Quantum spin models with exact dimer ground states

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(Received 5 February 2002; published 28 June 2002)

Inspired by the exact solution of the Majumdar-Ghosh model, a family of one-dimensional, translationally invariant spin Hamiltonians is constructed. The exchange coupling in these models is antiferromagnetic, and decreases linearly with the separation between the spins. The coupling becomes identically zero beyond a certain distance. It is rigorously proved that the dimer configuration is an exact, superstable ground-state configuration of all the members of the family on a periodic chain. The ground state is twofold degenerate, and there exists an energy gap above the ground state. The Majumdar-Ghosh Hamiltonian with a twofold degenerate dimer ground state is just the first member of the family. The scheme of construction is generalized to two and three dimensions, and illustrated with the help of some concrete examples. The first member in two dimensions is the Shastry-Sutherland model. Many of these models have exponentially degenerate, exact dimer ground states.

DOI: 10.1103/PhysRevB.66.024406 PACS number(s): 75.10.Jm, 75.30.Kz

I. INTRODUCTION

The studies of quantum spin models are of great current interest. These studies help us in getting some understanding of the magnetic properties of the real, physical systems. Studies of the magnetic properties described by the dimer (or valence bond) configurations has been a subject of continuing research activities, and has been of particular interest recently. A recent example is the Shastry-Sutherland[1,2] type models used in understanding the physical properties of $SrCu_2(BO_3)_2$.[3] This system is a magnetic insulator with dimer ground state, which is topologically equivalent to that of the Shastry-Sutherland (SS) model. Then there are studies related to the *kagomé* antiferromagnet,[4,5] where magnetic excitations are gapped, and this gap is filled with a large number of low-lying singlet excitations whose number grows as exponential in the number of sites. It is believed that the low-energy physics of the Heisenberg antiferromagnet on the *kagomé* lattice is of the resonating valence bond (RVB) type. The quantum dimer models have been applied to study antiferromagnets on triangular lattices.[6] Again, the idea employed is that of resonating dimer coverings of the lattice. The idea of doping the RVB ground state to achieve superconductivity has been the subject of great consideration in the context of high-T_c superconductors.[7,8] Though the low-temperature behavior of the undoped high-T_c materials does not show up RVB-like magnetic properties, the idea is still interesting, and motivates the search for a doped RVB superconductor.[9] All these studies clearly show the importance of understanding the physics governed by valence bond configurations. It makes the search for and the studies of models with dimer ground state particularly desirable.

The Majumdar-Ghosh (MG) model[10,11] is a one dimensional quantum spin model with the nearest- and next-nearest-neighbor exchange interactions. It is exactly solvable for a particular ratio of these exchange couplings, and has a twofold degenerate dimer ground state. Though this model has been studied for anisotropic exchange and general spin S, we will consider only isotropic exchange and the $S=1/2$ case. The exact solution of the MG model guides us in constructing models with exact dimer ground states which is the subject of the present work. The Hamiltonian for the MG model is written as

$$H_{MG}=J\sum_{i=1}^{L}(2\mathbf{S}_i\cdot\mathbf{S}_{i+1}+\alpha\mathbf{S}_i\cdot\mathbf{S}_{i+2}), \tag{1}$$

where $J>0$, and L is the number of sites in a one-dimensional (1D) lattice with periodic boundary condition (PBC). It is a well studied model, and shows a quantum phase transition from an ordered phase to a disordered, spin-liquid-like phase as α is increased from zero to some value greater than 0.482.[12] At $\alpha=1$, and for L being even, the *bond-singlet* (dimer) configurations, as shown in Figs. 1(a) and 1(b), form an exact, twofold degenerate ground subspace. Let us refer to these dimer configurations as $|\psi_1\rangle$ and $|\psi_2\rangle$ which are given below:

$$|\psi_1\rangle=[1,2]\otimes[3,4]\otimes[5,6]\otimes\cdots\otimes[L-1,L], \tag{2}$$

$$|\psi_2\rangle=[2,3]\otimes[4,5]\otimes[6,7]\otimes\cdots\otimes[L,1], \tag{3}$$

where $[l,m]=(|\uparrow_l\downarrow_m\rangle-|\downarrow_l\uparrow_m\rangle)/\sqrt{2}$ is the singlet state of a pair of spins, sitting at sites l and m, representing a double bond in the chemical sense. The ground-state energy, in units of the nearest-neighbor exchange $(2J)$, is $-(3/8)L$.

In the following sections, we will discuss briefly why H_{MG} is exactly solvable for its ground state at $\alpha=1$. It will

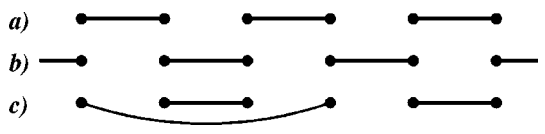

FIG. 1. (a) and (b) are two exact ground-state configurations of the family of 1D models with linear exchange coupling, as discussed in the text. The solid line joining two lattice points represents a singlet state between spins sitting at corresponding sites. These ground-state configurations are referred to as $|\psi_1\rangle$ and $|\psi_2\rangle$ in the text. (c) is one of the singlet configurations which is not an eigenconfiguration of the Hamiltonian given by Eq. (4).

guide us in constructing the general family of spin models on a 1D lattice with PBC. It will be rigorously proved that the dimer configurations, $|\psi_1\rangle$ and $|\psi_2\rangle$, form an exact, twofold degenerate ground subspace for the whole family. Various features of this family of 1D models will be discussed in some detail. Then, the generalization to two and three spatial dimensions will be discussed. Our scheme allows us to construct higher dimensional models with exact knowledge of the ground state. The exponential degeneracy in the ground subspace of such models will be discussed. Finally, we will conclude with some general discussion and remarks.

II. LESSONS FROM THE MG HAMILTONIAN'S SOLUTION

On a chain with PBC, and for $\alpha=1$, the MG Hamiltonian can be rewritten as: $H_{MG}=J\Sigma_{i=1}^{L}(\mathbf{S}_i\cdot\mathbf{S}_{i+1}+\mathbf{S}_i\cdot\mathbf{S}_{i+2}+\mathbf{S}_{i+1}\cdot\mathbf{S}_{i+2})=J\Sigma_{i=1}^{L}h_i$. The Hamiltonian $h_i=\mathbf{S}_i\cdot\mathbf{S}_{i+1}+\mathbf{S}_i\cdot\mathbf{S}_{i+2}+\mathbf{S}_{i+1}\cdot\mathbf{S}_{i+2}$, is that of a block of three spins, the ith spin and its next two neighbors, coupled to each other identically. Let us refer to these blocks as $\mathfrak{B}_3$ where $\mathfrak{B}$ stands for a block of completely connected spins, and the subscript 3 refers to the number of spins in the block. Spins within a block are understood to be identically coupled, unless specified. The minimum eigenvalue of h_i, $e_i^{min}=-3/4$, for $S=1/2$. This is easy to see if we rewrite h_i as $\frac{1}{2}[(\mathbf{S}_i+\mathbf{S}_{i+1}+\mathbf{S}_{i+2})^2-\mathbf{S}_i^2-\mathbf{S}_{i+1}^2-\mathbf{S}_{i+2}^2]$. The minimum-energy spin configuration for $\mathfrak{B}_3$ has one free spin and the other two spins form a singlet. For example, $[i,i+1]\otimes|\uparrow_{i+2}\rangle$ is one such eigenconfiguration. There are two linearly independent ways of forming such configurations for which the total $\mathfrak{B}_3$ spin is 1/2.

Since the MG chain is made up of $\mathfrak{B}_3$ units, the ground configuration of the MG chain can be constructed in such a way that it is also the lowest energy eigenconfiguration of $\mathfrak{B}_3$. This is not possible in general. Interestingly, this is possible for the MG chain because the minimum-energy configuration of $\mathfrak{B}_3$ has strictly one bond singlet and a *free* spin. This ''free'' spin is free in the sense that it can bond with the ''outside'' world, and the new composite configuration is still the eigenconfiguration of the block Hamiltonian with energy, $-3/4$, provided the other two spins of the block form a singlet. Since every spin on the MG chain has identical exchange connectivity, the above considerations imply that the ground-state configuration of H_{MG} is the one where every three neighboring spins share exactly one bond singlet. The key observation to make is the fact that the fundamental block has an odd number of spins, and its minimum-energy configuration contains exactly one free spin while the rest form a singlet. All this straightforwardly leads to the exact solution of the ground state of the MG chain, already mentioned in the Introduction, and can be proved rigorously using the inequality $\langle\phi|H_{MG}|\phi\rangle\geqslant E_g\geqslant J\Sigma_{i=1}^{L}e_i^{min}(\mathfrak{B}_3)=-(3J/4)L$ and the identities $\mathbf{S}_k\cdot(\mathbf{S}_l+\mathbf{S}_m)[l,m]=0\ \forall\ k\neq l$ and m; $\mathbf{S}_l\cdot\mathbf{S}_m[l,m]=-S(S+1)[l,m]$. If $|\phi\rangle$ is such an eigenstate of the Hamiltonian that the upper bound of the inequality equals the lower bound, then $|\phi\rangle$ is also the ground state of the Hamiltonian. In other words, this inequality ensures that an eigenstate of the Hamiltonian, which is also the ground state of the basic building blocks, is the ground state of the Hamiltonian. In the next section, we will construct a general family of one-dimensional spin Hamiltonians, and show that $|\psi_1\rangle$ and $|\psi_2\rangle$ are the exact ground-state configurations for the whole family.

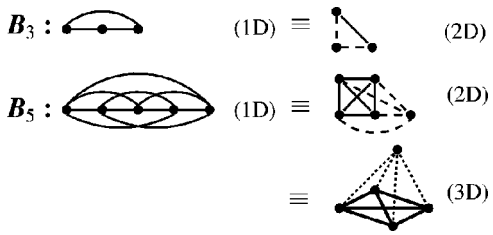

FIG. 2. The $\mathfrak{B}_{2\nu+1}$ blocks for $\nu=1$ and 2, in one, two, and three dimensions. The dashed coupling in 2D blocks is unity while the solid one is α. In the 3D block, the coupling of the *apex* to the square *face* is unity while the coupling within the face is α.

III. FAMILY OF 1D MODELS WITH DIMER GROUND STATE

The model. Let us consider a block $\mathfrak{B}_{2\nu+1}$ of identically and completely connected $(2\nu+1)$ spins where ν is a positive integer. The identical and complete connectedness of spins means that every spin in a block is coupled to every other, with the same strength (which is taken to be unity), as shown in Fig. 2 for $\nu=1$ and 2. On a spin chain, any site i and its next 2ν neighbors will form a $\mathfrak{B}_{2\nu+1}$ unit once they are identically and completely coupled. The Hamiltonian corresponding to the ith such block on a 1D lattice is written as: $h_i(\mathfrak{B}_{2\nu+1})=\{\mathbf{S}_i\cdot(\mathbf{S}_{i+1}+\cdots+\mathbf{S}_{i+2\nu})+\mathbf{S}_{i+1}\cdot(\mathbf{S}_{i+2}+\cdots+\mathbf{S}_{i+2\nu})+\cdots+\mathbf{S}_{i+2\nu-2}\cdot(\mathbf{S}_{i+2\nu-1}+\mathbf{S}_{i+2\nu})+\mathbf{S}_{i+2\nu-1}\cdot\mathbf{S}_{i+2\nu}\}$. Adding all such block Hamiltonians gives the total Hamiltonian for the spin chain made up of $\mathfrak{B}_{2\nu+1}$ units. There are exactly 2ν number of first-neighbor (nearest-neighbor) pairs, $(2\nu-1)$ number of second-neighbor pairs, and so on, within each $\mathfrak{B}_{2\nu+1}$ unit. Therefore the total Hamiltonian for a $\mathfrak{B}_{2\nu+1}$ spin chain with PBC is

$$\begin{aligned}H[\mathfrak{B}_{2\nu+1}]&=J\sum_{i=1}^{L}[2\nu\mathbf{S}_i\cdot\mathbf{S}_{i+1}+(2\nu-1)\mathbf{S}_i\cdot\mathbf{S}_{i+2}\\&\quad+\cdots+2\mathbf{S}_i\cdot\mathbf{S}_{i+2\nu-1}+\mathbf{S}_i\cdot\mathbf{S}_{i+2\nu}]\\&=J\sum_{i=1}^{L}\sum_{j=1}^{2\nu}(2\nu+1-j)\mathbf{S}_i\cdot\mathbf{S}_{i+j}.\end{aligned}\tag{4}$$

Thus, on a $\mathfrak{B}_{2\nu+1}$ spin chain, the ith spin is coupled to next 2ν neighbors with linearly decreasing exchange coupling. Note that the Hamiltonian corresponding to $\nu=1$ is just the MG Hamiltonian with dimer ground state. We will show that $H[\mathfrak{B}_{2\nu+1}]$, for any ν, has the same ground state. Therefore ν can be regarded as the label for the members in our family of 1D spin models with twofold degenerate dimer ground state.

The ground state. In order to find the ground-state energy and the corresponding eigenconfigurations of $H[\mathfrak{B}_{2\nu+1}]$, consider the minimum-energy configurations for a $\mathfrak{B}_{2\nu+1}$

unit. Rewriting $h_i(\mathfrak{B}_{2\nu+1})$ as $\frac{1}{2}[(\mathbf{S}_i+\mathbf{S}_{i+1}+\cdots+\mathbf{S}_{i+2\nu})^2-(\mathbf{S}_i^2+\mathbf{S}_{i+1}^2+\cdots+\mathbf{S}_{i+2\nu}^2)]$ tells us that the total spin $\Sigma_{j=0}^{2\nu}\mathbf{S}_{i+j}$ being minimum corresponds to the lowest energy of the block. For $S=1/2$, the minimum energy is $-3\nu/4$, and the corresponding spin configurations are such that there is exactly one free spin while the remaining 2ν spins form a *block singlet*. A block singlet can be described in terms of the bond singlets (valence bond or dimers, as we often call them). Since there are many, independent ways of doing that, a block singlet has an intrinsic degeneracy of valence bond configurations. For example, $\mathfrak{B}_5$'s minimum-energy configuration consists of a free spin, and a block singlet of four spins. For $S=1/2$, there are exactly two linearly independent valence bond configurations for a block singlet of four spins. We will see later that in certain other models, this intrinsic degeneracy leads to a ground-state with exponential degeneracy, and hence finite entropy density in the ground state. In the following, we try to construct the exact ground state configurations of $H[\mathfrak{B}_{2\nu+1}]$, for all values of ν.

If we can find a configuration where every block of successive $(2\nu+1)$ spins share exactly one block singlet of 2ν spins, then it is the ground state configuration of $H[\mathfrak{B}_{2\nu+1}]$. This is ensured by the inequality

$$\langle\phi|H[\mathfrak{B}_{2\nu+1}]|\phi\rangle \geqslant E_g \geqslant \sum_{i=1}^{L} e_i^{min}(\mathfrak{B}_{2\nu+1}). \tag{5}$$

The following identities are used in establishing the fact that such a construction will form an eigenconfiguration of the $H[\mathfrak{B}_{2\nu+1}]$:

$$(\mathbf{S}_{i_1}+\cdots+\mathbf{S}_{i_{2\nu}})\cdot\mathbf{S}_{i_{2\nu+1}}[i_1,i_2,\ldots,i_{2\nu}]=0, \tag{6}$$

$$\sum_{j_1=1}^{2\nu-1}\sum_{j_2>j_1}^{2\nu}\mathbf{S}_{i_{j_1}}\cdot\mathbf{S}_{i_{j_2}}[i_1,i_2,\ldots,i_{2\nu}]=-\frac{3}{4}\nu. \tag{7}$$

In Eq. (6), $i_{2\nu+1}\neq\{i_1,i_2,\ldots,i_{2\nu}\}$. The above identities are straightforward generalizations of what were used in finding the exact ground state for the MG model for $S=1/2$. For general spin S, the right-hand side of Eq. (7) will be $-\nu S(S+1)$. The notation $[i_1,i_2,\ldots,i_{2\nu}]$ denotes a block singlet of 2ν spins. Just to illustrate this notation, consider four sites labeled 1, 2, 3, and 4. Then [1,2,3,4] denotes all the singlet configurations made up of spins sitting at these sites. Thus $[1,2,3,4]=[1,2]\otimes[3,4]$ or $[2,3]\otimes[4,1]$. The indices $\{i_1,i_2,\ldots,i_{2\nu+1}\}$ take the values from the set $\{i,i+1,\ldots,i+2\nu\}$ while considering the ith $\mathfrak{B}_{2\nu+1}$ block on a spin chain. Also, each one of the indices, $\{i_1,i_2,\ldots,i_{2\nu+1}\}$, is distinct. Among the allowed dimer representations of $[i_1,i_2,\ldots,i_{2\nu}]$, one is simply $[i,i+1]\otimes[i+2,i+3]\otimes\cdots\otimes[i+2\nu-2,i+2\nu-1]$. This is like the dimers we have already seen. There are many different types of them. Some will be of the type, say, $[i,[i+1,i+2],i+3]\otimes[i+4,i+5]\otimes\cdots\otimes[i+2\nu-2,i+2\nu-1]$. Here, the notation $[i,[i+1,i+2],i+3]$ refers to a configuration where $\mathbf{S}_i$ and $\mathbf{S}_{i+3}$ pair up to form a singlet while $\mathbf{S}_{i+1}$ and $\mathbf{S}_{i+2}$ do the same. Each of these dimer representations for a 2ν-spin block singlet contains exactly ν dimers. The fact that every site has identical exchange connectivity suppresses the choices allowed by the intrinsic degeneracy of block singlets. The only configurations which satisfy the condition that every block of neighboring $(2\nu+1)$ spins on a chain has exactly ν dimers are $|\psi_1\rangle$ and $|\psi_2\rangle$. In fact, the other configurations are not even the eigenconfigurations of $H[\mathfrak{B}_{2\nu+1}]$. One such configuration is shown in Fig. 1(c). This is so because such configurations always find some building blocks whose minimum-energy configuration is not satisfied, whereas $|\psi_1\rangle$ and $|\psi_2\rangle$ sastisfy all building blocks' minimum-energy configurations. Therefore $|\psi_1\rangle$ and $|\psi_2\rangle$ are the exact ground-state configurations of the family of spin Hamiltonians given in Eq. (4), and the ground-state energy is $-(3\nu J/4)L$.

In the following, we give another proof of $|\psi_1\rangle$ and $|\psi_2\rangle$ being the ground-state configurations of $H[\mathfrak{B}_{2\nu+1}]$ for all values of ν, despite the fact that we have already shown it. This proof gives an independent existence to the Hamiltonian $H[\mathfrak{B}_{2\nu+1}]$. The proof is based on the principle of mathematical induction, and brings out an interesting property of *superstability*[13] which the dimer configurations possess. An eigenstate $|\phi\rangle$ of some Hamiltonian H is called superstable if it is also the eigenstate of the operator $H+V$, for a certain operator V where the commutator $[H,V]\neq 0$. To make a definitive statement about the superstability, we should clearly understand the relationship between successive Hamiltonians of the family. The proof by induction is based on the understanding of such relationships, and hence illustrates the superstability of the dimer configurations as the ground state of our family of Hamiltonians, in a rigorous way.

Proof by induction. For $\nu=1$, the Hamiltonian operator is $H[\mathfrak{B}_3]=H_{MG}$. It is known that $H_{MG}|\psi_{1,2}\rangle=-(3J/4)L|\psi_{1,2}\rangle$ with $|\psi_{1,2}\rangle$ being the ground-state configuration. Here, $|\psi_{1,2}\rangle$ refers to $|\psi_1\rangle$ and $|\psi_2\rangle$. Assume that $|\psi_{1,2}\rangle$ is the eigenconfiguration of the $(\nu-1)$th member of the family, with eigenenergy $E_{\nu-1}$ (for the sake of this proof ignore the fact that we have already proved it!). That is, $H[\mathfrak{B}_{2\nu-1}]|\psi_{1,2}\rangle=E_{\nu-1}|\psi_{1,2}\rangle$. Now, we check for the νth member. The Hamiltonian for the νth member, where $\nu>1$, can be rewritten in the following way:

$$\begin{aligned}
H[\mathfrak{B}_{2\nu+1}] &= J\sum_{i=1}^{L}\sum_{j=1}^{2\nu}(2\nu+1-j)\mathbf{S}_i\cdot\mathbf{S}_{i+j}\\
&= H[\mathfrak{B}_{2\nu-1}]+J\sum_{i=1}^{L}\left\{\mathbf{S}_i\cdot\mathbf{S}_{i+2\nu}+2\sum_{j=1}^{2\nu-1}\mathbf{S}_i\cdot\mathbf{S}_{i+j}\right\}\\
&= H[\mathfrak{B}_{2\nu-1}]+H_{MG}+J\sum_{i=1}^{L}\{\mathbf{S}_i\cdot\mathbf{S}_{i+2}\\
&\quad +2\mathbf{S}_i\cdot(\mathbf{S}_{i+3}+\cdots+\mathbf{S}_{i+2\nu-1})+\mathbf{S}_i\cdot\mathbf{S}_{i+2\nu}\}\\
&= H[\mathfrak{B}_{2\nu-1}]+H_{MG}\\
&\quad +J\sum_{i=1}^{L}(\mathbf{S}_i+\mathbf{S}_{i+1})\cdot(\mathbf{S}_{i+3}+\cdots+\mathbf{S}_{i+2\nu}).
\end{aligned} \tag{8}$$

Clearly, from Eq. (8), $H[\mathfrak{B}_{2\nu+1}]|\psi_{1,2}\rangle=[E_{\nu-1}-(3J/4)L]|\psi_{1,2}\rangle$, as one can easily show that

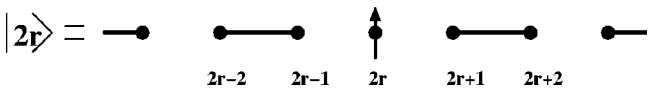

FIG. 3. This is one of the defect configurations with dangling spin being ↑, at site $2r$. One is allowed to have defect configurations with dangling spin polarization being ↓ without affecting the dispersion relation.

$$\sum_{i=1}^{L} (\mathbf{S}_i + \mathbf{S}_{i+1}) \cdot (\mathbf{S}_{i+3} + \cdots + \mathbf{S}_{i+2\nu}) |\psi_{1,2}\rangle = 0. \quad (9)$$

This proves that $|\psi_1\rangle$ and $|\psi_2\rangle$ are the eigenconfigurations of the νth member of the family, with eigenenergy $E_\nu = E_{\nu-1} - (3J/4)L$ for any $\nu > 1$ [with $E_1 = -(3J/4)L$]. This gives $E_\nu = -(3\nu J/4)L$. Since the E_ν we get here is same as the lower bound of the inequality [Eq. (5)], the $|\psi_1\rangle$ and $|\psi_2\rangle$ are also the ground states, and not just the eigenstates. Therefore we have been able to prove that the dimer configurations $|\psi_1\rangle$ and $|\psi_2\rangle$ form a superstable, twofold degenerate ground subspace of the whole family of $H[\mathfrak{B}_{2\nu+1}]$, parametrized by ν, on a one-dimensional lattice with PBC.

The summary of our findings, in the context of one-dimensional spin systems, is the following:

- A family of spin models with exact dimer ground state is constructed. The ground subspace is twofold degenerate.
- The general Hamiltonian of the family is: $H[\mathfrak{B}_{2\nu+1}] = J\Sigma_{i=1}^{L}\Sigma_{j=1}^{2\nu}(2\nu+1-j)\mathbf{S}_i \cdot \mathbf{S}_{i+j}$. Here, $J>0$, ν is a positive integer, and $\mathfrak{B}_{2\nu+1}$ refers to the fundamental building blocks for different members of the family. Note that the Hamiltonian has translational invariance, the exchange coupling decreases linearly with distance between the coupled neighbors, and for a given ν, every spin is coupled only up to 2νth neighbor, starting from the nearest one.
- The ground-state energy, in units of the strongest exchange coupling (the nearest-neighbor coupling $2\nu J$), is $-\frac{3}{8}L$, for all members of the family.
- The dimer states $|\psi_1\rangle$ and $|\psi_2\rangle$ are superstable ground-state configurations with respect to all members of the family.

Having described in detail the construction, and the exact ground state of this family of one-dimensional spin models, let us briefly discuss the nature of elementary excitations of the same.

Energy gap in the excitation spectrum. The MG model has gap in the excitation spectrum with respect to the dimer ground state. This was illustrated by a variational calculation of the dispersion of the defect (the dangling spin as shown in Fig. 3) boundary between two exact ground-state configurations.[14] Later, it was exactly proved that there is an energy gap in the excitation above the exact dimer ground states of the MG model.[15] A method to calculate the lower bound for the energy gap was also developed, and applied to certain quantum antiferromagnets.[16] We have calculated the dispersion relation for the propagating defect, variationally, for $\mathfrak{B}_5$ and $\mathfrak{B}_7$ chains. We find that the energy gap towards such solitonic excitations exists for each of these members of the family, and seems to increase monotonically for higher members in the family. The dispersion relation for $H[\mathfrak{B}_3]$ (the MG model) is $\mathcal{E}_3(k) = \frac{5}{4} + \cos(2k)$. This is already known from Ref. 14. Here, the lattice parameter a is taken to be unity, and $-\pi/2 \leqslant k \leqslant \pi/2$. The single defect-boundary excitation energies for the next two members of the family, that is $H[\mathfrak{B}_5]$ and $H[\mathfrak{B}_7]$, are the following:

$$\mathcal{E}_5(k) = 2 + \frac{3}{4}\cos(2k) - \frac{1}{2}\cos(4k), \quad (10)$$

$$\mathcal{E}_7(k) = \frac{11}{4} + \frac{3}{4}\cos(2k) - \frac{3}{8}\cos(4k) + \frac{1}{4}\cos(6k). \quad (11)$$

Here, $\mathcal{E}_{2\nu+1} = \langle k|(H[\mathfrak{B}_{2\nu+1}] - E_\nu)|k\rangle / \langle k|k\rangle$. $E_\nu = -\frac{3}{4}\nu N$; N is the total number of $\mathfrak{B}_{2\nu+1}$ units forming a finite chain of L sites. $N = L$ for a chain with PBC, and $N = L - 2\nu$ for an open chain. Actually, we have considered the chains with an odd number of spins, $L = 4M + 1$, in order to find the defect's dispersion relation, and taken the limit $M \to \infty$. The ket, $|k\rangle$ is defined as

$$|k\rangle = \frac{1}{\sqrt{2M+1}} \sum_{r=-M}^{M} e^{ik\mathbf{r}} |2r\rangle. \quad (12)$$

The ket, $|2r\rangle$, is a configuration where there is a dangling spin at $2r$th site, and the rest forming the dimer configuration of $|\psi_1\rangle$ type on one side of the dangling spin and of $|\psi_2\rangle$ type on the other side, as shown in Fig. 3. The dangling spin can have either ↑ polarization or ↓ polarization. These defect configurations are nonorthogonal. Since $\{|2r\rangle\}$ do not form a complete set of states, the propagating defect state $|k\rangle$ is only a variational choice. Nevertheless, it gives us some idea of the nature of excitations above the ground state. From the dispersion relations for the defect, we find that the energy gap for $\mathfrak{B}_3$, $\mathfrak{B}_5$, and $\mathfrak{B}_7$ chains, in units of the nearest-neighbor exchange, is 1/8, 3/16, and 11/48, respectively. Presently, we are unable to identify any simple relation between the members of the family and the corresponding energy gaps, nonetheless, we see that there is a gap, and it seems to be increasing as we go up in the hierarchy. One can also consider the case where there are many such dangling spins in the dimer background, and consider the possibility of the bound states. We will not do it here.

Connection with the Coulomb problem in one dimension. It is important to observe that the exchange coupling of the models constructed in the present work, $J_{ij} \propto (R - |i-j|)$, is exactly like the Coulomb interaction in one dimension, albeit with a range R.[17] From the Rth neighbor onwards, the exchange coupling is zero. But there is no restriction on the range R, and it can be anything. Hence what we have found, essentially, is a quantum spin analog of the Coulomb problem in one dimension which was studied exactly by Lenard and Baxter long ago.[18] It is an interesting and unexpected connection. Analogous to the Coulomb problem, one would expect plasmonlike gapped excitations in a spin model with infinitely long-ranged, linear exchange coupling. For antiferromagnetic exchange coupling, as we saw just now, the en-

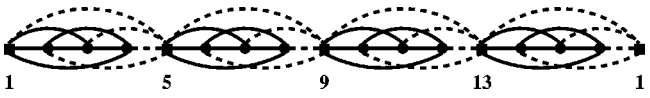

FIG. 4. The exchange connectivity of a one-dimensional spin model with exact dimer ground state whose degeneracy is exponential in the number of sites. Shown in the figure is an example of a 16-site model on a closed chain geometry. The sites where the $\mathfrak{B}_5$ unit repeats itself are numbered explicitly. The exchange couplings are: solid line $\equiv \alpha J$ and dashed line $\equiv J$.

ergy gap increases with R. But the analogy has interesting consequences for the spin models with ferromagnetic linear exchange coupling.

The Hamiltonian for the ferromagnetic case is written as: $H=-J\Sigma_{i=1}^{L}\Sigma_{m=1}^{R-1}(R-m)\mathbf{S}_i\cdot\mathbf{S}_{i+m}$. Here, $J>0$, and R can be even as well as odd unlike the antiferromagnetic case discussed in the present work. The ground-state energy for a general spin S is $E_g=-JS^2R(R-1)L/2$. And the exact one magnon dispersion with respect to the ground-state energy is: $\mathcal{E}(k)=JS\{R^2-[1-\cos(Rk)]/[1-\cos(k)]\}$, $\quad k=2n\pi/L$, where is n is an integer. The wave number k takes values between $-\pi$ and π. For any finite R, $\mathcal{E}(k)\to 0$ quadratically, as $k\to 0$. Hence the excitations are gapless. In order to consider the analog of Coulomb problem, R should be of the order of L, and let L go to infinity. We put $R=L/2$, and rescale J to J/R^2 (as the ground state as well as the magnon excitation energy goes as R^2 for large R). Then, $\mathcal{E}(k)=JS\{1-4[1-(-)^n]/L^2[1-\cos(k)]\}$. Therefore, in the thermodynamic limit, the magnon excitation has a gap of value JS, and a totally flat dispersion. Hence a ferromagnet with infinite ranged linear exchange coupling is gapped. Next, we discuss another type of 1D spin models where the dimer ground state is exponentially degenerate. These models are also constructed of $\mathfrak{B}_{2\nu+1}$ units.

1D models with exponentially degenerate dimer ground state. As mentioned earlier, the block singlets made up of four or more spins always have degenerate dimer representations. This intrinsic degeneracy at the block level, however, could not be greatly exploited in our previous construction which led to the class of linear exchange spin models on a chain. The dimer ground state of this class of models is only twofold degenerate. Now, we construct another type of spin model, on a closed chain, whose ground state has an exponentially large number of degenerate dimer configurations. Here, the degeneracy of the ground state is exponential in the number of lattice sites L. To illustrate this class of model, we describe a particular construction using $\mathfrak{B}_5$ units.

Consider a closed chain with even number of sites. Connect spins at the 1st, 2nd, 3rd, and 4th sites identically among themselves with exchange coupling αJ. Then connect each of these four spins to the spin at the 5th site with exchange coupling J, as shown in Fig. 4. Again, connect spins from the 5th to 8th site identically with coupling αJ, and then connect these four spins to the spin at 9th site with coupling J. Repeat this procedure for further spins, starting with the 13th site, the 17th, and so on. Thus we construct a spin chain of $\mathfrak{B}_5$ units. These $\mathfrak{B}_5$ units are slightly different from the ones we have used earlier. Here we have two exchange couplings J and αJ unlike the earlier considerations where coupling was identically J. The corresponding Hamiltonian can be written as: $H=J\Sigma_{l=1}^{L/4}h'_{4l-3}(\mathfrak{B}_5)$, with $J>0$. The block Hamiltonian,

$$\begin{aligned} h'_{4l-3}(\mathfrak{B}_5)=\alpha\{&\mathbf{S}_{4l-3}\cdot(\mathbf{S}_{4l-2}+\mathbf{S}_{4l-1}+\mathbf{S}_{4l})\\ &+\mathbf{S}_{4l-2}\cdot(\mathbf{S}_{4l-1}+\mathbf{S}_{4l})+\mathbf{S}_{4l-1}\cdot\mathbf{S}_{4l}\}\\ &+\mathbf{S}_{4l+1}\cdot(\mathbf{S}_{4l-3}+\mathbf{S}_{4l-2}+\mathbf{S}_{4l-1}+\mathbf{S}_{4l}), \end{aligned} \tag{13}$$

is a slightly generalized version of the previously defined $\mathfrak{B}_5$ block Hamiltonian, $h(\mathfrak{B}_5)$. When $\alpha=1$, $h'(\mathfrak{B}_5)$ is the same as $h(\mathfrak{B}_5)$.

For $\alpha>1$, $h'_{4l-3}(\mathfrak{B}_5)$'s minimum energy is equal to $-3\alpha/2$ for $S=1/2$. Spins $\mathbf{S}_{4l-3}$, $\mathbf{S}_{4l-2}$, $\mathbf{S}_{4l-1}$, and $\mathbf{S}_{4l}$ form a twofold degenerate block singlet corresponding to the block Hamiltonian's lowest eigenenergy, and spin $\mathbf{S}_{4l+1}$ remains ''free'' to be a part of the next block singlet. Therefore the ground-state energy of this spin model is $-3\alpha JL/8$, and the ground-state configurations can be written as

$$\begin{aligned} |\psi_g\rangle=&[1,2,3,4]\otimes[5,6,7,8]\\ &\otimes\cdots\otimes[L-3,L-2,L-1,L]. \end{aligned} \tag{14}$$

Since each of these four spin block singlets, $[4l-3,4l-2,4l-1,4l]$, has two dimer representations, $|\psi_g\rangle$ represents $2^{L/4}$ degenerate, dimer configurations forming the ground subspace. For $\alpha=1$, the ground-state energy is still given by the above expression, but the degeneracy of the ground state is $2(2^{L/4})$. This is so because, for $\alpha=1$, any four spins of a $\mathfrak{B}_5$ block can form a singlet in the minimum-energy configuration. This allows the following configurations, together with $|\psi_g\rangle$, in the ground state,

$$|\psi'_g\rangle=[2,3,4,5]\otimes[6,7,8,9]\otimes\cdots\otimes[L-2,L-1,L,1]. \tag{15}$$

And hence the degeneracy is doubled. The procedure, described here in detail, can be directly applied to construct models using bigger spin blocks. Thus we are able to construct quantum spin chains with exponentially degenerate dimer ground state, and the interesting thing to note is that the ground state is exactly known for $\alpha\geqslant 1$, unlike the linear exchange models where α is strictly 1. Or, in other words, the dimer configurations are superstable for all values of α greater than or equal to 1. We will come across the same features once more in the following section, while considering the higher dimensional generalization of our scheme.

In this section, we have developed the concepts, and explicitly used them in constructing one-dimensional spin models. In the next section, we will use this understanding to construct models with an exact dimer ground state in two and three dimensions. Before going to the next section, we must mention that recently there have been some generalizations of MG and SS models.[19–21] In this regard, we would like to stress the point that our scheme provides a very general framework to construct all such models using only two things: (i) the basic building blocks, and (ii) arranging them in such a way that each block is able to satisfy its minimum-energy configuration even when being a part of the full assembly. This we have already seen working in one dimen-

sion. We will illustrate this in the following section by constructing various models in two and three dimensions.

IV. GENERALIZATION TO TWO AND THREE DIMENSIONS

The SS model is a two-dimensional spin model with a specific exchange connectivity.[1] It has an exact dimer ground state with no degeneracy. It is, in some sense, the two-dimensional analog of the MG model. The SS model, like the MG model, is made up of fundamental blocks of three spins, that is, $\mathfrak{B}_3$ units in our language. Hence the SS model is the first member of the family of 2D spin models with an exact dimer ground state, if labeled according to our scheme.

This correspondence motivates us to construct spin models in higher dimensions whose ground states can be known exactly. In fact, we can construct a whole lot of them. The rules are very simple. Pick one spin of a $\mathfrak{B}_{2\nu+1}$ unit as free. Leave the rest to form a block singlet of size 2ν. Each of the 2ν spins of the singlet forming block can act as the ''free'' spin for the neighboring blocks. And any given block singlet can be fully or partially shared by other ''free'' spins. This allows us to extend the network in higher dimensions. The Hamiltonian for any such model is just the sum of all the block Hamiltonians. Only those dimer configurations which satisfy each building block's minimum-energy configuration form the exact ground-state configurations. This set of rules seem sufficiently general for constructing models with exact dimer ground states. For example, one gets the SS model by applying these rules to make a 2D model using $\mathfrak{B}_3$ units.

We consider a generalized $\mathfrak{B}_{2\nu+1}$ unit, where one of the spins is picked up to be exclusive such that it is coupled to the rest by unit exchange strength while the rest are completely coupled among themselves by some exchange α. These basic spin blocks are still completely connected, but the couplings are not identical. This choice is exactly like the one we had for constructing 1D models with exponentially degenerate dimer ground states; see Fig. 2, where $\mathfrak{B}_3$ and $\mathfrak{B}_5$ units are shown in one, two, and three spatial dimensions. The ''block singlet + free spin'' is the lowest energy configuration of the $\mathfrak{B}_{2\nu+1}$ unit for $\alpha \geqslant 1$. We can construct models in two dimensions (and also in three dimensions) in such a way that *spatially disjoint block-singlets* form the exact ground-state configurations for $\alpha \geqslant 1$. It is interesting to note that the domain of superstability for the dimer ground states in these models is $\alpha \geqslant 1$, unlike the case of translationally invariant 1D models constructed in the previous section. There, the dimer configurations form a superstable ground state only at $\alpha = 1$. Also, the spatial disjointness preserves the intrinsic degeneracy of the block singlets which makes the ground-state exponentially degenerate. Thus we are able to construct *spin models with finite entropy density in the ground state*. To illustrate all this, we describe models made up of $\mathfrak{B}_5$ units.

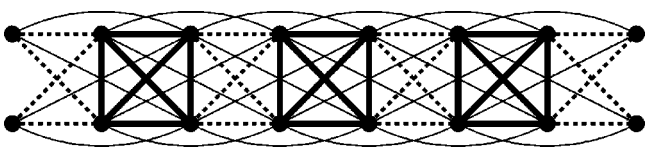

FIG. 5. The ladder made up of $\mathfrak{B}_5$ blocks. The exchange couplings are: thin solid line $\equiv J$, dashed line $\equiv 2J$, and thick solid line $\equiv 4\alpha J$.

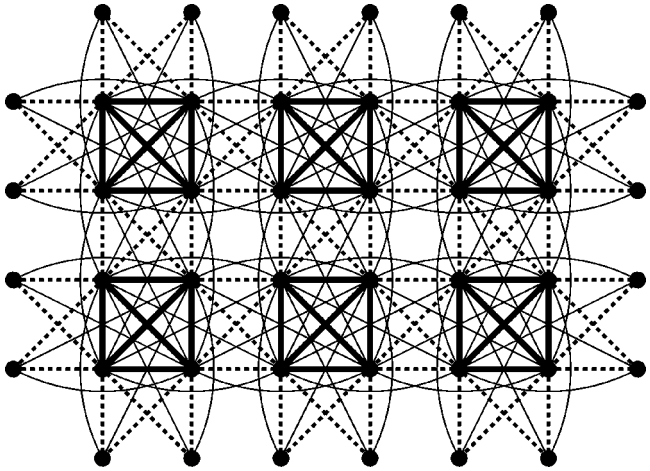

FIG. 6. A 2D model with a certain exchange connectivity, made up of $\mathfrak{B}_5$ blocks. The exchange couplings are: thin solid line $\equiv J$, dashed line $\equiv 2J$, and thick solid line $\equiv 8\alpha J$.

Following the rules stated above, a ladder model, as shown in Fig. 5, is constructed with $\mathfrak{B}_5$ units. (We could have constructed a simpler ladder with $\mathfrak{B}_3$ units, but the exponentially degenerate ground state cannot be realized there. The dimers along the rungs form the exact ground-state configuration of a $\mathfrak{B}_3$ ladder.) These ladders properly arranged on a plane give rise to a certain 2D spin model as shown in Fig. 6. Assuming a ribbonlike geometry for the ladder and the toroidal geometry for the 2D model, we can easily find their ground-state energies. In our construction, each lattice point of the ladder model contributes exactly one $\mathfrak{B}_5$ unit, whereas each site on the 2D model contributes two such units. Therefore the ground-state energies, for $\alpha \geqslant 1$, of the ladder and the 2D model are $-\frac{3}{2}\alpha JL$ and $-3\alpha JL$, respectively. Here, L is the total number of sites. The ground-state energy per site, in units of the strongest exchange ($4\alpha J$ for the ladder, and $8\alpha J$ for the 2D model), is just $-\frac{3}{8}$. The blocks of spins, connected with thick lines as shown in Figs. 5 and 6, forming singlets is the exact ground-state configuration. For the ladder, the ground-state configuration is shown in Fig. 7. The ground state for the 2D model is similar

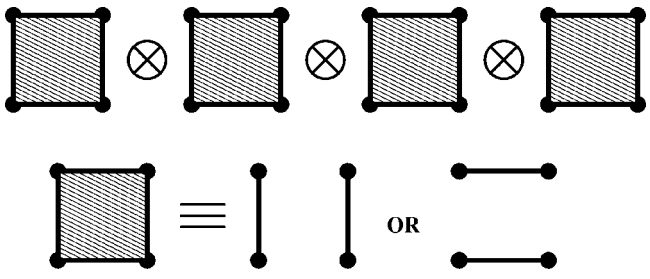

FIG. 7. The ground-state configuration of the ladder model. Each of the square blocks is a singlet which has two linearly independent dimer representations as shown by vertical and horizontal dimers.

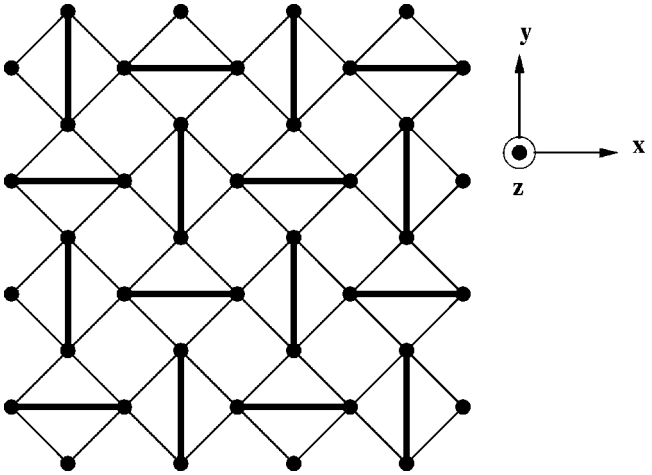

FIG. 8. This is a projection on an x-y plane of the ground-state configuration of a layer of the corner sharing octahedra made out of square pyramids (the $\mathfrak{B}_5$ units in three dimensions). Thick lines here represent the common basal plane of two $\mathfrak{B}_5$ pyramids making an octahedra. In the ground state, four spins lying on the common basal plane form a twofold degenerate block singlet. These block singlets lie in x-z or y-z planes. It is interesting to observe the topological equivalence of the arrangement of block singlets on orthogonal planes, here, to that of the dimers in the exact ground state of the SS model.

to that of the ladder. The twofold intrinsic degeneracy of each block singlet makes the ground subspace $2^{L/4}$-fold degenerate. The entropy density in the ground state is $\frac{1}{4}\ln(2)$, just one-fourth of that of a paramagnet. Each block singlet has two independent bond-singlet configurations, therefore the ground subspace consists of $2^{L/4}$ distinct dimer or valence bond configurations. For example, two of these configurations are the columnar dimer states which are the exact ground states of a certain model constructed by Bose and Mitra.[22] Thus the ground state of models discussed here is a spin liquid as well as a dimer liquid, as the dimers within a block singlet are correlated, but there is no correlation between dimers belonging to different block singlets. One can easily construct certain other models, using the same rules, whose ground states are dimer solids, though we will not discuss these models explicitly.

The generalizations of the SS model, considered in Ref. 19, provide good examples of the higher dimensional models made up of $\mathfrak{B}_3$ units. So, we will consider the next block, that is, $\mathfrak{B}_5$. The three-dimensional (3D) $\mathfrak{B}_5$ block is shown in Fig. 2. There can be many ways of constructing 3D models using $\mathfrak{B}_5$ pyramids. We consider a case where two such pyramids share the basal plane to form an octahedra. For $\alpha \geqslant 1$, the four spins of the common basal plane form a block singlet. Layers of the corner-sharing octahedra, stacked in certain ways, form one such model. The projection of one such layer on an x-y plane is shown in Fig. 8. The ground-state configurations are such that the singlet blocks are lying in orthogonal planes. It is topologically similar to the orthogonal arrangement of dimers in the exact ground state of the 2D SS model except that these are block singlets, and are lying in orthogonal planes. The spatial disjointness of the block singlets belonging to different planes gives rise to an exponentially degenerate dimer ground state.

The possibilities for making such models are enormous, and therefore cannot be exhausted in one place. But one thing must be emphasized. The basis for constructing models with exact dimer ground states, as discussed in the present work, is very general. The model Hamiltonians that may be constructed with these rules may be of definite importance in understanding some physics of the spin systems.

V. CONCLUSIONS

We first summarize the main results. We have constructed a family of one-dimensional spin models with linearly decreasing exchange coupling. The range of coupling is the family parameter which is decided by the size of the basic building block. The fundamental building blocks are completely connected blocks of an odd number of identical spins. All the members in the family have an exact, twofold degenerate dimer ground state. There exists an energy gap in the excitation spectrum above the dimer ground states, and it seems to be increasing monotonically for the higher members in the family. The interesting analogy of this family of models, with the Coulomb problem in one dimension, is discussed. One-dimensional spin models with exponentially degenerate dimer ground states are also discussed.

This way of constructing models is easily generalized to higher dimensions. We are able to construct models in two and three dimensions, with exact dimer ground states. The ground state of many such models is exponentially degenerate. The degeneracy in the ground state of these models arises due to many degenerate dimer representations of the block singlets forming the ground-state configuration. These models thus provide explicit examples of the systems with finite entropy density in the ground state, and hence violating the third law of thermodynamics. Also, these are the models with *finite ranged* RVB-type ground states, spanned by an exponentially large subset of the full set of linearly independent valence bond configurations. The range of the valence bonds is decided by the size of the building block. The rules for constructing such models are explicitly discussed.

We hope that the general class of models proposed here may be useful in studying the quantum phase transitions in the frustrated spin systems. The exact knowledge of the ground state with exponential degeneracy, in these models, is particularly remarkable. The building block way of looking at these models is quite in the spirit of synthetic chemists. It may be possible to realize such models.

Note added in proof. We are thankful to Professor Indrani Bose who brought to our notice an earlier work of Takano[23] which has some overlap with our present work. Takano had also arrived at the linear exchange spin chains with dimer ground state, and discussed its generalization in two dimensions. Though there are interesting similarities between Takano's and our present work, there are also clear differences in the approach and the emphases in the two.

ACKNOWLEDGMENTS

The author wishes to thank Professor B. Sriram Shastry and Professor Diptiman Sen for fruitful discussions and com ments, and likes to acknowledge the Council of Scientific and Industrial Research (CSIR), India, for financial support.

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