M_d \sin(\pi \xi/2). \quad (60)$$

This result is valid for any sign of $h - h_c$. We note that $S^{zz}(\omega, q, 0)$ vanishes as $q \rightarrow 0$ because the z component of the total magnetization, $S_1^z + S_2^z = \int dx I^3(x)$, is conserved. In the limit $K \rightarrow 1$ the two-particle contribution to Eq. (59) reduces to the form (58) as it should, if we identify M_d with m_d .

Exactly at the Gaussian criticality ($h = h_c$) the longitudinal structure factor is given by the well-known ‘‘chiral anomaly’’ formula (see, e.g., Ref. 27):

$$S^{zz}(\omega, q, 0) = K(\alpha/v)(qv)^2 \delta(\omega^2 - q^2 v^2). \quad (61)$$

2. Transverse structure factor

Now we turn to dynamical correlations of the transverse total current, $I^1 = -i\xi_\nu^2 \xi_\nu^3$. As before we will first adopt the free-fermion approximation and then discuss how the results are affected by the interaction between the doublet modes.

After some algebras we find the following expression for $S^{xx}(\omega, q, 0)$ (assuming that $\omega > 0$):

$$S^{xx}(\omega, q, 0) = \frac{\alpha}{2v} \frac{m_+^2 q^2 (s^2 - m_-^2) + m_-^2 (s^2 + q^2) (s^2 - m_+^2)}{s^4 \sqrt{(s^2 - m_+^2)(s^2 - m_-^2)}}, \quad (62)$$

$$m_\pm = m_3 \pm m_d, \quad s^2 \geq \max\{m_+^2, m_-^2\}.$$

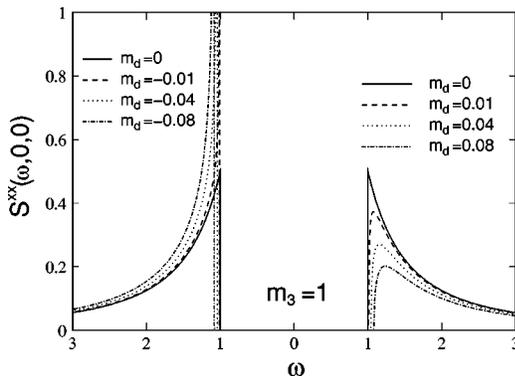


FIG. 3. Transverse structure factor at $q = q_\perp = 0$.

The case $q = 0$ is shown in Fig. 3. Apart from the $1/\omega^2$ decay at $\omega > m_3$, the behavior of $S^{xx}(\omega, q, 0)$ close to the threshold is similar to that of $S^{xx}(\omega, \pi + q, 0)$ (cf. Fig. 2).

Although at $q = 0$ the transverse structure factor is finite for $h < h_c$ ($m_d \geq 0$), at arbitrarily small q it is divergent at the threshold (see Fig. 4),

$$S^{xx}(\omega, q, 0) \sim \frac{q^2 (m_+^2 - m_-^2)^{1/2}}{m_+^2 (m_+^2 + q^2)^{1/4}} \frac{1}{\sqrt{\delta}},$$

where $\delta = \omega - \sqrt{m_+^2 + q^2} > 0$.

For $h < h_c$ and at $q \neq 0$ the curves shown in Fig. 4 display an interesting feature. The $1/\sqrt{\delta}$ drop of the structure factor slightly above the threshold is followed by an upturn which is a property of $S^{xx}(\omega, 0, 0)$. The maximum occurs at

$$\delta \sim \frac{(m_+^2 - m_-^2) q^2}{m_+ m_-^2}.$$

At $h > h_c$ ($m_d < 0$) the singularity at the threshold is of the form

$$S^{xx}(\omega, q, 0) \sim \frac{(m_-^2 - m_+^2)^{1/2} (m_-^2 + q^2)^{3/4}}{m_-^2} \frac{1}{\sqrt{\delta}}.$$

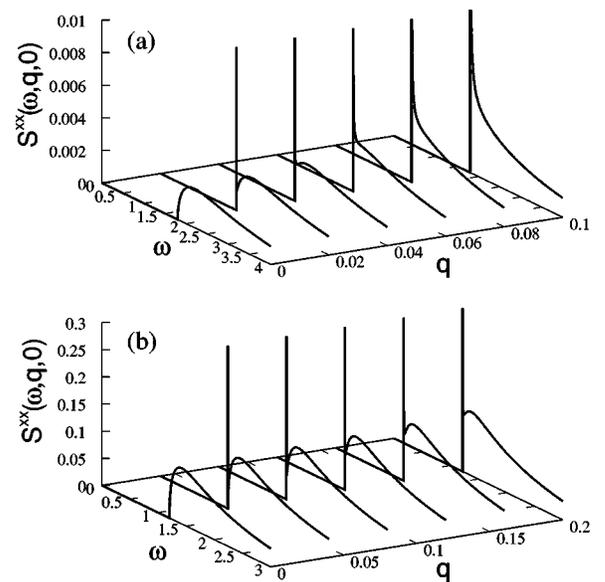


FIG. 4. Transverse structure factor at small q with parameter (a) $m_d = 0.8m_3$ and (b) $m_d = 0.2m_3$, $m_3 = 1$.

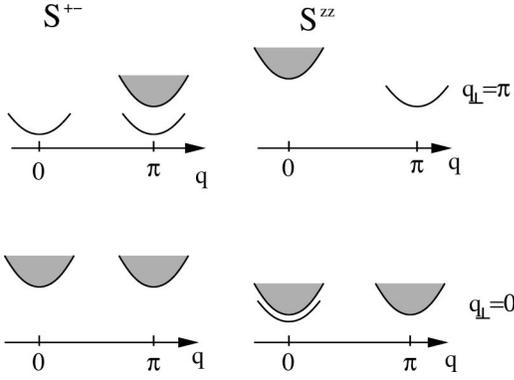


FIG. 5. Structure of low-energy magnetic excitations in the weak-coupling limit when $h < h_c$.

Now it survives the limit $q \rightarrow 0$ and disappears only at the critical point.

The free-fermion approximation is reliable as long as $h \neq h_c$. The derivation of the structure of $S^{xx}(\omega, q, 0)$ at criticality is given in Appendix D. Using Eqs. (D6) and (D7) we find that just above the threshold $\omega = \sqrt{q^2 + m_3^2}$ the structure factor $S^{xx}(\omega, q, 0)|_{\text{crit}}$ only differs from the expression (55a) by an extra factor $(\omega^2 + q^2 v^2)/m_3^2$,

$$S^{xx}(\omega, q, 0)|_{\text{crit}} = \mathcal{C}(\vartheta) \left(\frac{\omega^2 + q^2 v^2}{m_3^2} \right) \left(\frac{s^2 - m_3^2}{m_3^2} \right)^{2\vartheta}. \quad (63)$$

The discontinuity of $S^{xx}(\omega, q, 0)$ at the threshold, obtained in the free-fermion approximation, is rounded by the interaction in the doublet sector. However, due to the smallness of ϑ , the transverse structure factor rapidly increases beyond the threshold and reaches values of the order of 1 exponentially close to the threshold, $\omega - \sqrt{q^2 v^2 + m_3^2} \sim \alpha^{-1} e^{-1/2\vartheta}$.

E. Summary of the structure of the magnetic excitation spectrum at weak coupling

At this point it would appear to be useful to briefly summarize our results concerning the behavior of the dynamical structure factor in the weak-coupling regime derived above. We do this in a simple pictorial way, having in mind to illustrate the qualitative changes in the spin excitation spectrum associated with the transition induced by the staggered magnetic field.

In Figs. 5 and 6 curves denote coherent single-particle excitations whereas shaded areas correspond to incoherent multiparticle scattering continua. The main changes as we tune the staggered field through the transition concern $S^{zz}(\omega, q, \pi)$. The coherent mode visible in $S^{zz}(\omega, q \approx \pi, \pi)$ below the transition is replaced by an incoherent scattering continuum for $h > h_c$. In the vicinity of $q = 0$ exactly the opposite happens: the incoherent scattering continuum present below h_c splits off a coherent mode for $h > h_c$. The transverse magnon mode visible in $S^{+-}(\omega, q, \pi)$ is well defined both below and above the transition.

At $q_{\perp} = 0$ the general structure of the excitation spectrum is similar for $h < h_c$ and $h > h_c$. However, the precise form

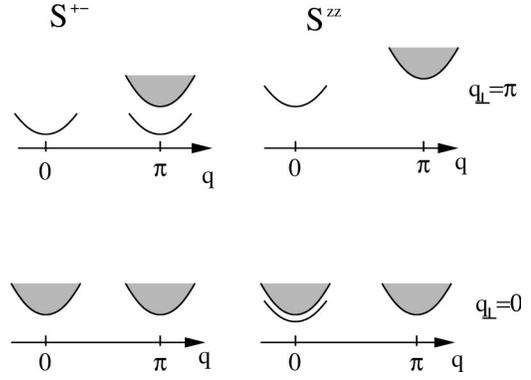


FIG. 6. Structure of low-energy magnetic excitations in the weak-coupling limit when $h > h_c$.

of the structure factor changes significantly, which of course has important consequences for the neutron scattering cross section.

F. Structure factor at $|m_s| \ll h \ll J$

This is the limit of two decoupled Heisenberg chains in a staggered magnetic field, described by a pair of the $\beta^2 = 2\pi$ sine-Gordon Hamiltonians (25). The problem basically reduces to the situation studied in Refs. 13–16. The elementary excitations are solitons (s) and antisolitons (\bar{s}) with a mass gap M (which scales as $M \sim h^{2/3}$) and two soliton-antisoliton “breather” bound states $B_{1,2}$ with gaps M and $\sqrt{3}M$, respectively. The dynamical structure factor has already been partially calculated by means of the form factor bootstrap approach in Ref. 14. We briefly review some important ingredients of this method in Sec. VI. The leading contributions to the various components of the structure factor are found to be

$$S^{+-}(\omega, \pi + q, q_{\perp}) = A_{\perp} \delta(s^2 - M^2) + \dots, \quad (64a)$$

$$S^{zz}(\omega, \pi + q, q_{\perp}) = A_{\parallel} \delta(s^2 - 3M^2) + \dots, \quad (64b)$$

$$S^{+-}(\omega, q, q_{\perp}) = B_{\perp} (\omega^2/M^2) \delta(s^2 - M^2) + \dots, \quad (64c)$$

$$S^{zz}(\omega, q, q_{\perp}) = B_{\parallel} (q^2/M^2) \delta(s^2 - M^2) + \dots. \quad (64d)$$

Here $A_{\perp, \parallel}$ and $B_{\perp, \parallel}$ are constants that can be calculated, $q_{\perp} = 0, \pi$, and the ellipses denote higher-energy contributions of multiparticle intermediate states. Both at $q \sim 0$ and $q \sim \pi$ the transverse structure factor shows a coherent δ -function peak due to one-soliton intermediate states. This suggests that there may be a coherent transverse soliton mode throughout the whole Brillouin zone. The longitudinal structure factor exhibits a coherent δ -function mode corresponding to the second breather B_2 with gap $\sqrt{3}M$ in the vicinity of $q = \pi$. For small momentum transfer along the chain direction the longitudinal structure factor exhibits a coherent δ -function peak which is a contribution of the first breather B_1 with gap M . However, the spectral weight is very small ($\propto q^2$).

Let us now compare these results to the ones found above for the weak-coupling limit $J \gg J_{\perp} \gg h > h_c$.

$q_{\perp} = \pi$: The transverse structure factor for $q \approx \pi$ is similar to Eq. (64a). The longitudinal structure factor at weak coupling is incoherent whereas for $h \gg J_{\perp}$ it is coherent. This is easily understood: increasing the staggered field eventually splits off a bound state from the incoherent continuum. Finally, the structure factors for $q \approx 0$ have the same general form as Eqs. (64c) and (64d), respectively.

$q_{\perp} = 0$: Here the situation at large staggered fields, $h \gg J_{\perp}$, is different from that found at $J_{\perp} \gg h > h_c$. The transverse structure factor around $q = \pi$ and $q = 0$ exhibits coherent modes for $h \gg J_{\perp}$, Eqs. (64a) and (64c), whereas it displays incoherent scattering continua for $J_{\perp} \gg h > h_c$. However, these continua are singular above the thresholds, and one can thus easily imagine that they are prone to split off coherent modes once h becomes sufficiently large. An analogous situation is encountered for $S^{zz}(\omega, q, 0)$ at small q . Finally, a coherent δ -function peak is found in $S^{zz}(\omega, \pi + q, 0)$ both at $h \gg J_{\perp}$ and $J_{\perp} \gg h > h_c$.

V. INDUCED STAGGERED MAGNETIZATION

In this section, we estimate the magnetic field dependence of the induced staggered magnetization \mathcal{M} in the Haldane spin-liquid phase of the spin ladder. Since at $h=0$ correlations of the total staggered magnetization $(-1)^n [S_{1n}^z + S_{2n}^z]$ are short ranged, the dependence $\mathcal{M}(h)$ will be linear in the limit $h \rightarrow 0$:

$$\mathcal{M}(h) = \chi_s(0)h \quad (h \rightarrow 0), \quad (65a)$$

$$\chi_s(0) = \frac{1}{v} \int d^2r \langle n_z^+(r) n_z^+(\mathbf{0}) \rangle_0 \sim \frac{J}{vJ_{\perp}}. \quad (65b)$$

On the other hand, when h is large ($h \gg J_{\perp}$), the transverse coupling between the two $S=1/2$ Heisenberg chains of the ladder can be neglected. Each chain is then described at low energies in terms of the $\beta^2 = 2\pi$ SGM (25). This leads to the result

$$\mathcal{M}(h) \propto \left(\frac{h\alpha}{v}\right)^{1/3}, \quad \chi_{\text{stag}}(h) \propto h^{-2/3}. \quad (66)$$

To estimate the singular part of $\mathcal{M}(h)$ at $|h - h_c| \rightarrow 0$, it is sufficient to know the h dependence of the ground-state energy density, $\mathcal{E}(h)$, of the low-energy effective Hamiltonian. Both in the weak-coupling and strong-coupling limits, the latter has the structure of the SGM (22) with the amplitude of the cosine term proportional to $|h - h_c|$. Using standard scaling arguments, we find that

$$\mathcal{E}(h) - \mathcal{E}(h_c) \sim -|h - h_c|^{2/(2-K)},$$

where K is given by Eq. (83). Higher-energy degrees of freedom which asymptotically decouple from the critical ones provide a finite contribution to $\mathcal{M}(h)$ at $h = h_c$. So

$$\begin{aligned} \mathcal{M}(h) &= \mathcal{M}(h_c) - \frac{\partial \mathcal{E}(h)}{\partial h} \\ &= m_{\text{stag}}(h_c) + \text{const} |h - h_c|^{K/(2-K)} \text{sgn}(h - h_c). \end{aligned} \quad (67)$$

The staggered susceptibility

$$\chi_s(h) \sim |h - h_c|^{-\nu}, \quad \nu = \frac{2(1-K)}{2-K} > 0, \quad (68)$$

is divergent at the transition. The nonuniversal exponent ν varies from $g_{\parallel}^c / \pi v$ at $J_{\perp} \ll J$ to values close to $2/5$ at $J_{\perp} \gg J$.

VI. STRONG-COUPLING LIMIT

In this section, we consider the model (1) in the strong-coupling limit $J_{\perp}, h \gg J$. It will be assumed that the number of lattice sites is even, $N = 2M$, and periodic boundary conditions are imposed.

In the zeroth-order approximation ($J=0$), the Hamiltonian describes a collection of even and odd rungs,

$$H_0 = \sum_{m=1}^M H_{2m}^0 + \sum_{m=0}^{M-1} H_{2m+1}^0, \quad (69a)$$

where

$$H_n^0 = J_{\perp} S_{1,n} \cdot S_{2,n} - h(-1)^n [S_{1,n}^z + S_{2,n}^z]. \quad (69b)$$

The spectrum of the even-rung Hamiltonian H_{2m}^0 consists of four levels,

$$E_{\pm} = \frac{J_{\perp}}{4} \mp h, \quad E_0 = \frac{J_{\perp}}{4}, \quad E_s = -\frac{3J_{\perp}}{4}, \quad (70)$$

corresponding to the triplet and singlet states

$$|+\rangle = |\uparrow\uparrow\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle], \quad |-\rangle = |\downarrow\downarrow\rangle,$$

$$|s\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle].$$

At $h < J_{\perp}$, the singlet state $|s\rangle$ is the lowest-energy state. At $h = J_{\perp}$ the $|+\rangle$ level crosses with the singlet $|s\rangle$ and at $h > J_{\perp}$ becomes the lowest-energy state. Focusing on the vicinity of the level crossing point ($|h - J_{\perp}| \ll J_{\perp}$), we will retain two low-energy states on each even rung:

$$|\uparrow\rangle_{2m} = |+\rangle_{2m}, \quad |\downarrow\rangle_{2m} = |s\rangle_{2m}. \quad (71)$$

Projecting the even-rung Hamiltonian onto the subspace of these two states, i.e., imposing the constraint $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = 1$, we obtain

$$PH_{2m}^0P = -\frac{1}{2} \left(\frac{J_{\perp}}{2} + h \right) - \frac{h - J_{\perp}}{2} \{ |\uparrow\rangle_{2m}\langle\uparrow|_{2m} - |\downarrow\rangle_{2m}\langle\downarrow|_{2m} \}. \quad (72)$$

The spectrum of the odd-rung Hamiltonian H_{2m+1}^0 is obtained from Eq. (70) by inverting the sign of h , which amounts to interchanging the states $|+\rangle$ and $|-\rangle$. Thus the lowest-energy states on odd rungs are

$$|\uparrow\rangle_{2m+1} = |s\rangle_{2m+1}, \quad |\downarrow\rangle_{2m+1} = |-\rangle_{2m+1}, \quad (73)$$

and the projected odd-rung Hamiltonian reads

$$PH_{2m+1}^0P = -\frac{1}{2}\left(\frac{J_\perp}{2} + h\right) + \frac{h - J_\perp}{2}\{|\uparrow\rangle_{2m+1}\langle\uparrow|_{2m+1} - |\downarrow\rangle_{2m+1}\langle\downarrow|_{2m+1}\}. \quad (74)$$

Introducing effective spin-1/2 operators T_n associated with the n th rung,

$$T_n^z = \frac{1}{2}(|\uparrow\rangle_n\langle\uparrow|_n - |\downarrow\rangle_n\langle\downarrow|_n), \\ T_n^+ = |\uparrow\rangle_n\langle\downarrow|_n, \quad T_n^- = |\downarrow\rangle_n\langle\uparrow|_n, \quad (75)$$

we see that, in the zeroth order in J , the low-energy Hamiltonian describes a collection of noninteracting spins 1/2 on a 1D lattice in an effective staggered magnetic field $h - J_\perp$:

$$PH_0P = \text{const} - (h - J_\perp) \sum_n (-1)^n T_n^z. \quad (76)$$

The exchange interaction between the spins T_n is mediated by the longitudinal (J) part of the Hamiltonian (1). In the two-dimensional low-energy subspaces of the even and odd rungs, the original spin operators $S_{j,n}$ ($j=1,2$) reduce to

$$S_{j,n}^z \rightarrow \frac{1}{2}\left[T_n^z + \frac{1}{2}(-1)^n\right], \\ S_{j,n}^\pm \rightarrow (-1)^{j+n} \frac{1}{\sqrt{2}} T_n^\pm. \quad (77)$$

As a result, in the first order in J , the interaction between neighboring rungs, e.g., $2m$ and $2m+1$ is given by

$$-J\left(T_{2m}^x T_{2m+1}^x + T_{2m}^y T_{2m+1}^y - \frac{1}{2} T_{2m}^z T_{2m+1}^z\right).$$

It is convenient to make a unitary transformation

$$T_n^\pm \rightarrow (-1)^n T_n^\pm, \quad T_n^z \rightarrow T_n^z, \quad (78)$$

under which formulas (77) transform to

$$S_{1n}^z + S_{2n}^z \rightarrow T_n^z + \frac{1}{2}(-1)^n, \quad S_{1n}^z - S_{2n}^z \rightarrow 0; \\ S_{1n}^+ + S_{2n}^+ \rightarrow 0, \quad S_{1n}^+ - S_{2n}^+ \rightarrow -\sqrt{2} T_n^+. \quad (79)$$

The low-energy effective Hamiltonian takes the form of an anisotropic spin-1/2 Heisenberg chain in a staggered magnetic field,

$$H_{\text{eff}} = \text{const} + J \sum_n (T_n^x T_{n+1}^x + T_n^y T_{n+1}^y + \Delta T_n^z T_{n+1}^z) \\ - h^* \sum_n (-1)^n T_n^z, \quad (80)$$

where

$$\Delta = \frac{1}{2}, \quad h^* = h - J_\perp + \frac{J}{2}. \quad (81)$$

Since $|\Delta| < 1$, the model (80) has a U(1) (Gaussian) critical line $h^* = 0$, i.e.,

$$h = J_\perp - \frac{J}{2} + O(J^2/J_\perp). \quad (82)$$

To be sure of this result, we must verify that higher-order terms ($\propto J^2/J_\perp$), originating from virtual transitions between the low-energy and high-energy states, do not introduce relevant perturbations at the U(1) criticality but only renormalize the parameters of H_{eff} in Eq. (80). Since the original Hamiltonian (1) is site-parity symmetric, a bond-alternating term cannot appear. Small corrections of the order J^2/J_\perp cannot make Δ greater than 1 and drive the XXZ chain to the massive Néel phase. Next-nearest-neighbor exchange interactions with small coupling constants are also known to be marginally irrelevant at low energies. Finally, with the definitions (71) and (73) of the spin-up and spin-down low-energy states, the effective model should be invariant under spin rotations around the staggered magnetic field, which is a property of the original model. This rules out a breakdown of the XY symmetry of the effective Hamiltonian, which would generate a finite gap. Thus, higher-order terms will slightly (in the order J^2/J_\perp) modify the parameters of the effective Hamiltonian (80) and, in particular, the critical line (82), without affecting the criticality itself. Small corrections to the zeroth-order parameter $\Delta = 1/2$ will keep the U(1) criticality far enough from the SU(2) critical point $\Delta = 1$.

The picture of the transition emerging in the strong-coupling case, $J_\perp, h \gg J$, is in full agreement with the one obtained in the weak-coupling limit, $J_\perp, h \ll J$ (see Sec. II). In both cases, the criticality is identified as that of an effective spin-1/2 XXZ chain. We therefore expect that the existence of the U(1) criticality in the antiferromagnetic $S=1/2$ two-leg ladder in a staggered magnetic field is a *universal* property of this system. The vicinity of the critical field is described by the SGM (22) with

$$K = \frac{1}{2\left(1 - \frac{1}{\pi} \arccos \Delta\right)}. \quad (83)$$

However, the equation for the critical line, as well as the value of Δ in the effective $S=1/2$ XXZ chain, are sensitive to the strength of the interchain coupling J_\perp . Being small at $J_\perp \ll J$, Δ increases with J_\perp and tends to 1/2 ($K \rightarrow 3/4$) in the strong-coupling limit. The scaling law for the critical line, $h_c \sim J_\perp^{3/2}$, valid for a weakly coupled ladder, can be easily obtained by comparing the mass gaps in the limiting cases

$h/J_{\perp} \rightarrow 0$ and $J_{\perp}/h \rightarrow 0$. In the former case the mass gap is linear in J_{\perp} (up to logarithmic corrections). In the second case we have two decoupled $S=1/2$ Heisenberg chains in a weak staggered magnetic field. Since the staggered magnetization has scaling dimension $1/2$, the mass gap scales as $h^{1/(2-d)} = h^{2/3}$. The condition $J_{\perp} \sim h^{2/3}$ brings us to Eq. (21). On increasing J_{\perp} , the power law $h_c \sim J_{\perp}^{3/2}$ gradually transforms to a linear dependence (82), valid at $J_{\perp} \gg J$ where the transition is governed by level crossing of the lowest-energy on-rung spin states.

The strong-coupling approach can be applied to the generalized ladder model (2) as well, provided that $|V| \ll J_{\perp}, h$. The effective low-energy Hamiltonian is again of the form (80), but its parameters are modified

$$J \rightarrow \bar{J} = J - \frac{V}{4}, \quad \Delta = \frac{1 - \frac{3V}{4J}}{2 \left(1 - \frac{V}{4J}\right)},$$

$$h^* = h - J_{\perp} + \frac{J}{2} + \frac{V}{8}. \quad (84)$$

VII. PHYSICAL PROPERTIES IN THE STRONG-COUPPLING LIMIT

In the weak-coupling case $J_{\perp} \ll J$ discussed above, the exact relationship between the parameters of the low-energy model of four massive Majorana fermions and those characterizing the original lattice spin ladder is unknown. For this reason, although the weak-coupling approach correctly captures the universal parts of all physical quantities in the vicinity of the critical point, the nonuniversal prefactors cannot be reliably estimated. On the other hand, in the strong-coupling limit ($J_{\perp} \gg J$), the mapping onto the effective XXZ spin-1/2 chain, Eq. (80), is exact in the sense that all its parameters can be found with any desired degree of accuracy in terms of the expansion in powers of J/J_{\perp} . Moreover, the projection of the spin-ladder operators $S_{j,n}^{\alpha}$ onto those of the XXZ chain, T_n^{α} , given by Eqs. (79), does not contain non-universal parameters. Therefore, one can take advantage of the fact that the SGM (22), describing the properties of the spin ladder in a staggered magnetic field close to its critical value, is integrable and employ the form factor bootstrap approach³⁵⁻³⁸ for a *quantitative* analysis of the spectral properties of the system.

For simplicity we only consider the case $V=0$ but, in view of Eqs. (84), the extension to a small nonzero V is straightforward.

In what follows we will use normalizations for the SGM that are slightly different from those used in, e.g., Eq. (25), but are more convenient for the calculations we need to carry out.

A. Scaling limit

In the scaling limit, the XXZ spin chain with exchange constant J and anisotropy

$$\Delta = -\cos \pi \gamma^2$$

is described by the Gaussian model with Lagrangian

$$\mathcal{L} = \frac{1}{16\pi} (\partial_{\mu} \phi)^2. \quad (85)$$

In the strong-coupling limit we have $\Delta = \frac{1}{2}$ and thus

$$\gamma^2 = 2/3.$$

The scaling limit is defined by

$$J \rightarrow \infty, \quad a_0 \rightarrow 0, \quad Ja_0 = 2 \frac{1 - \gamma^2}{\sin \pi \gamma^2} = \text{fixed}. \quad (86)$$

In these conventions a_0 is scaled in such a way that the spin velocity is set to 1. It is easily restored in the final results by dimensional analysis. Following Lukyanov,³⁸ we will normalize the field ϕ according to the short-distance OPE:

$$e^{i\gamma\phi(x)} e^{-i\gamma\phi(y)} \rightarrow |x-y|^{-4\gamma^2}, \quad |x-y| \rightarrow 0.$$

Note that this implies that the lattice spacing must be taken into account explicitly when relating lattice operators to field theory ones. For example, for the staggered components of the spin operators we have

$$2T_n^{\pm} \rightarrow (-1)^n a_0^{\gamma^2/2} \sqrt{\frac{F}{2}} \exp\left(\pm i \frac{\gamma}{2} \theta\right),$$

$$2T_n^z \rightarrow (-1)^n a_0^{1/2\gamma^2} \sqrt{2A} \cos\left(\frac{1}{2\gamma} \phi\right). \quad (87)$$

The nonuniversal constants A and F are known exactly:⁴¹

$$F = \frac{1}{2(1-\gamma^2)^2} \left[\frac{\Gamma\left(\frac{\gamma^2}{2-2\gamma^2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2-2\gamma^2}\right)} \right]^{\gamma^2}$$

$$\times \exp\left(-\int_0^{\infty} \frac{dt}{t} \left[\frac{\sinh(\gamma^2 t)}{\sinh t \cosh(t[1-\gamma^2])} - \gamma^2 e^{-2t} \right]\right), \quad (88a)$$

$$A = \frac{8}{\pi^2} \left[\frac{\Gamma\left(\frac{\gamma^2}{2-2\gamma^2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2-2\gamma^2}\right)} \right]^{1/\gamma^2}$$

$$\times \exp\left\{ \int_0^{\infty} \frac{dt}{t} \left[\frac{\sinh([2\gamma^2-1]t)}{\sinh(\gamma^2 t) \cosh(t[1-\gamma^2])} \right. \right.$$

$$\left. \left. - \left(2 - \frac{1}{\gamma^2}\right) e^{-2t} \right] \right\}. \quad (88b)$$

For $\gamma^2=2/3$ we obtain $F \approx 0.5360$ and $A \approx 0.4285$. When the staggered field term is added, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{16\pi} (\partial_\mu \phi)^2 - 2\mu \cos \bar{\beta} \phi, \quad (89)$$

where $\bar{\beta} = 1/(2\gamma) = \sqrt{3}/8$ and for $h^* \leq 0$,

$$2\mu = -\frac{h^*}{2} \sqrt{2A} a_0^2 \bar{\beta}^{2-1},$$

$$2T_n^\pm = (-1)^n a_0^{1/(8\bar{\beta}^2)} \sqrt{\frac{F}{2}} \exp\left(\pm i \frac{1}{4\bar{\beta}} \theta\right),$$

$$2T_n^z = -(-1)^n a_0^{2\bar{\beta}^2} \sqrt{2A} \cos \bar{\beta} \phi. \quad (90)$$

For $h^* > 0$ the signs of μ and of the expression for T_n^z need to be inverted. The spectrum of the SGM (89) at $\bar{\beta}^2 = 3/8$ consists of soliton and antisoliton with gap M_d and one soliton-antisoliton bound state called ‘‘breather’’ with gap

$$M_1 = 2M_d \sin(\pi\xi/2), \quad \xi = \frac{\bar{\beta}^2}{1-\bar{\beta}^2} = \frac{3}{5}. \quad (91)$$

The soliton gap can be expressed in terms of the scale μ by comparing the results of a thermodynamic Bethe ansatz calculation with those of a perturbative calculation valid at high energies.⁴²

$$\mu = \frac{\Gamma(\bar{\beta}^2)}{\pi\Gamma(1-\bar{\beta}^2)} \left[M_d \frac{\sqrt{\pi}}{2} \frac{\Gamma([1+\xi]/2)}{\Gamma(\xi/2)} \right]^{2-2\bar{\beta}^2}. \quad (92)$$

Combining Eq. (92) with Eqs. (90) we can express the soliton gap M_d in terms of the microscopic parameters of the lattice model:

$$\frac{M_d}{J} = \alpha \left(\frac{|h^*|}{J} \right)^{4/5}, \quad (93)$$

$$\alpha = \frac{3\sqrt{3}\Gamma\left(\frac{3}{15}\right)}{2\sqrt{\pi}\Gamma\left(\frac{8}{10}\right)} \left(\frac{\pi\Gamma\left(\frac{5}{8}\right)\sqrt{2A}}{3\sqrt{3}\Gamma\left(\frac{3}{8}\right)} \right)^{4/5} \approx 1.58424.$$

B. Staggered magnetization

The staggered magnetization is given by

$$\langle (-1)^n S_{j,n}^z \rangle = \frac{1}{4} + \frac{1}{2} \langle (-1)^n T_n^z \rangle. \quad (94)$$

The point to note here is that, in order to be close to criticality, the staggered field must be large, $h \approx J_\perp$, and consequently, the staggered magnetization is large as well. In the scaling limit, i.e., in the vicinity of the critical line $h^* = 0$, we can use the SGM to determine the deviation from 1/4:

$$\langle (-1)^n T_n^z \rangle = \text{sgn}(h^*) (a_0^{2\bar{\beta}^2}/2) \sqrt{2A} \langle \exp(i\bar{\beta}\phi) \rangle.$$

Expectation values of vertex operators have been determined in Ref. 43, and in particular we have

$$\langle \exp(i\bar{\beta}\phi) \rangle = \frac{(1+\xi)\pi\Gamma(1-\bar{\beta}^2)}{16\sin(\pi\xi)\Gamma(\bar{\beta}^2)}$$

$$\times \left(\frac{\Gamma\left(\frac{1}{2} + \frac{\xi}{2}\right)\Gamma\left(1 - \frac{\xi}{2}\right)}{4\sqrt{\pi}} \right)^{2\bar{\beta}^2-2} M_d^{2\bar{\beta}^2}. \quad (95)$$

Combining Eq. (95) with Eqs. (86) and (88b) we obtain

$$\langle (-1)^n T_n^z \rangle = 0.2897 \times \text{sgn}(h^*) \left(\frac{|h^*|}{J} \right)^{3/5}. \quad (96)$$

C. Dynamical structure factor

Our task is now to calculate the Fourier transform of the retarded dynamical correlation functions in the SGM. This is done by going to the spectral representation and then utilizing the integrability of the SGM to determine exactly the matrix elements of the specific operator under consideration between the ground state and various excited states. This method is known as the form factor bootstrap approach.³⁶ Let us review some of its relevant steps.

In order to utilize the spectral representation, we need to specify a basis of eigenstates of the Hamiltonian. Such a basis is given by scattering states of breathers, solitons, and antisolitons. To distinguish these, we introduce labels B, s, \bar{s} . As usual, for particles with relativistic dispersion it is convenient to introduce a rapidity variable θ to parametrize energy and momentum,

$$E_s(\theta) = M_d \cosh \theta, \quad P_s(\theta) = M_d \sinh \theta,$$

$$E_{\bar{s}}(\theta) = M_d \cosh \theta, \quad P_{\bar{s}}(\theta) = M_d \sinh \theta,$$

$$E_B(\theta) = M_1 \cosh \theta, \quad P_B(\theta) = M_1 \sinh \theta, \quad (97)$$

where the breather gap M_1 is given above. A basis of the scattering states can be constructed by means of the Zamolodchikov-Faddeev (ZF) algebra. The ZF algebra can be considered to be an extension of the algebra of creation and annihilation operators for free fermion or bosons to the case of interacting particles with factorizable scattering. This algebra is based on the knowledge of the exact spectrum and the scattering matrix.⁴⁴ For the SGM the ZF operators (and their Hermitian conjugates) satisfy the following algebra:

$$Z^{\epsilon_1}(\theta_1) Z^{\epsilon_2}(\theta_2) = S_{\epsilon_1', \epsilon_2'}^{\epsilon_1, \epsilon_2}(\theta_1 - \theta_2) Z^{\epsilon_2'}(\theta_2) Z^{\epsilon_1'}(\theta_1), \quad (98)$$

$$Z_{\epsilon_1}^\dagger(\theta_1) Z_{\epsilon_2}^\dagger(\theta_2) = Z_{\epsilon_2'}^\dagger(\theta_2) Z_{\epsilon_1'}^\dagger(\theta_1) S_{\epsilon_1', \epsilon_2'}^{\epsilon_1, \epsilon_2}(\theta_1 - \theta_2),$$

$$Z^{\epsilon_1}(\theta_1)Z_{\epsilon_2}^\dagger(\theta_2) = Z_{\epsilon_2}^\dagger(\theta_2)S_{\epsilon_2, \epsilon_1}^{\epsilon_2, \epsilon_1}(\theta_2 - \theta_1)Z^{\epsilon_1}(\theta_1) \\ + (2\pi)\delta_{\epsilon_2}^{\epsilon_1}\delta(\theta_1 - \theta_2). \quad (98)$$

Here $S_{\epsilon_1, \epsilon_2}^{\epsilon_1, \epsilon_2}(\theta)$ are the (factorizable) two-particle scattering matrices and $\epsilon_j = s, \bar{s}, B$.

Using the ZF generators, a Fock space of states can be constructed as follows. The vacuum is defined by

$$Z_{\epsilon_i}(\theta)|0\rangle = 0.$$

Multiparticle states are obtained by acting with strings of creation operators $Z_\epsilon^\dagger(\theta)$ on the vacuum:

$$|\theta_n \cdots \theta_1\rangle_{\epsilon_n \cdots \epsilon_1} = Z_{\epsilon_n}^\dagger(\theta_n) \cdots Z_{\epsilon_1}^\dagger(\theta_1)|0\rangle.$$

In term of this basis the resolution of the identity reads

$$\sum_{n=0}^{\infty} \sum_{\epsilon_i} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} \\ |\theta_n \cdots \theta_1\rangle_{\epsilon_n \cdots \epsilon_1} \langle \theta_1 \cdots \theta_n |_{\epsilon_1 \cdots \epsilon_n} = 1. \quad (99)$$

Inserting Eq. (99) between operators in a two-point correlation function we obtain the following spectral representation:

$$\langle \mathcal{O}(t, x) \mathcal{O}^\dagger(0, 0) \rangle \\ = \sum_{n=0}^{\infty} \sum_{\epsilon_i} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} \exp\left(i \sum_{j=1}^n (p_j x - e_j t)\right) \\ \times |\langle 0 | \mathcal{O}(0, 0) | \theta_n \cdots \theta_1 \rangle_{\epsilon_n \cdots \epsilon_1}|^2, \quad (100a)$$

where p_j and e_j are given by

$$p_j = M_{\epsilon_j} \sinh \theta_j, \quad e_j = M_{\epsilon_j} \cosh \theta_j, \quad (100b)$$

and

$$f^\mathcal{O}(\theta_1 \cdots \theta_n)_{\epsilon_1 \cdots \epsilon_n} \equiv \langle 0 | \mathcal{O}(0, 0) | \theta_n \cdots \theta_1 \rangle_{\epsilon_n \cdots \epsilon_1} \quad (100c)$$

are the form factors (FF's). Our conventions in Eq. (100b) are such that $M_s = M_{\bar{s}} = M_d$ and $M_B = M_1$. After carrying out the double Fourier transform we obtain the following expression for the dynamical structure factor:

$$S^\mathcal{O}(\omega, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt e^{i\omega t - iqx} \langle \mathcal{O}(t, x) \mathcal{O}^\dagger(0, 0) \rangle \\ = 2\pi \sum_{n=0}^{\infty} \sum_{\epsilon_i} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |f^\mathcal{O}(\theta_1 \cdots \theta_n)_{\epsilon_1 \cdots \epsilon_n}|^2 \\ \times \delta\left(q - \sum_j M_{\epsilon_j} \sinh \theta_j\right) \delta\left(\omega - \sum_j M_{\epsilon_j} \cosh \theta_j\right). \quad (101)$$

Let us now evaluate the leading contributions to Eq. (101) in the physically relevant cases.

D. Transverse structure factor at $q \approx \pi$

To determine the transverse structure factor, we need FF's of the operator

$$\exp\left(\frac{i}{4\beta} \theta\right).$$

The lowest-lying states to which it couples are one-soliton states, and the first nonvanishing FF is a constant:³⁹

$$\langle 0 | \exp\left(\frac{i}{4\beta} \theta(0, 0)\right) | \theta \rangle_- = \sqrt{Z_1(0)}. \quad (102a)$$

In our case we have

$$Z_1(0) = \left[\frac{4}{\xi} \exp\left(\int_0^\infty dt \frac{\sinh t \sinh(t[\xi - 1])}{\sinh(t\xi) \cosh^2 t}\right) \right]^{1/4} \\ \times \left[\frac{\sqrt{\pi} \Gamma\left(\frac{3}{2} + \frac{\xi}{2}\right)}{\Gamma\left(\frac{\xi}{2}\right)} \right]^{\gamma^2} M_d^{\gamma^2} \exp\left(2 \int_0^\infty dt \left[\frac{1}{4 \sinh(t\xi)} \right. \right. \\ \left. \left. + \frac{e^{(1+\xi)t} - 1}{4 \sinh(\xi t) \cosh t \sinh([1 + \xi]t)} - \frac{\gamma^2 e^{-2t}}{2} \right] \right) \\ \approx 4.01 M_d^{2/3}. \quad (102b)$$

The corresponding contribution to the dynamical structure factor is

$$S^{+-}(\omega, \pi + q) = \frac{F}{4} Z_1(0) a_0^{2/3} \delta(s^2 - M_d^2) \\ \approx 0.614 \left(\frac{|h^*|}{J}\right)^{8/15} \delta(s^2 - M_d^2). \quad (103)$$

The dynamical susceptibility of the original ladder model can easily be restored by means of the relations (79) between the T_n^α variables and the original spin operators $S_{n,j}^\alpha$:

$$S_{\text{ladder}}^{+-}(\omega, \pi + q, \pi) = S^{+-}(\omega, \pi + q), \\ S_{\text{ladder}}^{+-}(\omega, \pi + q, 0) = 0. \quad (104)$$

The nice thing about this result is that we are able to calculate the spectral weight in the δ function *exactly*. This result is exact up to frequencies $\omega = 3M_d$, where additional contributions from $s\bar{s}\bar{s}$ states arise. These can, in principle, be calculated by first determining the corresponding FF's from the relevant annihilation pole conditions (see, e.g., Ref. 35 for a similar calculation) and then by carrying out the remaining integrations over the rapidity variables numerically. The contribution of Eq. (103) to the total spectral weight at $q = \pi$ is proportional to $(|h^*|/J)^{-4/15}$ and therefore diverges as the field h^* goes to zero. At first sight this may look strange, but the same is true for the gapless XXZ chain, where

$$S_{XXZ}^{+-}(\omega, \pi) \propto \omega^{-2+\eta},$$

with $\eta = 1 - (1/\pi)\arccos \Delta$.

E. Longitudinal structure factor at $q \approx \pi$

In order to determine the longitudinal structure factor we need the FF's of the operator $\cos \bar{\beta}\phi$. The first nonvanishing FF is between the vacuum and two-particle $s\bar{s}$ states. The FF is given by³⁸

$$\langle 0 | \cos \bar{\beta}\phi(0,0) | \theta_1, \theta_2 \rangle_{-+} = \mathcal{G}_{\bar{\beta}} F(\theta_{12}), \quad (105a)$$

$$F(\theta) = \cot(\pi\xi/2) \frac{2i \cosh(\theta/2)}{\xi \sinh([\theta + i\pi]/2\xi)} \sinh(\theta/2) \\ \times \exp\left(\int_0^\infty \frac{dt}{t} \frac{\sinh^2(t[1 - i\theta/\pi]) \sinh(t[\xi - 1])}{\sinh 2t \sinh \xi t \cosh t} \right), \quad (105b)$$

$$\mathcal{G}_{\bar{\beta}} = \left[\frac{M_d \sqrt{\pi} \Gamma\left(\frac{1}{2 - 2\bar{\beta}^2}\right)}{2\Gamma\left(\frac{\bar{\beta}^2}{2 - 2\bar{\beta}^2}\right)} \right]^{2\bar{\beta}^2} \exp\left(\int_0^\infty \frac{dt}{t} f(t) \right),$$

$$f(t) = \frac{\sinh^2(2\bar{\beta}^2 t)}{2 \sinh(\bar{\beta}^2 t) \sinh t \cosh([1 - \bar{\beta}^2]t)} - 2\bar{\beta}^2 e^{-2t}, \quad (105c)$$

where $\theta_{12} = \theta_1 - \theta_2$. Carrying out the Fourier transformation, we obtain

$$S^{zz}(\omega, \pi + q) \approx \frac{A}{2} a_0^{4\bar{\beta}^2} |\mathcal{G}_{\bar{\beta}}|^2 \frac{2}{\pi} \frac{1}{s \sqrt{s^2 - 4M_d^2}} |F(2\theta_0)|^2 \\ \approx 1.13 \left(\frac{|h^*|}{J} \right)^{6/5} \frac{2}{\pi} \frac{1}{s \sqrt{s^2 - 4M_d^2}} |F(2\theta_0)|^2, \quad (106)$$

with $\theta_0 = \text{arccosh}(s/2M_d)$. We plot the result in Fig. 7.

Just above the threshold at $\omega = \sqrt{q^2 + 4M_d^2}$ the structure factor increases in a universal square-root fashion. This is easily seen by considering the limit $\theta_0 \ll 1$ of the function $|F(2\theta_0)|^2$:

$$|F(2\theta_0)|^2 \propto (s^2 - 4M_d^2)/M_d^2.$$

Using the relations (79) we finally obtain the longitudinal structure factor of the original ladder model:

$$S_{\text{ladder}}^{zz}(\omega, \pi + q, 0) = S^{zz}(\omega, \pi + q), \\ S_{\text{ladder}}^{zz}(\omega, \pi + q, \pi) = 0. \quad (107)$$

This result is exact for frequencies below $\omega = 4M_d$, where additional contributions due to intermediate states with two solitons and two antisolitons arise. Comparing the transverse

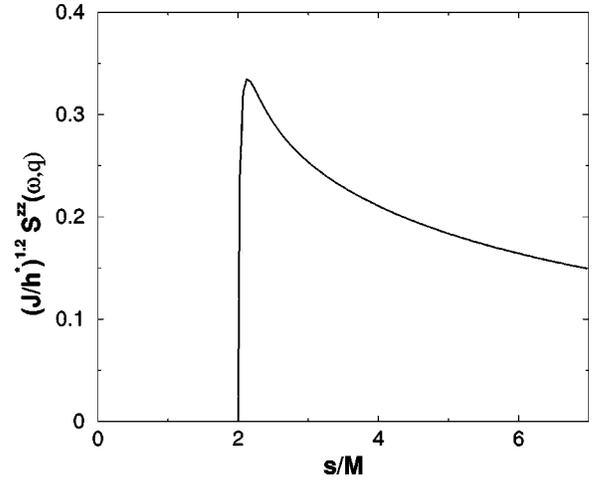


FIG. 7. Longitudinal staggered structure factor at $q_{\perp} = 0$.

structure factor to the longitudinal one, we see that, in the latter case, the contribution of the $s\bar{s}$ intermediate states to the total spectral weight is proportional to $(|h^*|/J)^{2/5}$ and thus becomes small at small fields h^* . Thus at low energies the coherent soliton modes in the transverse structure factor completely dominate the magnetic response around the anti-ferromagnetic wave number $q = \pi$.

F. Transverse structure factor at $q \approx 0$

The smooth components of the transverse spin operators in the XXZ chain are proportional to

$$\mathcal{J}^{\pm} = \exp\left(\pm \frac{i}{4\bar{\beta}} \theta \mp i\bar{\beta}\phi \right) + \exp\left(\pm \frac{i}{4\bar{\beta}} \theta \pm i\bar{\beta}\phi \right). \quad (108)$$

We note that the operator (108) reduces to the sum of the chiral SU(2) currents in the XXX case. Using the results of Ref. 39 one can determine the leading contribution to the transverse structure factor at small energies by means of the FF approach. The contribution of one soliton intermediate states is given by

$$\langle \mathcal{J}^-(\tau, x) \mathcal{J}^+(0) \rangle \propto K_0(M_d r) - \frac{x^2 - v^2 \tau^2}{x^2 + v^2 \tau^2} K_2(M_d r),$$

where $r^2 = x^2 + v^2 \tau^2$. Carrying out the Fourier transformation and analytically continuing we arrive at the following result for the dynamical structure factor:

$$S^{+-}(\omega, q, \pi) = \text{const} \frac{\omega^2}{M_d^2} \delta(s^2 - M_d^2) + \dots \quad (109)$$

G. Longitudinal structure factor at $q \approx 0$

The longitudinal structure factor around $q = 0$ can be calculated analogously. The relevant Fourier component of the spin operator is given by

$$T_n^z \simeq \frac{a_0 \bar{\beta}}{2\pi} \partial_x \phi.$$

Here $\partial_x \phi$ is the topological charge density in the SGM and its FF's are known.^{36,55,37} As we have mentioned before,

$$S^{zz}(\omega, q) \simeq \frac{0.0617}{J^2} \frac{q^2}{\sqrt{q^2 + M_1^2}} \delta(\omega - \sqrt{q^2 + M_1^2}) + \frac{16}{27\pi J^2} \frac{q^2 \sqrt{\omega^2 - q^2 - 4M_d^2}}{(\omega^2 - q^2)^{3/2}} \frac{1}{\cosh(2\theta_0/\xi) + \cos(\pi/\xi)} \times \exp\left(\int_0^\infty \frac{dt}{t} \frac{\sinh([1-\xi]t)[1 - \cosh 2t\cos(2\theta_0 t/\pi)]}{\sinh(2t)\cosh t \sinh(t\xi)}\right), \quad (110)$$

where we again have defined $\theta_0 = \text{arccosh}(s/2M_d)$. The relation of S^{zz} to the structure factor of the original ladder model is given by Eq. (107) with $\pi+q$ replaced by q . As expected the structure factor vanishes like q^2 as $q \rightarrow 0$. This behavior is completely fixed by the Lorentz invariance of the low-energy effective theory and the fact that the topological charge density is part of a Lorentz vector. The result (110) holds in the small-momentum region and it should be possible to resolve the structure of excitations in terms of the breather bound state and the $s\bar{s}$ scattering continuum by carrying out neutron scattering experiments at small momentum transfer.

Our results for the dynamical structure factor in the strong-coupling limit imply the structure of low-lying excited states shown in Fig. 8.

VIII. SPONTANEOUSLY DIMERIZED LADDER

Now we turn to the generalized spin-ladder model (2) in which the four-spin interaction V is superimposed on the antiferromagnetic exchange ($J_\perp > 0$) across the rungs. If V is positive and large enough, the ladder occurs in a non-Haldane, spontaneously dimerized phase with a fully inco-

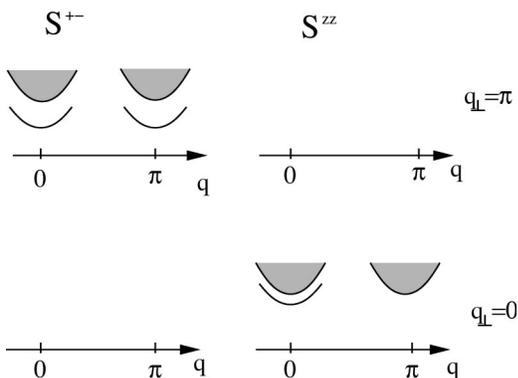


FIG. 8. Structure of low-energy magnetic excitations in the strong coupling limit. This picture is valid both at $h < h_c$ and $h > h_c$.

there exists a soliton-antisoliton bound state which gives rise to a *coherent* δ -function contribution to the longitudinal structure factor around $q=0$. The contribution with the next highest threshold in energy is due to a soliton-antisoliton scattering continuum. Taking only these two contributions into account we obtain

herent spectrum exhausted by pairs of massive dimerization kinks.^{22,45}

The tendency towards suppression of the spin-liquid phase upon increasing V is already seen in the strong-coupling limit; see formulas (84). Within the weak-coupling scheme ($J_\perp, V \ll J$), the transition to the spontaneously dimerized phase is associated with the sign reversal of the triplet mass. From Eqs. (10a) it then follows that, when all four Ising copies are ordered (the case $m_s, m_t < 0$), the ground state is dimerized and doubly degenerate, with $\langle \epsilon^- \rangle = \pm \epsilon_0$ being the order parameter, whereas the spin excitation spectrum represents a broad continuum with thresholds at $2|m_t|$ and $|m_t| + |m_s|$.

At the transition from the spin-liquid to the spontaneously dimerized phase, the Majorana triplet ξ becomes massless ($m_t = 0$), and the system becomes critical. The criticality belongs to the universality class of the level $k=2$ SU(2) WZNW model with central charge $C=3/2$ (see Ref. 22). When a staggered magnetic field is applied, then, in the parameter space (m_s, m_t, h) , the semi-infinite critical line $m_t = 0, m_s < 0$ splits in the direction of the field h into two critical surfaces: one corresponding to the U(1) criticality with central charge $C=1$, already considered in previous sections, and the other representing the Ising criticality with $C=1/2$ (see Fig. 1). The latter will be discussed in this section.

The existence of an Ising QCP in the generalized ladder model can be understood using an argument similar to the one in Sec. II. In the spontaneously dimerized phase all Ising copies are ordered ($m_t, m_s < 0$). Then, in the leading order, the interaction term in Eq. (11) can be replaced by $\tilde{h} \mu_3 \mu_4$, where $\tilde{h} \sim \tilde{h} \langle \sigma_1 \sigma_2 \rangle$. If the masses $m_3 = m_t$ and $m_4 = m_s$ were equal, the resulting model would be equivalent to the double-frequency SGM in which an Ising QCP has already been described in much detail.^{3,4} The existence of this transition can be easily visualized in the strong-coupling limit (large \tilde{h}). In this limit, the “relative” Ising degree of freedom, $\nu = \mu_3 \mu_4$, becomes effectively frozen out, while the “total” degrees of freedom, described, e.g., by μ_3 , can be tuned to criticality. This argument is still valid if the two Ising systems have different mass gaps, provided that they are in the

same (in this, case, ordered) phase.

Starting from the spontaneously dimerized phase with $m_t, m_s < 0$, switching on the staggered magnetic field, and assuming that $|m_s| \gg |m_t|$, we can integrate the singlet mode out to arrive again at the effective model (18), with the renormalized masses still given by Eqs. (19). The important difference with the previous case of the standard ladder is that the doublet mass gap now increases with h while the mass m_3 decreases and vanishes at the same critical value as before. But since this time only one Majorana mode becomes massless, the criticality is of the Ising type. The dimerization order parameter, which is nonzero at $h < h_c$, vanishes at the critical point as

$$\langle \epsilon^- \rangle \sim (h_c - h)^{1/8} \theta(h_c - h).$$

The (static) staggered magnetic susceptibility $\chi_{\text{stag}}(h)$ is constant in the zero-field limit,

$$\chi_{\text{stag}}(0) \sim (1/\alpha)(|m_t|/|m_s|)^{1/4}(|m_t| + |m_s|)^{-1}, \quad (111)$$

then increases with h and becomes logarithmically divergent at the transition

$$\chi_{\text{stag}}(h) \sim \ln(|h - h_c|/h_c). \quad (112)$$

Estimation of the large-distance asymptotics of the correlation functions is similar to what has been done in Sec. IV. One only has to keep in mind that the only mass which changes its sign across the transition is m_3 . Here we present the final results.

In the dimerized phase ($h < h_c$) the dynamical structure factor is entirely incoherent both in its longitudinal and transverse components:

$$S^{+-}(\omega, \pi + q, \pi) \propto \left(\frac{l_3}{l_s} \right)^{1/4} \frac{\theta[s^2 - (|m_d| + |m_3|)^2]}{\sqrt{s^2 - (|m_d| + |m_3|)^2}}, \quad (113a)$$

$$S^{zz}(\omega, \pi + q, \pi) \propto \left(\frac{l_d^2}{l_s l_3} \right)^{1/4} \frac{\theta(s^2 - 4m_d^2)}{\sqrt{s^2 - 4m_d^2}}. \quad (113b)$$

At the Ising criticality,

$$S^{+-}(\omega, \pi + q, \pi) \propto \left(\frac{\alpha^2}{l_s l_d} \right)^{1/4} \frac{\theta(s^2 - m_d^2)}{(s^2 - m_d^2)^{3/4}}, \quad (114a)$$

$$S^{zz}(\omega, \pi + q, \pi) \propto \left(\frac{l_d^3}{\alpha^2 l_s} \right)^{1/4} \frac{\theta(s^2 - 4m_d^2)}{(s^2 - 4m_d^2)^{1/4}}. \quad (114b)$$

In the region $h > h_c$, the longitudinal staggered spin fluctuations remain incoherent,

$$S^{zz}(\omega, \pi + q, \pi) \propto \left(\frac{l_d^2 l_3 (2l_2 + l_3)^2}{l_s \alpha^4} \right)^{1/4} \theta[s^2 - (2|m_d| + m_3)^2], \quad (115a)$$

whereas the transverse part of the dynamical structure factor displays a coherent δ -function peak,

$$S^{+-}(\omega, \pi + q, \pi) \propto \left(\frac{\alpha^4}{l_d^2 l_s l_3} \right)^{1/4} \delta(s^2 - m_d^2), \quad (115b)$$

describing a massive magnon with the spin projection $|S^z| = 1$. As in the standard ladder, at $h > h_c$ the dynamics of ϵ_- also displays a coherent mode with the mass $|m_3|$.

Thus, characterization of the massive phase occurring at $h > h_c$ in the spontaneously dimerized ladder coincides with that for the standard ladder. This phase occupies the region separated by the U(1) and Ising critical surfaces in Fig. 1.

IX. STRING ORDER PARAMETER

den Nijs and Rommelse⁴⁶ and Girvin and Arovas⁴⁷ have shown that the Haldane-gapped phase of the spin-1 chain is characterized by a nonlocal topological string order parameter,

$$\langle \mathcal{O}^\alpha \rangle = \lim_{|n-m| \rightarrow \infty} \left\langle S_n^\alpha \exp \left(i\pi \sum_{j=n+1}^{m-1} S_j^\alpha \right) S_m^\alpha \right\rangle$$

$$(\alpha = x, y, z),$$

whose nonzero value is associated with the breakdown of a hidden $Z_2 \times Z_2$ symmetry.⁴⁸ For a weakly coupled SU(2)-symmetric spin-1/2 Heisenberg ladder, the string order parameter, defined as

$$\mathcal{O}_{n,m}^\alpha = \prod_{j=n}^m (-4S_{1j}^\alpha S_{2j}^\alpha) = \exp \left[i\pi \sum_{j=n}^m (S_{1j}^\alpha + S_{2j}^\alpha) \right], \quad (116)$$

was discussed in Refs. 24 and 22 (see also Ref. 27). The description of the low-energy degrees of freedom of the spin ladder in terms of the Ising variables is especially efficient in this case because the operator (116) acquires a simple *local* form in terms of the Ising operators $\sigma_\alpha, \mu_\alpha$ ($\alpha = 1, 2, 3$), and the hidden $Z_2 \times Z_2$ symmetry becomes manifest. In the continuum limit,⁴⁹

$$\lim_{|x-y| \rightarrow \infty} \langle \mathcal{O}_\alpha(x, y) \rangle \equiv \langle \mathcal{O}_\alpha \rangle \sim \langle \sigma_\beta \sigma_\gamma \rangle^2 + \langle \mu_\beta \mu_\gamma \rangle^2$$

$$(\alpha \neq \beta \neq \gamma), \quad (117)$$

so that the string order parameter reveals the SU(2) symmetry and is nonzero both in the Haldane and dimerized phases just because the degenerate triplet of the Ising systems is either disordered ($\langle \sigma_\alpha \rangle = 0$, $\langle \mu_\alpha \rangle \neq 0$) or ordered ($\langle \sigma_\alpha \rangle \neq 0$, $\langle \mu_\alpha \rangle = 0$).

As we have seen, the staggered magnetic field removes the SO(3) degeneracy of the triplet modes, and in the massive phase located between the two critical surfaces shown in Fig. 1 (the case $h > h_c$), the signs of the masses $m_1 = m_2$ and m_3 are opposite. As a result, the dependence of the longitudinal and transverse components of the string order parameter on h becomes qualitatively different.

A. U(1) transition in the Haldane phase

Let us start with the longitudinal string order parameter \mathcal{O}_z . In the region of small fields, $0 < h \ll h_c$, one can adopt the picture of three independent triplet Ising copies as the zero-order approximation and take into account the effect of the staggered field as a small perturbation. According to Eqs. (19) and (20), the latter leads to splitting of the triplet masses and renormalization of the coupling constants, both being of the order of h^2 . As a result

$$\begin{aligned} \langle \mathcal{O}_z \rangle(h) &\simeq \langle \mu_1 \rangle^2 \langle \mu_2 \rangle^2 \\ &= \langle \mathcal{O}_z \rangle(0) \left[1 - \frac{C_1}{2} \left(\frac{l_t}{\alpha} \right) \left(\frac{\bar{h}}{m_s} \right)^2 + O(\bar{h}^4) \right], \end{aligned} \quad (118)$$

where $\langle \mathcal{O}_z \rangle(0) \sim (m_t \alpha / v)^{1/2}$. In vicinity of the critical point, $|h - h_c| \ll h_c$, the Ising doublet (1,2) becomes very soft and asymptotically decouples from the rest of the spectrum, being described in terms of the SGM (22). In this case the operator \mathcal{O}_z can be bosonized (see Appendix A). However, since $K \neq 1$, in formulas (A6a) and (A6b) for the products of Ising operators $\mu_1 \mu_2$ and $\sigma_1 \sigma_2$, one should rescale the field Φ : $\Phi \rightarrow \sqrt{K} \Phi$, so that

$$\langle \mathcal{O}_z \rangle \sim \langle \sin \sqrt{\pi K} \Phi \rangle^2 + \langle \cos \sqrt{\pi K} \Phi \rangle^2.$$

Since $m_d \sim h_c - h$, at $h < h_c$, $\langle \sin \sqrt{\pi K} \Phi \rangle = 0$, and

$$\langle \mathcal{O}_z \rangle \sim \langle \cos \sqrt{\pi K} \Phi \rangle^2 \sim M_d^{K/2} \sim (h_c - h)^{K/(4-2K)}. \quad (119)$$

At $h > h_c$, $\langle \cos \sqrt{\pi K} \Phi \rangle = 0$, and

$$\langle \mathcal{O}_z \rangle \sim \langle \sin \sqrt{\pi K} \Phi \rangle^2 \sim |M_d|^{K/2} \sim (h - h_c)^{K/(4-2K)}. \quad (120)$$

Formulas (119) and (120) determine the power law according to which the string order parameter $\langle \mathcal{O}_z \rangle$ vanishes at the transition.

Upon increasing the field in the region $h > h_c$, $\langle \mathcal{O}_z \rangle$ keeps growing. In the limit of strong fields, $h \gg h_c$, the system represents two identical copies of the Heisenberg $S=1/2$ chain in a staggered magnetic field, each of them being represented in the continuum limit by a $\beta^2 = 2\pi$ SGM with the nonlinear term proportional to $h \cos \sqrt{2\pi} \Phi_a$ ($a=1,2$). Clearly, the field Φ is a symmetric linear combination of Φ_1 and Φ_2 : $\Phi = (1/\sqrt{2})(\Phi_1 + \Phi_2)$. Therefore, in this limit,

$$\begin{aligned} \langle \mathcal{O}_z \rangle &\sim \langle \cos \sqrt{\pi/2} \Phi_1 \rangle^2 \langle \cos \sqrt{\pi/2} \Phi_2 \rangle^2 \\ &+ \langle \sin \sqrt{\pi/2} \Phi_1 \rangle^2 \langle \sin \sqrt{\pi/2} \Phi_2 \rangle^2 \\ &+ \langle \cos \sqrt{\pi/2} \Phi_1 \rangle^2 \langle \sin \sqrt{\pi/2} \Phi_2 \rangle^2 \\ &+ \langle \sin \sqrt{\pi/2} \Phi_1 \rangle^2 \langle \cos \sqrt{\pi/2} \Phi_2 \rangle^2. \end{aligned} \quad (121)$$

The minima of the potentials $h \cos \sqrt{2\pi} \Phi_a$ are $(\Phi_a)_m = \sqrt{2\pi}(m+1/2)$ at $h > 0$ and $(\Phi_a)_m = \sqrt{2\pi}m$ at $h < 0$, where $m=0, \pm 1, \pm 2, \dots$. Therefore, the last two terms in Eq. (121) vanish, and for any sign of h the z component of the string order parameter grows as

$$\langle \mathcal{O}_z \rangle \sim |h|^{1/3}. \quad (122)$$

Consider now the transverse components of the string order parameter, $\langle \mathcal{O}_x \rangle = \langle \mathcal{O}_y \rangle$. At $h \ll h_c$ the behavior of $\langle \mathcal{O}_x \rangle$ is similar to that for $\langle \mathcal{O}_z \rangle$:

$$\begin{aligned} \langle \mathcal{O}_x \rangle(h) &\simeq \langle \mu_2 \rangle^2 \langle \mu_3 \rangle^2 \\ &= \langle \mathcal{O}_z \rangle(0) \left[1 - \frac{1}{4} \left(\frac{l_s}{\alpha} \right) \left(\frac{C_2}{\pi} + C_1 \frac{g_1}{\pi v} \frac{|m_s|}{m_t} \right) \right. \\ &\quad \left. \times \left(\frac{\bar{h}}{m_s} \right)^2 \ln \left(\frac{l_t}{\alpha} \right) + O(\bar{h}^4) \right]. \end{aligned} \quad (123)$$

Near the critical field, where the doublet of the Ising systems decouples from the third Ising component of the triplet, one finds that

$$\langle \mathcal{O}_x \rangle(h) \sim \left(\frac{M_d \alpha}{v} \right)^{1/4} \theta(h_c - h) \sim (h_c - h)^{1/4(2-K)} \theta(h_c - h). \quad (124)$$

Thus, due to the fact that at $h > h_c$ the Ising doublet becomes ordered ($\langle \mu_1 \rangle = \langle \mu_2 \rangle = 0$), in the large- h massive phase the transverse components of the string order parameter vanish.

B. Ising transition in the dimerized phase

In the spontaneously dimerized, non-Haldane phase the Ising doublet remains massive at any h . The mass gap $|m_d|$ is an increasing function of h and so is the longitudinal string order parameter. At $h \ll h_c$,

$$\begin{aligned} \langle \mathcal{O}_z \rangle(h) &\simeq \langle \sigma_1 \rangle^2 \langle \sigma_2 \rangle^2 \\ &= \langle \mathcal{O}_z \rangle(0) \left[1 + \frac{C_1}{2} \left(\frac{l_t}{\alpha} \right) \left(\frac{\bar{h}}{m_s} \right)^2 + O(\bar{h}^4) \right]. \end{aligned} \quad (125)$$

$\langle \mathcal{O}_z \rangle$ grows monotonically with h and at large h ($h \gg h_c$) crosses over to the $|h|^{1/3}$ behavior. At $h \ll h_c$ $\langle \mathcal{O}_x \rangle$ is still given by formula (123), whereas close to the Ising criticality,

$$\langle \mathcal{O}_x \rangle \sim (h_c - h)^{1/4} \theta(h_c - h). \quad (126)$$

The dependence of the string order parameters on the staggered field is schematically shown in Fig. 9.

X. CONCLUSIONS

In this paper, we have analyzed the properties of the two-leg antiferromagnetic spin-1/2 ladder in a staggered magnetic field. We have considered the spin-liquid phase of the standard ladder and the spontaneously dimerized phase of a generalized spin-ladder model. We have shown that in the former case the staggered field drives the system towards a Gaussian criticality, whereas in the latter case it induces an Ising transition. These two criticalities are associated with a softening of the transverse ($S^z = \pm 1$) and longitudinal ($S^z = 0$) collective modes, respectively, and are characterized by power-law and logarithmic divergencies of the staggered magnetic susceptibility.

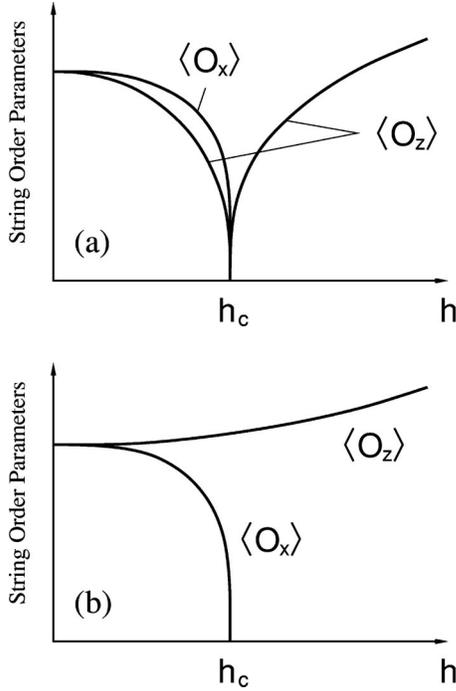


FIG. 9. String order parameters as functions of the staggered magnetic field across (a) the U(1) transition in the Haldane phase and (b) the Ising transition in the dimerized phase.

By comparing our results for weakly ($J_{\perp}/J \ll 1$) and strongly coupled ($J_{\perp}/J \gg 1$) ladders in a staggered field, we can identify certain universal features of the transition. The very existence of the Gaussian criticality is universal. At criticality the low-energy degrees of freedom can be described in terms of spin-1/2 *spinons* similar to those found in the anisotropic spin-1/2 Heisenberg XXZ chain. This is interesting and shows that the staggered field h leads to a destabilization of the magnons that form the low-lying part of the spectrum in the absence of h and eventually “deconfines” them into pairs of spinons.

Close to this criticality the low-energy part of the spectrum involves a doublet of interacting transverse modes. These are described in terms of an effective spin-1/2 XXZ chain in a weak staggered field, which vanishes at $h = h_c$. The transverse modes are identified with *quantum solitons* (with gap M_d) of an underlying SGM. These solitons together with soliton-antisoliton bound states determine the behavior of the dynamical structure factor in both massive phases ($h < h_c$ and $h > h_c$). The soliton reveals itself as a *coherent* δ -function peak in the transverse staggered structure factor $S^{\pm}(\omega, \pi + q, q_{\perp} = \pi)$.

On the other hand, the $q_{\perp} = 0$ part of the spin excitation spectrum is different in the weak- and strong-coupling cases. The same hold true for the $q_{\perp} = \pi$ part of the longitudinal structure factor: at weak coupling the $S^z = 0$ mode is still seen in the low-energy part of the spectrum, whereas it is pushed to very high energies in the strong-coupling regime.

In the absence of a staggered field the generalized ladder is spontaneously dimerized and its elementary excitations can be understood in terms of topological dimerization kinks.²² When a staggered field is applied the dimerization

diminishes but the qualitative picture remains unchanged until the field reaches its critical value h_c , where we find an Ising criticality. Here once again we may think of the elementary excitations in terms of pairs of spinons. The physical properties for $h > h_c$ are the same as in the $h > h_c$ phase of the standard ladder discussed above (see also Fig. 1). In particular, the coherent $S^z = \pm 1$ modes are recovered. This identification is confirmed by the behavior of the longitudinal and transverse string order parameters.

As we have seen, the physics of the spin-1/2 ladder in a staggered field is very rich. We think that it would be very interesting to explore it experimentally. An open question to analyze the crossover region between weak and strong coupling regimes may be addressed by, e.g., numerical methods.

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APPENDIX A: SOME FACTS ABOUT THE 2D ISING MODEL

In this appendix we briefly summarize those facts about the 2D Ising model which are used in the main part of the paper and other appendixes.

Close to criticality, $|T - T_c| \ll T_c$, the scaling properties of the 2D Ising model are described by a Lorentz-invariant (1+1)-dimensional quantum model of a massive real (Majorana) fermion.^{50,32} The corresponding 2D Euclidean action (written in complex notations $z = \tau + ix$, $\bar{z} = \tau - ix$, $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$; $v = 1$) reads⁵¹

$$S = \int d^2z (\xi_L \bar{\partial} \xi_L + \xi_R \partial \xi_R + im \xi_L \xi_R). \quad (\text{A1})$$

Here ξ_L and ξ_R are the holomorphic (left) and antiholomorphic (right) components of the fermionic field. The magnitude of the mass,

$$m \sim \left(\frac{v}{\alpha}\right) \left(\frac{T - T_c}{T_c}\right)$$

(α being a short-distance cutoff), determines the correlation length in the Ising model, $l_c \sim v/|m| \gg \alpha$, which diverges at criticality ($m = 0$), and the sign of the mass indicates whether the system is ordered ($m < 0$) or disordered ($m > 0$). The set of strongly fluctuating fields of the Ising model (at criticality these are known as primary fields of the conformal field theory with central charge $C = 1/2$) includes the fermion field (ξ_L, ξ_R), the mass bilinear (or energy density) $\varepsilon = i \xi_R \xi_L$, and order and disorder fields σ and μ . The latter two fields are nonlocal with respect to each other and also with respect to the Majorana field.

Two identical noninteracting copies of the 2D Ising models are described by a pair of Majorana fermions ξ^1 and ξ^2 ,

which can be combined into a complex (Dirac) massive fermion, $\psi = (\xi^1 + i\xi^2)/\sqrt{2}$. The latter can be bosonized.⁵⁰ If the two Ising copies are slightly noncritical, the resulting bosonic theory represents a quantum sine-Gordon model at the decoupling (or Luther-Emery) point $\beta^2 = 4\pi$:

$$S_{\text{SGM}} = \int d^2z \left[\frac{1}{2} (\partial_\mu \Phi)^2 - \frac{m}{\pi\alpha} \cos \sqrt{4\pi} \Phi \right]. \quad (\text{A2})$$

Below we give a list of bosonization rules for two Ising copies^{32,27,4} which are used in the main text.

(i) The chiral components of the U(1) current:

$$J_L(z) = i\xi_L^1(z)\xi_L^2(z) = +\frac{i}{\sqrt{\pi}} \partial \phi_L(z), \quad (\text{A3a})$$

$$J_R(\bar{z}) = i\xi_R^1(\bar{z})\xi_R^2(\bar{z}) = -\frac{i}{\sqrt{\pi}} \bar{\partial} \phi_R(\bar{z}). \quad (\text{A3b})$$

(ii) The total energy density:

$$\varepsilon_1(z, \bar{z}) + \varepsilon_2(z, \bar{z}) = \frac{1}{\pi\alpha} \cos \sqrt{4\pi} \Phi(z, \bar{z}). \quad (\text{A4})$$

(iii) The fermionic fields:

$$\xi_L^1(z) + i\xi_L^2(z) \simeq (\pi\alpha)^{-1/2} e^{-i\sqrt{4\pi}\phi_L(z)}, \quad (\text{A5a})$$

$$\xi_R^1(\bar{z}) + i\xi_R^2(\bar{z}) \simeq (\pi\alpha)^{-1/2} e^{+i\sqrt{4\pi}\phi_R(\bar{z})}. \quad (\text{A5b})$$

(iv) Mixed products of the order and disorder operators (a more accurate definition of these products includes Klein factors⁴):

$$\sigma_1 \sigma_2 \sim \sin \sqrt{\pi} \Phi, \quad \mu_1 \mu_2 \sim \cos \sqrt{\pi} \Phi, \quad (\text{A6a})$$

$$\sigma_1 \mu_2 \sim \cos \sqrt{\pi} \Theta, \quad \mu_1 \sigma_2 \sim \sin \sqrt{\pi} \Theta. \quad (\text{A6b})$$

In the above formulas,

$$\Phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z})$$

is the scalar field of the underlying SGM and

$$\Theta(z, \bar{z}) = \phi_L(z) - \phi_R(\bar{z})$$

is its dual counterpart.

Using these bosonization rules, one can easily recover all OPE's for a single Ising model. In particular, fusing the products of Ising operators in Eqs. (A6a) one derives the OPE's

$$\sigma(z, \bar{z}) \sigma(w, \bar{w}) \sim \frac{1}{\sqrt{2}} \left(\frac{\alpha}{|z-w|} \right)^{1/4} [1 - \pi |z-w| \varepsilon(w, \bar{w})], \quad (\text{A7a})$$

$$\mu(z, \bar{z}) \mu(w, \bar{w}) \sim \frac{1}{\sqrt{2}} \left(\frac{\alpha}{|z-w|} \right)^{1/4} [1 + \pi |z-w| \varepsilon(w, \bar{w})], \quad (\text{A7b})$$

used in Sec. II B. In the same way, from representation (A6b) one derives two more OPE's

$$\begin{aligned} \sigma(z, \bar{z}) \mu(w, \bar{w}) \\ \sim \sqrt{\frac{\pi}{2}} \frac{\gamma(z-w)^{1/2} \xi_L(w) + \gamma^*(\bar{z}-\bar{w})^{1/2} \xi_R(\bar{w})}{(|z-w|/\alpha)^{1/4}}, \end{aligned} \quad (\text{A8a})$$

$$\begin{aligned} \mu(z, \bar{z}) \sigma(w, \bar{w}) \\ \sim \sqrt{\frac{\pi}{2}} \frac{\gamma^*(z-w)^{1/2} \xi_L(w) + \gamma(\bar{z}-\bar{w})^{1/2} \xi_R(\bar{w})}{(|z-w|/\alpha)^{1/4}}, \end{aligned} \quad (\text{A8b})$$

where $\gamma = e^{i\pi/4}$.

Since the SGM (A2) occurs in a topologically ordered, massive phase, in the ground state the field Φ is locked in one of the infinitely degenerate minima of the potential $\mathcal{U} = -m \cos \sqrt{4\pi} \Phi$:

$$(\Phi)_n^{\text{vac}} = \sqrt{\pi} n, \quad \text{if } m > 0,$$

$$(\Phi)_n^{\text{vac}} = \sqrt{\pi}(n + 1/2), \quad \text{if } m < 0. \quad (\text{A9})$$

From Eqs. (A6a) it then follows that

$$\langle \sigma_{1,2} \rangle = 0, \quad \langle \mu_{1,2} \rangle \neq 0, \quad \text{if } m > 0,$$

$$\langle \sigma_{1,2} \rangle \neq 0, \quad \langle \mu_{1,2} \rangle = 0, \quad \text{if } m < 0. \quad (\text{A10})$$

Quantum solitons of the SGM (A2) are associated with the vacuum-vacuum transitions $\Phi \rightarrow \Phi \pm \sqrt{\pi}$, $\Theta \rightarrow \Theta$, which correspond to the following $Z_2 \times Z_2$ transformations of the fermionic fields and Ising operators:

$$\xi_L^a \rightarrow -\xi_L^a, \quad \xi_R^a \rightarrow -\xi_R^a \quad (a=1,2),$$

$$\sigma_1 \rightarrow \mp \sigma_1, \quad \sigma_2 \rightarrow \pm \sigma_2, \quad \mu_1 \rightarrow \pm \mu_1, \quad \mu_2 \rightarrow \mp \mu_2. \quad (\text{A11})$$

For any sign of m , this symmetry is spontaneously broken in one Ising copy and preserved in the other, leading to the conclusion that quantum solitons of the model (A2) describe kinks of a single ordered (disordered) Ising system that connect opposite values of the order (disorder) parameters.

Parity (or charge conjugation C) transformations

$$m > 0: \quad \xi_{R,L}^1 \rightarrow \xi_{R,L}^1, \quad \xi_{R,L}^2 \rightarrow -\xi_{R,L}^2,$$

$$\Phi \rightarrow -\Phi, \quad \Theta \rightarrow -\Theta,$$

$$\sigma_1 \sigma_2 \rightarrow -\sigma_1 \sigma_2, \quad \mu_1 \mu_2 \rightarrow \mu_1 \mu_2,$$

$$\sigma_1 \mu_2 \rightarrow \sigma_1 \mu_2, \quad \mu_1 \sigma_2 \rightarrow -\mu_1 \sigma_2 \quad (\text{A12})$$

and

$$m < 0: \quad \xi_{R,L}^1 \rightarrow \xi_{R,L}^1, \quad \xi_{R,L}^2 \rightarrow -\xi_{R,L}^2,$$

$$\Phi \rightarrow \sqrt{\pi} - \Phi, \quad \Theta \rightarrow \sqrt{\pi} - \Theta,$$

$$\begin{aligned} \sigma_1\sigma_2 \rightarrow \sigma_1\sigma_2, \quad \mu_1\mu_2 \rightarrow -\mu_1\mu_2, \\ \sigma_1\mu_2 \rightarrow -\sigma_1\mu_2, \quad \mu_1\sigma_2 \rightarrow \mu_1\sigma_2 \end{aligned} \quad (\text{A13})$$

keep invariant vacuum expectation values of the order (disorder) parameters at $m < 0$ ($m > 0$) and therefore serve as a tool to conclude whether a given correlation function in a broken-symmetry phase is nonzero. For example, consider the correlation function

$$K(\mathbf{r}) = \langle \mu(\mathbf{r}) \varepsilon(\mathbf{0}) \rangle$$

for a single *ordered* Ising model. For two identical and decoupled Ising models this correlator can be squared:

$$K_2(\mathbf{r}) = \langle \mu_1(\mathbf{r}) \mu_2(\mathbf{r}) \varepsilon_1(\mathbf{0}) \varepsilon_2(\mathbf{0}) \rangle = K^2(\mathbf{r}).$$

Under transformations (A13) the product $\varepsilon_1 \varepsilon_2$ stays intact but $\mu_1 \mu_2$ changes its sign. Therefore,

$$K_2(\mathbf{r}) = K^2(\mathbf{r}) = 0. \quad (\text{A14})$$

This fact has been used in Sec. II B.

APPENDIX B: FIELD-INDUCED ADMIXTURE BETWEEN THE SINGLET AND TRIPLET MODES

In this appendix we consider two, apparently ‘‘high-energy,’’ operators: the total staggered magnetization \mathbf{n}^+ and the smooth part of the relative magnetization \mathbf{K} defined in Eqs. (10a) and (9b), and find their projections onto the low-energy, triplet sector of the model.

1. Projecting \mathbf{n}^+

As follows from the comparison of Eq. (27) with formula (15), the low-energy projection of the operator $\mathcal{O}_0 = n_z^+$ has actually been found in Sec. II B, and the result is contained in the second-order correction to the effective action (17). Thus we arrive at Eq. (28a).

Consider now the operator $\mathcal{O}_0 = n_x^+$. Treating all Ising systems as decoupled, we have

$$\begin{aligned} \langle n_x^+(\mathbf{r}) n_z^+(\mathbf{r}_1) \rangle_s = \alpha^{-2} [\mu_1(\mathbf{r}) \sigma_1(\mathbf{r}_1)] [\sigma_2(\mathbf{r}_1) \sigma_2(\mathbf{r}_1)] \\ \times [\sigma_3(\mathbf{r}) \mu_3(\mathbf{r}_1)] \langle \mu_4(\mathbf{r}) \mu_4(\mathbf{r}_1) \rangle_s. \end{aligned}$$

The correlator $\langle \mu_4(\mathbf{r}) \mu_4(\mathbf{r}_1) \rangle_s$ is short ranged [see Eq. (16)]. Hence the products of operators in the square brackets, all of them defined in the triplet sector, are subject to fusion. Using the fusion rules (A7a), (A8a), and (A8b) and integrating over the relative coordinate $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}_1$, we arrive at the result (28b). In the same way one obtains the low-energy projection for n_y^+ given by Eq. (28c).

2. Projecting \mathbf{K}

Here we derive the low-energy projection of the operator

$$\mathbf{K} = i(\xi_R \xi_R^4 + \xi_L \xi_L^4). \quad (\text{B1})$$

Consider first the case $\mathcal{O}_0 = K_x$. The correlator in Eq. (27) reads

$$\begin{aligned} \langle K_x(\mathbf{r}) n_z^+(\mathbf{r}_1) \rangle_s = (i/\alpha) \sigma_2(\mathbf{r}_1) \mu_3(\mathbf{r}_1) \sum_{\nu=R,L} \xi_\nu^1(\mathbf{r}) \sigma_1(\mathbf{r}_1) \\ \times \langle \xi_\nu^4(\mathbf{r}) \mu_4(\mathbf{r}_1) \rangle_s. \end{aligned}$$

Keeping in mind that $m_s < 0$, first we note that the correlators $\langle \xi_\nu^4(\mathbf{r}) \mu_4(\mathbf{r}_1) \rangle_s$ are invariant under charge conjugation (A13) and, therefore, are nonzero. These correlators are chiral but otherwise short ranged, decaying exponentially at $|\mathbf{r} - \mathbf{r}_1| \sim l_s$. Since they serve as integral kernels, a qualitatively correct estimation of the integral in Eq. (27) can be obtained if one treats the product

$$[\xi_L^1(z) \xi_L^4(z) + \xi_R^1(\bar{z}) \xi_R^4(\bar{z})] \sigma_1(z_1, \bar{z}_1) \mu_4(z_1, \bar{z}_1)$$

by the OPE and then confines the integration region to the interval $0 < |\mathbf{r} - \mathbf{r}_1| \leq l_s$. To proceed further, one can bosonize two local products $\xi^1 \xi^4$ and $\sigma_1 \mu_4$, and then fuse these fields as those belonging to a (critical) Gaussian model. Since we are looking for a short-distance OPE, the relative sign of the Majorana masses m_1 and m_4 is unimportant, and bosonization rules (A6a) and (A6b) are perfectly applicable. We obtain

$$\begin{aligned} \frac{1}{\sqrt{\pi}} [\partial \phi_L(z) - \bar{\partial} \phi_R(\bar{z})] \cos \sqrt{\pi} \Theta(z_1, \bar{z}_1) \\ \sim \frac{1}{4\pi} \left(\frac{1}{z - z_1} + \frac{1}{\bar{z} - \bar{z}_1} \right) \sin \sqrt{\pi} \Phi(z_1, \bar{z}_1) \\ = \frac{1}{2\pi} \frac{\text{Re}(z - z_1)}{|z - z_1|^2} \mu_1(z_1, \bar{z}_1) \sigma_4(z, \bar{z}_1), \end{aligned} \quad (\text{B2})$$

and therefore

$$\langle K_x(\mathbf{r}) n_z^+(\mathbf{r}_1) \rangle_s \simeq \frac{i}{2\pi} \frac{\tau - \tau_1}{|\mathbf{r} - \mathbf{r}_1|^2} N_y^-(\mathbf{r}_1), \quad (\text{B3})$$

where $N_y^- \sim \alpha^{-1} \mu_1 \sigma_2 \mu_3 \langle \sigma_4 \rangle_s$ is the y component of relative staggered magnetization averaged over the high-energy singlet modes. According to the definition (27),

$$\delta K_x(\mathbf{r}) = \frac{i\hbar}{2\pi v} \int_{\rho < \xi_s} d^2 \boldsymbol{\rho} \left(\frac{\rho_0}{\rho^2} \right) N_y^-(\mathbf{r} - \boldsymbol{\rho}),$$

where $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}_1 = (\rho_0, \rho_1) = (v\tau, x)$. Expanding in ρ , the lowest-order projection of K_x onto the triplet sector is found to be given by formula (29a). Quite similarly one arrives at formula (29b) for the projection of K_y .

In the case $\mathcal{O}_0 = K_z$, we follow the same procedure and use Eqs. (A6a) to bosonize the product $\mu_3 \mu_4$. The corresponding OPE reads

$$\begin{aligned}
& [\xi_L^1(z) \xi_L^4(z) + \xi_R^1(\bar{z}) \xi_R^4(\bar{z})] \mu_3(z_1, \bar{z}_1) \mu_4(z_1, \bar{z}_1) \\
& \sim \frac{1}{\sqrt{\pi}} [\partial \phi(z) - \bar{\partial} \bar{\phi}(\bar{z})] \cos \sqrt{\pi} \Phi(z_1, \bar{z}_1) \\
& = \frac{1}{4\pi} \left(\frac{1}{z-z_1} - \frac{1}{\bar{z}-\bar{z}_1} \right) \sin \sqrt{\pi} \Phi(z_1, \bar{z}_1) \\
& = -\frac{i}{2\pi} \frac{\text{Im}(z-z_1)}{|z-z_1|^2} \sigma_3(z_1, \bar{z}_1) \sigma_4(z, \bar{z}_1), \quad (\text{B4})
\end{aligned}$$

and this eventually leads to formula (29c) for the low-energy projection of K_z .

APPENDIX C: ENERGY-DENSITY CORRELATIONS IN THE DOUBLET SECTOR

Here we estimate the Fourier transform of the two-point energy-density correlation function in the doublet sector,

$$K_d(q, \omega_n) = \int dx d\tau K_d(x, \tau) e^{-i(qx - \omega_n \tau)}, \quad (\text{C1})$$

$$K_d(x, \tau) = \langle T_\tau \varepsilon_d(x, \tau) \varepsilon_d(0, 0) \rangle, \quad \varepsilon_d = i \sum_{a=1,2} \xi_R^a \xi_L^a,$$

and find its analytical continuation ($i\omega_n \rightarrow \omega + i\delta$) that determines the structure factor $S^{zz}(\omega, \pi + q, 0)$. We will consider the case $m_d \neq 0$ and assume that the two massive Majorana fermions are decoupled. In this case $K_d(\mathbf{r}) = 2 \det \hat{G}(\mathbf{r})$, where $\hat{G}(\mathbf{r})$ is the real-space 2×2 Green's function matrix for a free massive fermion in the Nambu representation. We have (see Fig. 10)

$$\begin{aligned}
K_d(q, \omega_n) = 2 \int \frac{dk}{2\pi} \frac{d\varepsilon}{2\pi} [G_{RR}(k_+, \varepsilon_+) G_{LL}(-k_-, -\varepsilon_-) \\
- G_{RL}(k_+, \varepsilon_+) G_{LR}(-k_-, -\varepsilon_-)], \quad (\text{C2})
\end{aligned}$$

where

$$\hat{G}(k, \varepsilon) = \begin{pmatrix} G_{RR}(k, \varepsilon) & G_{RL}(k, \varepsilon) \\ G_{LR}(k, \varepsilon) & G_{LL}(k, \varepsilon) \end{pmatrix} = -\frac{i\varepsilon + kv\tau_3 + m_d \tau_2}{\varepsilon^2 + k^2 v^2 + m_d^2}, \quad (\text{C3})$$

and $k_\pm = k \pm q/2$, $\varepsilon_\pm = \varepsilon \pm \omega_n/2$.

Since the mass bilinear $\xi_R \xi_L$ is a Lorentz-invariant object, $K_d(q, \omega_n)$ depends only on $\omega_n^2 + q^2$. This makes it sufficient to estimate $K_d(0, \omega_n)$ in which case the calculations become especially simple. Integrating over ε in Eq. (C2) yields

$$K_d(0, \omega_n) = - \int \frac{dk}{2\pi} \frac{1}{i\omega_n E_k} \left(\frac{i\omega_n E_k - 2m_d^2}{i\omega_n - 2E_k} - \frac{i\omega_n E_k + 2m_d^2}{i\omega_n + 2E_k} \right),$$

where $E_k = \sqrt{k^2 v^2 + m_d^2}$. Analytically continuing this expression, then taking the imaginary part, and finally replacing ω^2 by $s^2 = \omega^2 - q^2 v^2$, we get ($\omega > 0$)

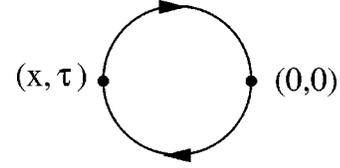


FIG. 10. Two-fermion bubble.

$$\text{Im} K_d(q, \omega + i\delta) = \frac{1}{2v} \frac{\sqrt{s^2 - 4m_d^2}}{s}. \quad (\text{C4})$$

This result has been used in Eq. (50).

APPENDIX D: TRANSVERSE STRUCTURE FACTOR AT SMALL MOMENTUM

In this appendix we provide some technical details concerning the small- q structure factor $S^{xx}(\omega, q, 0)$ at the critical point. We start from the real-space representation of the Matsubara polarization function $X^{xx}(x, \tau)$:

$$X^{xx}(x, \tau) = \langle T_\tau I^1(x, \tau) I^1(0, 0) \rangle = \text{Tr}[\hat{G}(\tau, x) \hat{G}(\tau, x)]. \quad (\text{D1})$$

Here \hat{G} and \hat{G} are the Green's function matrices for the massive and massless Majorana fields ξ^3 and ξ^2 , respectively. Due to the marginal interaction in the doublet (ξ^1, ξ^2) sector, \hat{G} has the structure of the single-particle Green's function of a spinless Tomonaga-Luttinger liquid with the interaction constant K . Using the explicit expressions for these two Green's functions,

$$G_{RR}(z, \bar{z}) = -\frac{m_3}{2\pi} \left(\frac{z}{\bar{z}} \right)^{1/2} K_1(m_3 |z|), \quad (\text{D2a})$$

$$G_{LL}(z, \bar{z}) = -\frac{m_3}{2\pi} \left(\frac{\bar{z}}{z} \right)^{1/2} K_1(m_3 |z|), \quad (\text{D2b})$$

$$G_{RR}(z, \bar{z}) = -\frac{1}{2\pi\alpha} \left(\frac{z}{\bar{z}} \right)^{1/2} \left(\frac{\alpha}{|z|} \right)^{(K+1/K)/2}, \quad (\text{D3a})$$

$$G_{LL}(z, \bar{z}) = -\frac{1}{2\pi\alpha} \left(\frac{\bar{z}}{z} \right)^{1/2} \left(\frac{\alpha}{|z|} \right)^{(K+1/K)/2}, \quad (\text{D3b})$$

$$G_{RL}(z, \bar{z}) = 0, \quad (\text{D3c})$$

we obtain

$$X(z, \bar{z}) = \frac{m_3 \alpha}{(2\pi\alpha)^2} \left(\frac{\alpha}{|z|} \right)^{(K+1/K)/2} \left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right) K_1(m_3 |z|).$$

Introducing two-dimensional real vectors, $\mathbf{q} = (\omega_n, qv)$ and $\mathbf{r} = (\tau, x/v)$ (with ω_n the Matsubara frequency), we pass to the Fourier transform $X^{xx}(\mathbf{q})$ and, after angular integration, obtain

$$X^{xx}(\mathbf{q}) = -\frac{1}{\pi} (m_3 \alpha)^{2\vartheta} \left(\frac{\omega_n^2 - q^2}{q^2} \right) \times \int_0^\infty dx x^{-2\vartheta} K_1(x) J_2\left(\frac{|q|}{m_3} x\right), \quad (\text{D4})$$

where $J_2(x)$ is the Bessel function, and

$$2\vartheta = \frac{1}{2} \left(K + \frac{1}{K} \right) - 1. \quad (\text{D5})$$

Since the expressions for the Tomonaga-Luttinger propagators, Eqs. (D3), are asymptotic (i.e., valid at $|z| > \alpha$), the lower integral cutoff must be finite, $\sim m_3 \alpha$. On the other hand, the integral in Eq. (D4) is convergent at $x \rightarrow 0$, if $\vartheta < 1$. As follows from Eq. (D5), this condition is satisfied not only in the weak-coupling case, where K is very close to 1, but also in the strong-coupling regime where $K = 3/4$ (in the latter case $\vartheta \approx 2 \times 10^{-2}$). This justifies the replacement of the lower integral limit by 0, in which case the result can be expressed in terms of a hypergeometric function

$$X^{xx}(\mathbf{q}) = -\frac{1}{\pi} \frac{\Gamma(2-\vartheta)\Gamma(1-\vartheta)}{2^{2(1+\vartheta)}} (m_3 \alpha)^{2\vartheta} \left(\frac{\omega_n^2 - q^2}{m_3^2} \right) \times F\left(1-\vartheta, 2-\vartheta; 3; -\frac{q^2}{m_3^2}\right).$$

To single out the leading singularity at the threshold in the interacting case, we use the transformation formula⁵²

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times F(c-a, c-b; c-a-b+1; 1-z),$$

and formally treat $1-z = 1 + \mathbf{q}^2/m_3^2$ as a small parameter. Then in the leading order

$$X^{xx}(\mathbf{q}) = -\frac{1}{\pi} \frac{\Gamma(-2\vartheta)}{2^{1+2\vartheta}} (m_3 \alpha)^{2\vartheta} \left(\frac{\omega_n^2 - q^2}{m_3^2} \right) \left(1 + \frac{\mathbf{q}^2}{m_3^2} \right)^{2\vartheta} \times \left[1 + O\left(1 + \frac{\mathbf{q}^2}{m_3^2} \right) \right]. \quad (\text{D6})$$

Performing analytical continuation ($i\omega_n \rightarrow \omega + i\delta$),

$$(\omega_n^2 + q^2 + m_3^2)^{2\vartheta} \rightarrow |s^2 - m_3^2|^{2\vartheta} [\theta(m_3^2 - s^2) + \theta(s^2 - m_3^2) \cos(2\pi\vartheta)] - i|s^2 - m_3^2|^{2\vartheta} \theta(s^2 - m_3^2) \sin(2\pi\vartheta) \text{sgn}(\omega),$$

and using the relation

$$\Gamma(-2\vartheta) \sin(2\pi\vartheta) = -\frac{\pi}{\Gamma(1+2\vartheta)}, \quad (\text{D7})$$

we arrive at the expression (63) of Sec. IV D 2.

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