Electron-electron interactions, quantum Coulomb gap, and dynamical scaling near integer quantum Hall transitions

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The effects of electron-electron interactions on tunneling into the bulk of a two-dimensional electron system are studied near the integer quantum Hall transitions. Taking into account the dynamical screening of the interactions in the critical conducting state, we show that the behavior of the tunneling density of states (TDOS) is significantly altered at low energies from its noninteracting counterpart. For the long-range Coulomb interaction, we demonstrate that the TDOS vanishes linearly at the Fermi level according to a quantum Coulomb gap form $\nu(\omega)=C_Q|\omega|/e^4$, with C_Q a nonuniversal coefficient of a quantum-mechanical origin. In the case of short-range or screened Coulomb interactions, the TDOS is found to follow a power law $|\omega|^\alpha$, with α proportional to the bare interaction strength. Since short-range interactions are known to be irrelevant perturbations at the noninteracting critical point, we predict that, upon scaling, the power law is smeared, leading to a finite zero-bias TDOS $\nu(\omega)/\nu(0)=1+(|\omega|/\omega_0)^\gamma$, where γ is a universal exponent determined by the scaling dimension of short-ranged interactions. We also consider the case of quasi-one-dimensional (1D) samples with edges, i.e., the long Hall bar geometry, and find that the TDOS becomes dependent on the Hall conductance due to an altered boundary condition for diffusion. For short-range interactions, the TDOS of a quasi-1D strip with edges is linear near the Fermi level, with a slope inversely proportional to ρ_{xx} in the perturbative limit. These results are in qualitative agreement with the findings of bulk tunneling experiments. We discuss recent developments in understanding the role played by electron-electron interactions at the integer quantum Hall transitions and the implications of these results on the dynamical scaling of the transition width. We argue that for long-range Coulomb interactions, the existence of the quantum Coulomb gap in the quantum critical regime of the transition gives rise to the observed dynamical exponent $z=1$.

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I. INTRODUCTION

A. Integer quantum Hall transitions and inadequacies of the noninteracting electron theory

The physics of disorder and interaction in strong magnetic fields is central to our understanding of the low-temperature, quantum-mechanical behaviors of novel electronic materials. One of the most important physical phenomena under such settings is the quantum Hall effect (QHE).[1,2] The QHE refers to the low-temperature magnetotransport properties of high-mobility two-dimensional electron systems in a strong transverse magnetic field.[1–3] The main part of the phenomenology can be summarized by (1) the existence of stable phases of matter, i.e., quantum Hall states, with vanishing dissipation and integer or fractional quantized Hall conductances; and (2) the existence of continuous, zero-temperature phase transitions between the quantum Hall states, which are often referred to as quantum Hall (plateau) transitions. The basic physics in (1) for the spin-polarized incompressible quantum Hall states and their low-energy excitations are well understood.[3,4] In contrast, (2) is yet an unresolved problem, which is the subject of this work.

In a nutshell, (2) is a metal-insulator transition problem of the Anderson-Mott type in a two-dimensional (2D) disordered system with strong time-reversal symmetry breaking. These transitions are generally believed to be prime examples of continuous quantum phase transitions, i.e., examples of quantum critical phenomena.[5,6] Although there are reasons to suspect that the critical phenomena are universal for both integer and fractional transitions,[7–11] here we focus on the integer quantum Hall transitions (IQHT) in samples with sufficiently strong disorder that fractional quantum Hall states do not intervene. In this case, the transitions are directly between adjacent integer quantized Hall plateaus. The experimental data, reviewed in Ref. 5, can be summarized as follows: (a) On either side of the transition the Hall conductivity σ_{xy} is quantized, and the dissipative conductivity has the limit $\sigma_{xx}\to 0$ at zero temperature. (b) At the transition, σ_{xy} is unquantized, and σ_{xx} remains finite at zero temperature, so that the disordered quantum critical state is conducting. Thus the quantum phase transition is an unusual insulator-to-insulator transition with no intervening metallic phase; only the critical point itself has a finite conductance.

In an experimental situation, the divergent length, i.e., the critical singularity, is cut off by the presence of a finite length scale, giving rise to a finite transition width within which σ_{xy} deviates from the quantized values and σ_{xx} is nonzero. The transition width, denoted as δ^*, follows the scaling form

$$\frac{\delta^*}{\delta_0}\sim \min\left[\left(\frac{L_0}{L}\right)^{1/\nu_{\rm loc}},\left(\frac{T}{T_0}\right)^{1/z_T\nu_{\rm loc}},\left(\frac{\omega}{\omega_0}\right)^{1/z_\omega\nu_{\rm loc}}\right], \quad (1.1)$$

where L, T, and ω are the finite system size, the temperature, and the measurement frequency in a specific experimental situation, and Δ_0, L_0, T_0, and ω_0 are microscopic scales. The various exponents in Eq. (1.1) have the usual meaning: $\nu_{\rm loc}$ is the static exponent of the single divergent length scale, the localization length $\xi\sim\delta^{-\nu_{\rm loc}}$, where δ is the distance to the quantum critical point; z_ω is the dynamical exponent de-

fining the length scale introduced by a finite frequency $L_\omega \sim \omega^{-1/z_\omega}$; and z_T is the thermal exponent governing a temperature-dependent length scale $L_T \sim T^{-1/z_T}$. In general, z_T and z_ω can be independent exponents,[12] but $z_T = z_\omega$ for a generic quantum phase transition.[5,6] The three scaling regimes in Eq. (1.1) have all been probed experimentally,[13–16] as well as the regime in which the electric-field strength sets the cutoff.[17] The critical exponents extracted from the experiments can be summarized as $\nu_{\rm loc} = 2.3 \pm 0.1$, $1/z_\omega \nu_{\rm loc} = 0.41 \pm 0.04$, and $1/z_T \nu_{\rm loc} = 0.42 \pm 0.04$. Thus we have ω/T scaling with $z_T = z_\omega = 1$, which is in conformity with the dynamical scaling description of a generic quantum phase transition.

The phase structure of the IQHT appears to be consistent with that of the noninteracting theory of disordered 2D electrons in a strong magnetic field.[18,19] In a single-particle description, all states are localized due to disorder, except for those at a single critical energy E_c near the center of each disorder-broadened Landau level. The IQHT takes place when the Fermi level E_F of the 2D electron system and one of the discrete set of the critical energy E_c cross, i.e., when $\delta \equiv |E_F - E_c|$ approaches zero. Moreover, numerical calculations based on the noninteracting theory give a localization length exponent $\nu_{\rm loc} \simeq 2.3$, which is remarkably close to the experimental value.

However, our understanding of the IQHT is far from complete. It has become increasingly clear that the noninteracting theory, reviewed in Ref. 20, contradicts the experimental findings in several ways. Three of these are as follows. (i) Recent experimental work has shown that the tunneling density of states (TDOS) vanishes linearly at the Fermi level,[21] in sharp contrast to the finite density of states in the noninteracting theory. (ii) It was pointed out recently that due to the peculiar phase structure involving a set of extended states that has a measure zero, the conductivity σ_{xx} in the noninteracting theory is rigorously zero in the limit of large sample size at all values of the magnetic field, including the critical values, for any nonzero temperature.[12] This is in direct contradiction to the experimental observations. (iii) The noninteracting theory does not offer a correct description of the dynamical scaling behavior observed experimentally. The dynamical exponent governing how the energy (temperature) scale relates to the length scale for noninteracting electrons is $z = d = 2$, which *disagrees* with the experimentally obtained values quoted above. In fact, the experimental findings of ω/T scaling with $z_T = z_\omega = 1$ is in conformity with the dynamical scaling description of a generic quantum phase transition in which the Coulomb interaction is relevant and scales to a finite value at the transition.[22,23] The failure of the noninteracting theory highlighted by (i)–(iii) puts serious constraints on the ability of the free-electron model to explain the IQHT in real materials, and necessitates the investigation of the effects of electronic interactions and their interplay with disorder and localization.

B. Recent theoretical developments on the effects of interactions and the focus of this work

A significant part of the recent theoretical studies on the role of Coulomb interactions near the IQHT has centered around the three interconnected issues (i)–(iii) raised above.[24,25,23,12,26–29] There were also formal approaches by Pruisken and co-workers that aimed at extending the topological nonlinear σ-model description of the noninteracting transition to include Coulomb interactions.[30]

The first quantitative study of the one-particle density of states (DOS) in the presence of Coulomb interactions was carried out by Yang and MacDonald. Using a self-consistent Hartree-Fock (HF) approach, in which disorder was treated exactly while the Coulomb interaction was treated by the HF approximation, they found that the TDOS vanishes linearly at the Fermi level at *all* filling factors in the lowest Landau level, even at the critical energy.[24] The linear Coulomb gap behavior, especially at the critical energy, is in sharp contrast to that expected of the noninteracting theory [see (i) above], and is in qualitative agreement with experimental findings.[21] In spite of the dramatic TDOS change due to Coulomb interactions, however, Yang, MacDonald, and Huckestein found that the value of the localization length exponent and the fractal dimension of the critical eigenstate wave functions remain unchanged from the noninteracting theory, as does the qualitative behavior of the conductivity.[25] It is important to emphasize a unique and important feature of the HF theory for the IQHT: the *noncritical* suppression of the single-particle DOS, i.e., it vanishes linearly at all filling fractions regardless of whether the system is at criticality or not.[23]

In order to understand the effects of Coulomb interactions from the point of view of critical phenomena, Lee and Wang carried out a stability analysis of the noninteracting fixed point (NIFP), which governs the noninteracting transition, by numerical calculations of the perturbative renormalization-group (RG) scaling dimensions for the interactions.[23] They found that interactions of sufficiently short range are perturbatively irrelevant at the NIFP, and scale to zero in the asymptotic limit. The NIFP is therefore stable against such screened interactions, and, as a result, $\nu_{\rm loc} \simeq 2.3$ and $z = 2$. Wang *et al.* showed that, although short-range interactions are irrelevant in the RG sense, they generate a nonzero critical value for the dissipative conductance, and thus remove the pathology (ii) of the noninteracting theory and control the temperature-scaling behavior of σ_{xx}.[12] They showed that in the presence of irrelevant interactions, the scaling theory for transport properties becomes unconventional, ω/T scaling breaks down, and a third independent critical exponent, the thermal exponent z_T in Eq. (1.1), emerges. The value of z_T is set by the scaling dimension $-\alpha < 0$ of the interaction strength through the finite temperature dephasing time in the critical regime, $\tau_\phi \sim T^{-p}$, where $p = 1 + 2\alpha/z$, leading to $z_T = 2z/(z + 2\alpha)$. They argued that quantum critical scaling behavior of this kind may be a generic feature of finite-temperature transport near quantum critical points, when interactions are (dangerously) irrelevant.[12]

In contrast to short-range, model interactions, true long-range Coulomb interactions are, on the other hand, found to be relevant perturbations at the NIFP, making the latter unstable.[23] Hence the true critical point must be interacting, corresponding to an interacting fixed point having a finite interaction strength. This is consistent with the fact that the

experimentally extracted dynamical exponents $z_T = z_\omega = 1$, which are typical of the charge dynamics at quantum criticality controlled by Coulomb interactions.[22] However, Lee and Wang proposed that the fixed point in the theory where Coulomb interaction is treated via the HF approximation may in fact be stable.[31] They introduced the concept of a HF fixed point (HFFP), and argued that it is the simplest possible interacting fixed point of the IQHT. Correlation effects are found to be marginal perturbations at the HFFP due to the linear Coulomb gap in the HF theory that degrades of the RG dimensions of the residual interactions. They conjectured that a change in the dynamical exponent (z) with no change in the static one ($\nu_{\rm loc}$) can be due to the noncritical linear suppression of the single-particle (tunneling) DOS induced by Coulomb interactions. The HF theory, in particular the HFFP of the plateau transition, presents itself as a concrete example. There are two important issues that must be resolved before this conjecture can be further substantiated.

First, the theory of Coulomb gap was derived largely on the basis of classical physics.[32] It applies directly to electronic systems with Fermi energy lying in an excitation gap such as semiconductors and insulators. Therefore it may not be completely surprising that a 2D Coulomb gap DOS exists away from the transition regime where the electronic states are strongly localized and where the transport is dominated by variable range hopping in the presence of a 2D Coulomb gap.[33] What is remarkably surprising is that the linearly vanishing Coulomb gap is found to pertain to the critical regime of the IQHT where the localization length is enormously large and the conductivity finite. This behavior is unprecedented, and it is natural to ask whether it is an artifact of the HF approximation that does not include the screening of the exchange interactions. Therefore, it is necessary to go beyond the HF theory in the critical conducting regime, and study the behavior of the TDOS when the screening of Coulomb interactions is taken into account. In ordinary disordered metals in zero or weak magnetic fields, the dynamical screening of the Coulomb interactions by the diffusive motion of the electrons is known to be very important.[34–36] It leads to *critical* corrections of the TDOS.[37] The natural question is whether the interplay between quantum diffusion and Coulomb interaction at the IQHT leads to a linear Coulomb gap beyond the HF theory.

This is the focus of the present work. In a recent paper,[29] we reported our findings that the quantum diffusive motion of the electrons, i.e., the diffusive dynamics, is too slow to effectively screen out the Coulomb singularity in the dynamical case. A nonperturbative resummation of the most singular corrections in the long time limit to the TDOS gives rise to a linearly vanishing TDOS for the critical conducting state. This behavior, termed the quantum Coulomb gap, can be thought as the quantum-mechanical analog of the classical Coulomb gap. It has a quantum origin and the slope of the gap is nonuniversal in contrast to the classical case. In this paper, we provide more physical and detailed theoretical derivations of the quantum Coulomb gap. We also study the TDOS behaviors for short-range interactions, both outside the scaling regime where a nonuniversal power-law TDOS is found, and in the scaling regime where the power law is smeared and a finite zero-bias TDOS recovered in accordance with the observation that short-range interactions are irrelevant perturbations in the RG sense. Interestingly, in the scaling regime, the change in the TDOS, $\delta\nu(\varepsilon)$, follows a power law with a universal exponent determined by the scaling dimension of short-range interactions and the frequency exponent $z=2$ in this case. In this paper we also address the issue of whether and how the bulk TDOS depends on the Hall conductance. To this end, we study the case of quasi-1D samples with edges, such as in the long Hall bar geometry, and find that the TDOS becomes dependent of the Hall conductance due to an altered boundary condition for diffusion in a finite magnetic field. It vanishes linearly at the Fermi level with a slope that is inversely proportional to the magnetic-field strength in the perturbative regime, in good qualitative agreement with recent bulk tunneling experiments.[21] These results will be summarized in Sec. I D.

The second issue has to do with the implications of the linear Coulomb gap on dynamical scaling. The linearly vanishing DOS in two dimensions means that the averaged energy-level spacing scales with the length of system according to $\Delta_E \sim 1/L$, leading to a dynamical scaling exponent $z=1$. However, one of the persistent mysteries remains, namely, it is not clear that this is the dynamical exponent measured by the transport experiments. The fact that quantum diffusion exists at the critical point of the transition implies a frequency-dependent length scale $L_\omega \sim [(dn/d\mu)\hbar\omega]^{-1/2}$ that is shorter than the dynamical length scales derived from the single-particle sector. Note that the relevant DOS in L_ω is the thermodynamic DOS or the compressibility $dn/d\mu$.[38] Although it is somewhat unnecessary to associate a critical exponent with diffusion, a value of $z_\omega = 2$ is directly implied and should govern the dynamics of diffusive transport in the asymptotic limit. In a recent attempt to substantiate our previous conjecture made in Ref. 23, Huckestein and Backhaus[26] evaluated the density-density response function near the IQHT within a time-dependent HF approximation (TDHFA), in an effort to determine z_ω from two-particle correlation functions. Their analysis gives $z_\omega = 1$, but, under the compressibility sum rule that relates $dn/d\mu$ to the static limit of the irreducible density response function, it appears to have resulted from using a linearly vanishing $dn/d\mu$ in L_ω. This result is at least counterintuitive, since $dn/d\mu$ is expected to be smooth and finite for a disordered system on general grounds. Moreover, a finite compressibility is necessary for observing the quantum Hall *transition* without the latter being interrupted by incipient quantization plateaus. If $dn/d\mu$ were indeed vanishing, the linear screening length would diverge and the screening properties of the critical state would be similar to those of an insulator. Recently, Yang, Wang, and MacDonald[27] pointed out that the controversial result may be a consequence of not accounting for the consistency of the exchange local fields and the disorder potential in the TDHFA used. Analyzing the charge redistribution following the insertion of an external test charge, they studied the screening properties in the long-wavelength limit of the self-consistent HF theory, and found that the thermodynamic DOS is finite in spite of the linearly vanishing tunneling

DOS in the critical conducting state. Therefore the question of whether or how the vanishing single-particle DOS affects the dynamics of transport near the quantum Hall critical point remains open. We will discuss this issue in more detail in Sec. VI.

The main part of this paper is devoted to understanding how a linearly vanishing TDOS in two dimensions is likely so long as the conductivity is finite. A similar analysis in the case of zero magnetic field was carried out recently by Kopietz[39] in connection to the 2D $B=0$ metal-insulator transition.[40] Our basic finding is that, in the presence of disorder, the Coulomb interaction is insufficiently screened by the quantum diffusive medium at finite frequencies. As a result, the single-particle DOS in the extended regime comes to resemble that in the localized regime, i.e., exhibiting a linear Coulomb gap, although the slope of the gap is different due to a different mechanism. After an understanding of the dynamics in the single-particle sector has been developed, we will turn to the important question of how the depletion of single-particle DOS relates to the larger issue of dynamical scaling near quantum phase transitions in disordered systems.

C. Interplay between disorder and interaction

At roughly the same time as the discovery of the integer quantum Hall effect, there were some remarkable developments in our understanding of quantum transport such as localization and metal-insulator transitions.[41] The weak-localization theory was developed as a perturbative approach to study the effects of disorder and interactions. Early works by Altshuler and Aronov and Altshuler, Aronov, and Lee[34] found several remarkable effects arising from the interplay of interaction and disorder: (1) the electron-electron scattering rate is enhanced due to the prolonged stay of electrons near one another; (2) there is a correction to conductivity comparable to the localization effect caused by quantum interference; and (3) most dramatically, the TDOS is significantly altered from its noninteracting counterpart near the Fermi energy.

For ordinary disordered metals, perturbative diagrammatic calculations show that in three dimensions the weak-localization correction to the TDOS, $\nu(\omega)$, is of the form $\delta\nu\sim\sqrt{\omega}$, a result largely confirmed by experiments in the early 1980s,[34] where ω is measured from the Fermi energy. In two dimensions, for long-range Coulomb interaction,

$$\delta\nu=-\frac{1}{8\pi^2\hbar D}\ln(\omega\tau_0)\ln(\omega\tau_1), \tag{1.2}$$

indicating the possibility of a vanishing $\nu(\omega)$ near the Fermi energy as $\omega\to 0$. In Eq. (1.2), D is the diffusion constant, τ_0 is the elastic scattering time, and, in terms of the inverse screening length $\kappa=2\pi e^2 dn/d\mu$, τ_1 is given by $1/\tau_1=\tau_0(D\kappa^2)^2$. Summations of all logarithmic terms are needed to find the limiting behavior. This was done first by Finkel'stein in a field-theoretic treatment of disorder and interaction.[42] Defining the dimensionless conductivity *in units of* $e^2/\hbar$ via the Einstein relation $\sigma=(dn/d\mu)D$, it was shown that

$$\nu(\omega)=\nu_0 e^{-(1/8\pi^2\sigma)\ln(\omega\tau_0)\ln(\omega\tau_1)}, \tag{1.3}$$

which is valid *only* for $\sigma\gg\ln(1/\omega\tau_0)$ such that the weak localization correction to σ can be ignored. The conductivity is given by $\sigma=\nu_0 D$ where ν_0 is the finite density of states in the self-consistent Born approximation (SCBA). It should be emphasized that Eq. (1.3) does not represent the asymptotic behavior of the TDOS at small bias ω, where the conductivity is strongly renormalized and becomes itself scale dependent.[42]

The behavior of $\nu(\omega)$ in metallic systems should be contrasted to the classical Coulomb gap behavior of the TDOS in disordered insulators. Efros and Shklovskii[32] (ES) showed that, when the long-range Coulomb interaction is unscreened, which is true in dielectric insulators, the single-particle DOS exhibits a universal Coulomb gap behavior,

$$\nu_{\mathrm{ES}}(\omega)=\alpha_d|\omega|^{d-1}/e^{2(d-1)}, \tag{1.4}$$

where $d=3$ and 2 is the dimensionality and α_d is a dimensionless constant of order unity. Thus in the insulating regime, one expects a linearly vanishing Coulomb gap in two dimensions; $\nu_{\mathrm{ES}}=\alpha_2|\omega|$. Since the long-range $1/r$ Coulomb singularity is crucial in the derivation of the classical Coulomb gap, it is only expected to be valid in the strongly localized regime where the screening of the interaction is weak and dielectriclike. In the quantum Hall effect, the latter corresponds to the regions far away from the quantum Hall transitions.

Perhaps less well known is that the same double-logarithmic correction to the TDOS as given in Eq. (1.2) was later derived *in the presence of a strong magnetic field* by Girvin, Jonson, and Lee[35] and Houghton, Senna, and Ying.[36] Diffusion in a strong magnetic field comes from the ''skipping'' of the semiclassical cyclotron orbits caused by impurity scattering.[43] In the SCBA, the diffusion constant in Eq. (1.2) is given by $D=\frac{1}{2}r_c^2\tau_0^{-1}$, where the cyclotron radius $r_c=(2N+1)^{1/2}l_B$, with l_B the magnetic length and N the Landau-level index. Note that in this case, D is proportional to the field-dependent scattering rate $1/\tau_0(B)\simeq[\omega_c/\tau_0(B=0)]^{1/2}$. In this work, we will derive the analog of Eq. (1.3) in strong magnetic fields by a nonperturbative resummation of the double-log divergences. Since the critical conductance is finite and scale invariant at the IQHT, it is possible for us to derive the true asymptotic behavior of the TDOS in the low-bias limit. We show that, in the presence of disorder, Coulomb interaction is insufficiently screened by the 2D quantum diffusive medium at finite frequencies. As a result, the TDOS exhibits a linearly vanishing quantum Coulomb gap behavior.

D. Main results—quantum Coulomb gap in the TDOS

The main results can be explained physically in a simple semiclassical picture. The electron-electron interaction $v(\mathbf{r}-\mathbf{r}')$ can be viewed as being mediated by a fluctuating po-

tential field $\Phi(\mathbf{r})$ with a distribution $P(\Phi) \sim e^{-(1/2)\Phi(\mathbf{r})v^{-1}(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r}')}$. At the crudest level, neglecting all dynamical effects, the presence of a potential field directly changes the energy levels of individual electrons and contributes a phase delay $e^{-i\int_0^\tau d\tau'\Phi[\mathbf{r}_{cl}(\tau')]}$ to the single-electron propagator, where $\mathbf{r}_{cl}(\tau')$ is the classical trajectory. Such a semiclassical phase approximation was recently used in the context of composite fermions coupled to a fluctuating gauge field to study *edge tunneling*.[44,45] Averaging the Φ field over different trajectories as well as the random potentials, we obtain the averaged phase lapse

$$e^{-W(\tau)} = \langle e^{-i\int_0^\tau d\tau'\Phi[\mathbf{r}_{cl}(\tau')]}\rangle \tag{1.5}$$

during a time interval $(0,\tau)$. This phase delay can be viewed as a Debye-Waller factor[42] for the impurity-averaged single-particle Green's function at a (tunneling) site $\mathbf{r}$, $G(\tau) = \langle \psi(\mathbf{r},\tau)\psi^*(r,0)\rangle$,

$$G(\tau) \simeq G_0(\tau)e^{-W(\tau)}, \tag{1.6}$$

where $G_0(\tau)\sim 1/\tau$ is the counterpart of $G(\tau)$ in the absence of interactions. Note that $G(\tau)$ no longer depends on the coordinate $\mathbf{r}$ after impurity averaging. The TDOS is given by

$$\nu(\omega) = -\frac{1}{\pi}\mathrm{Im}\int d\tau e^{i\omega_n\tau}G(\tau)\big|_{i\omega_n\to\omega+i\eta}. \tag{1.7}$$

Two factors, both resulting from the diffusive nature of the electron motion in the presence of disorder, lead to the divergence of the phase delay $W(\tau)$ at large τ, and subsequently to the vanishing of the single-particle DOS at the Fermi-energy: (1) electrons stay longer in the vicinity of one another at each encounter, and (2) the Coulomb potential is not completely screened at finite times. Since we address the effects of interactions only up to a phase delay, this part of the physics presumably can be set aside from the rest by performing a U(1) rotation.[42,29,46]

The specific form of the pseudogap in the TDOS depends on the type of the interaction and on the scaling behavior of the interaction strength. We shall consider both long- and short-range screened Coulomb interactions.

1. Long-range Coulomb interaction

In the case of a long-range Coulomb potential, the phase delay diverges at long times as $W(\tau)\sim \ln(\tau/\tau_0)\ln(\tau/\tau_1)$. We will show in detail that this double-log divergence renders the τ integral over $e^{-W(\tau)}$ convergent, thereby enabling an expansion in ω for $\nu(\omega)$. In the asymptotic low-frequency limit, this leads to a linearly vanishing TDOS at low temperatures:

$$\nu(\omega) = C_Q\hbar|\omega|/e^4. \tag{1.8}$$

We shall refer to Eq. (1.8) as the 2D *quantum Coulomb gap behavior*. In contrast to the 2D classical Coulomb gap behavior given in Eq. (1.4), the coefficient C_Q in the quantum Coulomb gap is not a universal number, but rather a quantity of quantum-mechanical origin. It depends on microscopic details of the sample such as the mobility. For large ω, $\nu(\omega)$ crosses over to the perturbative diagrammatic result in strong magnetic fields.[35,36]

Since Coulomb interaction is a relevant perturbation at the NIFP, the true transition must be governed by an interacting fixed point where the Coulomb interaction strength is finite. Thus we expect that the quantum Coulomb gap to be the true asymptotic behavior of the TDOS at the integer quantum Hall transitions. Note that the Coulomb gap TDOS that we obtained for the critical conducting state at the IQHT is qualitatively different from those obtained in the clean case[47] and in a weak magnetic field.[48]

2. Short-range interactions—prescaling regime

For simplicity, we consider the case of a δ-function interaction potential $v(\mathbf{r}-\mathbf{r}') = u\,\delta(\mathbf{r}-\mathbf{r}')$ as a prototype short-range interacting potential. Outside the scaling regime, the scale dependence of the interaction strength u can be ignored, i.e., u can be treated as a constant or equivalently as a marginal perturbation. In this case, dynamical screening of the interaction leads to a weaker, single-logarithmic divergence in the phase delay $W(\tau)\sim\ln(\tau/\tau_0)$. The integral of the Debye-Waller factor is no longer convergent, such that a power series expansion in ω becomes singular. This is similar to the situation encountered in the x-ray edge problem.[49] We find a pseudogap in the TDOS that takes the form of a power law,

$$\nu(\omega) \simeq \nu_0|\omega\tau_0|^\alpha, \tag{1.9}$$

where the exponent α is nonuniversal and depends on the interaction strength. It is well known that transport at the quantum Hall transition in the noninteracting theory exhibits anomalous diffusion,[50] i.e., the diffusion constant $D = D(q^2/\omega)\sim D_0(q^2\omega)^{\eta/2}$ when $D_0q^2 > \omega$, where η is a critical exponent related to the multifractal dimension $D_2 = 2-\eta$. We will show that taking into account the anomalous diffusion, which has no effect in the Coulomb case, only leads to a weak η dependence in the exponent α in Eq. (1.9).

3. Short-range interactions—scaling regime

Because short-range interactions are irrelevant perturbations at the NIFP,[23] the strength of the effective interaction u must scale to zero in the scaling regime according to $u_{\mathrm{eff}} \sim u\,\omega^{x_+/z}$, where $-x_+ \simeq -0.64$ is the dimension of the interaction and $z=2$ is the dynamic exponent at the stable NIFP. This makes the phase delay $W(\tau)$ converge in the large-τ limit. As a result, the power-law decay in Eq. (1.9) is smeared, resulting in a finite zero-bias TDOS,

$$\nu(\omega) = \nu(0)\left[1+\left(\frac{|\omega|}{\omega_0}\right)^\gamma\right], \tag{1.10}$$

where ω_0 is a frequency scale and $\gamma = x_+/z \simeq 0.32$ is a universal exponent. This result leads to several interesting predictions: (a) For short-range interactions, the TDOS is finite and nonuniversal at zero bias. (b) It can be shown that $\nu(0) \ll \nu_0$ if the bare interaction strength is strong, so short-range interactions irrelevant in the RG sense can still lead to

strong density of states suppression at low bias. (c) Equation (1.10) shows that the TDOS increases with ω according to a *universal* power law with an initial cusp singularity for our value of γ. These predictions can, in principle, be tested experimentally since the Coulomb interaction can be made short ranged by placing a metallic screening gate (ground plane) nearby.

4. Quasi-1D samples with edges

We also study whether and under what condition the bulk TDOS depends on the Hall conductance. That $\nu(\omega)$ in Eqs. (1.8)–(1.10) does not depend on σ_{xy} is a direct consequence of the fact that transverse force does not affect the charge spreading in the bulk of the sample. Thus any direct σ_{xy} dependence in tunneling must come from contributions at the boundary. It was shown by Khmel'nitskii and Yosefin[51] and by Xiong, Read, and Stone[52] that, in the presence of edges, the Hall conductance enters measurable quantities even in the perturbative limit. More recently, Shytov, Levitov, and Halperin studied the problem of *edge* tunneling into the fractional quantum Hall state, where the Hall conductivity dependence of the *I*-*V* characteristics also arises from the boundary condition at the tunneling edge.[45]

We considered a quasi-1D sample with its length L much greater than its width W, and with two reflecting edges. This condition is realized experimentally in the long Hall bar geometry. The edge effect becomes prominent in such a limit, since the boundary condition effectively changes the diffusion constant from D in two dimensions to $D_{1D}=D(1+\gamma_H^2)$, where $\gamma_H=\sigma_{xy}/\sigma_{xx}$ is the Hall ratio. For the case of δ-function interaction, we find that the asymptotic TDOS becomes linear in an infinite strip with edges:

$$\nu(\omega)=s|\omega|. \tag{1.11}$$

The slope of the linear gap is proportional to the inverse of the dissipative resistivity $s\sim\rho_{xx}^{-1}$. In the high-field limit where energy levels form Landau bands ($\omega_c\tau_0\gg 1$), the bare value of σ_{xx} in the SCBA at the center of the Landau levels is proportional to the Landau index N while the bare Hall ratio is of order 1. The above result then implies that, in the perturbative regime where the localization correction to the conductivity is much smaller than the SCBA conductivity, the slope of the linear density of states is proportional to N or $1/B$. Interestingly such a dependence was indeed observed in the high-field bulk tunneling experiments.[21] It remains to be seen whether the samples used in certain experimental setup can be qualified as being quasi-1D with edges, and whether the presence of ground planes indeed make the Coulomb interaction short ranged.

E. Organization of the paper

In Sec. II, we revisit the role of electronic interactions at the NIFP of the quantum Hall transition, and cast the results of Lee and Wang[23] for the RG dimensions of interactions in an analysis of the level spacing in the Hartree-Fock theory. In doing so, we make connections to the more conventional scaling theory of localization *a la* Wegner,[53] and motivate the study of the single-particle DOS. We then proceed, in Sec. III, to formulate the effective-field theory and the semiclassical phase approximation, and to derive the Debye-Waller factor in Eq. (1.6). The bulk TDOS in two dimensions is calculated in Sec. IV for various forms of interactions in the perturbative and the scaling regimes. The results summarized in Eqs (1.8)–(1.10) are derived in this section. The effect of the anomalous diffusion at the IQHT is also studied. Section V is devoted to the derivation of the TDOS in quasi-1D samples in the presence of edges [Eq. (1.11)]. A summary and discussions of the connection between the single-particle DOS and the dynamical scaling of the transition width are given in Sec. VI. We argue that the existence of the quantum Coulomb gap in the quantum critical regime of the transition gives rise, through the interplay between quasiparticle decay rate and level spacing, to the experimentally observed dynamical exponent $z=1$.

II. ROLE OF INTERACTIONS—HARTREE-FOCK ANALYSIS OF LEVEL SPACING

The Hamiltonian of interest describes interacting electrons moving in a two-dimensional random potential in the presence of a magnetic field,

$$H=\sum_i\left[\frac{1}{2m}\left(\mathbf{p}_i+\frac{e}{c}\mathbf{A}\right)^2+V(\mathbf{r}_i)\right]+H_{\rm int}, \tag{2.1}$$

where $\mathbf{A}$ is the external vector potential producing the strong transverse magnetic field, and $V(\mathbf{r})$ is the one-body impurity potential. The interacting part of the Hamiltonian $H_{\rm int}$ is given by the two-body interaction potential:

$$H_{\rm int}=\frac{1}{2}\sum_{i\neq j}v(\mathbf{r}_i-\mathbf{r}_j), \tag{2.2}$$

$$v(\mathbf{r}_i-\mathbf{r}_j)=\frac{u}{|\mathbf{r}_i-\mathbf{r}_j|^p}. \tag{2.3}$$

Here u and p control the strength and the range of the interaction. The Coulomb potential corresponds to $u=e^2/\epsilon$ and $p=1$, and a short-distance cutoff can be introduced for the case of $p\geqslant 4$.

The noninteracting fixed point of the IQHT is obtained by setting $u=0$. The effects of interactions can be studied in the framework of critical phenomena by analyzing the stability of the NIFP. Imagine starting with a system at the NIFP, and adiabatically switching on the interaction u. One can ask whether u is a relevant or irrelevant perturbation in the renormalization-group sense by calculating the RG scaling dimension of u. This has been done by Lee and Wang.[23] They analyzed the most singular contributions to the disordered averaged free-energy functional. Here we present an alternative treatment[54] from the point of view of the single-particle DOS, and cast the result of Lee and Wang in terms of the interaction induced corrections to the single-particle level spacing.

For this purpose, it is convenient to use the exact eigenstates approach[55] at criticality. Consider a finite system of

linear dimension L, and two adjacent one-electron eigenstates located on the two sides of the critical energy E_c with energy $E_1<E_c$ and $E_2>E_c$ and a separation $\omega=E_1-E_2$. In the noninteracting theory, the finite-size scaling behavior of the separation should follow that of the mean level spacing and scale with L according to

$$\Delta_{12}^0=\frac{1}{\nu_0 L^d}\propto\frac{1}{L^2}, \tag{2.4}$$

where ν_0 is the noncritical DOS in the noninteracting theory. Switching on the interaction u causes a mixing of the critical eigenstates, which results in shifting of the single-particle energy levels. The level spacing becomes

$$\Delta_{12}=\Delta_{12}^0+\delta\Delta_{12}, \tag{2.5}$$

where $\delta\Delta_{12}$ is the level shift of E_2 due to the mixing with level E_1. The question we would like to ask is, in the limit $\omega\to 0$ and the associated length scale $L_\omega=(\nu_0\omega)^{-1/d}\to\infty$, how the interaction correction to the level spacing $\delta\Delta_{12}$ scales with L as we approach the thermodynamic limit $L<L_\omega\to\infty$. If it falls off faster than the mean level spacing $\Delta_{12}^0\sim 1/L^2$ of the noninteracting system, the level statistics will be determined by that of the noninteracting eigenstates, and unaffected by the interaction in the thermodynamic limit. Thus the interaction would be an irrelevant perturbation in the RG sense. On the other hand, if $\delta\Delta_{12}$ falls off slower than $1/L^2$, the level spacing will be dominated by the interaction-induced level shifts as an increasing number of noninteracting eigenstates is mixed by the interaction with increasing system size L. The interaction would therefore be a relevant perturbation in this case, and the noninteracting fixed point would be unstable.

Let's calculate $\delta\Delta_{12}$ perturbatively, which is sufficient for the perturbative RG analysis. To first order in perturbation theory we have

$$\delta\Delta_{12}=P_1\langle 2|H_{\rm int}|2\rangle, \tag{2.6}$$

where P_1 is a projection operator that keeps exclusively the contribution from state $|1\rangle$ to the level shift of $|2\rangle$. The factorized interaction $H_{\rm int}$ can be written as

$$H_{\rm int}=\sum_\alpha \Sigma_\alpha^{HF} c_\alpha^\dagger c_\alpha, \tag{2.7}$$

where Σ_α^{HF} is precisely the Hartree-Fock self-energy correction to the single particle state $|\alpha\rangle=c_\alpha^\dagger|0\rangle$,

$$\Sigma_\alpha^{HF}=\sum_\beta^{\rm occ.}\int d^2\mathbf{r}d^2\mathbf{r}'[|\psi_\alpha(r)|^2|\psi_\beta(r')|^2 - \psi_\alpha(r)^*\psi_\alpha(r')\psi_\beta^*(r')\psi_\beta(r)]v(\mathbf{r}-\mathbf{r}'). \tag{2.8}$$

Here the summation is over all occupied states. To discuss the average energy shift, it is necessary to study the disorder average of the self-energy at the fixed energy E_2,

$$\Sigma_2=\frac{1}{\nu_0 L^2}\sum_\alpha \delta(E_2-E_\alpha)\Sigma_\alpha, \tag{2.9}$$

where ν_0 is the DOS per unit area for noninteracting electrons. Taking the disorder average of Eq. (2.6) and using Eqs. (2.7)–(2.9), we obtain the averaged level shifts,

$$\overline{\delta\Delta_{12}}=\left(\frac{1}{\nu_0 L^2}\right)^2\int d^2\mathbf{r}d^2\mathbf{r}'\overline{\sum_{\alpha\beta}[|\psi_\alpha(r)|^2|\psi_\beta(r')|^2-\psi_\alpha(r)^*\psi_\alpha(r')\psi_\beta^*(r')\psi_\beta(r)]\delta(E_1-E_2-\omega)v(\mathbf{r}-\mathbf{r}')}$$
$$=\left(\frac{1}{\nu_0 L^2}\right)^2\int d^2\mathbf{r}d^2\mathbf{r}'[O_A(\mathbf{r}-\mathbf{r}')-O_B(\mathbf{r}-\mathbf{r}')]v(\mathbf{r}-\mathbf{r}'). \tag{2.10}$$

In order calculate this quantity, we need to know the scaling behavior of the impurity-averaged products of four wave functions denoted by O_A and O_B in Eq. (2.10) in the limit $\omega\to 0$. The latter are functions of $\mathbf{r}-\mathbf{r}'$, since the translation symmetry is restored after impurity averaging.

It is instructive to follow Wegner's symmetry decomposition[53,56,57] and extract the orthogonal (eigen) scaling variables under the RG. Consider the most general four-field operator in the unitary ensemble,

$$O_4=\sum_{\alpha\beta\gamma\delta} v_{\beta\delta}^{\alpha\gamma}\psi_\alpha^*\psi_\beta\psi_\gamma^*\psi_\delta\equiv\sum_{\alpha\beta\gamma\delta} v_{\beta\delta}^{\alpha\gamma}O_{\alpha\gamma}^{\beta\delta}, \tag{2.11}$$

where $O_{\alpha\gamma}^{\beta\delta}=O_{\gamma\alpha}^{\delta\beta}$. The coefficients v obeys the traceless condition, corresponding to the subtraction of vacuum expectations:

$$\sum_i v_{i\delta}^{i\gamma}=\sum_j v_{\beta j}^{j\gamma}=0. \tag{2.12}$$

There are two irreducible representations for these operators, a symmetric one and an antisymmetric one, under permutations of indices of the rank-2 tensor. We can therefore decompose O_4 into independent scaling operators,

$$O_{\alpha\gamma}^{\beta\delta}=\tfrac{1}{2}O_+ +\tfrac{1}{2}O_-, \quad O_\pm=O_{\alpha\gamma}^{\beta\delta}\pm O_{\alpha\gamma}^{\delta\beta}. \tag{2.13}$$

The operators O_+ and O_-, having independent scaling dimensions $x_\pm$, describe the eigenscaling directions of the four-field operators under the RG. In the unitary universality class of the metal-insulator transition, i.e., the cases of weak

magnetic field and spin-flip scattering by magnetic impurities, $x_{\pm}=\pm\sqrt{2\epsilon}$ has been derived from perturbation theory in $2+\epsilon$ dimensions.[56]

At the IQHT, the RG dimensions of $O_{\pm}$ were determined numerically by Lee and Wang,[58,23] from the leading scaling operators associated with the fusion products of four fermion operators that are antisymmetric and symmetric under permutations, respectively. The scaling dimension of O_- is obtained from the product of two ''spin'' operators[58]

$$x_-=x_{2s}=-0.60\pm0.02, \tag{2.14}$$

whereas that of O_+ is extracted from the leading scaling operator fused by the product of two nearby density operators:[23]

$$x_+=x_{2\rho}=0.65\pm0.04. \tag{2.15}$$

A general four-field operator involves contributions from both O_- and O_+, but its leading scaling behavior will be dominated by that of O_-, since O_- is much more relevant than O_+. A good example[58] is the ensemble averaged inverse participation ratio introduced by Wegner[53] $P^{(2)}$. Expressed in terms of a four-field operator, its scaling dimension, which is also known as the multifractal dimension of the eigenstates $D(2)$, is governed by x_-, i.e., $D(2)=d+x_-=1.4\pm0.02$, indicating strong amplitude fluctuations of the critical eigenstates at plateau transitions. Indeed, the exponent η used by Chalker and Daniel[50] to describe the anomalous diffusion at the IQHT is given by $\eta=-x_-$. The scatter in the value of x_- is most likely due to uncertainties involved in different numerical approaches. By the same token, one can show that the scaling behavior of the ultrasonic attenuation, extensively studied in $2+\epsilon$ dimensions near the conventional metal-insulator transitions,[59] is controlled by O_- as well.

Now let us apply these results to the interaction-induced level shifts in Eq. (2.10). We will show that the density-density correlation is, in contrast to the inverse participation ratio and the anomalous diffusion coefficient, controlled by the symmetric operator O_+ with the scaling dimension x_+. Individually, operators O_A and O_B contain contributions from both O_- and O_+. Their leading scaling behavior is therefore dictated by that of the operator O_-. We have

$$O_A(\mathbf{r}-\mathbf{r}')\sim O_B(\mathbf{r}-\mathbf{r}')\sim\left(\frac{|r-r'|}{L}\right)^{x_-} \tag{2.16}$$

for $|r-r'/L\ll L_\omega$. However, the combination of O_A-O_B has precisely the symmetry of the symmetric operator O_+. Thus

$$O_A(\mathbf{r}-\mathbf{r}')-O_B(\mathbf{r}-\mathbf{r}')\sim\left(\frac{|r-r'|}{L}\right)^{x_+}. \tag{2.17}$$

We see that the scaling behavior of the interacting-induced level shift is determined by the fusion product of two density operators in the symmetric representation. The absence of pure powers of $|r-r'|$ in Eqs. (2.16) and (2.17) comes from the fact that bilinear field operators have dimensions

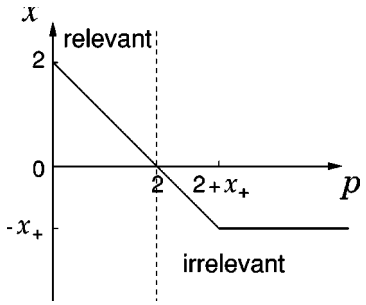

FIG. 1. The scaling dimension x of the interaction strength u in a $1/r^p$ potential as a function of p. The interaction is relevant for $x>0$ and irrelevant for $x<0$ in the RG sense.

zero,[58,23] consistent with the single-particle DOS being noncritical at the noninteracting critical point.

Substituting Eq. (2.17) into Eq. (2.10) and carring out the spatial integrals from a lattice cutoff a to the system size L, we obtain

$$\overline{\delta\Delta_{12}}=\frac{u}{L^p}\left[c_1+c_2\left(\frac{a}{L}\right)^{2+x_+-p}\right], \tag{2.18}$$

where c_1 and c_2 are nonuniversal constants. Note that for $p>2+x_+$, the integral depends on the lower cutoff and the second term in Eq. (2.18) diverges as $a\to0$ which must be absorbed into the renormalized interaction. We are now ready to determine the relevance of the interactions by comparing the scale dependence of $\overline{\delta\Delta_{12}}$ to the mean level spacing of the noninteracting system for large L. Defining the scaling dimension of the interaction u according to

$$x=\frac{d}{d\ln L}\left[\log\left(\frac{\overline{\delta\Delta_{12}}}{\overline{\Delta_{12}^0}}\right)\right], \tag{2.19}$$

we obtain, using Eq. (2.18) and $\overline{\Delta_{12}^0}\sim L^{-2}$,

$$x=\max(2-p,-x_+). \tag{2.20}$$

The behavior of x is shown in Fig. 1. For $p<2$, the interactions acquire a RG scaling dimension $x=2-p>0$ and are relevant. We can refer to these types of interactions as long-range interactions.[23] In this case, the interaction-induced level shift becomes much larger than the mean level spacing of the critical eigenstates in the noninteracting theory for large system sizes L. For the Coulomb interaction, $p=1$, u has a RG scaling dimension $x_{\text{Coul.}}=1$, and is therefore a relevant perturbation. The resulting flow away from the NIFP will lead to an interacting fixed point at which the effective interaction strength is finite. Presumably, the simplest version of the latter is the Hartree-Fock fixed point discussed in Sec. I. At the level of the Hartree-Fock theory, Eq. (2.18) shows that the level spacing is entirely dominated by the Coulomb interaction-induced level shift that scales as L^{-1} in the thermodynamic limit, consistent with the linear Coulomb gap DOS found in numerical calculations.[24]

On the other hand, for all values of $p>2$, we have $x<0$. The interactions are irrelevant, and can be referred to as short-range interactions. The dipole-dipole interaction, in

particular, having $p=3$, belongs to this class of interactions. The NIFP is therefore stable against short-range interactions. For screened Coulomb interaction with $2<p<2+x_+$ the RG dimension of u is $x=2-p$, while for $p>2+x_+$ it is $x=-x_+$. In both cases the interaction scales to zero at the transition in the asymptotic limit, although it controls the finite-temperature behavior of the conductances.[12] From Eq. (2.18), it is clear that the interaction-induced mixing between the critical eigenstates of the noninteracting theory only leads to level shifts that are much smaller than the mean level spacing in the thermodynamic limit. Thus the zero-bias DOS must be finite in the asymptotic limit for short-range interactions. However, as we will demonstrate later in this paper, even in this case the interactions lead to remarkable properties of the TDOS in the prescaling regime, which may have important experimental consequences.

III. FIELD THEORY FRAMEWORK AND SEMICLASSICAL APPROXIMATION

To include the screening of the Coulomb interactions by the diffusive electrons in the calculation of the TDOS beyond the Hartree-Fock theory, we will set up, in this section, the effective field theory and the semiclassical approximation in order to derive the Debye-Waller factor in Eq. (1.6).

A. General formalism

We consider the action for interacting electrons in a random potential and a magnetic field described by the Hamiltonian in Eq. (2.1),

$$S=\int_0^\beta d\tau d^2\mathbf{r}\mathcal{L}, \tag{3.1}$$

where

$$\mathcal{L}=\psi^*[\partial_\tau+H_0+V(\mathbf{r})]\psi + \frac{1}{2}\int d^2\mathbf{r}\psi^*(\mathbf{r})\psi(\mathbf{r})v(\mathbf{r},\mathbf{r}')\psi^*(\mathbf{r}')\psi(\mathbf{r}'). \tag{3.2}$$

Here, once again, $V(\mathbf{r})$ is the random external potential, $v(\mathbf{r}-\mathbf{r}')$ is the electron-electron interaction, and $H_0=(1/2m)(\partial_i+ieA_i)^2$, with $\mathbf{A}(\mathbf{r})$ being the vector potential of a uniform external magnetic field perpendicular to the 2D plane. In Eq. (3.2), ψ and ψ^* are independent Grassmann fields. The electron single-particle Green's function is defined as

$$G(\mathbf{r},\mathbf{r},\tau)=\mathcal{Z}^{-1}\int D[\psi^*]D[\psi]\psi(\mathbf{r},0)\psi^*(\mathbf{r},\tau)e^{-S}, \tag{3.3}$$

where $\mathcal{Z}$ is the partition function expressed in terms of the imaginary-time path integral:

$$\mathcal{Z}=\int D[\psi^*]D[\psi]e^{-S}. \tag{3.4}$$

The interaction between the electrons can be viewed as being mediated by a fluctuating scalar field Φ in the Coulomb gauge, for one can always rewrite the interaction term in the following way:

$$e^{-(1/2)\int d^2\mathbf{r}\int d^2\mathbf{r}\psi^*(\mathbf{r})\psi(\mathbf{r})v(\mathbf{r},\mathbf{r}')\psi^*(\mathbf{r}')\psi(\mathbf{r}')} = \int D[\Phi] \times e^{i\int d^2\mathbf{r}\psi^*(\mathbf{r})\Phi(\mathbf{r})\psi(\mathbf{r})-(1/2)\int d^2\mathbf{r}\int d^2\mathbf{r}'\Phi(\mathbf{r})v^{-1}(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r}')}. \tag{3.5}$$

To perform the averaging over random potentials, we use the replica trick, calculate $\mathcal{Z}^n$, where n is the number of replicas, and take the limit $n\to 0$ at the end. The ensemble-averaged single-particle Green's function can be obtained according to

$$\langle G(\tau)\rangle=\lim_{n\to 0}\int D[\Phi]\int D[V]P[V] \times\int D[\psi^*]D[\psi]\psi_{\alpha_1}(\mathbf{r},0)\psi^*_{\alpha_1}(\mathbf{r},\tau) \times\exp\left\{-\int_0^\beta d\tau\sum_{\alpha=0}^{n}\left[\int d^2\mathbf{r}\psi^*_\alpha(\mathbf{r},\tau)[\partial_\tau+H_0+V -i\Phi_\alpha(\mathbf{r},\tau)]\psi_\alpha(\mathbf{r},\tau)+\int d\mathbf{r} \times\int d\mathbf{r}'\Phi_\alpha(\mathbf{r},\tau)v^{-1}(\mathbf{r}-\mathbf{r}')\Phi_\alpha(\mathbf{r}',\tau)\right]\right\}. \tag{3.6}$$

In the above equation, $P[V]$ is the distribution of the random potential which is taken to be Gaussian white noise: $P[V]\sim e^{-(1/2g)V^2(\mathbf{r})}$ for the short-range correlated impurities, α is the replica index, and α_1 represents an arbitrary replica channel. As in the usual treatment of disordered systems, integrating over $V(\mathbf{r})$ in Eq. (3.6) introduces a four-point interaction term in the action that is local in space but nonlocal in time, $(g/2)\Sigma_\alpha\Sigma_{\alpha'}\int d^2\mathbf{r}\int_0^\beta d\tau\int_0^\beta d\tau'|\psi_\alpha(\mathbf{r},\tau)|^2|\psi_{\alpha'}(\mathbf{r},\tau')|^2$. The latter is usually decoupled by introducing an auxiliary field $Q_{\alpha,\alpha'}(\tau,\tau')$ by way of the Hubbard-Stratonovic transformation:

$$e^{(g/2)\int d^2\mathbf{r}\int_0^\beta d\tau\int_0^\beta d\tau'|\psi_\alpha(\mathbf{r},\tau)|^2|\psi_{\alpha'}(\mathbf{r},\tau')|^2}=\int D[Q]e^{-(1/2g)\int d^2\mathbf{r}\int_0^\beta d\tau\int_0^\beta d\tau' Q_{\alpha\alpha'}(\tau,\tau')Q_{\alpha'\alpha}(\tau',\tau)+i\int d^2\mathbf{r}\int_0^\beta d\tau\int_0^\beta d\tau'\psi^*_\alpha(\tau)Q_{\alpha\alpha'}(\tau,\tau')\psi_{\alpha'}(\tau')}. \tag{3.7}$$

This quench-averaging process leads to the following replicated action:

$$S(\psi^*,\psi,Q,\Phi)=\int d^2\mathbf{r}\int_0^\beta d\tau\Bigg[\int_0^\beta d\tau'\sum_{\alpha\alpha'}\Bigg\{\psi^*_\alpha(\mathbf{r},\tau)\{[\partial_\tau+H_0-\Phi_\alpha(\tau)]\delta(\tau-\tau')\delta_{\alpha\alpha'}-iQ_{\alpha,\alpha'}(\tau,\tau')\}\psi_{\alpha'}(\mathbf{r},\tau')+\frac{1}{2g}Q_{\alpha\alpha'}(\tau,\tau')Q_{\alpha'\alpha}(\tau'\tau)\Bigg\}+\frac{1}{2}\sum_\alpha\times\int d^2\mathbf{r}'\Phi_\alpha(\mathbf{r},\tau)v^{-1}(\mathbf{r},\mathbf{r}')\Phi_\alpha(\mathbf{r}',\tau)\Bigg]. \tag{3.8}$$

The impurity-averaged Green's function in Eq. (3.6) becomes

$$\langle G(\tau)\rangle=\lim_{n\to 0}\int D[\Phi]\int D[Q]\int D[\psi^*]D[\psi]\psi_{\alpha_1}(\mathbf{r},0)\times\psi^*_{\alpha_1}(\mathbf{r},\tau)e^{-S(\psi^*,\psi,Q,\Phi)}. \tag{3.9}$$

The rest of this section proceeds as follows: (1) We integrate out the Q field as well as the ψ field, and derive an effective action in terms of the Φ field: $S_{\rm eff}(\Phi)$. The Green's function $\langle G(\mathbf{r},\mathbf{r}';\tau)\rangle$ can be expressed in terms of the averaged noninteracting electron Green's function in the presence of the fluctuating potential Φ, which we denote by $\overline{G}(\Phi)$, weighted by $e^{-S_{\rm eff}(\Phi)}$. The effective action $S_{\rm eff}$ can be obtained systematically in a power series of Φ and $1/\sigma_{xx}$. (2) By appealing to the semiclassical approximation for the slowly varying part of the Φ field, we argue that $\overline{G}(\Phi)\approx\overline{G}(0)\exp(-i\int_0^\beta d\tau\int d^2\mathbf{r}\Phi\rho)$, where $\rho(\mathbf{r})$ is the diffusion propagator. (3) Keeping in $S_{\rm eff}$ up to quadratic terms in Φ^2 and integrating out the Φ field, we arrive at $\langle G\rangle\sim\overline{G}(0)e^{-W(\tau)}$.

B. Effective action

Let us define the effective action by formally integrating out the Q and the ψ field:

$$\mathcal{Z}[\Phi]=e^{-S_{\rm eff}(\Phi)}=\int D[Q]\int D[\psi^*]D[\psi]e^{-S(\psi^*,\psi,Q,\Phi)}. \tag{3.10}$$

Carrying out the functional integral explicitly over the ψ field, we obtain

$$\mathcal{Z}[\Phi]=e^{-S_{\rm eff}(\Phi)}=\int D[Q]e^{-S_Q(\Phi)-S_2^{(0)}(\Phi)}, \tag{3.11}$$

where

$$S_Q(\Phi)=-\frac{1}{2g}{\rm Tr}\,Q^2+{\rm Tr}\ln[i\omega-H_0+i\Phi+iQ], \tag{3.12}$$

$$S_2^{(0)}(\Phi)=\frac{1}{2}{\rm Tr}\,\Phi(\mathbf{r})v^{-1}(\mathbf{r},\mathbf{r}')\Phi(\mathbf{r}'). \tag{3.13}$$

In Eq. (3.12), ω is the fermion Matsubara frequency, and "Tr" stands for the trace over the replica, spatial, and Matsubara indices. Expanding ${\rm Tr}\ln[i\omega-H_0+i\Phi+iQ]$ in power series of Φ, we obtain

$$S_Q=-\frac{1}{2g}{\rm Tr}\,Q^2+{\rm Tr}\ln[i\omega-H_0+iQ]+{\rm Tr}\left(\frac{1}{i\omega-H_0+iQ}i\Phi\right)-\frac{1}{2}{\rm Tr}\left(\frac{1}{i\omega-H_0+iQ}i\Phi\frac{1}{i\omega-H_0+iQ}i\Phi\right)+\mathcal{O}(\Phi^3). \tag{3.14}$$

We group the terms in $S_Q(\Phi)$, which are zeroth order in Φ as S_σ,

$$S_\sigma=-\frac{1}{2g}{\rm Tr}\,Q^2+{\rm Tr}\ln[i\omega-H_0+iQ], \tag{3.15}$$

the terms linear in Φ as $\Delta S_1(\Phi)$,

$$\Delta S_1={\rm Tr}\left[\frac{1}{i\omega-H_0+iQ}i\Phi\right], \tag{3.16}$$

and the Φ^2 term as $\Delta S_2(\Phi)$,

$$\Delta S_2=-\frac{1}{2}{\rm Tr}\left[\frac{1}{i\omega-H_0+iQ}i\Phi\frac{1}{i\omega-H_0+iQ}i\Phi\right]. \tag{3.17}$$

Clearly, S_σ is nothing but the transport action for electrons in a random potential in the absence of the Coulomb interaction. The standard procedure[42] is to expand around the saddle point of S_σ. The self-consistent saddle-point equation is given by

$$iQ(\mathbf{r})=-g\left\langle\mathbf{r}\left|\frac{1}{i\omega-H_0+iQ}\right|\mathbf{r}\right\rangle. \tag{3.18}$$

The saddle-point solution is given by

$$i[Q_0]^{mn}_{\alpha\beta}=q_0\delta_{\alpha\beta}\delta_{mn}+\frac{i}{2\tau_0}\delta_{mn}\delta_{\alpha\beta}{\rm sgn}(n), \tag{3.19}$$

where τ_0 is the elastic scattering time. In general τ_0 depends on the magnetic field B. In the weak-magnetic-field limit, the Landau levels overlap due to disorder broadening and $\tau_0(B\to 0)=1/2\pi\nu_0 g$, where ν_0 is the density of states. In the strong-field limit $\omega_c\tau_0(0)\gg 1$, where ω_c is the cyclotron frequency. The diffusion comes from the "skipping" of the semi-classical cyclotron orbits caused by impurity scattering. One must solve Eq. (3.18) in the presence of quantized Landau levels. Such a solution renders the famous semicircle density of states by Ando and co-workers,[43] usually referred to as the result of the SCBA. Let us denote the saddle-point Green's function as

$$[G_{sp}]^{mn}_{\alpha\beta}=\left\langle \mathbf{r}\left|\frac{1}{i\omega-H_0+iQ_0}\right|\mathbf{r}\right\rangle \delta_{\alpha\beta}\delta_{mn}, \qquad (3.20)$$

which is also called the SCBA Green's function. Using G_{sp} one can calculate the bare parameters of the theory, i.e., the transport coefficients σ_{xx} and σ_{xy} at the level of the SCBA. Without going into the details of the SCBA calculations, which can be found in Refs. 43, 35, 36, and 52, we point out the following key results: (1) In the strong-field limit the elastic-scattering time τ_0 is of order $\sqrt{\tau_0(0)/\omega_c}$, the mean free path $l=v_f\tau_0$ becomes the cyclotron radius R_c, and the diffusion constant $D=\frac{1}{2}R_c^2/\tau_0$ depends on the magnetic field. (2) σ_{xx} in the center of the Nth Landau level is approximately given by $\sigma_{xx}^{SCBA}\simeq(N+1/2)e^2/h$; therefore, the perturbative expansion in $1/\sigma_{xx}$ is valid as long as $N>1$.

The single-particle Green's function in Eq. (3.18), and hence the saddle-point solution for Q has, quite generally, a branch cut at $\omega=0$. Taking this into account, the saddle-point solution assumes the generic form $iQ_0=q_0+(1/2\tau_0)\Lambda$, where Λ is a diagonal matrix, $\Lambda^{nm}_{\alpha\beta}=\delta_{nm}\delta_{\alpha\beta}\mathrm{sgn}(n)$, in the space spanned by the replica (α,β) and the Matsubara frequency (n,m) indices. The nonlinear σ model is obtained by including the gapless, long-distance fluctuations around the saddle-point manifold of the form

$$\tilde{Q}(\mathbf{r})=U^{-1}(\mathbf{r})Q_0U(\mathbf{r}), \qquad (3.21)$$

where U is a unitary matrix $U\in U(M)$, and M is the product of the number of replicas and that of the frequencies. Defining a dimensionless matrix field

$$Q(\mathbf{r})=U^{-1}(\mathbf{r})\Lambda U(\mathbf{r}), \qquad (3.22)$$

it is straightforward to show that

$$\Delta S_1=\pi\nu_0\mathrm{Tr}(Q\Phi)+i\frac{1}{2}\mathrm{Tr}[(G_{sp}+G^*_{sp})\Phi]+\mathcal{O}(Q\nabla Q), \qquad (3.23)$$

$$\Delta S_2=\pi\nu_0\frac{\tau_0}{\hbar}\mathrm{Tr}(\Phi^2-Q\Phi Q\Phi)+\frac{1}{8}\mathrm{Tr}[(G_{sp}+G^*_{sp})\Phi(G_{sp}+G^*_{sp})\Phi]+\mathcal{O}(Q\nabla Q). \qquad (3.24)$$

It is now convenient to denote the quantum average over Q under the statistical weight $e^{-S_\sigma(Q)}$ by $\langle\cdots\rangle_\sigma$. From Eqs. (3.11)–(3.17) we obtain the effective action

$$S_{\mathrm{eff}}(\Phi)=\langle\Delta S_1\rangle_\sigma+\langle\Delta S_2\rangle_\sigma-\frac{1}{2}[\langle(\Delta S_1)^2\rangle_\sigma-\langle\Delta S_1\rangle^2_\sigma]+S_2^{(0)}(\Phi)+\mathcal{O}(\Phi^3) \qquad (3.25)$$

The second and third Φ^2 terms in Eq. (3.25) define the (density) polarization function Π:

$$\langle\Delta S_2\rangle_\sigma-\frac{1}{2}[\langle\Delta S_1^2\rangle_\sigma-\langle\Delta S_1\rangle^2_\sigma]=\frac{1}{2}\sum_n\int d^2\mathbf{r}\int d\mathbf{r}'\,\Pi(\mathbf{r},\mathbf{r}';\omega_n)\Phi_n(\mathbf{r})\Phi_{-n}(\mathbf{r}'). \qquad (3.26)$$

The polarization function can be calculated in power series of $1/\sigma_{xx}$. To leading order in $1/\sigma_{xx}$, we recover the result of the ladder approximation:

$$\Pi(\mathbf{q},i\omega_n)=\nu_0\frac{Dq^2}{Dq^2+|\omega_n|}. \qquad (3.27)$$

Higher-order interference corrections presumably renormalizes the diffusion constant D, and the thermodynamic DOS $\nu_0\to dn/d\mu$ in Eq. (3.27). Thus we have derived the effective action to order Φ^2:

$$S_{\mathrm{eff}}\approx\frac{T}{2}\sum_n\Phi_n(\mathbf{r})[v^{-1}(\mathbf{r}-\mathbf{r}')+\Pi(\mathbf{r}-\mathbf{r}';i\omega_n)]\Phi_n(\mathbf{r}'). \qquad (3.28)$$

The scalar field Φ precisely mediates the diffusion-screened electron-electron Coulomb interaction.[34]

C. Semiclassical phase approximation

Now we turn to the evaluation of the impurity averaged single-particle Green's function given in Eq. (3.9). By a simple reordering of functional integrals,[44,29] we have

$$\langle G(\mathbf{r},\mathbf{r},\tau)\rangle=\int D[\Phi]\bar{G}(\mathbf{r},\mathbf{r},\tau,\Phi)e^{-S_{\mathrm{eff}}(\Phi)}, \qquad (3.29)$$

where $\bar{G}$ is the averaged Green's function in a fixed configuration of the scalar potential Φ:

$$\bar{G}(\mathbf{r},\mathbf{r},\Phi)=\lim_{n\to0}\int D[Q]\times\int D[\bar{\psi}]D[\psi]\psi_{\alpha_1}\psi^*_{\alpha_1}e^{-S(\psi^*,\psi,Q,\Phi)}/\mathcal{Z}[\Phi]. \qquad (3.30)$$

We now make an approximation regarding $\bar{G}(\Phi)$ that takes into account exclusively the important interference effects between the phases of the electron wave functions. The amplitude fluctuations are small for the slowly varying fluctuations of the Φ field that dominate the contributions to the effective action in Eq. (3.28). Since these fluctuations are spatially smooth on the scale of the elastic mean free path l, i.e., $\nabla\Phi l/E_f\ll1$, they do not significantly alter the classical trajectory of the electrons. This is a unique feature of the slow diffusive dynamics of the electrons in a random media. Appealing now to the semiclassical approximation, the single-electron propagator in the presence of interactions is modified by a U(1) phase factor,

$$\bar{G}(\mathbf{r},\mathbf{r};\tau;\Phi)\approx\bar{G}(\mathbf{r},\mathbf{r};\tau;0)e^{-\Delta S_{cl}(\Phi)}, \qquad (3.31)$$

where ΔS_{cl} is the change of the action caused by sampling the potential Φ along the classical path,

$$\Delta S_{cl}(\Phi) = -i \int_0^\tau d\tau' \Phi[\mathbf{r}'_{cl}(\tau'), \tau'], \tag{3.32}$$

with $\mathbf{r}'_{cl}(\tau')$ being the classical trajectory that starts and ends at $\mathbf{r}$ in the presence of random potential but in the absence of the Φ field. Upon averaging over the random potentials, the classical trajectory can be described by a random walk. Let $\rho(\mathbf{r},\tau)$ be the probability of a particle being at $\mathbf{r}$ at time τ:

$$\Phi[\mathbf{r}'_{cl}(\tau'), \tau'] = \int d\mathbf{r}' \rho(\mathbf{r}', \tau') \Phi(\mathbf{r}', \tau'). \tag{3.33}$$

Since the critical conductivity is finite at the IQHT, the charge spreading is expected to be described by (anomalous) diffusion. The probability density ρ then satisfies the diffusion equation

$$[-D(\nabla')^2 + \partial_{\tau'}]\rho(\mathbf{r}', \tau') = [\delta(\tau') - \delta(\tau' - \tau)]\delta(\mathbf{r}' - \mathbf{r}), \tag{3.34}$$

where the δ functions on the right-hand side result from the boundary conditions imposed on the original trajectory and correspond to injecting an electron at $\mathbf{r}$ and time 0 and removing it at time τ. The associated current density is given by

$$\mathbf{J} = -D(\nabla - \gamma_H \hat{z} \times \nabla)\rho, \tag{3.35}$$

where $\gamma_H = \sigma_{xy}/\sigma_{xx}$ is the Hall ratio. Note that γ_H does not enter diffusion equation (3.34) because the transverse force does not affect the charge spreading which is described by $\nabla \cdot \mathbf{J}$ in the continuity equation. Solving Eq. (3.34) *in the bulk* of systems without edges, we find,

$$\rho(q, i\omega_n) = \frac{1 - e^{i\omega_n \tau}}{Dq^2 + |\omega_n|}. \tag{3.36}$$

Later we will show that γ_H does enter in the presence physical edges. In this case, the diffusion equation must be solved with the appropriate spatial boundary conditions.

Inserting the results of Eqs. (3.36), (3.32), and (3.33) into Eq. (3.31), we have

$$\overline{G}(\mathbf{r}, \mathbf{r}; \tau; \Phi) \approx \overline{G}(\mathbf{r}, \mathbf{r}; \tau; 0) e^{i \int_0^\tau d\tau' \int d\mathbf{r}' \Phi(\mathbf{r}', \tau') \rho(\mathbf{r}', \tau')}. \tag{3.37}$$

Note that the above is but a special case of the more general phase approximation in the presence of U(1) gauge fields.[29,44,45] The quantum interference effects can be included by the renormalization of the diffusion constant D and other parameters of the theory. In fact, it was argued recently in Ref. 46 that the phase approximation of Eq. (3.37), along with the effective action of the screened potential of Eq. (3.28) can be derived by seeking a temporally and spatially varying saddle-point solution $Q_0(\mathbf{r}, \tau, \Phi)$ of the action $S_Q(\Phi)$ for each $\Phi(\mathbf{r}, \tau)$. Quantum interference can be treated systematically by considering fluctuations around such saddle-point solutions. We note in passing that, at criticality, another complication arises: due to the multifractal behavior of the critical eigenstates, the diffusion is anomalous, i.e., D becomes dependent on the length and the time scales. These subtleties will be addressed in later sections.

The final step is to substitute Eq. (3.37) into Eq. (3.29) and carry out the functional integral. Taking into account the Gaussian fluctuations in Φ captured by the effective action in Eq. (3.28), we obtain the central result

$$\langle G(\mathbf{r}, \mathbf{r}; \tau) \rangle \approx \overline{G}(\mathbf{r}, \mathbf{r}; \tau; 0) e^{-W(\tau)}, \tag{3.38}$$

where the Debye-Waller phase-delay factor is

$$W(\tau) = \frac{T}{2} \sum_n \int \frac{d^2q}{(2\pi)^2} \rho(\mathbf{q}, i\omega_n) v_{\mathrm{sc}}(q, i\omega_n) \rho(-\mathbf{q}, -i\omega_n), \tag{3.39}$$

and v_{sc} is the dynamically screened interaction implied in the effective action in Eq. (3.28):

$$v_{\mathrm{sc}}(\mathbf{q}, i\omega_n) = \frac{v(q)}{1 + v(q)\Pi(q, \omega_n)} = \frac{v(q)}{1 + \dfrac{dn}{d\mu} v(q) \dfrac{Dq^2}{Dq^2 + |\omega_n|}}. \tag{3.40}$$

In Eq. (3.38), $\overline{G}(\tau)$ corresponds to the SCBA Green's function $G_{sp}(\tau)$ defined in Eq. (3.20):

$$\overline{G}(\mathbf{r}, \mathbf{r}, \tau) = -i\pi\nu_0 \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \mathrm{sign}(\omega_n) = -\nu_0 \frac{\pi/\beta}{\sin(\tau\pi/\beta)}. \tag{3.41}$$

After carrying out the sum over Matsubara frequency in Eq. (3.39), the details of which are given in Appendix A, we obtain the finite-temperature expression

$$W(\tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} [f(-i\omega) - f(i\omega)] \frac{1 - e^{\omega\tau}}{e^{\beta\omega} - 1}, \tag{3.42}$$

where

$$f(-i\omega) = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \left(\frac{1}{Dq^2 - i\omega} \right)^2 v_{\mathrm{sc}}(q, -i\omega). \tag{3.43}$$

The interaction correction to the TDOS is determined by the behavior of the phase factor $W(\tau)$, which depends on the nature of the dynamically screened interaction v_{sc}.

IV. BULK TUNNELING DENSITY OF STATES IN TWO DIMENSIONS

We now derive the TDOS at $T = 0$. It is necessary to perform the following analytical continuation:

$$\nu(\omega) = -\frac{1}{\pi}\mathrm{Im}\, G(\mathbf{r},\mathbf{r};i\omega_n)|_{i\omega_n\to\omega+i\delta} = -\frac{1}{\pi}\mathrm{Im}\left[\int d\tau e^{i\omega_n\tau}\overline{G}(\mathbf{r},\mathbf{r},\tau)e^{-W(\tau)}\right]_{i\omega_n\to\omega+i\delta}. \tag{4.1}$$

The procedure turns out to be quite nontrivial. Since we could not find discussions of the technique in the literature, we elect to include the details of the analytical continuation in Appendix B, where we show, in the limit $T\to 0$,

$$\nu(\omega)\approx\frac{2}{\pi}\nu_0\int_0^\infty\frac{\sin(|\omega|t)}{t}e^{-W(it)}, \tag{4.2}$$

where ν_0 is the noninteracting TDOS near the Fermi level. Taking the $T=0$ limit of Eq. (3.42), we obtain

$$W(it)=\int_0^\infty\frac{d\omega}{2\pi i}[f(-i\omega)-f(i\omega)](1-e^{-i\omega t}), \tag{4.3}$$

with the function f given in Eq. (3.43). The term with the oscillatory factor $e^{-i\omega t}$ averages to zero upon integration except for $\omega\ll 1/t$ where $e^{-i\omega t}\approx 1$. Therefore we can effectively leave out the $e^{-i\omega t}$ term and introduce a lower cutoff $\hbar/t$ to the integral:

$$W(it)=\int_{1/t}^{1/\tau_0}d\omega\frac{1}{2\pi i}[f(-i\omega)-f(i\omega)]. \tag{4.4}$$

The upper cutoff of the integral in the above equation arises from the fact that the diffusive picture becomes invalid at time scales shorter than the elastic scattering time. Using Eq. (3.43), we obtain

$$W(it)=\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int\frac{d^2q}{(2\pi)^2}\mathrm{Im}\left[v_{\mathrm{sc}}(q,-i\omega)\left(\frac{1}{Dq^2-i\omega}\right)^2\right]. \tag{4.5}$$

We next turn to the evaluation of the most singular contributions to W and thus to the TDOS for different forms of interactions.

A. Long-range Coulomb interaction

The singularity in the TDOS arises from the physics of dynamical screening. For Coulomb interaction, $v(q)=2\pi e^2/q$. The dynamical screened interaction in Eq. (3.40) becomes

$$v_{\mathrm{sc}}(q,-i\omega)=\frac{2\pi e^2}{q+\dfrac{\kappa Dq^2}{Dq^2-i\omega}}, \tag{4.6}$$

where $\kappa=2\pi e^2 dn/d\mu$ is the inverse screening length at the transition. It is important to note that, in the presence of disorder, the range of validity for static screening is quite small.[38] Since diffusion is a relatively slow process, at nonzero frequency the long distance singularity associated with the long-range Coulomb interaction is not screened, as can be seen from Eq. (4.6). In fact, in the region where $Dq^2<\omega<D\kappa q$, the effective interaction has the most singular form

$$v_{\mathrm{sc}}(q,-i\omega)\sim\frac{1}{D\kappa q^2},\quad Dq^2<\omega<D\kappa q, \tag{4.7}$$

which gives the main contribution to the wave-vector integral in Eq. (4.5):

$$W(it)=\frac{1}{4\pi^2\sigma_{xx}}\int_{1/t}^{1/\tau_0}\frac{d\omega}{\omega}\ln\left(\frac{\omega}{D\kappa^2}\right). \tag{4.8}$$

Note that, in this region, the diffusion coefficient D is a constant. The anomalous diffusive behavior[50] in the regime $Dq^2\gg|\omega|$ does not affect the leading contribution. The remaining frequency integral generates the double-logarithmic dependence in time,

$$W(it)\approx\frac{1}{8\pi^2\sigma_{xx}}\ln\left(\frac{t}{\tau_0}\right)\ln\left(\frac{t}{\tau_1}\right), \tag{4.9}$$

where $\tau_1=1/\tau_0(D\kappa^2)^2$, and $\sigma_{xx}=Ddn/d\mu$ is the conductivity defined via the Einstein relation. Near the Landau-level centers, one can show in the SCBA that $\tau_1/\tau_0=(1/4\pi^4\sigma_{xx})(k_f a_B)^2/k_f l\ll 1$. This double-logarithmic form is the dominant behavior of the Debye-Waller phase factor in the long-time limit. Next-order corrections are of the order $\{1/\sigma_{xx},1/\sigma_{xx}^2\}\ln(t/\tau_0)$. The contributions from all six different integration regions in the (ω,q)-plane are discussed in detail in Appendix C. Substituting Eq. (4.9) into Eq. (4.2), we obtain the zero-temperature TDOS in the Coulomb case:

$$\nu(\omega)=\frac{2\nu_0}{\pi}\int_0^\infty dt\frac{\sin(|\omega|t)}{t}e^{-(1/8\pi^2\sigma_{xx})\ln(t/\tau_0)\ln(t/\tau_1)}. \tag{4.10}$$

Keeping in mind that in deriving this result we have assumed a frequency-independent conductivity σ_{xx}, i.e., we have neglected the quantum interference effects. In general, σ_{xx} is renormalized by localization effects, of leading order $(1/\sigma_{xx})\ln\omega\tau_0$ in the unitary ensemble, and by interaction effects of leading order $\ln\omega\tau_0$ in strong magnetic field.[35,36] Thus σ_{xx} takes on the frequency-independent SCBA value only if $|\ln(\varepsilon\tau_0)|\ll\sigma_{xx}$.

1. High-frequency regime: $|\ln(\omega\tau_0)|\ll\sigma_{xx}$

In this regime, the weak localization correction to the conductivity can be neglected. If in addition, $|\ln(\omega\tau_0)|\ll\sqrt{\sigma_{xx}}$, we can expand the exponential in Eq. (4.10) to leading order in $1/\sigma_{xx}$, and obtain

$$\nu(\omega)=\nu_0\left[1-\frac{1}{8\pi^2\sigma_{xx}}\ln(|\omega|\tau_0)\ln(|\omega|\tau_1)\right]. \tag{4.11}$$

This reproduces the high-field perturbative diagrammatic result of Girvin, Jonson, and Lee[35] and Houghton, Senna, and Ying.[36] For frequencies in the range $\sqrt{\sigma_{xx}}\ll|\ln(|\omega|\tau_0)|\ll\sigma_{xx}$, the integral in Eq. (4.10) can be evaluated by the stationary

point/instanton method, leading to a nonperturbative resummation of the double-log divergences in Eq. (4.11):

$$\nu(\omega)=\nu_0\exp\left[-\frac{1}{8\pi^2\sigma_{xx}}\ln(|\omega|\tau_0)\ln(|\omega|\tau_1)\right]. \tag{4.12}$$

In the zero-magnetic-field case, such a nonperturbative resummation of the perturbative double-log divergences was carried out by Finkel'stein,[42] and recently reexamined using different approaches.[60,46,39] Our result of Eq. (4.12) can be regarded as an extension of the latter to the strong-magnetic-field case.

2. Low-frequency regime: $|\ln(\omega\tau_0)|\gg\sigma_{xx}$

Here the quantum interference effects will, in general, lead to a frequency-dependent conductivity. However, at the IQHT, the critical conductivity σ_c is finite and of the order of $e^2/\hbar$. This experimental fact was shown numerically for both noninteracting electrons and interacting electrons in the HF theory.[61,50,62,63,25,26] Thus the validity of our analysis, i.e., the structure of the double-log divergence at long times, can be extended into the regime of small ω, provided that σ_{xx} in Eq. (4.10) is replaced by the critical conductivity $\sigma_c\simeq 0.5/2\pi$. Note that due to the double-log term in the exponent, Eq. (4.10) implies

$$\lim_{t\to\infty} W(it)=+\infty,$$

and consequently a zero-bias anomaly in the TDOS

$$\nu(\omega=0)=0.$$

To obtain the limiting behavior of $\nu(\omega)$ for small ω, we expand the $\sin(\omega t)$ factor in Eq. (4.10) in a power series in ωt. It is important to emphasize that this is possible because of the double-log contribution which makes the time integral over $e^{-W(it)}$ converge fast enough such that the TDOS becomes analytic at small ω. The claims made by Polyakov and Samokhin[28] that the TDOS falls off faster than any power law in ω is in fact incorrect. Since the signs of the expansion-coefficients alternate, the series is asymptotic, i.e., it can be infinitely accurate at small ω. To first order in ω,

$$\nu(\omega)=\nu_0|\omega|\frac{2}{\pi}\int_0^\infty dt e^{-(1/8\pi^2\sigma_c)\ln(t/\tau_0)\ln(t/\tau_1)}. \tag{4.13}$$

Performing this integral, and using the fact that the compressibility is only weakly renormalized, i.e., $dn/d\mu\simeq\nu_0$, we obtain the 2D quantum Coulomb gap behavior given in Eq. (1.8) in Sec. I, i.e.,

$$\nu(\omega)=C_Q\hbar|\omega|/e^4. \tag{4.14}$$

In contrast to the 2D classical Coulomb gap, the slope C_Q is not a universal number. It is given by

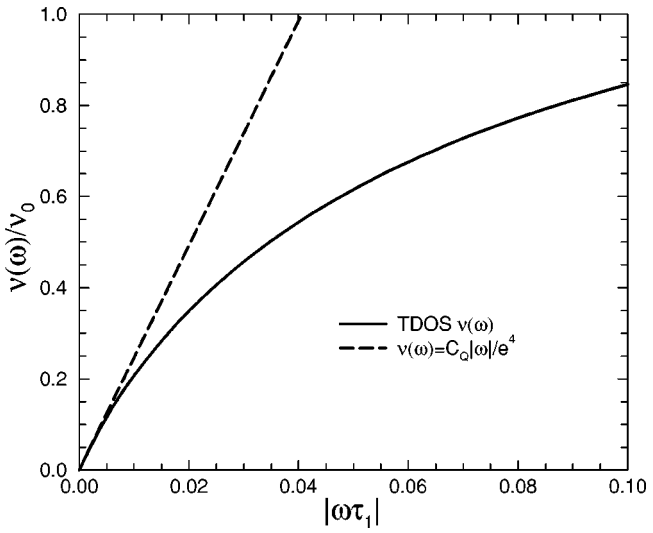

FIG. 2. The TDOS in Eq. (4.10) in the case of long-range Coulomb interaction, showing an asymptotic linear Coulomb pseudogap behavior at small $|\omega|$. The parameters are $\sigma_{xx}=\sigma_c$ and $\tau_0/\tau_1=10$.

$$C_Q=\sqrt{\frac{1}{2\pi^3\sigma_c}}[1+\mathbf{\Phi}(\sqrt{2\pi^2\sigma_c})]e^{(2\pi^2\sigma_c)+(1/8\pi^2\sigma_c)\log^2 R}, \tag{4.15}$$

where $\mathbf{\Phi}(\mathbf{x})$ is the error function and $R=\sqrt{\tau_1/\tau_0}=1/D\kappa^2\tau_0$ is a quantity that depends on the degree of disorder. The latter can be written in terms of more familiar quantities according to

$$R=\frac{1}{4\pi^4\sigma_c}\frac{(k_f a_B)^2}{k_f l_0}, \tag{4.16}$$

where a_B is the Bohr radius and l_0 is the *zero-field* mean free path.

It is easy to verify that the next term in the expansion is of the order $\omega(\omega\tau_0)^2e^{\sigma_{xx}}$, which is small in this regime. The results of numerical integration of Eq. (4.10) is plotted in Fig. 2, which shows the crossover from the high-frequency behavior described by Eq. (4.12) to the asymptotic linear Coulomb gap of Eq. (4.14) at low frequencies. Since the real transition must be governed by an interacting fixed point where the Coulomb interaction strength is finite, we conclude that the true asymptotic behavior of the bulk TDOS exhibits the quantum Coulomb gap at the IQHT.

B. Short-range interactions

In this subsection, we address the question of how short-range interactions, such as screened Coulomb interactions, which are irrelevant perturbations at the NIFP in the RG sense, cause depletion of the TDOS near the Fermi level. For simplicity, we focus on the local interactions described by the prototype short-range interacting potential $v(\mathbf{r}-\mathbf{r}')=u\,\delta(\mathbf{r}-\mathbf{r}')$ and $v(q)=u$. The screened interaction in Eq. (3.40) becomes,

$$v_{sc}(q,i\omega_n)=\frac{u}{1+u\dfrac{dn}{d\mu}\dfrac{Dq^2}{Dq^2+|\omega_n|}}. \tag{4.17}$$

Inserting this expression into Eq. (4.5) for $W(it)$, one finds

$$W_{sr}(it)=\int_0^{1/\tau_0}\frac{d\omega}{2\pi}(1-e^{-i\omega t})\int\frac{d^2q}{(2\pi)^2} \times \mathrm{Im}\left[\frac{1}{Dq^2-i\omega}\frac{1}{D'q^2-i\omega}\right], \tag{4.18}$$

where $D'=D+u\sigma_{xx}$. In contrast to the long-range Coulomb case, the contributions to the q integral from the $\omega>Dq^2$ and $\omega<Dq^2$ regimes are now comparable.

1. Prescaling regime

Let us first ignore the quantum interference effect and focus on the perturbative regime appropriate when $|\ln\omega\tau_0| \ll\sqrt{\sigma_{xx}}$. In this case we can treat the diffusion coefficient D as a constant, and the interaction strength u as a marginal perturbation (a scale invariant constant) in Eq. (4.18). Carrying out integrations, we arrive at

$$W_{sr}(it)=\int_0^{1/\tau_0}d\omega\frac{\alpha}{\omega}(1-e^{-i\omega t}), \tag{4.19}$$

where α is a nonuniversal dimensionless quantity dependent on the interaction strength,

$$\alpha=\lambda\frac{1}{8\pi^2\sigma_{xx}}\frac{2+\lambda}{(1+\lambda)^2}(1+\ln\sqrt{1+\lambda}), \tag{4.20}$$

with $\lambda=u\nu_0$. Thus $W(it)$ diverges in the long-time limit only logarithmically. The situation is completely analogous to the classic x-ray edge problem.[49] Taking care of the short-time behavior in Eq. (4.19) as in the x-ray edge problem, we obtain

$$W_{sr}(it)\simeq-\alpha\ln(1+t/\tau_0). \tag{4.21}$$

Substituting Eq. (4.21) into Eq. (4.2), we have, for the TDOS,

$$\frac{\nu_{sr}(\omega)}{\nu_0}=\frac{2}{\pi}\nu_0\int_0^{\tau_0}dt\frac{\sin(|\omega t|)}{t}\frac{1}{(1+t/\tau_0)^\alpha}=C_\alpha|\omega\tau_0|^\alpha, \tag{4.22}$$

where $C_\alpha=(2\pi)\int_0^\infty dy(\sin y/y)[1/(1+y)^\alpha]$ is a dimensionless numerical constant. Thus we conclude that a nonuniversal power-law suppression of the TDOS prevails in the prescaling regime for short-range interactions.

2. Scaling regime

On approaching the scaling regime of the IQHT, it is necessary to take into account the scaling behaviors of (1) the diffusion coefficient D, and (2) the interaction strength u. It is known from the work of Chalker and Daniel[50] that the multifractality of the critical eigenstates leads to anomalous diffusion in the regime $Dq^2>\omega$. The diffusion constant becomes a function of q^2/ω,

$$D(q,\omega)=D(qL_\omega)^{-\eta}, \tag{4.23}$$

where as before, $L_\omega=\sqrt{D/\omega}$ and $\eta=-x_-$ which is given in Eq. (2.14), and $D_2=2-\eta$ is the multifractal dimension. The critical conductivity σ_c is once again finite and will be taken as scale independent. The Debye-Waller phase factor $W_{sr}(it)$ in Eq. (4.19) is now modified by replacing $\alpha\to\alpha'$, where

$$\alpha'=\frac{1}{8\pi^2\sigma_c}\frac{2+\lambda}{(1+\lambda)^2}[c_\eta+\ln\sqrt{1+\lambda}], \tag{4.24}$$

with $c_\eta=1/2+2/(4-3\eta)$. This modification due to the multifractal behavior alone would lead to, repeating the calculations above, the same behavior of the TDOS as in Eq. (4.22) in the prescaling regime, except the exponent α is replaced by α'.

Next we must take into account the fact that u is an irrelevant perturbation. The effective interaction scales to zero according to[23,12] $u_{\mathrm{eff}}\sim u\omega^{x_+/z}$, where $-x_+$ is the scaling dimension for short-range interactions discussed in Sec. II, and $z=2$ is the dynamical exponent at the NIFP. As a result, the quantity α' obeys the following scaling relation:

$$\alpha'(u,\omega)=\mathcal{A}(u\omega^{x_+/z}). \tag{4.25}$$

The fact that $\mathcal{A}(u\to0,\omega)=0$ implies, together with Eq. (4.24), the leading scaling behavior for α':

$$\alpha'\simeq A\lambda(\omega\tau_0)^{x_+/z},\quad A=c_\eta/4\pi^2\sigma_c. \tag{4.26}$$

Substituting this result into Eq. (4.19), we find

$$W_{sr}(it)=A\lambda\gamma^{-1}\left[\left(\frac{\tau_0}{t}\right)^\gamma-1\right], \tag{4.27}$$

where $\gamma=x_+/z\simeq0.32$. That $W_{sr}(it)$ converges now in the long-time limit should be contrasted with the long-range Coulomb case, and is a consequence of the short-range interactions being irrelevant, i.e., $\gamma>0$. An immediate implication is that the TDOS would be finite at zero bias and the level spacing scales as $1/L^2$ as in the noninteracting theory. However, we shall show below that if the bare interaction λ is large, it still leads to strong suppression of the TDOS at low energies.

From Eqs. (4.2) and (4.27), the TDOS is given by

$$\nu_{sr}(\omega)=\nu(0)\frac{2}{\pi}\int_{\tau_0}^\infty dt\frac{\sin(|\omega|t)}{t}e^{(A\lambda/\gamma)(\tau_0/t)^\gamma}, \tag{4.28}$$

where $\nu(0)=\nu_0e^{-A\lambda/\gamma}<\nu_0$. Performing the integral using the saddle-point/instanton approximation, we find that, at low frequencies $\omega<\omega_0$, the TDOS is given by

$$\nu_{sr}(\omega)=\nu(0)\left[1+\left(\frac{|\omega|}{\omega_0}\right)^\gamma\right], \tag{4.29}$$

where $\omega_0=\tau_0^{-1}(A\lambda/\gamma)^{-1/\gamma}$ is an energy scale. We see that upon scaling, the irrelevance of short-range interactions leads

to a smearing of the power-law behavior in the perturbative regime, giving rise to a finite and nonuniversal TDOS at zero bias. However, although short-range interactions are irrelevant in the RG sense, since $\nu(0) \ll \nu_0$ for large λ, they still lead to a strong density-of-state suppression at low bias. What is remarkable is that Eq. (4.29) predicts an increase of the TDOS with energy that follows a *universal* power law, with an initial cusp singularity for our value of γ. These predictions can, in principle, be tested experimentally by deliberately screening out the long-ranged Coulomb interaction using metallic gates or ground planes.

C. General interacting potential: $v(q)=u/q^{2-p}$

It is interesting to consider a general interacting potential of the form $v(q)=u/q^{2-p}$. We find that for $p<2$, the dominant contribution to $W(it)$ in the long-time limit comes from the same double-log term as in the case of Coulomb interaction (corresponding to $p=1$). We conclude that, for $p<2$, the density of states in the asymptotic $\omega \to 0$ limit is of the form of the linear gap $\nu(\omega) \sim \omega$. For $p>2$, the phase-delay factor $W(it)$ approaches a constant $W_\infty > 0$ for $t \gg \tau_0$. In this case the density of states does not vanish, but rather develops a shallow dip at $\omega=0$, where $\nu(0)=\nu_0 e^{-W_\infty}$. The borderline case is that of $p=2$, corresponding to the δ-function interaction studied. It can be shown from either the $p>2$ or the $p<2$ side that as $p \to 2$, a single-log term emerges and dominates the contributions in $W(it)$, leading to the power-law density of states of Eq. (4.22) in the perturbative regime, and to the finite zero bias TDOS obtained in Eq. (4.29). Details of this analysis can be found in Appendix C.

V. TDOS IN QUASI-1D SYSTEMS WITH EDGES

In the cases studied above, the *bulk* TDOS does not depend on the Hall conductance. This is in keeping with the fact that the bulk diffusion equation is the same with or without time-reversal symmetry. The traverse force induced by a magnetic field does not affect the diffusive charge spreading. It is well known that in the noninteracting theory of the IQHT, the term in the action that depends on σ_{xy} is topological and nonperturbative.[18] However, it was discovered recently that in the presence of edges, the Hall conductance enters measurable quantities even in the perturbative limit. The topological term gives rise to a tilted boundary condition for diffusion, and at more subtle levels affects the quantum interference processes. For example, it was shown by Khmel'nitskii and Yosefin[51] and by Xiong, Read, and Stone[52] that mesoscopic conductance fluctuations in phase-coherent samples become dependent on the Hall conductance in the presence of edges. More recently Shytov, Levitov, and Halperin demonstrated that the *I-V* curves for *edge tunneling* into the 1D Luttinger-liquid-like edge excitations of fractional quantum Hall liquids can be obtained from the point of view of bulk composite fermions by using a similar phase approximation in treating the effects of gauge fluctuations.[45] There the σ_{xy} dependence in the exponent of the power-law tunneling conductance also arises from a boundary condition of the source current at the edge of a semi-infinite sample. Our case differs from and is simpler than that of the composite fermions, in the sense that we need to consider only pure potential fluctuations mediated interactions in the integer quantum Hall regime.

To study how the physical boundaries bring the Hall ratio into the *bulk* TDOS, we consider, instead of the half-plane geometry,[45] a quasi-1D sample with its length L much greater than its width W, exposing two reflecting edges along its width. This condition can be realized experimentally in the long Hall bar geometry, and is the same as that considered by Xiong, Read, and Stone[52] in their study of the edge effects on mesoscopic conductance fluctuations in strong magnetic fields.

Because the incident current is at an angle with the reflecting edges, the presence of the magnetic field affects the diffusion process through a modified boundary condition which depends on the Hall ratio $\gamma_H=\sigma_{xy}/\sigma_{xx}$:

$$[\partial_n + \gamma_H \partial_t]\rho = 0. \tag{5.1}$$

Here the subscript n denotes the directions normal to the edge, and t denotes the tangential direction. Strictly speaking, if boundary condition (5.1) is taken into account, the diffusion propagator as well as the screened interactions will depend on the Hall conductivity. Such an effect is minimal if the sample is wider than it is long, and becomes pronounced only in the quasi-1D limit when $L \gg W$. For simplicity, we consider the case of the δ-function interaction encountered in Sec. IV B and the geometry of an infinite strip with hard walls at $y=0$ and $y=W$. We also limit ourselves to the perturbative regime, and neglect scaling corrections to the conductivities and the interaction strength.

The diffusive modes that are solutions of diffusion equation (3.34) and satisfy the boundary condition (5.1) can be obtained as follows:[52]

$$\phi^L_{k,q} = a_k e^{iqx}\left[\frac{k\pi}{W}\cos\left(\frac{k\pi y}{W}\right) - i\gamma_H q \sin\left(\frac{k\pi y}{W}\right)\right] \quad \text{for} \quad k \neq 0,$$

$$\phi^L_{0,q} = a_0 e^{iqx - i\gamma_H q y} \quad \text{for} \quad k=0. \tag{5.2}$$

Since the boundary condition is not self-adjoint, there is also a set of right eigenfunctions (with the same eigenvalues) that satisfy the boundary condition under parity transformation:

$$[\partial_n - \gamma_H \partial_t]\phi^R = 0. \tag{5.3}$$

They are given by

$$\phi^R_{k,q} = a_k e^{iqx}\left[\frac{k\pi}{W}\cos\left(\frac{k\pi y}{W}\right) + i\gamma_H q \sin\left(\frac{k\pi y}{W}\right)\right] \quad \text{for} \quad k \neq 0,$$

$$\phi^R_{0,q} = a_0 e^{iqx + i\gamma_H q y} \quad \text{for} \quad k=0. \tag{5.4}$$

$\{\phi^L, \phi^R\}$ are the eigenmodes of the Laplacian operator,

$$-D\nabla^2 \phi^{L,R}_q(x,y) = \Lambda_{k,q}\phi^{L,R}_q(x,y), \tag{5.5}$$

where

$$\Lambda_{q,k}=D\left[\frac{k^2\pi^2}{W^2}+q^2\right] \quad \text{for} \quad k\neq 0,$$

$$\Lambda_{q,0}=D(1+\gamma_H^2)q^2 \quad \text{for} \quad k=0. \tag{5.6}$$

Using the biorthogonality relation and the completeness condition, we can express the Debye-Waller phase factor $W(it)$ in Eq. (4.5) in terms of the sum over the eigenmodes in the transverse channels:

$$W(it)=\frac{1}{2}\int_{1/t}^{\tau_0}\frac{d\omega}{(2\pi)}\sum_{q,k}\ \phi_{k,q}^{L}(\mathbf{r})[\,\phi_{k,q}^{R}(\mathbf{r})]^{*}2\ \mathrm{Re}$$

$$\times\left\{\frac{1}{(\Lambda_{k,q}-i\omega)^2}\,\frac{u}{1+u\nu_0\dfrac{\Lambda_{k,q}}{\Lambda_{k,q}-i\omega}}\right\}. \tag{5.7}$$

In the limit $Dt\gg W^2$, contributions from $k>0$ modes can be ignored. For the consideration of $\nu(\omega)$, this condition translates into $L_\omega\gg W$, i.e., $\hbar\omega\gg D/W^2$. Strictly speaking, $W(it)$ becomes dependent on the spatial position, but we do not expect any spatial singularity. It is therefore justifiable to average $W(it)$ over the entire strip. We obtain

$$W(it)\approx\frac{4}{3\pi^2}\,\frac{u}{\hbar\sqrt{D_1}}\,\frac{1}{1+\lambda}(\sqrt{t}-\sqrt{\tau_0}), \tag{5.8}$$

where $D_1=(1+\gamma_H^2)D$. Thus, the phase factor is dominated in the quasi-1D case by the $\sqrt{t}$ divergence in the long-time limit. Rewriting the prefactor in Eq. (5.8) as

$$\sqrt{\omega_B}=\frac{4}{3\pi^2}\,\frac{e\sqrt{\nu_0}}{\hbar^{3/2}\sqrt{(1+\gamma_H^2)\sigma_{xx}}}\,\frac{\lambda}{1+\lambda}, \tag{5.9}$$

we have $W(it)=\sqrt{\omega_B t}-\sqrt{\omega_B\tau_0}$. Upon substitution of $W(it)$ into Eq. (4.2), the TDOS in the quasi-1D case is given by

$$\nu_{\mathrm{q1D}}(\omega)=\nu_0\frac{2}{\pi}\int_{\tau_0}^{\infty}dt\frac{\sin(|\omega|t)}{t}e^{-\sqrt{\omega_B}(\sqrt{t}-\sqrt{\tau_0})}. \tag{5.10}$$

Approximating the time integral by the stationary point/instanton solution, we find that, for $\omega\ll\omega_B$,

$$\nu_{\mathrm{q1D}}(\omega)\propto\nu_0 e^{-\omega_B/4|\omega|}. \tag{5.11}$$

Thus the TDOS is strongly suppressed at low energies in quasi-1D systems in a manner that is sensitive to the applied magnetic field through the energy scale ω_B. If one naively extends the behavior of Eq. (5.11) to arbitrarily small frequencies, one could conclude that the TDOS goes to zero on the Fermi-surface faster than any power law, if the renormalization of the conductivity and the interaction strength at low energies are ignored. This is in fact incorrect, because Eq. (5.11) is only valid at intermediate frequencies. Since the Debye-Waller factor $e^{-W(it)}$ converges fast in the long-time limit, the low-energy behavior of $\nu_{\mathrm{q1D}}(\omega)$ is actually described by an asymptotic series expansion in powers of ω. We find

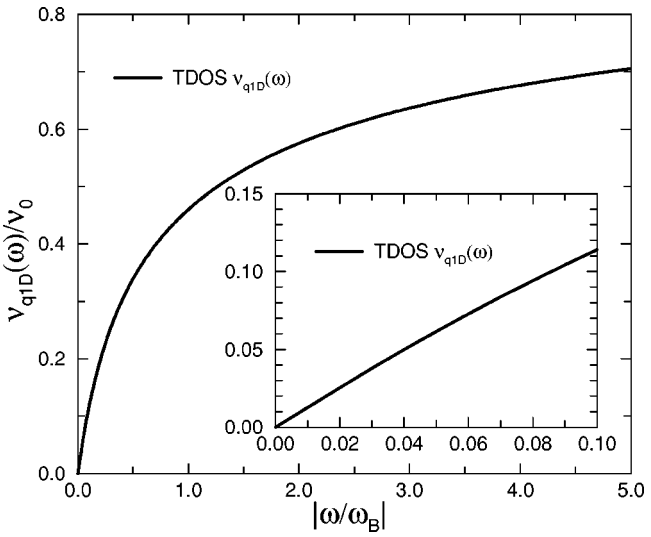

FIG. 3. The TDOS in quasi-1D strips with edges obtained from Eq. (5.10) for short-range interactions. The inset shows the asymptotic low-energy behavior of a linear pseudogap given by Eq. (5.13).

$$\nu_{\mathrm{q1D}}(\omega)\approx\nu_0\frac{|\omega|}{\omega_B}\sum_{n=0}^{\infty}(-1)^n\frac{(4n+1)!}{(2n+1)!}\left(\frac{\omega}{\omega_B}\right)^{2n}. \tag{5.12}$$

The TDOS is therefore dominated by the linear term near zero bias:

$$\nu_{\mathrm{q1D}}(\omega)\approx\nu_0\frac{\omega}{\omega_B}=s|\omega|. \tag{5.13}$$

The magnetic field dependent slope is given by

$$s=\frac{9\pi^4}{16u^2}\,\frac{1}{\rho_{xx}}(1+\lambda)^2, \tag{5.14}$$

where $\rho_{xx}=\sigma_{xx}(1+\gamma_H^2)$ is the dissipative resistivity. Note that this result is valid at small frequencies such that $|\omega|/\omega_B>e^{\omega_B/4|\omega|}$, i.e., for $|\omega|<0.12\omega_B$. In Fig. 3, we plot the TDOS obtained by numerical integration of Eq. (5.10) as a function of ω. The asymptotic linear pseudogap behavior of the TDOS at low bias is shown in the inset. Therefore, we conclude that the TDOS of a quasi-1D quantum Hall strip with reflecting edges exhibits a linearly vanishing pseudogap near the Fermi level, with a slope proportional to ρ_{xx}^{-1} in the perturbative regime. Within the SCBA, the values of both σ_{xx} and σ_{xy} at the center of the Landau levels are proportional to the Landau-level index N.[43] This leads to a Hall ratio γ_H of order 1 and a longitudinal resistivity $\rho_{xx}\sim 1/N\sim B$. We see that, for a fixed interaction strength u, the slope is inversely proportional to the magnetic field $s\sim 1/B$. It is interesting to remark that bulk tunneling measurements under the quantum Hall conditions using time-domain capacitance spectroscopy[21] indeed reveals a linearly vanishing pseudogap TDOS with a slope that scales with $1/B$. In addition, the measured slope of the tunneling pseudogap oscillates weakly as a function of filling fraction, which mimics

the oscillatory behavior of the SCBA conductance. However, although the experimental setup[21] allows screening of the Coulomb interaction by the metallic gates (electrodes), such that the interactions may be short ranged, it is not clear at present if the the sample used can be effectively qualified as being quasi-one-dimensional. It is also interesting to note that the results for the quasi-1D bulk TDOS in Eqs. (5.13) and (5.11) depends strongly on the interaction strength u. This is in contrast to the case of tunneling into a single fractional quantum Hall edge in the composite fermion description, where the TDOS was found to be a power law with an exponent that depends only weakly on the interaction strength.[45]

VI. DISCUSSIONS: QUANTUM COULOMB GAP AND DYNAMICAL SCALING OF TRANSITION WIDTH

Understanding how interactions introduce new physics into the single-particle sector near the IQHT is an important step toward a more complete understanding of the interplay between disorder and correlation and its effects on the transport properties in the quantum Hall regime. A great part of this paper is devoted to demonstrating how various types of interaction-induced anomalies in the TDOS at low energy are likely to occur so long as the 2D conductivity is finite, which occurs near the quantum Hall transitions. Our basic finding is that, in the presence of disorder, the range of validity for static screening of the Coulomb interaction is very small, whereas at finite frequencies the diffusive dynamics is too slow to effectively screen out the Coulomb interaction at long distances. This leads to a Debye-Waller phase-delay factor in the averaged single-particle Green's function that vanishes in the long-time limit. As a result, the TDOS in the extended regime comes to resemble that in the localized regime, exhibiting a linearly vanishing quantum Coulomb gap behavior.

It is important to emphasize the coexistence of the vanishing Coulomb gap in the *tunneling* DOS with a finite thermodynamic DOS. The double-log singularity in Eq. (4.10), arising from the correlations of the single-particle phases, will not show up in the two-particle density-density correlation function that determines the compressibility or the thermodynamic DOS in the static limit. This point was recently demonstrated explicitly in the self-consistent Hartree-Fock theory,[27] where it is shown that the charge redistribution induced by a test charge inserted into the quantum Hall critical state is consistent with the presence of a finite screening length. The finite critical conductivity then implies that the uniform diffusion constant must be finite.

It is also important to understand how the depletion of TDOS relates to the larger issue of dynamical scaling near the IQHT. This is more challenging due to the possible existence of several different time scales that control different dynamical processes: charging, charge spreading, inelastic phase breaking, etc. While a linearly vanishing single-particle DOS in two dimensions does imply, through the quasiparticle level spacing $\Delta \sim 1/L$, *a* dynamical scaling exponent $z=1$, it has not been shown that this z, which obviously controls the charging dynamics, is also the one that enters the conductivity scaling at finite temperatures or frequencies in the transport measurements.

The conventional wisdom, at least for dc transport, has been that the hopping transport in insulators is determined by the single-particle DOS, whereas the diffusive transport in metals is directly related to the thermodynamic DOS (compressibility). Generalized to finite frequencies, this would imply that the dynamical aspects of the transport in insulators are controlled by that of charging, and in metals by that of diffusive charge spreading. In ordinary disordered metals, this problem can be better quantified. As noted by Finkel'stein[42] in the calculation of the conductivity, there is a great degree of cancelation between the corrections to the Green's functions (which causes the anomalous behavior in the TDOS) and to the vertices. Therefore the interaction effects that cause the depletion of the the single-particle DOS $\nu(\omega)$ do not appear to influence directly the dc transport properties. At finite frequencies, the coupled scaling equations in the RG calculation based on the nonlinear-σ model involve the conductivity, the interaction strength, and the frequency renormalization Z_ω that enters in the diffusion kernel as $1/(Dq^2 - iZ_\omega\omega)$. Note that $Z_\omega = 1$ corresponds to the noninteracting case and a dynamical scaling exponent $z_\omega = \dim[D] + 2 = d$ in d dimensions. In conventional Anderson-Mott metal-insulator transitions in $d = 2 + \epsilon$ dimensions,[37] the critical conductivity is zero. The lack of quantum diffusion at the transition is accompanied by the interaction induced frequency renormalization, i.e., $Z_\omega \sim L^{-\zeta}$. For both the spin scattering and the spin-polarized case, one-loop calculations give $\zeta = \epsilon/2$. As a result, the dynamical scaling exponent relevant for metallic transport departs from the noninteracting value: $z_\omega = d - \zeta$. It is very difficult to extend the same quantitative analysis to the quantum Hall problem because the perturbative approach is no longer valid due to the presence of the topological θ term in the nonlinear-σ model action. However, the fact that both $dn/d\mu$ and the conductivity are finite at the quantum Hall transition in two dimensions ensures that the frequency in the diffusion propagator remains unrenormalized and its associated exponent $z_\omega = 2$.

The above analysis conveys a simple but important point, i.e., since it is Z_ω and not the single-particle DOS that enters scaling and controls the dynamics of the diffusive transport from the metallic side, within the existing framework,[42,64] it is natural to suspect that the suppression of the TDOS plays no role in the dynamical scaling behavior of the conductivity. Therefore, the linear Coulomb gap may not by itself appear to be an explanation for why $z = 1$ at the quantum Hall transitions. In the following, we argue that it is indeed the interplay between the quasiparticle inelastic dephasing (level broadening) and the level spacing that controls the transition width, contrary to common perceptions.

Since the relevant phenomenon here is transport, this state of affairs naturally translates into the question of which exponent determines the dephasing length L_φ. In the standard procedure,[65] the system is divided into $L_\varphi \times L_\varphi$ phase-coherent blocks. Transport within each block can be described by phase-coherent transport from the underlying noninteracting theory, and the relevant conductivity is given by

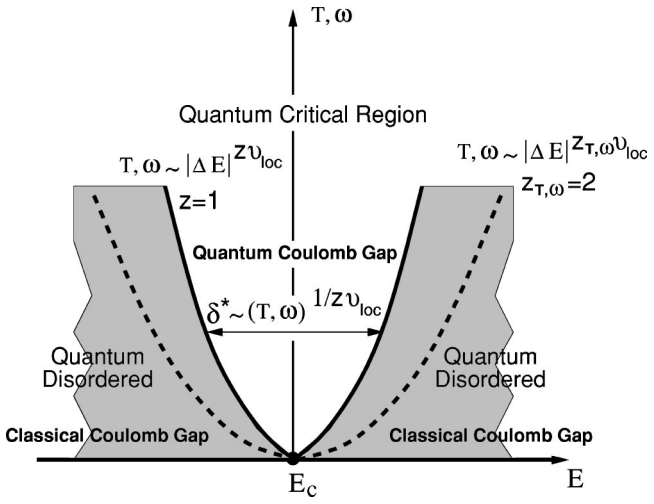

FIG. 4. Schematic phase diagram near the quantum critical point (E_c) of the IQHT. The dashed line describes the ''would-be'' crossover between phase coherence, the diffusive metallic transport of the quantum critical region, and the localized quantum-disordered regime at $L_{T,\omega}\sim\xi$. The associated thermal and frequency exponents are $z_T=z_\omega=2$. The solid line describes the true crossover from diffusive, metallic to single-particle, insulatorlike transport (shaded area) that takes place when $L_\varphi\sim\xi$. The observed transition width δ^* is narrower than $\delta^*_{\text{would-be}}$, and has a scaling behavior controlled by the dynamical exponent $z=1$, arising from the quantum Coulomb gap behavior.

the disorder average over the phase-coherent blocks. The outcome is that the scaling variable for the conductivity becomes L_φ/ξ in the presence of interactions instead of L/ξ in the noninteracting case, where L is the sample size and ξ is the localization length:

$$\sigma_{xx}=\frac{e^2}{h}F\left(\frac{L_\varphi}{\xi}\right). \tag{6.1}$$

The scaling function $F(x)$ has the limiting behavior

$$F(x)=\begin{cases}\sigma_c, & x\to 0\\ 0, & x\to\infty.\end{cases} \tag{6.2}$$

The conducting critical regime at $L_\varphi\ll\xi$ and the insulating regime at $L_\varphi\gg\xi$ are separated by a crossover at $L_\varphi\sim\xi$, where the scaling variable in Eq. (6.1) is of order 1, giving rise to a transition width $\delta^*\sim L_\varphi^{-1/\nu_{\text{loc}}}$. Physically, the transition width can be viewed as the width of the energy window of states whose localization length exceeds the phase-coherence length. In the language of quantum critical phenomenon, the latter corresponds to the width of the quantum critical region. This is a generic property associated with the quantum critical point. The only peculiarity is that on either side of the quantum Hall critical point the ground states are insulators which, drawing analogy to quantum spin systems,[66] are quantum disordered (see Fig. 4). The renormalized classical regimes (metallic phases) are absent.

Now we examine the conventional view of finding L_φ. For a generic quantum phase transition[5,6,12] the critical regime is characterized by the only time scale $\hbar/T$, and thus the dephasing time $\tau_\varphi\sim 1/T$. If quantum diffusion is all that matters, the associated length scale, i.e., the thermal diffusion length would be set by $L_T=(D\tau_\phi)^{1/2}\sim 1/T^{1/2}$, leading to a thermal exponent $z_T=2$. Similarly, the length associated with a finite frequency is $L_\omega=(D\hbar\omega)^{-1/2}$ such that the dynamic exponent $z_\omega=2$. The conventional approach is to identify $L_{T,\omega}$ with the quasiparticle dephasing length L_φ in the scaling function in Eq. (6.1). As a result, the transition width in this picture is determined by the crossover of length scales set by $L_{T,\omega}\sim\xi$, indicated in Fig. 4 by the dashed line, which leads to

$$\delta^*_{\text{would-be}}\sim(T^{1/z_T\nu_{\text{loc}}},\omega^{1/z_\omega\nu_{\text{loc}}}). \tag{6.3}$$

As emphasized in Sec. I, with the values $\nu\simeq 2.3$ and $z_T=z_\omega=2$, Eq. (6.3) does not agree with the scaling behavior of the transition width measured by transport experiments.[13,16] From the theoretical point of view, $\delta^*_{\text{would-be}}$ would be the width of the phase coherent, diffusive metallic transport regime, provided that Coulomb interaction effects are not too strong to induce single-particle localization of the quasiparticle states.

We now point out the problem with this picture which is commonly used to describe metal-insulator transitions. The use of $L_{T,\omega}$ as L_φ in Eq. (6.1) completely ignores the important Mott physics in the single-particle sector, namely, the tendency toward the single-particle insulator behavior induced by Coulomb interaction. Physically, the inelastic dephasing time is related to the interaction-induced quasiparticle decay rate or level broadening $\Gamma\sim\hbar/\tau_\varphi$. The dephasing length L_φ, on the other hand, can be determined only if the underlying transport mechanism is specified, ballistic or diffusive, insulating or metallic. Clearly, diffusive metallic transport is possible only if there is a significant overlapping of the quasiparticle levels within Γ, i.e., the level broadening must be larger than the interaction-induced level spacing, $\Gamma>\Delta$. In the opposite limit, $\Gamma<\Delta$, the single-particle states are essentially gapped as a result of electron-electron interaction and the transport would be controlled by the localization in the single-particle sector similar to that in a Mott insulator. Diffusion would be impossible in this case, and one would expect variable range hopping to be the dominant mechanism of transport. It is therefore more appropriate to define the quasiparticle dephasing length by the condition $\Gamma\sim\Delta(L_\varphi)$. The presence of the quantum Coulomb gap [Eq. (4.14)], in the critical regime implies that, near the Fermi level,

$$\Delta(L_\varphi)\simeq\frac{e^2}{\sqrt{C_Q}L_\varphi}, \tag{6.4}$$

which leads to $L_\varphi\sim\tau_\varphi\sim 1/T$ and the dynamical exponent $z=1$. Note that $L_\varphi/L_T\sim T^{-1/2}$, i.e., $L_\varphi\gg L_T$ at low temperatures. However, it is L_φ that controls the crossover to the quantum-disordered insulating regime since when $L_\varphi>\xi$, the level spacing within a $\xi\times\xi$ volume becomes larger than the level broadening and the diffusive metallic transport becomes impossible. It must be replaced by hopping-dominated transport similar to the Coulomb blockade regime

in quantum dots. This part of the physics was emphasized by Polyakov and Shklovskii[33] and by Polyakov and Samokhin[28] in terms of the classical Coulomb gap.

Using this L_φ, the scaling function in Eq. (6.1) now describes the true crossover line, set by $L_\varphi \sim \xi$ (the solid line in Fig. 4), that separates the diffusive, metallic transport from the single-particle, insulatorlike transport. The scaling behavior of the width of the critical conducting regime is therefore given by

$$\delta^* \sim (T,\omega)^{1/z\nu_{\rm loc}}, \tag{6.5}$$

with $z=1$ as observed in transport measurements. In this theory, the presence of the quantum Coulomb gap behavior is central to the emergence of the $z=1$ scaling of the transition width. It eliminates the difficulty associated with invoking the classical Coulomb gap or the bare charging energy due to the unscreened Coulomb interaction[33,28] which is only valid deep in the insulating regime.

As shown in Fig. 4, the single-particle DOS obeys the quantum Coulomb gap behavior in the quantum critical regime, whereas, deep in the quantum disordered, insulating regime, it is expected that the Coulomb interaction reinstates the classical Coulomb gap of Efros and Shklovskii.[32] Thus it is remarkable that in the presence of Coulomb interactions, the crossover between quantum critical and quantum disordered regimes is accompanied by a crossover in the behavior of the TDOS—from the quantum to the classical Coulomb gap. Comparing Eqs. (1.4) and (1.8), we see that the crossover is simply described by a crossover in the slope of the linear gap near the Fermi level. Such a crossover should, in principle, be detectable experimentally by sitting at a fixed distance to the critical point of the transition. In this case, as the temperature or frequency is lowered, one should observe a linearly vanishing gap with an initial nonuniversal slope that turns into a universal number in the low-temperature/frequency limit.

We emphasize that the linear *quantum* Coulomb gap behavior results from the combined effects of (i) two dimensionality, (ii) long-range Coulomb potential, and (iii) quantum diffusion, i.e., a finite conductivity at $T=0$. It is expected to pertain to other metal-insulator transitions in 2D amorphous electron systems, provided that the critical conductivity is finite. The physics discussed here is quite generic of the 2D disordered metal-insulator quantum critical point. A recent example is the 2D $B=0$ metal-insulator transition.[40] In this case, Fig. 4 needs to be modified to include the renormalized classical, i.e., the metallic region. It is our hope that the present work will stimulate further experimental investigations on the nature of dynamical scaling in the quantum Hall effect and in other metal-insulator transitions.

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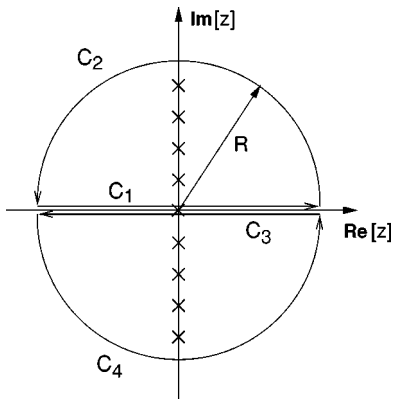

FIG. 5. The contours of integration for the functions $F^{\pm}(z)$ defined in Eq. (A6). The crosses indicate the locations of the poles at Matsubara frequencies $\omega_n = 2\pi n/\beta$.

APPENDIX A: THE MATSUBARA SUM

In this appendix, we carry out the discrete frequency sum in the phase delay given in Eq. (3.39),

$$W(\tau) = \frac{1}{\beta}\sum_n [1-e^{i\omega_n\tau}] f(|\omega_n|), \tag{A1}$$

where $\omega_n = 2\pi n/\beta$, $n=0,\pm 1,\pm 2,\ldots$ is the boson Matsubara frequency, and

$$f(|\omega_n|) = \left[\frac{1}{|\omega_n| + Dq^2}\right]^2 \frac{v(q)}{1+v(q)\nu_0 \dfrac{Dq^2}{|\omega_n|+Dq^2}}. \tag{A2}$$

To perform the Matsubara sum, we first separate the positive and the negative frequencies by writing

$$W(\tau) = W^+(\tau) + W^-(\tau), \tag{A3}$$

where

$$W^+(\tau) = \sum_{n>0} \frac{1}{\beta}[1-e^{i\omega_n\tau}] f(\omega_n), \tag{A4}$$

$$W^-(\tau) = \sum_{n<0} \frac{1}{\beta}[1-e^{i\omega_n\tau}] f(-\omega_n). \tag{A5}$$

Next we define a function on the complex plane:

$$F^{\pm}(z) = \frac{1-e^{z\tau}}{e^{\beta z}-1} f(\mp iz). \tag{A6}$$

We consider the integrals of $F(z)$ along contours in the upper (C_1+C_2) half-plane and the lower (C_3+C_4) half-plane as shown in Fig. 5. The results are given, respectively, by

$$\oint_{C_1+C_2} F^+(z)dz = \frac{2\pi i}{\beta}\sum_{n>0}[1-e^{i\omega_n\tau}]f(\omega_n) + 2\pi i \times[\text{residues from } f(-iz)],$$

$$\oint_{C_3+C_4} F^+(z)dz = \frac{2\pi i}{\beta}\sum_{n<0}[1-e^{i\omega_n\tau}]f(-\omega_n) + 2\pi i \times[\text{residues from } f(iz)]. \quad \text{(A7)}$$

It can be shown straightforwardly that the integrals of $F(z)$ along both semicircles C_2 and C_4 (with $|z|=R$) vanish at infinite radius $R\to\infty$, provided that $0<\tau<\beta$. Since residues of $f(-iz)$ lie in the lower half-plane while those of $f(iz)$ lie in the upper half-plane, they do not contribute to the contour integrals as we defined. Therefore, summing up the integrals along C_1 and C_3, we obtain

$$W(\tau) = \int_{-\infty}^{+\infty} d\epsilon \frac{1-e^{\tau\epsilon}}{e^{\beta\epsilon}-1}\frac{1}{2\pi i}[f(-i\epsilon)-f(i\epsilon)]. \quad \text{(A8)}$$

Note that since the fermionic Green's functions are antiperiodic in τ, i.e., $G(\tau+\beta)=-G(\tau)$, this implies that, through Eq. (3.38), a periodic phase factor $W(\beta+\tau)=W(\tau)$ which is indeed satisfied by Eq. (A8).

APPENDIX B: ANALYTICAL CONTINUATION

In this appendix, we describe one of the technical subtleties encountered when taking the analytical continuation in Eq. (4.1). We show how to obtain $G(\omega+i0^+)$ directly from the time-ordered imaginary-time Green's function $G(\tau)$ by analytically continuing $\tau\to it+0^+$. We begin with the Fourier transform of the fermion Green's function,

$$G(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}G(\tau), \quad \text{(B1)}$$

which satisfies the antiperiodic boundary condition $G(\tau+\beta)=-G(\tau)$. As a result $G(i\omega_n)$ is non-zero only for odd Matsubara frequencies, i.e., for $\omega_n=[(2n+1)\pi/\beta]$.

To perform the integration in Eq. (B1) and the analytical continuation to the real frequency, we extend τ to the complex z plane with $\mathrm{Re}[z]=\tau$ and $i\,\mathrm{Im}[z]=it$. We seek to analytically continue the integral in the segment bounded by $(0,\beta)$ on the real axis to integrals along the vertical axis at $\tau=0,\beta$. To this end, consider the closed-path integral along the contour shown in Fig. 6, chosen to lie in the upper half-place for $\omega_n>0$. Since $G(\tau)$ is nonanalytic at $\tau=0,\beta$, the vertical segments of the contours are shifted infinitesimally such that $0<\mathrm{Re}\,z<\beta$. The analytical continuation is possible when $G(z)$ is analytic, and has no poles encircled by the contour:

$$\oint dz e^{i\omega_n z}G(z) = 0. \quad \text{(B2)}$$

Since the integral along the $|z|\to\infty$ segment of the contour in Fig. 6 vanishes for $\omega_n>0$, we have

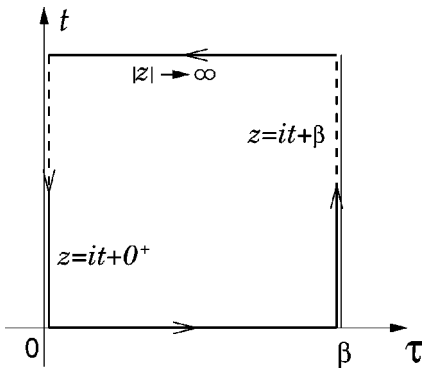

FIG. 6. The contour of integration used in Eq. (B2) for carrying out the analytical continuation. The real axis corresponds to the imaginary time τ, whereas the imaginary axis is labeled by the real time t.

$$G(i\omega_n) = i\int_0^\infty dt e^{-\omega_n t}[G(it+0^+)-G(it+\beta)]. \quad \text{(B3)}$$

Using the antiperiodic property, we obtain

$$G(i\omega_n) = 2i\int_0^\infty dt e^{-|\omega_n|t}G(it+0^+), \quad \text{(B4)}$$

where we have included the result for $\omega_n<0$, in which case, the integration contour was chosen to lie in the low half-plane.

Next we take the analytical continuation in frequency, $i\omega_n\to\omega+i0^+$, and obtain

$$G(\omega+i\delta) = 2i\int_0^\infty dt e^{i\omega t}G(it). \simeq 2i\int_{\tau_0}^\infty e^{i\omega t}\overline{G}_0 e^{-W(it)}, \quad \text{(B5)}$$

where we have used Eq. (3.38) for the Green's function $G(it)$ in our semiclassical phase approximation. Substituting the expression of the SCBA Green's function $\overline{G}_0$ in Eq. (3.41), we obtain the TDOS at finite temperatures:

$$\nu(\omega) = -\frac{1}{\pi}\mathrm{Im}\,G(\omega+i\delta) = \nu_0\frac{2}{\beta}\int_{\tau_0}^\infty \frac{\sin(|\omega|t)}{\sinh(\pi t/\beta)}e^{-W(it)}. \quad \text{(B6)}$$

Note that an overall factor stemming from the Fermi distribution function has not been included in the definition of $\nu(\omega)$ at finite temperatures, since it is, at any rate, unimportant at low temperatures.

APPENDIX C: SIX INTEGRATION REGIONS

In this appendix, we perform integrations over the frequency and wave vector in Eq. (4.5) to obtain the phase-delay factor $W(it)$. We consider the interaction potential of the general form $v(q)=u(1/q^{2-p})$ in momentum space or $v(r)=u/r^p$ in real space. We can rewrite Eq. (4.5) as

$$W(it) = \int_{1/t}^\infty d\omega\frac{d\omega}{2\pi i}\int\frac{dq^2}{4\pi^2}Q(q,\omega), \quad \text{(C1)}$$

where the integrand with $\kappa^{2-p}=\nu_0 u$ is given by

$$Q(q,\omega)=\frac{2u\omega(2Dq^2+Dq^p\kappa^{2p})}{[(Dq^2)^2+\omega^2][(Dq^2+Dq^p\kappa^{2-p})^2+\omega^2]q^{2-p}}. \tag{C2}$$

There are six characteristic integration regions differed by the ordering of Dq^2, $Dq^p\kappa^{2-p}$, and ω. The details of the integration are discussed below. For all $p<2$ we find the leading contribution is the double-log term appeared in the long-range Coulomb case ($p=1$) in Sec. IV A. For $p>2$ the leading contribution is of the form $W_\infty\sim 1/t^\delta$ with $\delta>0$, and $\delta\to 0$ as $p\to 2$.

1. Regime $Dq^2<\omega<Dq^p\kappa^{2-p}$

The condition $Dq^2<\omega<Dq^p\kappa^{2-p}$ requires $\omega<D\kappa^2$ for $p<2$; and $\omega>D\kappa^2$ for $p>2$. In this case, the integrand Q in Eq. (C2) can be approximated by

$$Q(p,\omega)\approx\frac{2}{D\nu_0\omega q^2}. \tag{C3}$$

For $p<2$ we obtain

$$\int\frac{d^2q}{4\pi^2}Q(q,\omega)=-\frac{1}{2\pi}\frac{1}{D\nu_0}\frac{2-p}{p}\log\omega\tau_s \tag{C4}$$

where we have introduced τ_s via $1/\tau_s=D\kappa^2$. Performing the remaining ω integral, we obtain

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int\frac{d^2q}{4\pi^2}Q(q,\omega)=\frac{1}{8\pi^2}\frac{1}{D\nu_0}\frac{2-p}{p}\log t/\tau_0\log t/\tau_1, \tag{C5}$$

where $\tau_1^2=\tau_0/D\kappa^2$. For $p>2$, due to the requirement that $\omega>D\kappa^2$, this regime does not have any time-dependent contribution in the long-time limit when $t>\tau_s$.

2. Regime $Dq^2<Dq^p\kappa^{2-p}<\omega$

This regime requires $q<q_0(\omega)\equiv\min[\kappa,(\omega\tau_s)^{1/p}\kappa]$ for $p<2$ and $\kappa<q<(\omega\tau_s)^{1/p}\kappa$ for $p>2$. The latter case requires $\omega>1/\tau_s$. This means that, for $p>2$, this regime does not contribute in the long-time limit in a time-dependent way.

For $p<2$, we have

$$Q(q,\omega)\approx\frac{2}{\omega^3}\nu_0u^2Dq^{2p-2}, \tag{C6}$$

such that

$$\int\frac{d^2q}{4\pi^2}Q(q,\omega)=\frac{1}{\pi\omega^3}\nu_0u^2D\int_0^{q_0(\omega)}dqq^{2p-1}=\frac{1}{\pi}\nu_0u^2D\frac{q_0^{2p}(\omega)}{2p}. \tag{C7}$$

Usually $t>\tau_0>\tau_s$, therefore, $\omega<\tau_s^{-1}$. We obtain

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int\frac{d^2q}{4\pi^2}Q(q,\omega)=\frac{1}{2\pi^2}\nu_0u^2D\frac{1}{2p}\kappa^{2p}\tau_s^2\log(t/\tau_0)=\frac{u}{2\pi^2}\frac{1}{2p}\tau_s\kappa^p\log(t/\tau_0). \tag{C8}$$

This single-log term is subleading when compared to the leading double-log contribution in regime 1 in the long-time limit.

3. Regime $Dq^pk^{2-p}<Dq^2<\omega$

In this case, we have

$$Q(q,\omega)\approx\frac{4Du}{\omega^3}q^p. \tag{C9}$$

The limits for the q and ω integrals are $\kappa<q<\sqrt{\omega/D}$, $\omega>D\kappa^2=1/\tau_s$ for $p<2$, and $q<\min[\kappa,\sqrt{\omega/D}]$ for $p>2$. The $p<2$ case is of no interest in this regime since the lower cutoff of the frequency integral is $1/\tau_s$ and is time independent in the limit $t>\tau_s$.

We now discuss the $p>2$ case. For most of the physical systems the mean free path l is greater than the screening length κ; therefore, $\min\{\kappa,\sqrt{\omega/D}\}=\sqrt{\omega/D}$. The integrals can be carried out according to

$$\int\frac{d^2q}{4\pi^4}Q(q,\omega)=\frac{2Du}{\pi}\frac{1}{\omega^3}\int_0^{\sqrt{\omega/D}}qdqq^p=\frac{2u}{\pi}\frac{1}{p+2}\frac{1}{D^{p/2}}(\omega)^{p/2-2}, \tag{C10}$$

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int\frac{d^2q}{4\pi^2}Q(q,\omega)=\frac{u}{\pi^2D^{p/2}}\frac{1}{p+2}\frac{2}{p-2}\times\left[\frac{1}{\tau_0^{p/2-1}}-\frac{1}{t^{p/2-1}}\right]. \tag{C11}$$

4. Regime $Dq^p\kappa^{2-p}<\omega<Dq^2$

In this regime, we have

$$Q(q,\omega)\approx\frac{4\omega u}{D^3}\frac{1}{q^{8-p}}.$$

The limits of the integrations are $(\omega/D)^{1/2}<q<\kappa(\omega\tau_s)^{1/p}$, $\omega>1/\tau_s$ for $p<2$, and $\omega<1/\tau_s$ for $p>2$. The $p<2$ case only produces time-independent contributions in the limit $t>\tau_s$. For the case of $p>2$, it is straightforward to obtain

$$\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{2u\omega}{D^3\pi}\int_{(\omega/D)^{1/2}}^{\kappa(\omega\tau_s)^{1/p}} dq \frac{1}{q^{7-p}} = \frac{2u}{D^3\pi}\frac{1}{6-p}\left(\frac{1}{\omega^{2-p/2}D^{p/2-3}} - \frac{1}{\omega^{6/p-2}\tau_s^{6/p-1}\kappa^{6-p}}\right), \tag{C12}$$

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{u}{\pi^2}\frac{1}{6-p}\left\{\frac{2\tau_0}{(p-2)l^p}[1-(\tau_0/t)^{p/2-1}] - \frac{p\tau_s\kappa^p}{(3p-6)(\kappa l)^6(\tau_s/\tau_0)^{6/p}}[1-(\tau_0/t)^{3-6/p}]\right\}, \tag{C13}$$

where $l=(2D\tau_0)^{1/2}$ is the mean free path.

5. Regime $\omega<Dq^p\kappa^{2-p}<Dq^2$

In this regime, the integrand Q in Eq. (C2) can be approximated by

$$Q(q,\omega)\approx \frac{4u\omega}{D^3}\frac{1}{q^{8-p}}. \tag{C14}$$

For $\omega<1/\tau_0<1/\tau_s$, the limits of the integrations are $q>\kappa$ for $p<2$ and $(\omega\tau_s)^{1/p}\kappa<q<\kappa$ for $p>2$.

In the case of $p<2$, we have

$$\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{2u\omega}{D^3\pi}\int_\kappa^\infty dq \frac{1}{q^{7-p}} = \frac{2u\omega}{D^3\pi}\frac{1}{6-p}\frac{1}{\kappa^{6-p}}, \tag{C15}$$

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{u\tau_s^3\kappa^p}{2\pi^2\tau_0^2(6-p)}[1-(\tau_0/t)^2]. \tag{C16}$$

On the other hand, for $p>2$, we have

$$\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{2u\omega}{D^3\pi}\int_\kappa^\infty dq \frac{1}{q^{7-p}} = \frac{2u\omega}{D^3\pi}\frac{1}{(6-p)\kappa^{6-p}}\left[\frac{1}{(\omega\tau_s)^{(6-p)/p}} - 1\right], \tag{C17}$$

leading to

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{u}{\pi^2}\frac{1}{6-p}\frac{p\tau_s\kappa^p}{(3p-6)(\kappa l)^6(\tau_s/\tau_0)^{6/p}} \times[1-(\tau_0/t)^{3-6/p}] - \frac{u\tau_s^2\kappa^p}{2\pi^2\tau_0^2(6-p)}[1-(\tau_0/t)^2]. \tag{C18}$$

6. Regime $\omega<Dq^2<Dq^p\kappa^{2-p}$

Finally, in regime 6, we have

$$Q(q,\omega)\approx \frac{2\omega}{\nu_0(Dq^2)^3}. \tag{C19}$$

The limits of the integrations are $(\omega/D)^{1/2}q<\kappa$ for $p<2$ and $q>\kappa$ for $p>2$.

In the $p<2$ case, integrals give

$$\int \frac{d^2q}{4\pi} Q(q,\omega) = \frac{1}{4\pi}\frac{\omega}{\nu_0 D^3}\left[\frac{D^2}{\omega^2} - \frac{1}{\kappa^4}\right], \tag{C20}$$

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{1}{8\pi^2\nu_0 D}\left\{\log(t/\tau_0) - \frac{1}{2D^2\kappa^4}[1/\tau_0^2 - 1/t^2]\right\}. \tag{C21}$$

The $p>2$ case, on the other hand, gives,

$$\int_{1/t}^{1/\tau_0}\frac{d\omega}{2\pi}\int \frac{d^2q}{4\pi^2} Q(q,\omega) = \frac{1}{16\pi^2\nu_0 D}\tau_s^2[1/\tau_0^2 - 1/t^2]. \tag{C22}$$

Note that the δ potential considered in Sec. IV B corresponds to the $p=0$ case. The result in Eq. (4.21) can be obtained from either the $p>2$ case or the $p<2$ case by taking the limit $p\to 2$ using $\lim_{x\to 0}(1/x)(1-y^x) = -\ln y$ and $\lim_{p\to 2}(p-2)\ln(\tau_1) = \ln(\nu_0 u)$.

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