Quantum and classical localization, the spin quantum Hall effect, and generalizations

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We consider network models for localization problems belonging to symmetry class C. This symmetry class arises in a description of the dynamics of quasiparticles for disordered spin-singlet superconductors which have a Bogoliubov–de Gennes Hamiltonian that is invariant under spin rotations but not under time reversal. Our models include but also generalize the one studied previously in the context of the spin quantum Hall effect. For these systems we express the disorder-averaged conductance and density of states in terms of sums over certain classical random walks, which are self-avoiding and have attractive interactions. A transition between localized and extended phases of the quantum system maps in this way to a similar transition for the classical walks. In the case of the spin quantum Hall effect, the classical walks are the hulls of percolation clusters, and our approach provides an alternative derivation of a mapping first established by Gruzberg, Ludwig, and Read [Phys. Rev. Lett. **82**, 4524 (1999)].

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I. INTRODUCTION

Localization of a particle moving in a random environment may occur both quantum mechanically and with classical dynamics, but the phenomenon is very different in the two cases. In this paper we discuss a class of quantum-mechanical localization problems for which some physical quantities can be expressed exactly in terms of averages taken in a classical counterpart. The equivalence holds despite the fact that interference effects dominate the behavior of the quantum systems.

Disordered quantum systems can in general be classified according to their symmetries under time reversal and spin rotation. Three such symmetry classes are represented by the Wigner-Dyson random-matrix ensembles, while an additional seven have been identified more recently. The models we study here belong to one of these additional classes, termed class C by Altland and Zirnbauer.[1] One feature which distinguishes systems belonging to each of the additional symmetry classes from those in the Wigner-Dyson classes is that they have a special energy in their spectrum, with eigenstates occurring in pairs either side of this energy. Some of the additional classes have realizations as Bogoliubov–de Gennes Hamiltonians for quasiparticles in disordered superconductors, where pairing interactions are treated at the mean-field level. Here the special energy is the chemical potential in the superconductor and eigenstates are related in pairs by a particle-hole transformation. In particular, class C arises for quasiparticles in a spin-singlet superconductor in which time-reversal symmetry is broken for orbital motion but Zeeman splitting is negligible.[1] Since quasiparticle charge is not conserved in a superconductor, experiments to investigate quasiparticle dynamics in these systems must probe thermal or spin transport. Moreover, as the characteristic features of the symmetry class appear only close to the chemical potential, it is particularly gapless superconductors that are interesting: cuprate superconductors in the mixed state constitute a conspicuous example.

The models we study are especially simple realizations of their symmetry class. They are obtained as generalizations of the network model originally introduced to describe localization in the context of the integer quantum Hall plateau transition.[2] Thus they are formulated in the language of scattering theory and represent quantum particles, in general with N-component wave functions, propagating on the directed links, or edges, of a lattice and scattering between links at nodes. The symmetry of class C restricts N to even values, while our approach requires that all nodes of the lattice have two ingoing and two outgoing links. For these models, we are concerned with the density of states, obtained from the time-evolution operator, and with the disorder-averaged conductance of a finite sample, calculated from the Landauer formula. In both cases, our starting point is an expansion for the Green function as a sum over Feynman paths. Our central result is that the terms in this sum which survive after disorder averaging can be interpreted as self-avoiding classical random walks with attractive, short-range interactions.

A particular network model from class C, in two dimensions and with $N=2$, has been studied previously.[3–7] It shows the so-called spin quantum Hall effect, having two insulating phases, with quantized values of the Hall conductance differing by an integer, separated in the phase diagram by a delocalization transition which is analogous to the quantum Hall plateau transition. In a remarkable paper, using supersymmetry to perform disorder averages, Gruzberg, Ludwig, and Read[5] (GLR) showed that many physical quantities of interest in this model can be determined from the properties of the perimeters, or hulls, of classical percolation clusters in two dimensions. The approach we describe here provides an alternative derivation of their results, using more elementary, nonsupersymmetric methods, as well as an extension to other lattices, including ones in more than two dimensions and irregular lattices for which transfer matrix methods are inappropriate. It also extends to any even integer N. Our expressions give disorder-averaged physical quantities for the quantum system in terms of averages over classical random walks on the same lattice. In the case

treated by GLR these walks are simply percolation hulls, for which many analytical results are available. By contrast, in the general case the properties of the classical walks are not known. Nevertheless, the classical problem is much simpler than the original quantum problem, and we are able to construct further examples for which it is tractable. We remark that a different type of connection between quantum Hall plateau transitions and percolation, based on the classical limit, was discussed recently in Refs. 8 and 9.

There are some important qualitative differences between the properties of systems from Wigner-Dyson classes and those from the additional symmetry classes. In particular, while single-particle quantities such as the density of states are smooth functions of energy in the former case, in the latter case they may have singularities at the special energy, which we take to be zero in the following. This is illustrated by previous results on the behavior of models from class C, obtained using a variety of techniques.[10–16] Random-matrix ensembles with this symmetry, representing the zero-dimensional limit appropriate for quantum dots, have a density of states that vanishes quadratically in energy at energies much smaller than the mean level spacing.[1,10] A similar behavior is expected for finite-dimensional systems if states are Anderson localized, on the grounds that random-matrix theory should describe states within a localization volume.[13] Calculations for one-dimensional systems from class C, using either supersymmetry[11,13] or the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation,[14] confirm this idea.

Existing information on localization in class C systems is also provided in part by renormalization-group treatments[12] of the appropriate nonlinear sigma model at weak coupling, corresponding to weak disorder. These calculations identify two as the lower critical dimension. Thus, as in the Wigner-Dyson symmetry classes, it is only in more than two dimensions that a transition occurs between localized and metallic phases as a function of disorder strength, while in one dimension even weak disorder is sufficient to localize all states. In addition, for two-dimensional systems with broken time-reversal symmetry, including both ones from the Wigner-Dyson unitary class and ones from class C, a delocalization transition of the quantum Hall type is possible, and it is this transition that has been the focus of past work on class C network models.[3–7]

Many of these aspects, including the form of the density of states in a localized phase and the possibility of a quantum Hall plateau transition, emerge naturally from the approach we describe here, which is presented as follows. In Sec. II we introduce in detail the models that we are concerned with. In Sec. III we set out our general results, relating the density of states and average conductance for a network model to averages over certain classical random walks, and present proofs of these results. We describe applications of these general results to the spin quantum Hall effect, to random-matrix theory, and to localization on a Cayley tree, in Sec. IV. Open questions and future prospects are discussed in Sec. V.

II. MODELS

We shall be concerned with models both for closed systems and (in connection with the Landauer formula for conductance) for open systems, but initially we restrict definitions to closed systems. Consider a graph $\mathcal{G}$ consisting of directed edges e connecting nodes n, each of degree 4, with the restriction that at each node two directed edges enter and two leave. An N-component wave function propagates on each edge. This propagation may be described by a unitary evolution operator $\mathcal{U}$, which evolves the wave function one unit forward in time as the particle moves from a given edge to a neighboring one. The evolution operator plays the same role in defining the network model as does the Hamiltonian in the case, for example, of a tight-binding model. This was discussed for the $U(1)$ network model in Refs. 17 and 18. In general, it is constructed from two ingredients. First, with each edge e is associated a unitary $N\times N$ matrix U_e. This matrix specifies the (generalized) phase acquired on traversing the link. Second, with each node n is associated an S matrix of the form

$$S_n = \mathbb{1}\otimes\begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix}, \tag{1}$$

where $\mathbb{1}$ is the $N\times N$ unit matrix. This S matrix describes scattering at the node from the incoming edges to the outgoing ones. If $\mathcal{G}$ has E edges (and therefore $E/2$ nodes), then $\mathcal{U}$ is an $\mathcal{N}\times\mathcal{N}$ matrix, with $\mathcal{N}=EN$. It consists of $E/2$ blocks, each associated with a particular node and of size $2N\times 2N$. The block at the node n has the form

$$\begin{pmatrix} U_3^{1/2} & 0 \\ 0 & U_4^{1/2} \end{pmatrix} S_n \begin{pmatrix} U_1^{1/2} & 0 \\ 0 & U_2^{1/2} \end{pmatrix} \tag{2}$$

where (1,2) and (3,4) label the edges which are, respectively, incoming and outgoing at this node.

So far, the symmetry class of the network model has not been fixed, except that propagation along directed links breaks time-reversal symmetry. To identify network models from class C, one starts[4] from the defining property of a Hamiltonian $\mathcal{H}$ with this symmetry, which is[1]

$$\mathcal{H}^* = -\sigma_y \mathcal{H} \sigma_y , \tag{3}$$

where σ_y denotes the conventional Pauli matrix acting on spin variables and $\mathcal{H}^*$ is the complex conjugate of $\mathcal{H}$. Applying this to $\mathcal{U}$, interpreted as $\mathcal{U}=e^{i\mathcal{H}}$, the number of wave-function components N must be even, so that the space of states on each link may be viewed as consisting of $N/2$ two-component subspaces, within which σ_y operates. Then Eq. (3) becomes

$$\mathcal{U} = \sigma_y \mathcal{U}^* \sigma_y , \tag{4}$$

and from this an equivalent restriction follows on the edge phases, $U_e = \sigma_y U_e^* \sigma_y$, which are therefore unitary $\mathrm{Sp}(N)$ matrices, equivalent for $N=2$ to SU(2) matrices.

Randomness is introduced into these models via the edge phases. We take them to be independent random variables drawn from a distribution which is uniform on the invariant (Haar) measure of $\mathrm{Sp}(N)$. The quenched average of a given quantity in the network model, denoted by $\langle\cdots\rangle$, is the mean with respect to this measure.

An open system is constructed from a closed system of this type by ''cutting open'' a number of edges. The two halves of each edge cut into two in this way constitute one new edge directed into the system and one new edge directed out of the system. We may consider a conductance experiment between two ''contacts'' by grouping a subset of these of the incoming edges $\{e_{\rm in}\}$ to form one contact and another subset of the outgoing edges $\{e_{\rm out}\}$ to form the other. The transmission matrix t between these two contacts is a rectangular matrix whose elements are $\langle e_{\rm out}|(1-\mathcal{U})^{-1}|e_{\rm in}\rangle$. The spin conductance measured between the two contacts in units of $(\hbar/2)^2/h$ is

$$g=\mathrm{Tr}\, t^\dagger t. \tag{5}$$

from the multichannel Landauer formula.

Clearly, a great variety of specific models can be constructed within this framework, by making different choices for the graph $\mathcal{G}$ and for the number of channels N. We defer discussion of particular examples to Sec. IV.

III. GENERAL RESULTS

In this section we state and prove our results for a general graph $\mathcal{G}$. We consider first the particular case of $N=2$, and discuss the extension to general N in Sec. III E.

A. Green function, Feynman path expansion and classical walks

The Green function for propagation from edge e' to edge e is (with $N=2$) a 2×2 matrix

$$G(e,e';z)\equiv\langle e|(1-z\mathcal{U})^{-1}|e'\rangle, \tag{6}$$

where $|e\rangle$ is a state in the two-component space of wave functions for a particle located on the edge e. For $|z|<1$, Eq. (6) may be expanded as a sum over Feynman paths on $\mathcal{G}$ which begin on e' and end on e: each path gives an ordered product of factors zU_j along edges traversed by the path, weighted by appropriate factors of $\cos\theta_n$ and $\pm\sin\theta_n$ for each node through which it passes. Alternatively, we may rewrite Eq. (6) as

$$G(e,e';z)=-\langle e|z^{-1}\mathcal{U}^\dagger(1-z^{-1}\mathcal{U}^\dagger)^{-1}|e'\rangle, \tag{7}$$

obtaining instead a series convergent for $|z|>1$, involving an ordered product of factors $z^{-1}U_j^\dagger$ and an overall negative sign for each Feynman path.

Our central result is an expression for the disorder-averaged, Sp(2) trace of the Green function, $\mathrm{Tr}\langle G(e,e,z)\rangle$, in terms of classical paths. To state this result, we define on the same graph $\mathcal{G}$ a *classical scattering* problem as follows. Each node may be decomposed into two disconnected pieces in two ways, viz. (13,24) or (14,23), as illustrated in Fig. 1.

Theorem 1. The average Green function is given for $|z|<1$ by the generating function for the probability $P(e;L)$ in the classical problem that the edge e belongs to a loop of a given length L. Explicitly,

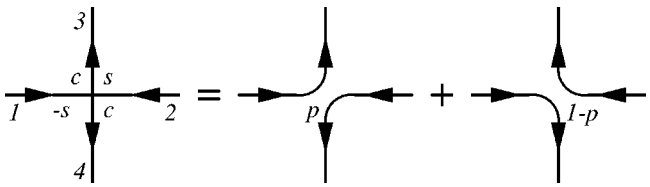

FIG. 1. Decomposition of a given node. In the network model, S-matrix elements $\cos\theta_n$ and $\pm\sin\theta_n$ are associated with the transitions $(1,2)\rightarrow(3,4)$ as indicated on the left. Each decomposition of the node is then weighted with factors $p_n=\cos^2\theta_n$ and $1-p_n=\sin^2\theta_n$, as indicated.

$$\mathrm{Tr}\langle G(e,e;z)\rangle=2-\sum_{L>0}P(e;L)z^{2L}. \tag{8}$$

For $|z|>1$, it is given instead by

$$\mathrm{Tr}\langle G(e,e;z)\rangle=\sum_{L>0}P(e;L)z^{-2L}. \tag{9}$$

B. Density of states

The eigenvalues of the evolution operator $\mathcal{U}$ for a closed graph lie on the unit circle in the complex plane and may be written as $\exp(i\epsilon_j)$, with eigenphases $-\pi<\epsilon_j\leqslant\pi$ for $j=1\ldots\mathcal{N}$. These eigenphases are analogous for the network model to the energy eigenvalues of a system specified by its Hamiltonian. We define the density of states to be

$$\rho(\epsilon)\equiv\frac{1}{\mathcal{N}}\sum_j\langle\delta(\epsilon-\epsilon_j)\rangle. \tag{10}$$

A consequence of the symmetry of Eq. (4) is that $\rho(\epsilon)=\rho(-\epsilon)$.

Defining $P(L)$, the edge average of $P(e;L)$, by

$$P(L)=\frac{1}{E}\sum_e P(e;L), \tag{11}$$

it follows from Eqs. (8) and (9) that

$$\rho(\epsilon)=\frac{1}{2\pi}\left[1-\sum_{L>0}P(L)\cos(2L\epsilon)\right]. \tag{12}$$

C. Conductance

The classical scattering problem introduced in Sec. III A may also be considered for an open system. For an open system with M cut links, each decomposition breaks $\mathcal{G}$ into M directed paths which each run from an ingoing edge to an outgoing one, together with a number (possibly zero) of closed loops. Let $P(e,e')$ be the probability that a path runs from the ingoing edge e' to the outgoing edge e.

Theorem 2. The disorder average of the conductance defined in Eq. (5) is given by

$$\langle g\rangle=2\sum_{e\in 1,e'\in 2}P(e,e'), \tag{13}$$

where sets 1 and 2 denote, respectively, the edges incident on the first contact from $\mathcal{G}$, and those incident on $\mathcal{G}$ from the second contact.

D. Proofs

For a closed system it is useful to introduce the resolvent

$$R(z)\equiv\sum_e \mathrm{Tr}\, G(e,e;z) \tag{14}$$

[where Tr again indicates an Sp(2) trace] and to generalize this to the case where the parameter z takes independent values z_e on each edge e. The expansion of $R(\{z\})$ as a sum over paths then yields a multinomial expression in all the z_e. We now require two lemmas about Sp(2) matrices.

Lemma 1. If $U\in \mathrm{Sp}(2)$, then its mean qth moment, $\langle U^q\rangle$, is zero unless $q=0$ or $q=2$, in which case it takes the value $\mathbf{1}$ or $-\frac{1}{2}\mathbf{1}$, respectively. This can be shown using the representation $U=\exp(i\alpha\vec{n}\cdot\vec{\sigma})$, where $\vec{n}$ is a unit real 3 vector and the σ's are the Pauli matrices, so that

$$U^q=\cos q\alpha\mathbf{1}+i\sin q\alpha\vec{n}\cdot\vec{\sigma}. \tag{15}$$

The result now follows from the observation that the invariant measure for Sp(2) is the uniform measure on the group manifold S_3, and therefore has the form $\int dU=\pi^{-1}\int_0^\pi(1-\cos 2\alpha)d\alpha\int d\vec{n}$.

Lemma 2. If G is a real linear combination of Sp(2) matrices, it is itself proportional to an Sp(2) matrix, with a real scalar constant of proportionality. This follows directly from the above representation for each matrix.

The main argument in the proof of theorem 1 now proceeds as follows. As discussed above, for each realization of the randomness, $R(\{z\})$ is a sum over closed directed paths on $\mathcal{G}$. For a particular path, each link or node may be traversed an arbitrary number of times. We first show that it is sufficient, in calculating the mean $\langle R(\{z\})\rangle$, to restrict this sum to those paths which traverse each edge exactly twice or not at all. Let us consider the sum of all paths in which a particular edge e is traversed exactly q times. This has the form

$$z_e^q\mathrm{Tr}[U_eA(e,e)]^q \tag{16}$$

where $A(e,e)$ denotes the sum over *all* weighted paths which begin and end on e, but do not themselves traverse e. By lemma 2, this is proportional to an Sp(2) matrix,[19] so it may be written as

$$A(e,e)=|A(e,e)|\tilde{A}(e,e), \tag{17}$$

where $|A(e,e)|$ is real and $\tilde{A}(e,e)\in\mathrm{Sp}(2)$. Defining $U'_e\equiv U_e\tilde{A}(e,e)$, Eq. (16) may be written

$$z_e^q\mathrm{Tr}\, U_e'^q|A(e,e)|^q. \tag{18}$$

The invariant integration over U_e is equivalent to that over U'_e, so that, by lemma 1, the result will vanish unless $q=0$ or $q=2$. Since $\langle R(\{z\})\rangle$ is a multinomial expression in the parameters $\{z\}$, the argument may be applied to each edge in turn to show that the only allowed powers of any z_e entering this expression are 0 or 2. This establishes the first part of the proof.

For paths in which each node is visited only 0 or 2 times, the main result follows immediately. For such a path must traverse a closed loop in the decomposition of $\mathcal{G}$ exactly twice. At each node there will be factors of $\cos^2\theta_n$ or $(\pm\sin\theta_n)^2$, giving precisely the correct weighting for this loop to appear in the decomposition. The product of Sp(2) matrices along the loop will have the form

$$\mathrm{Tr}\langle(U_1U_2\ldots)(U_1U_2\ldots)\rangle \tag{19}$$

Thus, defining $U'_1\equiv U_1U_2\ldots$, this is equivalent to averaging $U_1'^2$, which gives $-\frac{1}{2}$ by Lemma 1. Finally, the trace gives a factor of 2.

Further effort is required to treat paths which visit some nodes more than twice. A little thought shows that such a node must be visited exactly four times, entering and leaving exactly twice along each directed edge, if the contribution to $R(z)$ is not to vanish on averaging. Consider such a node n, and label the incoming and outgoing edges as (1,2), (3,4) respectively (see Fig. 1). We show that the sum over all paths visiting this node four times may be written in terms of the two ways of decomposing this node, with precisely the correct weights. Let $A(i,j)$ be the sum over all paths from edge $j\in(3,4)$ to edge $i\in(1,2)$, which do not use any of these four edges. The sum over paths visiting the node four times may be decomposed into eighteen different contributions, depending on the order in which the edges are visited. Each contains a product of four factors $A(i,j)$ as well as Sp(2) matrices U_i and U_j. There are in fact two types of contribution: in the first type, two of the $A(i,j)$ appear twice and the others not at all; while in the second type all four $A(i,j)$ appear once each. There are six of the first type and they have the forms

$$\mathrm{Tr}\, U_1A(1,3)U_3U_2A(2,4)U_4U_1A(1,3)U_3U_2A(2,4)U_4s^4, \tag{20}$$

$$\mathrm{Tr}\, U_1A(1,3)U_3U_1A(1,3)U_3U_2A(2,4)U_4U_2A(2,4)U_4c^2(-s^2), \tag{21}$$

$$\mathrm{Tr}\, U_1A(1,3)U_3U_2A(2,4)U_4U_2A(2,4)U_4U_1A(1,3)U_3c^2(-s^2) \tag{22}$$

and

$$\mathrm{Tr}\, U_1 A(1,4) U_4 U_2 A(2,3) U_3 U_1 A(1,4) U_4 U_2 A(2,3) U_3 c^4, \tag{23}$$

$$\mathrm{Tr}\, U_1 A(1,4) U_4 U_1 A(1,4) U_4 U_2 A(2,3) U_3 U_2 A(2,3) U_3 c^2(-s^2), \tag{24}$$

$$\mathrm{Tr}\, U_1 A(1,4) U_4 U_2 A(2,3) U_3 U_2 A(2,3) U_3 U_1 A(1,4) U_4 c^2(-s^2), \tag{25}$$

where we have introduced the shorthand $s \equiv \sin\theta_n$ and $c \equiv \cos\theta_n$. [Note that Eqs. (21) and (22) give separate contributions to $R(\{z\})$; the case is similar for Eq. (24) and (25)]. As before, we write $A(i,j) = |A(i,j)|\tilde{A}(i,j)$ and note that, by a change of integration variable, Eq. (20) is, on averaging, equivalent to the average of U_1^2 (that is, $-\frac{1}{2}$), multiplied by $|A(1,3)|^2|A(2,4)|^2 s^4$. By a similar argument, Eqs. (21) and (22) are equal to the average of $U_1^2 U_2^2$ [that is, $(-\frac{1}{2})^2$], multiplied by $|A(1,3)|^2|A(2,4)|^2 c^2(-s^2)$. The total of Eqs. (20)–(22) is therefore

$$\begin{aligned} &-|A(1,3)|^2|A(2,4)|^2[s^4 + 2(-\tfrac{1}{2})c^2(-s^2)] \\ &\quad = -|A(1,3)|^2|A(2,4)|^2 \sin^2\theta_n . \end{aligned} \tag{26}$$

Similarly, Eqs. (23)–(25) sum up to

$$-|A(1,4)|^2|A(2,3)|^2 \cos^2\theta_n . \tag{27}$$

Now consider the other twelve contributions, in which $A(1,3)$, $A(2,4)$, $A(1,4)$, and $A(2,3)$ each appear exactly once. Six of these are

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_1 A(1,4) U_4 U_2 A(2,4) U_4 U_2 A(2,3) U_3 c^4, \tag{28}$$

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_2 A(2,3) U_3 U_2 A(2,4) U_4 U_1 A(1,4) U_4 s^4, \tag{29}$$

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_1 A(1,4) U_4 U_2 A(2,3) U_3 U_2 A(2,4) U_4 c^2(-s^2), \tag{30}$$

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_2 A(2,3) U_3 U_1 A(1,4) U_4 U_2 A(2,4) U_4 c^2(-s^2), \tag{31}$$

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_2 A(2,4) U_4 U_1 A(1,4) U_4 U_2 A(2,3) U_3 c^2(-s^2), \tag{32}$$

$$\mathrm{Tr}\, U_1 A(1,3) U_3 U_2 A(2,4) U_4 U_2 A(2,3) U_3 U_1 A(1,4) U_4 c^2(-s^2). \tag{33}$$

In addition, there are six equal contributions in which the factors are cyclically permuted so that each begins $U_1 A(1,4) \ldots$. Each term is proportional to $|A(1,3)||A(2,4)||A(1,4)||A(2,3)|$. The remainder of the expressions may be simplified, for example, by redefining $U_1\tilde{A}(1,3) = U_1'$ and $U_2\tilde{A}(2,4) = U_2'$. This reduces Eqs. (28)–(33) to

$$\mathrm{Tr}\, U_1' U_3 U_1' B U_4 U_2' U_4 U_2' C U_3 c^4, \tag{34}$$

$$\mathrm{Tr}\, U_1' U_3 U_2' C U_3 U_2' U_4 U_1' B U_4 s^4, \tag{35}$$

$$\mathrm{Tr}\, U_1' U_3 U_1' B U_4 U_2' C U_3 U_2' U_4 c^2(-s^2), \tag{36}$$

$$\mathrm{Tr}\, U_1' U_3 U_2' C U_3 U_1' B U_4 U_2' U_4 c^2(-s^2), \tag{37}$$

$$\mathrm{Tr}\, U_1' U_3 U_2' U_4 U_1' B U_4 U_2' C U_3 c^2(-s^2) \tag{38}$$

$$\mathrm{Tr}\, U_1' U_3 U_2' U_4 U_2' C U_3 U_1' B U_4 c^2(-s^2), \tag{39}$$

where $B \equiv \tilde{A}(1,3)^{-1}\tilde{A}(1,4)$ and $C \equiv \tilde{A}(2,4)^{-1}\tilde{A}(2,3)$. By making suitable changes of integration variables, as before, we find that Eqs. (34) and (35) give factors of $(-\frac{1}{2})^2 \mathrm{Tr}(BC)$, while Eqs. (36)–(39) give $(-\frac{1}{2})^3 \mathrm{Tr}(BC)$. The sum of all 12 such contributions is therefore

$$\begin{aligned} X &\equiv \frac{1}{2}\mathrm{Tr}(BC)|A(1,3)||A(2,4)||A(1,4)||A(2,3)| \\ &\quad \times (c^4 + s^4 + 2c^2 s^2) \\ &= \frac{1}{2}\mathrm{Tr}\, A(1,3)^\dagger A(1,4) A(2,4)^\dagger A(2,3). \end{aligned} \tag{40}$$

The important feature of this result is that it is *independent of* θ_n. It may be written, trivially, as

$$X = p_n X + (1-p_n)X, \tag{41}$$

while the first six contributions have the form [as given in Eqs. (26) and (27)]

$$-p_n|A(1,4)|^2|A(2,3)|^2 - (1-p_n)|A(1,3)|^2|A(2,4)|^2. \tag{42}$$

Therefore we can obtain the same total result, after averaging over $U_1, \ldots, U_4$, if we decompose the node n in each of the two possible ways, and weight the two decompositions with probabilities p_n and $1-p_n$ respectively (see Fig. 1). For in

the first case (13,24) we find $-|A(1,4)|^2|A(2,3)|^2+X$, while in the second case of (14,23) we obtain $-|A(1,3)|^2|A(2,4)|^2+X$.

Now we may simply go through $\mathcal{G}$, decomposing it node by node. The result is that $\langle R(\{z\})\rangle$ is given by the weighted sum over the same quantity calculated on graphs arising from all possible decompositions of $\mathcal{G}$; such graphs consist of closed loops with no remaining nodes. In each decomposition, a given edge e belongs to just one loop, and, for that loop, of length L say, the contribution to $\mathrm{Tr}\,G(e,e;z)$ is just $-z^{2L}$. Thus the mean $\mathrm{Tr}\langle G(e,e;z)$ is given by the average of this quantity over the ensemble of loops corresponding to the decomposition of $\mathcal{G}$. This completes the proof of theorem 1.

So far we have considered unitary evolution on a closed system. However, since the proof is constructed to work for arbitrary values of the parameters $\{z_e\}$, it generalizes straightforwardly to an open system where probability is not conserved at the nodes which are connected to external leads. For example, if in Fig. 1, edges (1,3) correspond to external leads which carry no current, then we should consider only those Feynman paths through n which pass directly from 4 $\rightarrow$2, with amplitude $\cos\theta_n$. In the above argument, this may be taken into account by regarding this node as a single edge, carrying an Sp(2) matrix U_4U_2, and with fugacity $z_2z_4\cos\theta_n$. With this modification, the proof of theorem 1 goes through as before.

Now consider conductance measurements on an open graph $\mathcal{G}$, with external contacts $\{e_{\mathrm{in}}\}$ and $\{e_{\mathrm{out}}\}$, as described in Sec. II. Consider the graph $\mathcal{G}'$ formed from $\mathcal{G}$ by joining particular outgoing and incoming external edges e_{out} and e_{in} to create a new internal edge e. Observe that, in calculating $\langle G(e,e;z)\rangle$, the edge e is always traversed exactly twice. Therefore,

$$\langle G(e_{\mathrm{out}},e_{\mathrm{in}};z)^2\rangle_{\mathcal{G}}=\langle G(e,e;z)\rangle_{\mathcal{G}'}\big|_{z_e=1} \tag{43}$$

Since $G(e_{\mathrm{out}},e_{\mathrm{in}})$ is given by a sum over Sp(2) matrices, by lemmas 1 and 2, $\langle G(e_{\mathrm{out}},e_{\mathrm{in}};z)^2\rangle=-\frac{1}{2}\langle|G(e_{\mathrm{out}},e_{\mathrm{in}})|^2\rangle$. Now apply theorem 1: it follows that $\langle|G(e_{\mathrm{out}},e_{\mathrm{in}})|^2\rangle$ is twice the probability that e_{in} and e_{out} are connected by a path in the decomposition of $\mathcal{G}$. Summing over all the edges e_{in} and e_{out} as required by Landauer formula (5) then gives the result stated in theorem 2.

E. General N

Our approach to the Sp(N) model with even $N>2$ is based on the fact that a general Sp(N) rotation can be written as a product of rotations derived from a limited number of generators. Moreover, under rather weak conditions, the probability distribution of such a product containing random factors will converge as the number of factors increases to the Haar measure on Sp(N): the requirement is simply that no subspace is left invariant by the ensemble of rotations.

We therefore build the Sp(N) model on $\mathcal{G}$ by taking $N/2$ copies of the Sp(2) model, each individually defined on $\mathcal{G}$, and coupling the copies together. This coupling takes the form of a product of many noncommuting matrices. Successive factors in the product are of two kinds. One kind is block diagonal, with $N/2$ random Sp(2) blocks, resulting in intracopy scattering; the other kind must produce intercopy scattering and may be chosen to be nonrandom. It is convenient to restrict the intercopy scattering to be between corresponding links in each copy. For example, in the case of Sp(4) each pair of corresponding links is coupled as shown in Fig. 2. The evolution matrix for this pair of links consists of a product S of a large number of factors, each of the form

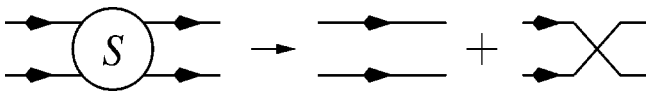

FIG. 2. Implementation of a link in the Sp(4) model. It consists of pairs of incoming and outgoing Sp(2) links, connected through a random S matrix which mixes the two channels. After averaging, this is equivalent to the decomposition shown on the right-hand side, equally weighted.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}, \tag{44}$$

where U_1 and U_2 are random Sp(2) matrices, chosen independently for each term in the matrix product.

The advantage of this choice is that we may immediately apply our main theorem to show that, after averaging over the random Sp(2) matrices, the mean density of states and mean conductance are given in terms of a decomposition of each node as before, and a decomposition of each link as shown in Fig. 2, with equal probabilities for each term. This construction may be generalized to Sp(N) for arbitrary even N: the result is that, in the classical system, the $N/2$ exit trajectories are a random permutation of the incident ones, with all permutations having equal weight, independently for each edge.

IV. APPLICATIONS OF GENERAL RESULTS

There are three obvious classes of behavior possible in the classical scattering problem which we have arrived at, corresponding to loops, or closed classical walks, that are localized, extended or critical.

By *localized* classical walks, we mean that only a vanishing fraction of walks have infinite length, and that the number of walks longer than a characteristic size ξ decreases rapidly with ξ. One might have, for example, $P(L)\sim\exp(-L/\xi)$ for $L\gg\xi$. Then the fraction of loops longer than L, given by

$$f(L)=1-\sum_{l=1}^{L}P(l), \tag{45}$$

approaches zero as $L\rightarrow\infty$. As a consequence, $\rho(\epsilon)\rightarrow0$ as $\epsilon\rightarrow0$. Introducing the mean-square length of loops,

$$\langle L^2\rangle=\sum_{L>0}P(L)L^2, \tag{46}$$

provided $\langle L^2 \rangle$ is finite, $\rho(\epsilon)$ vanishes quadratically, varying as

$$\rho(\epsilon) = (\pi)^{-1} \langle L^2 \rangle \epsilon^2 + O(\epsilon^4). \tag{47}$$

This is the behavior expected in the localized phase of the quantum problem. Moreover, if classical walks have a characteristic size ξ and if $\mathcal{G}$ is embedded in Euclidean space, then we expect the conductance to decrease rapidly with increasing contact separation, for separations larger than ξ.

Alternatively, classical walks are *extended* if a finite fraction has infinite length, so that $\lim_{L\to\infty} f(L) > 0$. In this case, $\rho(0) = (2\pi)^{-1} f(\infty)$ is nonzero, which we expect in the extended phase of the quantum problem. Clearly, the form of $\rho(\epsilon)$ at small ϵ is determined by that of $f(L)$ at large L: if $f(L) - f(\infty) \sim L^{-x}$ with $0 < x < 2$ then $\rho(\epsilon) - \rho(0) \propto |\epsilon|^x$, while if $f(L) - f(\infty)$ falls faster than L^{-2} at large L, then $\rho(\epsilon) - \rho(0) \sim \epsilon^2$. It is an interesting consequence of Eq. (12) that $\rho(\epsilon)$ is bounded, and so a divergence in the density of states, as occurs for example in class D localization problems, is excluded in class C network models, whatever the choice for $\mathcal{G}$. In the extended phase for a system in d-dimensional Euclidean space, Ohm's law dictates that the two-terminal conductance should vary with the separation, $l_\parallel$, between terminals and their cross-sectional area, $l_\perp$ as $l_\perp^{d-1}/l_\parallel$ for large $l_\parallel, l_\perp$; this places strong constraints on the behavior in the corresponding ensemble of classical walks, suggesting that, on large distance scales, they should behave like free random walks.

Classical walks that are *critical* have no characteristic loop size, and a vanishing fraction that are of infinite length. Then $f(\infty) = 0$, and a possible behavior for $P(L)$ is $P(L) \sim L^{-y}$ at large L, with $y > 1$. The resulting quantum density of states has the critical behavior $\rho(\epsilon) \sim |\epsilon|^{y-1}$ for small ϵ. The conductance of such a system is expected to depend on the geometry of sample and contacts, but to be unchanged under a rescaling of all spatial dimensions.

The task that remains, given a particular model for quantum localization, specified by the graph $\mathcal{G}$ and values for the node probabilities p_n, is to determine the behavior of the corresponding classical walks. In general, this remains a challenging open problem with connections to previously studied random-walk problems which we summarize in Sec. V. In particular instances, however, relevant properties of the classical walks can be calculated; we describe three examples in the remainder of this section.

A. Spin quantum Hall effect

A two-dimensional model exhibiting the spin quantum Hall effect is obtained by taking $\mathcal{G}$ to be the L lattice, illustrated in Fig. 3. As GLR have shown, the two possible classical decompositions of a node may be associated with the presence or absence of a bond, with probabilities p and $1-p$, between neighboring sites on an associated square lattice. This associated lattice is rotated by 45° relative to the L lattice, and has a larger lattice spacing by a factor $\sqrt{2}$. In this way, closed loops of the classical problem form interior or exterior hulls of bond percolation clusters on the larger lattice. It is known that such loops are finite with characteristic size ξ except at the critical point, $p = p_c$, which for bond percolation on the square lattice occurs at $p_c = 1/2$. On approaching the critical point, ξ diverges as $\xi \sim |p - p_c|^{-\nu}$ with $\nu = 4/3$, while at the critical point the distribution of hull lengths is $P(L) \sim L^{-y}$ at large L, with $y = 8/7$. In this way one finds that the quantum localization length diverges with the same exponent value, $\nu = 4/3$, as the plateau transition is approached, and that the density of states $\rho(\epsilon)$ varies for small ϵ as $\rho(\epsilon) \sim \epsilon^2$ in the localized phase, and as $\rho(\epsilon) \sim |\epsilon|^{1/7}$ at the critical point.

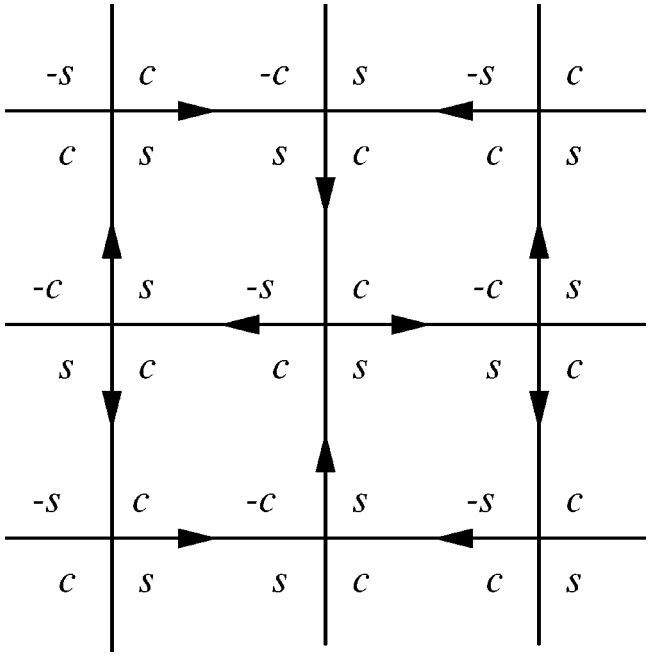

FIG. 3. The network model for the spin quantum Hall effect, defined using the L lattice. Scattering amplitudes of $\pm\cos\theta_n$ and $\pm\sin\theta_n$ are indicated by $\pm c$ and $\pm s$.

B. Random matrix theory

The simplest application of our discussion of Sp(N) models with general N is to random matrix theory. To this end, we take $\mathcal{G}$ to consist of a single edge closed on itself. Then the evolution operator $\mathcal{U}$ is a random Sp(N) matrix, chosen with the Haar measure. In this case the density of states $\rho(\epsilon)$ is the eigenphase density for the Sp(N) random-matrix ensemble, which has been determined previously by Zirnbauer[10] using supersymmetry methods.

Applying to this problem the approach we have described in Sec. III E, we must consider $N/2$ copies of an edge, which are closed by connecting outgoing ends to a permutation of ingoing ends. For example, the case $N = 4$ corresponds to joining opposite ends of the two possibilities shown in Fig. 2, thus producing with equal probability either two loops, each of length 1, or a single loop of length 2. For general N, all possible lengths L up to $N/2$ are possible, with equal probability. Thus $P(L) = 2/N$ for $1 \leqslant L \leqslant N/2$, and $P(L) = 0$ otherwise. Using Eq. (12) we thus find

$$\rho(\epsilon) = \frac{N+1}{2\pi N}\left[1 - \frac{\sin(N+1)\epsilon}{(N+1)\sin\epsilon}\right], \tag{48}$$

in agreement with Ref. 10.

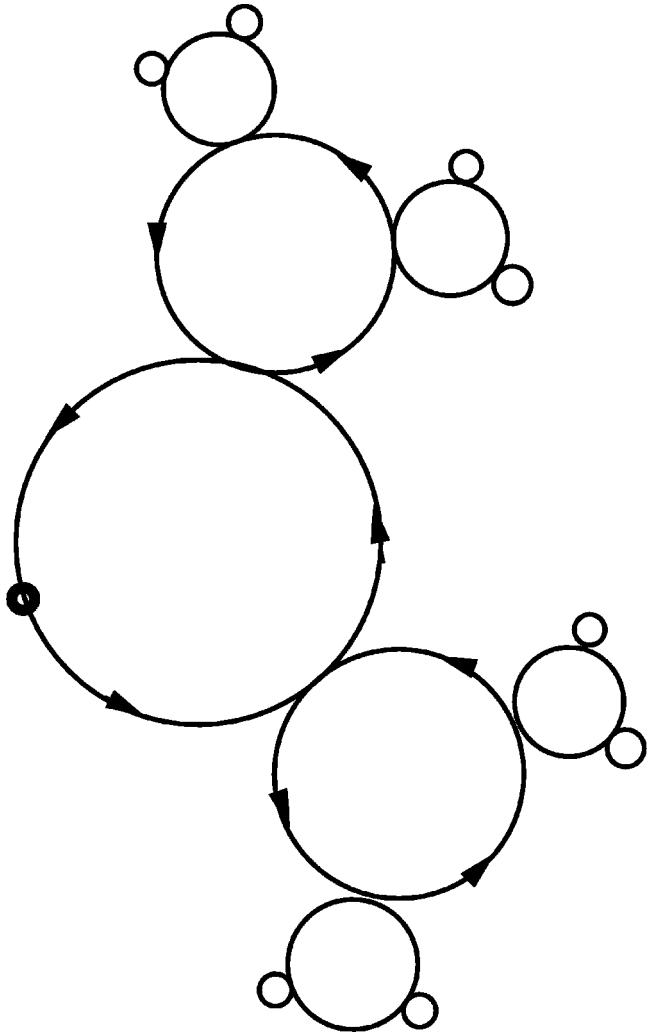

FIG. 4. The graph $\mathcal{G}$ for a network model based on a rooted Cayley tree. An example with coordination number $q=3$ and four generations is shown; the system size is increased by increasing the number of generations. *S*-matrix amplitudes at nodes are $\cos\theta$ to remain at the same generation, and $\pm\sin\theta$ to change generations.

C. Cayley tree

A solvable model which illustrates each of the types of behavior—localized, extended, and critical—for classical loops is based on the geometry of the Cayley tree. Specifically, we take $\mathcal{G}$ to be a graph of the type illustrated in Fig. 4; the U(1) network model on such a tree has been studied previously in Ref. 20. We restrict attention to $N=2$ and consider first coordination number $q=3$. As previously, we define $P(e;L)$ to be the probability that the edge e of the root, far from the surface of the tree, lies on a given closed loop of length L. Then

$$P(e;3L)=p^2\delta_{L,1}+2p(1-p)P(e;3L-3)+(1-p)^2\sum_{m=1}^{L-2}P(e;3m)P(e;3L-3m-3). \tag{49}$$

This can be solved by using the generating function $G(z)=\Sigma_{r\geqslant 1}z^rP(e;3r)$: we obtain

$$G(z)=zp^2+2zp(1-p)G(z)+(1-p)^2zG(z)^2. \tag{50}$$

We take the branch of $G(z)$ such that $G(0)=0$; thus

$$G(z)=\frac{1}{2z(1-p)^2}[1-2zp(1-p)-\sqrt{1-4zp(1-p)}]. \tag{51}$$

When $z=1$, $\sqrt{1-4zp(1-p)}$ is equal to $1-2p$ for $p<1/2$ and $2p-1$ for $p>1/2$. Therefore, for $p>1/2$, $G(1)=1$, so that all walks are localized. The characteristic loop size is given by

$$\langle L\rangle=3\left.\frac{\partial G}{\partial z}\right|_{z=1}=\frac{3}{2p-1}. \tag{52}$$

However, when $p<1/2$, $G(1)=[p/(1-p)]^2<1$. Thus in this case a proportion $1-G(1)$ of the walks do not close; they are extended. For such walks, Eq. (49) is invalid since it describes only closed walks of a finite length. For $p=1/2$ and large $L\in\mathbb{Z}$, we find the critical behavior $P(e;3L)\sim L^{-3/2}$. The corresponding density of states in the quantum system (defined locally on an edge far from the surface, to avoid surface effects) has the behavior

$$\rho(\epsilon)\propto\begin{cases}\epsilon^2, & p>\frac{1}{2}\\ |\epsilon|^{1/2}, & p=\frac{1}{2}\\ \text{const} & p<\frac{1}{2}\end{cases} \tag{53}$$

We can extend this analysis to consider coordination number $q>3$; then

$$G(z)=z[p+(1-p)G(z)]^{q-1}. \tag{54}$$

Thus $z_b=(1-p)^{-1}(q-1)^{1-q}[(q-2)/p]^{q-2}$ locates the unique branch point of $G(z)$. The complex roots of the two branches of $G(z)$ corresponding to $G(1)=1$ (all paths of finite length) and $G(1)=W$ (where W is the fraction of paths which are infinite), coincide at $p=(q-2)/(q-1)$. We conclude that for $p>(q-2)/(q-1)$ all walks are localized, whereas for $p<(q-2)/(q-1)$ there exists a fraction W of extended walks. In addition, we find that the critical behavior $P(e;qL)\sim L^{-3/2}$ holds for all $q\geqslant 3$. In the localized phase, the typical loop length is given by

$$\langle L\rangle=\frac{q}{p-(1-p)(q-2)}. \tag{55}$$

Thus, with the critical value $p_c=1/2$ replaced by $p_c=(q-2)/(q-1)$, the behavior of the density of states is the same as in the $q=3$ case. The critical exponents are therefore universal and independent of q. In fact, we note that there is a one-to-one correspondence between walks on the Cayley cactus and percolation clusters on the Cayley tree; therefore the critical exponents are related to those of percolation.

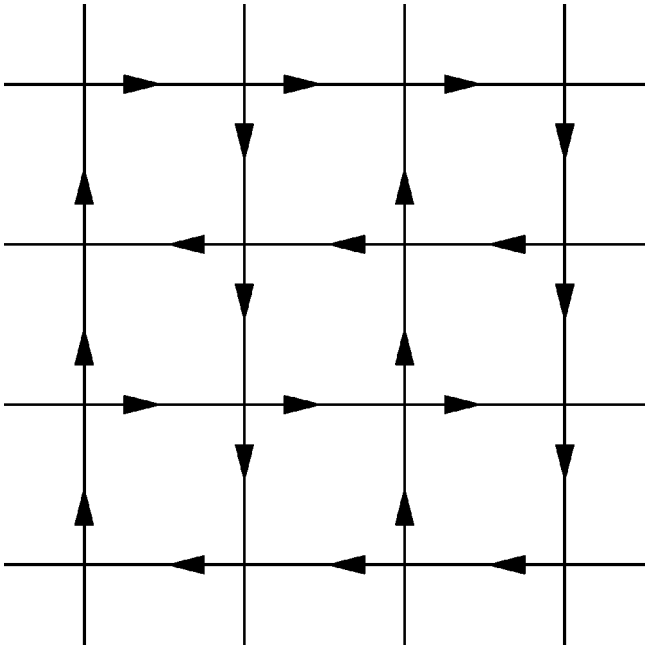

FIG. 5. The Manhattan lattice. In the network model, *S*-matrix elements of $\cos\theta$ and $\pm\sin\theta$ are associated, respectively, with 90° turns at nodes and with propagation in a straight line.

V. DISCUSSION

In this section we outline directions for future work. These are of two kinds. First, one can imagine a variety of models which are likely to exhibit phenomena not shown in the examples treated in Sec. IV. Second, one can hope to make use of connections between the classical random walks that arise in our approach and statistical problems studied previously. Further work on both these aspects is in progress.[21,22] In general, the equivalent classical problems correspond to self-attractive random walks of various kinds. They are attractive because the weight for passing through a given node twice is p, rather than p^2. However, the actual behavior is expected to depend very much on the individual lattice. Many of these problems correspond to the classical scattering of light by random arrays of mirrors: for example, in each decomposition of Fig. 1, a two-sided mirror can be placed so that the classical trajectory reflects off it. Such problems have been studied extensively.[23–26] They may also be realized in terms of *history-dependent kinetic* random walks, in the sense that the walker may be thought of as placing a mirror at random the first time it reaches a node: if it revisits the node, however, the mirror is already in place. This puts such models in the class of so-called ''true'' self-avoiding walks, which have been studied using field-theoretic renormalization-group methods in the limit of weak scattering.[27] Interestingly, such studies indicate a critical dimension of two, just as for Anderson localization.

A two-dimensional model with behavior which we expect to contrast with that found in the spin quantum Hall effect can be obtained by taking $\mathcal{G}$ to be the Manhattan lattice, illustrated in Fig. 5, in place of the L lattice. The crucial distinction is that, for the network model on the L lattice, two distinct phases can be identified from the different natures of edge states in the limits $p\to 0$ and $p\to 1$, while for the Manhattan lattice there are no edge states in either limit. Equivalently, the Hall conductance of the model on the L lattice may be nonzero, being determined at short distances by the value of p, but quantized at large distances, while on the Manhattan lattice the Hall conductance always has an average value zero. Consequently, one expects from the scaling flow diagram for quantum Hall systems[28] and the renormalization-group calculations for the class C nonlinear sigma model at weak coupling[12] that states of the Manhattan model should be localized for all $p>0$. Since classical trajectories on the Manhattan lattice cross at nodes with probability p, they are not the hulls of percolation clusters. Nevertheless, it is straightforward to use bond percolation on a lattice which has a lattice spacing larger by a factor $\sqrt{2}$ and is rotated by 45° relative to the Manhattan lattice to set an upper bound on loop sizes: this is sufficient to prove localization for $p>1/2$, and for $p=1$ classical loops are simply the elementary plaquettes of $\mathcal{G}$. Conversely, at $p=0$ classical trajectories are simply straight lines, and so one expects the localization length to diverge as $p\to 0$.

A second variant on the spin quantum Hall effect can be constructed by considering the Sp(N) model on the L lattice, but with $N>2$. It is natural to anticipate from the scaling flow diagram proposed for quantum Hall systems[28] that this model should have $N/2$ delocalization transitions as the node parameter p increases from $p=0$ to 1. Between transitions, states are localized and the Hall conductance is quantized. As a transition is approached, the localization length diverges, and on passing through a transition, the number of edge states and the quantized value of the Hall conductance both change by two. It remains a challenge to understand how or whether such a behavior arises in the language of classical walks, and might be interesting to relate the large-N limit of the lattice problem to the field theory discussed in Ref. 16

A third direction is to consider models defined on graphs in three or more dimensions. A three-dimensional version of the U(1) network model has been studied previously using a layered system[29] and models with cubic symmetry may also be constructed. For systems in three and higher dimensions, one expects that both localized and metallic phases should be possible, each existing over a range of values for node parameter, p, with a transition between the two at a critical value $p=p_c$. Since in two dimensions it is known that the theta-point transition between collapsed and swollen phases of a self-attracting polymer chain has the same exponents as those of percolation hulls,[30] it is tempting to suggest that in higher dimensions this delocalization transition might also be in the theta-point universality class. If so, we would expect a mean-field behavior, with logarithmic corrections for $d=3$.

ACKNOWLEDGMENTS

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