Coordinate representation of the one-spinon one-holon wave function and spinon-holon interaction

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By deriving and studying the coordinate representation for the one-spinon one-holon wave function we show that spinons and holons in the supersymmetric *t-J* model with $1/r^2$ interaction attract each other. The interaction causes a probability enhancement in the one-spinon one-holon wave function at short separation between the particles. We express the hole spectral function for a finite lattice in terms of the probability enhancement, given by the one-spinon one-holon wave function at zero separation. In the thermodynamic limit, the spinon-holon attraction turns into the square-root divergence in the hole spectral function.

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I. INTRODUCTION

Landau's Fermi-liquid theory applies to interacting electron systems that can be adiabatically deformed to a Fermi gas. If the interaction is smoothly switched off, the spectrum of a Fermi liquid reduces to the spectrum of a noninteracting fermionic system. The excitations of a Fermi-liquid are given by quasiparticles and quasiholes. Although their lifetime may be short, it always becomes infinite at the Fermi surface.[1] As one is concerned only with energies close to the Fermi surface, Fermi-liquid's picture applies to a wide class of correlated systems. Experimentally, Landau quasiparticles are observed as a resonant peak at the Fermi surface in the spectral density of states measured at fixed momentum.

Nevertheless, there are several low-dimensional strongly correlated systems where Landau's picture breaks down. In Luttinger liquids the spin and charge degrees of freedom "separate" and the quasiparticles and quasiholes are no longer elementary excitations.[2,3] The same phenomenon was discovered by means of Bethe-ansatz-like techniques in exactly solvable models, e.g., the supersymmetric *t-J* model with $1/r^2$ interaction, which we study in this paper.[4]

Physically a particle or a hole injected in a strongly correlated chain breaks up into particles carrying spin, but no charge (spinons) and charge, but no spin (holons). The quantum numbers of particles and holes "fractionalize." Spinons and holons are the true elementary excitations of the strongly correlated chains.[5,6] When widely separated one from each other, spinons and holons propagate independently, in general, with different velocities. Such a phenomenon is usually referred to as "spin-charge separation." As a consequence of spin-charge separation, the quasiparticle peak disappears and is substituted by a broad spectrum. This was experimentally detected in angle-resolved photoemission spectroscopy experiments on quasi-1D (one-dimensional) samples.[7]

Using Bethe-ansatz solutions of 1D models, one can work out the energy of a many-spinon many-holon state. In the thermodynamic limit the total energy is the sum of the energies of each isolated particle. However, this does not imply that spinons and holons do not interact. In recent papers[8,9] we carefully studied spinon interaction in an exact solution of the Haldane-Shastry model (HSM),[10,11] a model for a strongly correlated 1D system with no charge degrees of freedom. We showed that spinons interact, although in the thermodynamic limit the energy of a many-spinon solution is additive. We also showed that their interaction is responsible for the lack of integrity of spin waves against decay into spinons, the true low-energy excitations of the system.

In this paper we generalize the formalism introduced in Ref. 9 to study the interaction between spinons and holons in an exact closed-form solution of the supersymmetric extension of the HSM: the supersymmetric *t-J* model with $1/r^2$ interaction [Kuramoto-Yokoyama (KY) model[12]]. The KY-model is a system of electrons located at the sites of a circular lattice, where double occupancy of a site is forbidden by strong Coulomb repulsion. Charge hopping, Coulomb interaction, and spin-spin antiferromagnetic interaction are all inversely proportional to the square of the chord between the corresponding sites. Charge vacancies at some sites (holes) may be created by filling the system with fewer electrons than the number of sites. In the KYM one is concerned with both spin and charge degrees of freedom.

We investigate the basic features of the KY model at half-filling by employing a formalism based on analytic variables on the unit radius circle. As in the case of the HSM, it is easier to construct and visualize the spinon and holon excitation by using a real-space formalism than by using the Bethe-ansatz formalism. Within our formalism we derive a "real-space" representation of the one-spinon one-holon wave function as a solution of an appropriate equation of motion.

By taking the thermodynamic limit of the exact solution of the equation of motion, we find that the probability of finding a spinon and a holon at large separations from each other is independent of the distance between the particles, while it is greatly enhanced when the two particles are on top of each other, a phenomenon we refer to as "short-distance spinon-holon probability enhancement." In the thermodynamic limit, the spinon-holon interaction assumes the same form as the spinon-spinon interaction derived in Ref. 8, although the corresponding equations of motion are completely different. The physical interpretation is the same as in the case of the spinon-spinon interaction: a spinon and a holon do not interact when they are widely separated from each other, while they exhibit a short-range attraction at short separations.[13]

To show the effects of spinon-holon interaction on the hole spectral density, $A_h(\omega,q)$, we exactly calculate the contribution to $A_h(\omega,q)$ from one-spinon one-holon states, $A_h^{\mathrm{sp,ho}}(\omega,q)$. The spinon-holon interaction has important consequences on the functional form of $A_h^{\mathrm{sp,ho}}(\omega,q)$. The probability enhancement causes the overlap between the wave function for the localized hole and that for a spinon-holon pair to be significant, although not enough to form a spinon-holon bound state. The corresponding matrix element is enhanced—despite the fact that low-energy density of states is uniform at lower energies—so as to make the hole excitation fully unstable to decay into a spinon-holon pair. Taking the thermodynamic limit of our result, we show that, as the size of the system increases, spinon-holon interaction turns into a square-root singularity at the one-spinon one-holon creation threshold. Correspondingly, $A_h^{\mathrm{sp,ho}}(\omega,q)$ shows no Landau quasiparticle's peak, but it rather exhibits a sharp singular threshold, followed by a broad branch cut.[14] Similar features have also been found in the electron addition spectral function of the KYM at generic filling.[15]

The paper is organized as follows: In Sec. II we shortly review the KY Hamiltonian and its supersymmetry; in Sec. III we introduce the ground state of the KY model at half-filling and its representation as a function of analytic variables on the unit circle. At half-filling, the ground state is the same as the ground state of the HS model—a disordered spin singlet. We will briefly review some properties of the ground state, already discussed at length in Ref. 9. In Sec. IV we analyze the one-spinon solution and review its relevant properties. In Sec. V we focus on the one-holon solution and derive its relevant properties. In Sec. VI we derive the action of $\mathcal{H}_{KY}$ on the one-spinon one-holon states, the energy eigenvalues, the corresponding eigenvectors, and their norm; in Sec. VII we write the Schrödinger equation for the one-spinon one-holon wave function, whose solutions are simple polynomials. From the behavior of the one-spinon one-holon wave function, we infer the nature of the interaction between spinons and holons: a short-range attraction. The physical consequences of such an interaction are discussed at length in Sec. VIII, where we rederive an exact expression for the contribution of one-spinon one-holon states to the hole spectral function in terms of the spinon-holon wave functions and rigorously prove that this contribution is completely determined by the spinon-holon interaction. In the thermodynamic limit spinon-holon interaction turns into the square-root divergence in the hole spectral function, obtained in Ref. 14. In Sec. IX we provide our main conclusions.

II. KURAMOTO-YOKOYAMA HAMILTONIAN

The Kuramoto-Yokoyama model is defined on a lattice with periodic boundary conditions. Sites are parametrized by the Nth roots of unity, z_α $(\alpha=1,\ldots,N.)$ Strong electron repulsion forbids double occupancy at each site. Therefore, sites can be occupied by an ↑ or a ↓ electron, or they can be empty. The number of empty sites can be tuned by means of an external chemical potential that fixes the total charge of the system. The Kuramoto-Yokoyama Hamiltonian[12] is a generalization of the Haldane-Shastry Hamiltonian,[10,11] where charge dynamics is also taken into account. It takes the form

$$\mathcal{H}_{KY}=J\left(\frac{2\pi}{N}\right)^2\sum_{\alpha<\beta}^{N}\frac{1}{|z_\alpha-z_\beta|^2}P\Bigg\{\vec{S}_\alpha\cdot\vec{S}_\beta-\frac{1}{2}\sum_{\sigma}\,(c^\dagger_{\alpha\sigma}c_{\beta\sigma})+\frac{1}{2}(n_\alpha+n_\beta)-\frac{1}{4}n_\alpha n_\beta-\frac{3}{4}\Bigg\}P, \tag{1}$$

where the Gutzwiller's projector

$$P=\prod_\alpha\,(1-c^\dagger_{\alpha\uparrow}c^\dagger_{\alpha\downarrow}c_{\alpha\downarrow}c_{\alpha\uparrow}) \tag{2}$$

accounts for the no-double-occupancy constraint.

The site occupation and spin operators are given by

$$n_\alpha=c^\dagger_{\alpha\uparrow}c_{\alpha\uparrow}+c^\dagger_{\alpha\downarrow}c_{\alpha\downarrow}, \tag{3}$$

$$S^a_\alpha=\frac{1}{2}\sum_{\sigma,\sigma'}c^\dagger_{\alpha\sigma}\tau^a_{\sigma\sigma'}c_{\alpha\sigma'}, \tag{4}$$

where $\tau^a, a=x,y,z$, are Pauli matrices. An empty site corresponds to a charge-1 hole that can tunnel to nearby sites by means of the same inverse-square matrix element characterizing the spin exchange and the charge-repulsion term in $\mathcal{H}_{KY}$.

Usual bosonic symmetries of a $(t$-$J)$-like model are total spin, corresponding to the operator $\vec{S}=\Sigma_\alpha\vec{S}_\alpha$, and total charge, corresponding to the operator $N=\Sigma_\alpha n_\alpha$. The equivalence of energy scales for magnetism, charge transport, and charge interaction causes the KY Hamiltonian to be supersymmetric, in the sense that it commutes with the electron or hole injection operators $\mathcal{Q}_\sigma=\Sigma_\alpha Pc_{\alpha\sigma}P, \mathcal{Q}^\dagger_\sigma=\Sigma_\alpha Pc^\dagger_{\alpha\sigma}P$.

As in the Haldane-Shastry Hamiltonian, since the complex variable z lies on the unit circle $(z^*=z^{-1})$, the interaction is an analytic function of the coordinates, that is,

$$\frac{1}{|z_\alpha-z_\beta|^2}=-\frac{z_\alpha z_\beta}{(z_\alpha-z_\beta)^2}.$$

Throughout the paper we use the representation in terms of the analytic variables z_α.[16,9] This turns out to be very useful for describing the properties of spinons and holons in real space.

III. GROUND STATE

In this section and in the following one we discuss the ground-state and the one-spinon eigenstates of the KY model at half-filling. At half-filling the KY model reduces to the HS Hamiltonian[10,11]

$$\mathcal{H}_{HS}=J\left(\frac{2\pi}{N}\right)^2\sum_{\alpha<\beta}^{N}\frac{\vec{S}_\alpha\cdot\vec{S}_\beta}{|z_\alpha-z_\beta|^2}. \tag{5}$$

Both the ground-state and the one-spinon eigenstates are the same as for the HS model. Since in Ref. 9 we have already applied our formalism to study basic properties of the HS model, here we will only briefly review the main results in view of their extension to states where holon excitations are present.

A. Ground-state wave function

Let N be even. We first give the representation of the ground state $|\Psi_{GS}\rangle$ in terms of the z coordinates and then derive its energy. $|\Psi_{GS}\rangle$ is defined in terms of its projection onto the set of states with $M=N/2$ spins up and the remaining spins down. If $z_1,\ldots,z_M$ are the coordinates of the up-spins, one defines the state $|z_1,\ldots,z_M\rangle$ as: $|z_1,\ldots,z_M\rangle=\Pi_{j=1}^{M}S_j^+\Pi_{\alpha=1}^{N}c_{\alpha\downarrow}^\dagger|0\rangle$ where $|0\rangle$ is the empty state. The projections are given by

$$\Psi_{GS}(z_1,\ldots,z_M)=\prod_{j<k}^{M}(z_j-z_k)^2\prod_{j=1}^{M}z_j, \tag{6}$$

where Ψ_{GS} is a polynomial in the analytic variables $z_1,\ldots,z_M$. Its norm was first computed by Wilson[17] by using the following identity:

$$\begin{aligned}C_M&=\sum_{z_1,\ldots,z_M}\prod_{i<j}^{M}|z_i-z_j|^4\\&=\left(\frac{N}{2\pi i}\right)^M\oint\frac{dz_1}{z_1}\cdots\oint\frac{dz_M}{z_M}\prod_{i\neq j}^{M}\left(1-\frac{z_i}{z_j}\right)^2\\&=N^M\frac{(2M)!}{2^M}.\end{aligned} \tag{7}$$

Basic properties of Ψ_{GS} follow in this section, together with their derivation.

B. Singlet state

The ground state is a spin singlet. $|\Psi_{GS}\rangle$ is annihilated by both S^z and S^-. $S^z|\Psi_{GS}\rangle=0$ because $|\Psi_{GS}\rangle$ has an equal number of ↑ and ↓ spins, while

$$\begin{aligned}[S^-\Psi_{GS}](z_2,\ldots,z_M)&=\sum_{\alpha=1}^{N}\langle z_2,\ldots,z_M|S_\alpha^-|\Psi_{GS}\rangle\\&=\lim_{z_1\to 0}\sum_{l=1}^{N-1}\frac{1}{l!}\left\{\sum_{\alpha=1}^{N}z_\alpha^l\right\}\\&\quad\times\frac{\partial^l}{\partial z_1^l}\Psi_{GS}(z_1,\ldots,z_M)=0,\end{aligned} \tag{8}$$

since $\Sigma_{\alpha=1}^{N}z_\alpha^l=N\delta_{l0}(\mathrm{mod}\ N)$.

As Ψ_{GS} is a spin singlet, it takes exactly the same form if expressed either in terms of the ↑-spin coordinates $z_1,\ldots,z_M$, or of the ↓-spin coordinates, $\eta_1,\ldots,\eta_M$, that is,

$$\Psi_{GS}(z_1,\ldots,z_M)=\Psi_{GS}(\eta_1,\ldots,\eta_M). \tag{9}$$

Equation (9) is proved in Appendix A, where we derive the formulas to relate the representation of the states of the system in terms of ↑-spin coordinates to the representation in terms of ↓-spin coordinates.

In the thermodynamic limit, half-odd spin chains exhibit a gapless spectrum, although they are not allowed to order.[18] Accordingly, $|\Psi_{GS}\rangle$ is a disordered spin liquid state, and the spin-spin correlation function, $\chi(z_\alpha)=\langle\Psi_{GS}|S_0^+S_\alpha^-|\Psi_{GS}\rangle/\langle\Psi_{GS}|\Psi_{GS}\rangle$, falls off with the distance as $(-1)^x/x$, thus showing absence of spin order.[9]

C. Ground-state energy

At half-filling, $|\Psi_{GS}\rangle$ is the ground state of $\mathcal{H}_{KY}$, with eigenvalue

$$\mathcal{H}_{KY}|\Psi_{GS}\rangle=\mathcal{H}_{HS}|\Psi_{GS}\rangle=-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}\right)|\Psi_{GS}\rangle. \tag{10}$$

Equation (10) has been derived originally by Haldane and Shastry.[10,11] In Ref. 9, we rederived it by means of our own technique, consisting in substituting sums over spins on the lattice with derivative operators acting on the analytic extension of $\Psi_{GS}(z_1,\ldots,z_M)$. The z_j's are allowed to take any value on the unit circle. After computing the derivatives, we constrain them again to lattice sites. In this section we review our technique, in view of its generalization to the case where the filling is $\neq 1/2$ and the dynamics of the system is described by the full KY Hamiltonian.

Since $[S_\alpha^+S_\beta^-\Psi_{GS}](z_1,\ldots,z_M)$ is identically zero unless one of the arguments $z_1,\ldots,z_M$ equals z_α, we have

$$\begin{aligned}&\left[\left\{\sum_{\beta\neq\alpha}^{N}\frac{S_\alpha^+S_\beta^-}{|z_\alpha-z_\beta|^2}\right\}\Psi_{GS}\right](z_1,\ldots,z_M)\\&=\sum_{j=1}^{M}\sum_{\beta\neq j}^{N}\frac{1}{|z_j-z_\beta|^2}\Psi_{GS}(z_1,\ldots,z_{j-1},z_\beta,z_{j+1},\ldots,z_M)\\&=\sum_{l=0}^{N-2}\sum_{j=1}^{M}\frac{z_j^{l+1}}{l}A_l\left(\frac{\partial^l}{\partial z_j^l}\right)\left\{\frac{\Psi_{GS}(z_1,\ldots,z_M)}{z_j}\right\}.\end{aligned} \tag{11}$$

The coefficients A_l are calculated in Ref. 9. They are zero for $N>l>2$. Therefore, Eq. (11) can be rewritten as

$$\sum_{j=1}^{M}\left\{\frac{(N-1)(N-5)}{12}z_j-\frac{N-3}{2}z_j^2\frac{\partial}{\partial z_j}+\frac{1}{2}z_j^3\frac{\partial^2}{\partial z_j^2}\right\}\left\{\frac{\Psi_{GS}(z_1,\ldots,z_M)}{z_j}\right\}$$
$$=\left\{\frac{N(N-1)(N-5)}{24}-\frac{N-3}{2}\sum_{j\neq k}^{M}\frac{2z_j}{z_j-z_k}+\sum_{j\neq k\neq m}^{M}\frac{2z_j^2}{(z_j-z_k)(z_j-z_m)}+\sum_{j\neq k}^{M}\frac{z_j^2}{(z_j-z_k)^2}\right\}\times\Psi_{GS}(z_1,\ldots,z_M)$$
$$=\left\{-\frac{N}{8}-\sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\right\}\Psi_{GS}(z_1,\ldots,z_M). \quad (12)$$

The "Ising-spin term," on the other hand, provides:

$$\left[\left\{\sum_{\beta\neq\alpha}^{N}\frac{S_\alpha^zS_\beta^z}{|z_\alpha-z_\beta|^2}\right\}\Psi_{GS}\right](z_1,\ldots,z_M)=\left\{-\frac{N(N^2-1)}{48}+\sum_{j\neq k}^{M}\frac{1}{|z_j-z_k|^2}\right\}\Psi_{GS}(z_1,\ldots,z_M). \quad (13)$$

Adding up Eqs. (12) and (13) provides (10).

Using the factorization property of $\mathcal{H}_{HS}$, Shastry proved that $|\Psi_{GS}\rangle$ is the actual ground state of $\mathcal{H}_{KY}$ at half-filling.[19] The same proof can be rephrased within our formalism, as discussed in Ref. 9.

The crystal momentum of the state, q, is defined (mod 2π) by the equation

$$\Psi_{GS}(z_1z,\ldots,z_Mz)=e^{iq}\Psi_{GS}(z_1,\ldots,z_M), \quad (14)$$

where $z=\exp(2\pi i/N)$. From Eq. (14) q can be either 0 or π, according to whether N is divisible by 4 or not. In the former case Ψ_{GS} equals itself, when translated by one lattice constant, while it equals negative value of itself in the latter case.

Other relevant properties of $|\Psi_{GS}\rangle$ are discussed in detail in Ref. 9 and will not be analyzed here.

IV. ONE-SPINON WAVE FUNCTION

The elementary excitations above the ground state of a correlated 1D electron system are not Landau's quasiparticles, but rather spinons and holons. Spinons have been identified as the elementary excitations of a spin-(1/2) 1D antiferromagnet. They can be thought of as localized spin defects carrying total spin-(1/2), embedded in an otherwise featureless disordered singlet spin sea. As spinon excitations appear in the KY chain at any filling, we study spinons at filling-(1/2), when the KY model reduces back to the HS model. One-spinon states appear as states of the chain with an odd number of sites. In this case, the minimum possible value for the total spin is 1/2, and the state for a localized spinon at s is created by constraining the spin at s to be $\downarrow$, in a surrounding spin singlet sea.[10,11,9] Since correlations in the ground state are short ranged, in the thermodynamic limit it makes no difference whether one begins with an odd or an even number of sites. Therefore, in the thermodynamic limit, there is no way to distinguish between chains with odd number of sites or chains with even number of sites. States with half-odd spin are alleged eigenstates of $\mathcal{H}_{KY}$ at half-filling with an odd number of spinons and no holons. In this section we briefly review the one-spinon wave function and discuss its properties. This was first studied by Haldane and Shastry,[10,11] and discussed at length in Ref. 9 within the framework of the formalism of analytic variables.

A. One-spinon spin doublet

Let N be odd and $M=(N-1)/2$. The wave function for a localized spinon at s takes Haldane's form[10,11]

$$\Psi_s^{\rm sp}(z_1,\ldots,z_M)=\prod_{j=1}^{M}(z_j-s)z_j\prod_{i<j}^{M}(z_i-z_j)^2, \quad (15)$$

where $z_1,\ldots,z_M$ denote again the position of the $\uparrow$-spins and s is the coordinate of a lattice site where the spin is fixed to be $\downarrow$.

By definition, $\Psi_s^{\rm sp}$ is an eigenstate of S^z with eigenvalue $-1/2$. In order to prove that it is a spin-(1/2) state, we need to show that S^- annihilates it. Indeed, per Eq. (8) we have

$$\sum_{z_\beta\neq z_s}^{N}S_\beta^-\Psi_s^{\rm sp}=0, \quad (16)$$

which proves that $\Psi_s^{\rm sp}$ is the $\downarrow$-spin component of a spin doublet.

$\Psi_s^{\rm sp}(z_1,\ldots,z_M)$ is a polynomial of degree less than $N+1$ in each variables z_j. Therefore, we may again apply Taylor's expansion technique used to calculate the ground-state energy. Doing so, we find

$$\mathcal{H}_{KY}\Psi_s^{\rm sp}=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left\{\lambda+\frac{N}{48}(N^2-1)+\frac{M}{6}(4M^2-1)-\frac{N}{2}M^2\right\}\Psi_s^{\rm sp}, \quad (17)$$

provided that λ satisfies the following eigenvalue equation for $\Phi_s^{\rm sp}=\Pi_j^M(z_j-s)$:

$$\left\{M(M-1)-s^2\frac{\partial^2}{\partial s^2}-\frac{N-3}{2}\left[M-s\frac{\partial}{\partial s}\right]\right\}\Phi_s^{\rm sp}=\lambda\Phi_s^{\rm sp}. \quad (18)$$

One-$\downarrow$-spinon energy eigenstates are given by propagating one-spinon plane waves,

$$\Psi_m^{\rm sp}(z_1,\ldots,z_M)=\frac{1}{N}\sum_s(s^*)^m\Psi_s^{\rm sp}(z_1,\ldots,z_M). \quad (19)$$

The energy eigenvalue is

$$\mathcal{H}_{KY}|\Psi_m^{\rm sp}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N-\frac{1}{N}\right)+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\times m\left(\frac{N-1}{2}-m\right)\right\}|\Psi_m^{\rm sp}\rangle \tag{20}$$

with $0\leqslant m\leqslant(N-1)/2$ and $\lambda=m[(N-1)/2-m]$.

As the total crystal momentum of the state $|\Psi_m^{\rm sp}\rangle$ is given by

$$q_m^{\rm sp}=\frac{\pi}{2}N-\frac{2\pi}{N}\left(m+\frac{1}{4}\right)\pmod{2\pi}, \tag{21}$$

the total energy may be rewritten as

$$\mathcal{H}_{KY}|\Psi_m^{\rm sp}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}-\frac{3}{N^2}\right)+E(q_m^{\rm sp})\right\}|\Psi_m^{\rm sp}\rangle, \tag{22}$$

that is, the sum of a ground-state contribution, plus the kinetic energy of the propagating spinon. The one-spinon dispersion relation is correspondingly provided by

$$E(q_m^{\rm sp})=\frac{J}{2}\left[\left(\frac{\pi}{2}\right)^2-(q_m^{\rm sp})^2\right]\pmod{\pi}, \tag{23}$$

As extensively discussed in Refs. 10, 11, and 9, the one-spinon dispersion relation shows the typical features of spinon excitations. It spans only the inner or outer half of the Brillouin zone, depending on whether $N-1$ is divisible by 4 or not, which corresponds to the absence of negative-energy states, i.e., to the absence of "antispinons." The spinon dispersion at low energies is linear in q with a velocity

$$v_{\rm spinon}=\frac{\pi}{2}J. \tag{24}$$

The half-bands of single elementary excitations for odd N are the only $S=1/2$ states without extra degeneracies. The ground state of the odd-N spin chain is fourfold degenerate and is given by $|\Psi_m^{\rm sp}\rangle$ for $m=0$ and $(N-1)/2$ and their $\uparrow$ counterparts. This corresponds physically to a "leftover" spinon with momentum $\pm\pi$.

The spin density in the state $\Psi_s^{\rm sp}$ as a function of the spinon position is uniformly zero, as appropriate for the disordered spin singlet, except for an abrupt dip centered at $z=s$.[9] The dip is identified with a localized spinon at s. Therefore, $\Psi_s^{\rm sp}$ may be thought of as the wave function for a localized spinon at s. Starting from such an interpretation, in Ref. 9 we showed that, although spinons are collective excitations of a strongly correlated system, they can still be treated as real quantum-mechanical particles. In this paper we will generalize our formalism to states where both spinons and holons are present.

B. The norm

The squared norm of the one-spinon energy eigenstates is defined as the scalar product

$$\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle=\sum_{z_1,\ldots,z_M}|\Psi_m^{\rm sp}(z_1,\ldots,z_M)|^2. \tag{25}$$

For the Haldane-Shastry model, we derived the formula for Eq. (25) in Ref. 9. By employing a recursion relation between $\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle$ and $\langle\Psi_{m-1}^{\rm sp}|\Psi_{m-1}^{\rm sp}\rangle$, we expressed all the norms in terms of m and of the constant C_M introduced in Eq. (7). The induction relation is

$$\frac{\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle}{\langle\Psi_{m-1}^{\rm sp}|\Psi_{m-1}^{\rm sp}\rangle}=\frac{\left(m-\frac{1}{2}\right)(M-m+1)}{m\left(M-m+\frac{1}{2}\right)}, \tag{26}$$

which recursively gives

$$\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle=\frac{\Gamma[M+1]\Gamma\left[m+\frac{1}{2}\right]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma\left[M+\frac{1}{2}\right]\Gamma[m+1]\Gamma[M-m+1]}C_M. \tag{27}$$

V. ONE-HOLON WAVE FUNCTION

Holons are charged, spin-0 elementary excitations of the Kuramoto-Yokoyama Hamiltonian. They are constructed by removing an electron from the center of a spinon. The state for a localized holon at h_0 is given by $c_{h_0\downarrow}|\Psi_{s=h_0}^{\rm sp}\rangle$, where $|\Psi_s^{\rm sp}\rangle$ is the state defined in Eq. (15). By construction, in the KY model, the holon is the supersymmetric partner of the spinon. However, unlike in the spinon case, the Brillouin zone for one-holon states is not halved, as both negative- and positive-energy holons can be constructed. In this section, we are concerned mainly with holon "kinematics." We derive the one-holon eigenstates, their norm, their energy, and their crystal momentum. In particular, we focus on negative-energy one-holon states, since these states are the ones relevant to the spinon-holon interaction. In the following sections we analyze holon and spinon dynamics—the interaction between spinons and holons and its relation to the instability of the hole excitations in the KY model.

A. One-holon spin singlet

Let N be odd and $M=(N-1)/2$. The wave function for a propagating, negative-energy holon is given by

$$\Psi_n^{\rm ho}(z_1,\ldots,z_M|h)=(h)^n\prod_j^M(z_j-h)z_j\prod_{j<k}^M(z_j-z_k)^2, \tag{28}$$

where $z_1,\ldots,z_M$ denote the positions of the $\uparrow$ sites and h denotes the position of the empty site, all others being $\downarrow$. Also, $0\leqslant n\leqslant(N+1)/2$. Different from the spinon case, in Eq. (28) the holon coordinate h is not a quantum number but a coordinate variable. Therefore, unlike localized one-spinon eigenstates, $\Psi_n^{\rm ho}$ takes a well-defined crystal momentum, as we will show later.

$\Psi_n^{\rm ho}$ is a spin-singlet state. Indeed, by definition its total component of the spin along z is zero. Following the same steps leading to Eq. (16), we also get

$$S^z\Psi_n^{\rm ho}=S^-\Psi_n^{\rm ho}=0, \tag{29}$$

which proves that $\Psi_n^{\rm ho}$ is a spin singlet.

B. Negative one-holon energy eigenstates

$\Psi_n^{\rm ho}$ is an eigenstate of $\mathcal{H}_{KY}$ with energy eigenvalue given by

$$\mathcal{H}_{KY}|\Psi_n^{\rm ho}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N-\frac{1}{N}\right)+\frac{J}{2}\left(\frac{2\pi}{N}\right)^2 n\left(n-\frac{N+1}{2}\right)\right\}|\Psi_n^{\rm ho}\rangle, \tag{30}$$

where $0\leqslant n\leqslant(N+1)/2$.

In order to prove Eq. (30), let us first split H_{KY} as follows:

$$\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2}=h_S^T+h_S^V+h_N+h_Q^{\downarrow}+h_Q^{\uparrow}. \tag{31}$$

We define the various terms when calculating their contributions to the total energy.

Spin-exchange term

$$[h_S^T\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\left[\sum_{\alpha\neq\beta}\frac{PS_\alpha^+S_\beta^-P}{|z_\alpha-z_\beta|^2}\Psi_n^{\rm ho}\right](z_1,\dots,z_M|h)=\left\{\left[\frac{1-N^2}{24}-\sum_{i\neq j}^M\frac{1}{|z_i-z_j|^2}\right]h^n+\frac{N-3}{2}h^{n+1}\frac{\partial}{\partial h}-h^{n+2}\frac{\partial^2}{\partial h^2}\right\}\left\{\frac{\Psi_n^{\rm ho}(z_1,\dots,z_M|h)}{h^n}\right\}. \tag{32}$$

Spin-Ising term

$$[h_S^V\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\left[\sum_{\alpha\neq\beta}\frac{PS_\alpha^zS_\beta^zP}{|z_\alpha-z_\beta|^2}\Psi_n^{\rm ho}\right](z_1,\dots,z_M|h)=\left\{\sum_{i\neq j}^M\frac{1}{|z_i-z_j|^2}+\sum_j^M\frac{1}{|z_j-h|^2}-\frac{N(N^2-1)}{48}\right\}\times\Psi_n^{\rm ho}(z_1,\dots,z_M|h). \tag{33}$$

Electrostatic repulsion energy

$$[h_N\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\left\{\sum_{\alpha\neq\beta}\frac{1}{|z_\alpha-z_\beta|^2}P\left[\frac{1}{2}(n_\alpha+n_\beta)-\frac{1}{4}n_\alpha n_\beta-\frac{3}{4}\right]P\Psi_n^{\rm ho}\right\}(z_1,\dots,z_M|h)=\frac{1-N^2}{24}\Psi_n^{\rm ho}(z_1,\dots,z_M|h). \tag{34}$$

↓-charge kinetic energy

$$[h_Q^{\downarrow}\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\sum_{\alpha\neq\beta}\left[\frac{Pc_{\alpha\downarrow}c_{\beta\downarrow}^\dagger}{|z_\alpha-z_\beta|^2}\Psi_n^{\rm ho}\right](z_1,\dots,z_M|h)=\sum_{z_\beta\neq h}\sum_{k=0}^{M+1}\frac{z_\beta^n(z_\beta-h)^k}{k!|z_\beta-h|^2}\left(\frac{\partial}{\partial h}\right)^k\left\{\frac{\Psi_n^{\rm ho}(z_1,\dots,z_M|h)}{h^n}\right\}=\left\{\left[\frac{N^2-1}{12}+\frac{n(n-N)}{2}\right]h^n-\left[\frac{N-1}{2}-n\right]h^{n+1}\frac{\partial}{\partial h}+\frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\right\}\left\{\frac{\Psi_n^{\rm ho}(z_1,\dots,z_M|h)}{h^n}\right\}. \tag{35}$$

↑-charge kinetic energy

$$[h_Q^{\uparrow}\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\sum_{\alpha\neq\beta}\left[\frac{Pc_{\alpha\uparrow}c_{\beta\uparrow}^\dagger}{|z_\alpha-z_\beta|^2}\Psi_n^{\rm ho}\right](z_1,\dots,z_M|h). \tag{36}$$

To properly work out the contribution in Eq. (36), we have to express $\Psi_n^{\rm ho}$ in terms of the ↓-spin coordinates, $\eta_1,\dots,\eta_M$. In Appendix A we prove that

$$\Psi_n^{\rm ho}(z_1,\dots,z_M|h)=\Psi_n^{\rm ho}(\eta_1,\dots,\eta_M|h). \tag{37}$$

Therefore, we obtain

$$[h_Q^{\uparrow}\Psi_n^{\rm ho}](z_1,\dots,z_M|h)=\left\{\left[\frac{N^2-1}{12}+\frac{n(n-N)}{2}\right]h^n-\left[\frac{N-1}{2}-n\right]h^{n+1}\frac{\partial}{\partial h}+\frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\right\}\left\{\frac{\Psi_n^{\rm ho}(\eta_1,\dots,\eta_M|h)}{h^n}\right\}. \tag{38}$$

By using the identity

$$h^{n+1}\frac{\partial}{\partial h}\left\{\frac{\Psi_n^{\rm ho}(\eta_1,\dots,\eta_M|h)}{h^n}\right\}=\left\{\frac{N-1}{2}-\sum_j^M\frac{h}{h-z_j}\right\}\Psi_n^{\rm ho}(z_1,\dots,z_M|h)=\left\{\left(\frac{N-1}{2}\right)h^n-h^{n+1}\frac{\partial}{\partial h}\right\}\left\{\frac{\Psi_n^{\rm ho}(z_1,\dots,z_M|h)}{h^n}\right\}, \tag{39}$$

and the identity

$$h^{n+2}\frac{\partial}{\partial h^2}\left\{\frac{\Psi_n^{\rm ho}(\eta_1,\ldots,\eta_M|h)}{h^n}\right\} = \left\{\left[\frac{(N-1)(N-2)}{3}+2\sum_j^M\left(\frac{h}{h-z_j}\right)^2\right]h^n - (N-1)h^{n+1}\frac{\partial}{\partial h} + h^{n+2}\frac{\partial^2}{\partial h^2}\right\}\left\{\frac{\Psi_n^{\rm ho}(z_1,\ldots,z_M|h)}{h^n}\right\}, \quad (40)$$

we finally derive

$$[h_Q^{\uparrow}\Psi_n^{\rm ho}](z_1,\ldots,z_M|h) = \left\{\frac{n(n-1)}{2}h^n+\sum_j^M\frac{h^{n+2}}{(h-z_j)^2}-nh^{n+1}\frac{\partial}{\partial h} + \frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\right\}\Psi_n^{\rm ho}(z_1,\ldots,z_M|h). \quad (41)$$

Adding Eqs. (32)–(35) and Eq. (41) all together, we find that

$$[H_{KY}\Psi_n^{\rm ho}](z_1,\ldots,z_M|h) = \frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left\{-\frac{N(N^2-1)}{48}+n(n-M-1)\right\} \times\Psi_n^{\rm ho}(z_1,\ldots,z_M|h). \quad (42)$$

This is the formula we had to prove.

C. Crystal momentum

The state $|\Psi_n^{\rm ho}\rangle$ is a propagating holon with crystal momentum

$$q_n^{\rm ho}=\frac{\pi}{2}N+\frac{2\pi}{N}\left(n-\frac{1}{4}\right)(\text{mod } 2\pi), \quad (43)$$

with the definition

$$\Psi_n^{\rm ho}(z_1z,\ldots,z_Mz|hz)=\exp(iq_n^{\rm ho})\Psi_n^{\rm ho}(z_1,\ldots,z_M|h). \quad (44)$$

Rewriting the eigenvalue as

$$\mathcal{H}_{KY}|\Psi_n^{\rm ho}\rangle=\left\{-J\left(\frac{\pi^2}{24}\right)\left(N+\frac{5}{N}-\frac{3}{N^2}\right)+E(q_n^{\rm ho})\right\}|\Psi_n^{\rm ho}\rangle, \quad (45)$$

we obtain the dispersion relation

$$E(q_n^{\rm ho})=-\frac{J}{2}\left[\left(\frac{\pi}{2}\right)^2-(q_n^{\rm ho})^2\right]\ (\text{mod } \pi). \quad (46)$$

We worked out Eq. (46) for the case of negative-energy one-holon eigenstates. Unlike in the spinon case, the holon dispersion relation around the band minimum is quadratic in q, while it is linear in q near $\pi/2$, where the band closes.

Positive-energy one-holon eigenstates may be constructed by supersymmetrically rotating one-spinon eigenstates.[16] Their energy is given by $|E(q_n^{\rm ho})|$ and the momentum spans the remaining half of the Brillouin zone. The corresponding dispersion relation is plotted in Figs. 1 and 2. Since for the purpose of studying spinon-holon interaction we need negative-energy holon states only, we do not discuss here positive-energy holon states. We also need the wave function for a localized holon at site h_0, $|\Psi_{h_0}^{\rm ho}\rangle$, which is obtained by Fourier transforming the propagating holon wave function to real space,

$$\Psi_{h_0}^{\rm ho}=\sum_{n=0}^{(N+1)/2}h_0^{-n}\Psi_n^{\rm ho}. \quad (47)$$

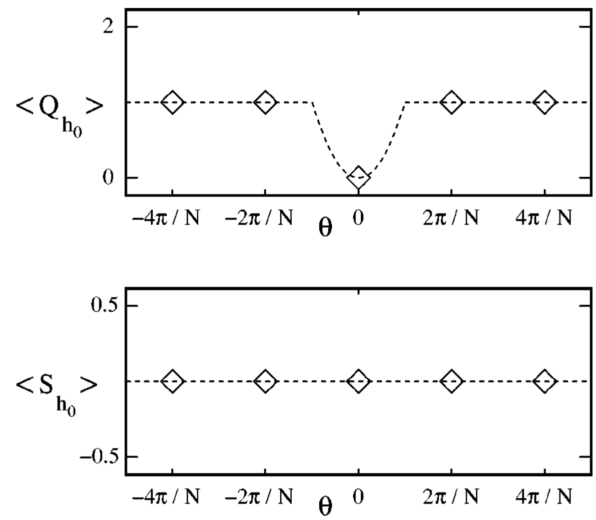

FIG. 1. Spin and charge profiles of the localized holon $|\Psi_{h_0}^{\rm ho}\rangle$ defined by Eq. (47). θ is defined as $\theta=-i\ln(z/h_0)$, where z is the independent variable.

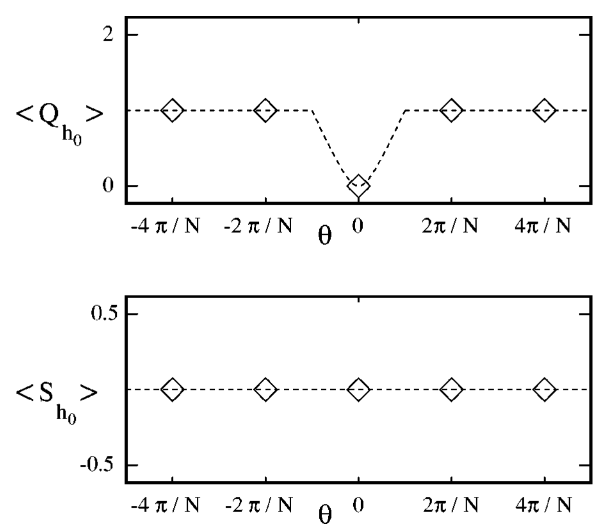

FIG. 2. Upper graph, holon dispersion relation. Positive-energy holons are unstable towards decay into spinons; Lower, configuration space for $N=9$ and $N=11$ sites

D. The norm

The norm of the state Ψ_n^{ho} is defined as

$$\langle \Psi_n^{\mathrm{ho}} | \Psi_n^{\mathrm{ho}} \rangle = \sum_{z_1, \ldots, z_M; h} |\Psi_n(z_1, \ldots, z_M | h)|^2. \quad (48)$$

From the definition of the norm it immediately follows that $\langle \Psi_n^{\mathrm{ho}} | \Psi_n^{\mathrm{ho}} \rangle$ does not depend on n. The formula for $\langle \Psi_n^{\mathrm{ho}} | \Psi_n^{\mathrm{ho}} \rangle$ is worked out in Appendix B and is given by

$$\langle \Psi_n^{\mathrm{ho}} | \Psi_n^{\mathrm{ho}} \rangle = N^{M+1} \frac{(2M)!}{2^M} \frac{\Gamma\left[\frac{1}{2}\right]\Gamma[M+1]}{\Gamma\left[M+\frac{1}{2}\right]}. \quad (49)$$

VI. ONE-SPINON ONE-HOLON WAVE FUNCTION

As for many-spinon configurations, spinons and holons maintain their integrity when many of them are present in the same state. Therefore, we can diagonalize $\mathcal{H}_{KY}$ within subspaces with a fixed number of spinons and holons.

In this section we derive the action of $\mathcal{H}_{KY}$ on the one-spinon one-holon eigenstates, diagonalize the corresponding matrix, and work out the norm of the states.

A. Action of $\mathcal{H}_{KY}$ on one-spinon one-holon states

To construct one-spinon one-holon eigenstates of $\mathcal{H}_{KY}$ we start with states in a mixed representation Ψ_n^s, where the spinon is localized at site s, but the holon is propagating with momentum q_n^{ho}. For N even and $M = N/2 - 1$, we have

$$\Psi_s^n(z_1, \ldots, z_M | h) = h^n \prod_j^M (z_j - s)(z_j - h) z_j \prod_{j<k}^M (z_j - z_k)^2, \quad (50)$$

where $1 \leqslant n \leqslant M+2$. To derive the action of the KY Hamiltonian on Ψ_n^s, we split it as in Eq. (31).

The action of spin-exchange term on Ψ_s^n provides

$$[h_S^T \Psi_s^n](z_1, \ldots, z_M | h)$$
$$= -\sum_{j=\uparrow} \sum_{z_\beta \neq z_j} \frac{z_j z_\beta^2}{(z_j - z_\beta)^2} \Psi_s^n(z_1, \ldots, z_\beta, \ldots | h)$$
$$+ \frac{M}{12}(-2N+5)\Psi_s^n - \sum_{i \neq j = \uparrow} \frac{1}{|z_i - z_j|^2} \Psi_s^n + (M-1)$$
$$\times \left[s \frac{\partial}{\partial s} \Psi_s^n + h^{n+1} \frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right) \right]$$
$$- s^2 \frac{\partial^2}{\partial s^2} \Psi_s^n - h^{n+2} \frac{\partial^2}{\partial h^2}\left(\frac{\Psi_s^n}{h^n}\right) + \frac{1}{2}\left(\frac{h+s}{h-s}\right)$$
$$\times \left[s \frac{\partial}{\partial s} \Psi_s^n - h^{n+1} \frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right) \right]. \quad (51)$$

The Ising term gives

$$[h_S^V \Psi_s^n](z_1, \ldots, z_M | h) = \left\{ \sum_{i \neq j = \uparrow} \frac{1}{|z_i - z_j|^2} - \frac{M}{2} \frac{N^2 - 1}{12} + \sum_{j = \uparrow} \frac{1}{|z_j - h|^2} \right\} \Psi_s^n. \quad (52)$$

The Coulomb potential term acting on the one-spinon one-holon wave function reads

$$[h_N \Psi_s^n](z_1, \ldots, z_M | h) = \frac{1 - N^2}{24} \Psi_s^n. \quad (53)$$

The $\downarrow$-spin contribution to the charge kinetic energy gives

$$[h_Q^\downarrow \Psi_s^n](z_1, \ldots, z_M | h)$$
$$= -\sum_{z_\beta \neq h} \frac{z_\beta h}{(z_\beta - h)^2} \Psi_s^n(z_1, \ldots, z_M | z_\beta)$$
$$= \left[\frac{N^2 - 1}{12} + \frac{n(n-N)}{2} \right] \Psi_s^n - \left[\frac{N-1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right)$$
$$+ \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2}\left(\frac{\Psi_s^n}{h^n}\right). \quad (54)$$

In order to manipulate the $\uparrow$-spin contribution to the charge kinetic energy, we need to express $\Psi_s^n(z_1, \ldots, z_M | h)$ in terms of the $\downarrow$-spin coordinates η. As in the one-holon case, we obtain

$$\Psi_s^n(z_1, \ldots, z_M | h) = \Psi_s^n(\eta_1, \ldots, \eta_M | h).$$

Therefore, we have

$$[h_Q^\uparrow \Psi_s^n](\eta_1, \ldots, \eta_M | h)$$
$$= -\sum_{z_\beta \neq h} \frac{h z_\beta}{(h - z_\beta)^2} \Psi_s^n(\eta_1, \ldots, \eta_M | z_\beta)$$
$$+ \frac{sh}{(s-h)^2} \Psi_s^n(\eta_1, \ldots, \eta_M | s)$$
$$= \left[\frac{N^2 - 1}{12} + \frac{n(n-N)}{2} \right] \Psi_s^n - \left[\frac{N-1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right)$$
$$+ \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2}\left(\frac{\Psi_s^n}{h^n}\right) + \frac{sh}{(s-h)^2} \Psi_s^n(\eta_1, \ldots, \eta_M | s). \quad (55)$$

By going back to $\uparrow$-spin coordinates z, Eq. (55) provides

$$\frac{n^2-2n+1}{2}\Psi_s^n+\left(\frac{1-n}{2}\right)\frac{h+s}{h-s}\Psi_s^n+\left(\frac{3}{2}-n\right)h^{n+1}\frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right)+\frac{hs}{(h-s)^2}\Psi_s^n+\frac{1}{2}\left(\frac{h+s}{h-s}\right)h^{n+1}\frac{\partial}{\partial h}\left(\frac{\Psi_s^n}{h^n}\right)+\frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\left(\frac{\Psi_s^n}{h^n}\right)-\sum_{z_j}\frac{1}{|z_j-h|^2}\Psi_s^n-\frac{sh}{(s-h)^2}\left(\frac{s}{h}\right)^{n-1}\Psi_h^n. \tag{56}$$

Adding up all the contributions together, we obtain

$$h_{KY}\Psi_s^n(z_1,\ldots,z_M|h)=\left\{\frac{-N^3+19N}{48}+\left(n-\frac{N}{2}-1\right)\right\}\Psi_s^n+(M-1)s\frac{\partial}{\partial s}\Psi_s^n-s^2\frac{\partial^2}{\partial s^2}\Psi_s^n+\frac{1}{2}\frac{h+s}{h-s}\left(s\frac{\partial}{\partial s}\Psi_s^n+(1-n)\Psi_s^n\right)+\frac{hs}{(h-s)^2}\left[\Psi_s^n-\left(\frac{s}{h}\right)^{n-1}\Psi_h^n\right]. \tag{57}$$

In the following section we will will solve Eq. (57) by working out the basis of one-spinon one-holon states that diagonalize h_{KY}.

B. One-spinon one-holon energy eigenstates

To diagonalize h_{KY}, we introduce the propagating one-spinon one-holon energy eigenstates

$$\Psi_{mn}(z_1,\ldots,z_M|h)=\sum_s\frac{s^{-m}}{N}\Psi_s^n(z_1,\ldots,z_M|h). \tag{58}$$

The second and the third rows of Eq. (57) are diagonal in the basis of the states Ψ_{mn}, and their contribution is given by

$$\left[\epsilon_0+n\left(n-1-\frac{N}{2}\right)+m(M-m)\right]\Psi_{mn}, \tag{59}$$

where

$$\epsilon_0=\frac{-N^3+19N}{48}.$$

On the other hand, the diagonalization of the "interaction" term, given by the fourth and the fifth rows of Eq. (57), needs further work. By using a straightforward, although tedious, application of basic identities proved in the Appendix of Ref. 9, one obtains

$$\sum_{s\in S^N}s^{-m}\left\{\frac{1}{2}\left(\frac{h+s}{h-s}\right)\left[s\frac{\partial}{\partial s}\Psi_s^n+(1-n)\Psi_s^n\right]+\frac{hs}{(h-s)^2}\left[\Psi_s^n-\left(\frac{s}{h}\right)^{n-1}\Psi_h^n\right]\right\}$$
$$=\sum_{s\neq h}s^{-m}\left\{\frac{1}{2}\left(\frac{h+s}{h-s}\right)\left[s\frac{\partial}{\partial s}\Psi_s^n+(1-n)\Psi_s^n\right]+\frac{hs}{(h-s)^2}\left[\Psi_s^n-\left(\frac{s}{h}\right)^{n-1}\Psi_h^n\right]\right\}+\lim_{s\to h}s^{-m}\left\{\frac{1}{2}\left(\frac{h+s}{h-s}\right)\times\left[s\frac{\partial}{\partial s}\Psi_s^n+(1-n)\Psi_s^n\right]+\frac{hs}{(h-s)^2}\left[\Psi_s^n-\left(\frac{s}{h}\right)^{n-1}\Psi_h^n\right]\right\}$$
$$=-\frac{N}{2}(m-n+1)\Psi_{mn}+\sum_{j=0}^{m}N(m-n+1)\Psi_{m-j,n-j} \tag{60}$$

if $n-m-1>0$, and

$$-\frac{N}{2}(n-m-1)\Psi_{mn}+\sum_{j=0}^{M-m}N(n-m-1)\Psi_{m+j,n+j} \tag{61}$$

if $n-m-1\leqslant 0$. Therefore the action of h_{KY} on Ψ_{mn} is given by

$$[h_{KY}\Psi_{mn}]=\left[\epsilon_0+m(M-m)+n\left(n-1-\frac{N}{2}\right)\right]\Psi_{mn}-\frac{1}{2}(n-m-1)\Psi_{mn}-(n-m-1)\sum_{j=1}^{m}\Psi_{m-j,n-j} \tag{62}$$

if $m-n+1<0$, and

$$[h_{KY}\Psi_{mn}]=\left[\epsilon_0+m(M-m)+n\left(n-1-\frac{N}{2}\right)\right]\Psi_{mn}-\frac{1}{2}(m-n+1)\Psi_{mn}-(m-n+1)\sum_{j=1}^{M-m}\Psi_{m+j,n+j}, \tag{63}$$

if $m-n+1\geqslant 0$. Since the Hamiltonian matrix is upper (or lower) triangular, complete diagonalization is possible. The energy eigenstates will be linear combinations of Ψ_{mn}

$$\Phi_{mn}=\sum_{j=0}^{m}a_j\Psi_{m-j,n-j} \tag{64}$$

if $m-n+1<0$, and

$$\Phi_{mn}=\sum_{j=0}^{M-m} a_j\Psi_{m+j,n+j} \tag{65}$$

if $m-n+1\geqslant 0$, corresponding to the energy eigenvalues

$$E_{mn}^{+}=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left[\epsilon_0+m(M-m)+n\left(n-1-\frac{N}{2}\right)-\frac{1}{2}(n-m-1)\right]$$

if $m-n+1<0$, and

$$E_{mn}^{-}=\frac{J}{2}\left(\frac{2\pi}{N}\right)^2\left[\epsilon_0+m(M-m)+n\left(n-1-\frac{N}{2}\right)+\frac{1}{2}(n-m-1)\right]$$

if $m-n+1\geqslant 0$. The coefficients a_l are defined by the recursion relation

$$a_l=-\frac{1}{2l}\sum_{k=0}^{l-1} a_k a_0=1. \tag{66}$$

In terms of spinon and holon momenta, E_{mn}^{+} and E_{mn}^{-} take the same form E_{mn}, given by

$$E_{mn}=E_{GS}+E(q_m^{\rm sp})+E(q_n^{\rm ho})-\frac{\pi J}{N}\frac{|q_m^{\rm sp}-q_n^{\rm ho}|}{2}. \tag{67}$$

E_{mn} is the sum of the ground-state energy, the energies of an isolated spinon and an isolated holon plus a negative interaction contribution that becomes negligibly small in the thermodynamic limit. Equations (64) and (65) can be inverted. The result is

$$\Psi_{mn}=\sum_{j=0}^{m} b_j\Phi_{m-j,n-j} \tag{68}$$

if $m-n+1<0$, and

$$\Psi_{mn}=\sum_{j=0}^{M-m} b_j\Phi_{m+j,n+j} \tag{69}$$

if $m-n+1\geqslant 0$. The coefficients are given by

$$b_j=\frac{\Gamma\left[j+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[j+1]}. \tag{70}$$

C. The norm

The squared norm of the state Ψ_{mn} is defined as

$$\langle\Phi_{mn}|\Phi_{mn}\rangle=\sum_{z_1,\ldots,z_M}|\Phi_{mn}(z_1,\ldots,z_M|)|^2. \tag{71}$$

In Fig. 1 we plot the charge and spin profiles for a one-holon state. In a similar fashion to the two-spinon case discussed in Ref. 9, we compute the norm of the one-spinon one-holon states by means of mathematical induction. The calculation is presented in detail in Appendix C. The basic induction relation in the case $n-k+1<0$ is given by

$$\frac{\langle\Phi_{kn}|\Phi_{kn}\rangle}{\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle}=\frac{\left(k-\frac{1}{2}\right)\left(M-k+\frac{3}{2}\right)}{k(M-k+1)}, \tag{72}$$

which provides the formula for the norm of the one-spinon one-holon energy eigenstates

$$\langle\Phi_{kn}|\Phi_{kn}\rangle=N^{M+1}\frac{(2M)!}{2^M}\left(M+\frac{1}{2}\right)\frac{\Gamma[M-k+1]\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[M-k+\frac{3}{2}\right]\Gamma[k+1]}. \tag{73}$$

In the complementary case, $n-k+1\geqslant 0$, we obtain

$$\langle\Phi_{kn}|\Phi_{kn}\rangle=\langle\Phi_{M-k,M-n}|\Phi_{M-k,M-n}\rangle=N^{M+1}\frac{(2M)!}{2^M}\left(M+\frac{1}{2}\right)\times\frac{\Gamma\left[M-k+\frac{1}{2}\right]\Gamma[k+1]}{\Gamma[M-k+1]\Gamma\left[k+\frac{3}{2}\right]}. \tag{74}$$

VII. SPINON-HOLON ATTRACTION

In this section we analyze the interaction between a spinon and a holon by constructing the real-space representation of the one-spinon one-holon wave function, and by studying the behavior of the corresponding probability as a function of the separation between the two particles. Our exact results show that a spinon and a holon interact through a short-range attraction identical, in the thermodynamic limit, to the atraction between two spinons.

The state for a localized spinon at site s and a localized holon at site h_0,Ψ_{sh_0} is defined as the Fourier transform of Ψ_s^n back to coordinate space

$$\Psi_{sh_0}=\sum_{n=1}^{M+2} h_0^{-n}\Psi_s^n. \tag{75}$$

Following the same steps as for the two-spinon wave functions,[8,9] we define the real-space coordinate representation for a spinon-holon pair, $s^m h_0^{-n}p_{mn}(s/h_0)$, as follows:

$$\begin{aligned}\Psi_{sh_0}&=\sum_{n=1}^{M+2}\sum_{m=0}^{n-2}s^m h_0^{-n}\sum_{j=0}^{m} b_j\Phi_{m-j,n-j}+\sum_{n=1}^{M+2}\sum_{m=n-1}^{M}s^m h_0^{-n}\sum_{j=0}^{M-m} b_j\Phi_{m+j,n+j}\\&=\sum_{n=1}^{M+2}\sum_{m=0}^{n-2}s^m h_0^{-n}p_{mn}\left(\frac{s}{h_0}\right)\Phi_{mn}+\sum_{n=1}^{M+2}\sum_{m=n-1}^{M}s^m h_0^{-n}p'_{mn}\left(\frac{s}{h_0}\right)\Phi_{mn},\end{aligned} \tag{76}$$

where $|\Phi_{mn}\rangle$ is an eigenstate of $\mathcal{H}_{KY}$ with eigenvalue E_{mn}, that is

$$\langle\Phi_{mn}|H_{KY}|\Psi_{sh_0}\rangle = E_{mn}\langle\Phi_{mn}|\Psi_{sh_0}\rangle. \tag{77}$$

The matrix element $\langle\Phi_{mn}|\mathcal{H}_{KY}|\Psi_{sh_0}\rangle$ can be written as a differential operator acting on the analytic extension of $\langle\Phi_{mn}|\Psi_{sh_0}\rangle$, where s and h_0 are understood to take any value on the unit circle. Therefore, by equating $\langle\Phi_{mn}|\mathcal{H}_{KY}|\Psi_{sh_0}\rangle$ to $E_{mn}\langle\Phi_{mn}|\Psi_{sh_0}\rangle$, it is straightforward to write down the equation of motion for the one-spinon one-holon wave function, which reads

$$\begin{aligned}(E_{mn}-E_{GS})\langle\Phi_{mn}|\Psi_{sh_0}\rangle &= \langle\Phi_{mn}|(H_{KY}-E_{GS})|\Psi_{sh_0}\rangle\\ &= J\left(\frac{2\pi}{N}\right)^2\Bigg\{\Bigg[\left(M-s\frac{\partial}{\partial s}\right)s\frac{\partial}{\partial s}\\ &\quad + h_0\frac{\partial}{\partial h_0}\left(1+\frac{N}{2}+h_0\frac{\partial}{\partial h_0}\right)+\frac{1}{2}\left(\frac{h_0+s}{h_0-s}\right)\\ &\quad \times\left(s\frac{\partial}{\partial s}+h_0\frac{\partial}{\partial h_0}+1\right)\Bigg]\langle\Phi_{mn}|\Psi_{sh_0}\rangle\\ &\quad + \frac{h_0}{s-h_0}\left(\frac{s}{h_0}\right)^{\nu}\langle\Phi_{mn}|\Psi_{h_0h_0}\rangle\Bigg\},\end{aligned} \tag{78}$$

where $\nu=M$ if $m-n+1<0$, $\nu=0$ otherwise. In the differential operator in Eq. (78), we recognize the sum of the energies of the free spinon and holon, a velocity-dependent interaction, which diverges at small spinon-holon separation, and another term that takes into account the correction for the case when the spinon and the holon are at the same position. By using Eqs. (76)–(78), we find the following equations for the "relative wave functions" $p_{mn}(z)$ and $p'_{mn}(z)$ $(z=s/h_0)$:

$$\left[2\frac{d}{dz}-\frac{1}{(1-z)}\right]p_{mn}(z)+\frac{z^{M-m-1}}{(1-z)}p_{mn}(1)=0 \tag{79}$$

if $m-n+1<0$, and

$$\left[2\frac{d}{d\left(\frac{1}{z}\right)}-\frac{1}{\left(1-\frac{1}{z}\right)}\right]p'_{mn}(z)+\frac{\left(\frac{1}{z}\right)^m}{\left(1-\frac{1}{z}\right)}p'_{mn}(1)=0 \tag{80}$$

if $m-n+1\geqslant 0$.

Equations (79) and (80) are first-order "Dirac-like" equations. They are first order because the spinon and the holon energy bands have opposite curvature. In this respect, they differ from the differential equation obtained in the two-spinon case, which was second order.[8,9] The corresponding solutions are given by

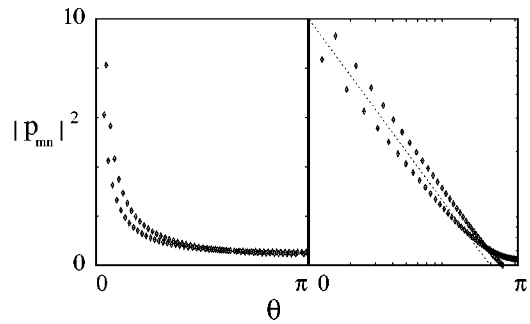

FIG. 3. Left panel, $|p_{mn}(e^{i\theta})|^2$ versus θ for $m=M, n=0$; right panel, the same plot on a log-log scale. The dashed straight line is a plot of $1/\theta$. The points show small oscillations about $1/\theta$. Oscillations decrease with the separation between spinon and holon, as it may be seen by comparison with the dashed line.

$$p_{mn}(z)=\sum_{k=0}^{M-m-1}\frac{\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[k+1]}z^k, \tag{81}$$

for Eq. (79) and

$$p'_{mn}(z)=\sum_{k=0}^{m}\frac{\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[k+1]}\left(\frac{1}{z}\right)^k \tag{82}$$

for Eq. (80).

The value of the spinon-holon wave function at zero separation between the particles is derived in Appendix D. It is given by

$$p_{mn}(1)=2\frac{\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[M-m]} \tag{83}$$

and

$$p'_{mn}(1)=2\frac{\Gamma\left[m+\frac{3}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[m+1]}. \tag{84}$$

Within the framework of our formalism, it is possible to treat spinons and holons, collective excitations of strongly correlated one-dimensional electron systems, as actual quantum-mechanical particles. We were first able to associate a two-particle wave function to a spinon-holon pair and to write down the corresponding equation of motion [Eqs. (76) and (78)]. We then worked out the exact wave functions corresponding to each energy eigenvalue [Eqs. (81) and (82)].

The squared modulus for the spinon-holon wave function, $|p_{mn}(z)|^2$, gives the probability for a spinon and a holon configuration as a function of the separation between the two particles. In Fig. 3 we plot $|p_{mn}(e^{i\theta})|^2$ versus the distance between the spinon and the holon, θ. From Fig. 3 the nature of the interaction between a spinon and a holon may be easily inferred. While at large separations $|p_{mn}(e^{i\theta})|^2$ does not depend on θ, as it is appropriate for noninteracting particles, at small separations it shows a remarkable enhancement. This corresponds to a huge increase in the probability of configurations with the spinon and the holon on top of each other. The enhancement does not depend on the holon momentum. As N gets larger, the probability enhancement peaks up. It survives the thermodynamic limit, even though the interaction energy goes to zero, and the total energy becomes the sum of the energies of the isolated spinon and holon. However, the attraction is not strong enough to create a spinon-holon bound state, even in the thermodynamic limit. This corresponds to the absence of a low-energy stable hole excitation, and it is what causes the quasiparticle peak to disappear.

An intriguing feature of the spinon-holon interaction in the thermodynamic limit is that it has the same power-law form as the spinon-spinon interaction derived in Refs. 8 and 9. In the right panel of Fig. 3 we plot $|p_{mn}(e^{i\theta})|^2$ on a log-log scale and compare it with $1/\theta$. The probability falls off as the first power of the separation between the two particles. This shows that, although the equation of motion for a spinon and a holon is quite different from the one for two spinons, the interaction in both cases results in a short-range attraction, and its effects on the corresponding two-particle wave function are basically the same.

VIII. HOLE SPECTRAL FUNCTION

In this section we work out $A_h^{\text{sp, ho}}(\omega,q)$, the one-spinon one-holon contribution to the hole spectral function $A_h(\omega,q)$. We show that this contribution [which provides quite a good approximation to $A_h(\omega,q)$ for $q\sim 0$] depends only on the p_{mn}'s and the p'_{mn}'s calculated at $z=1$ (that is, as the spinon and the holon lay at the same site). This allows us to obtain for any finite N a simple closed-form expression for $A_h^{\text{sp, ho}}(\omega,q)$, and to relate it to the spinon-holon interaction. In the thermodynamic limit, we obtain the previously known formula for the contribution of the one-spinon one-holon states to $A_h^{\text{sp, ho}}(\omega,q)$.[14] The formula in Ref. 14 shows that there is no low-energy hole pole in the hole spectral function, but rather a sharp square-root singularity followed by a branch cut. These features have also been experimentally detcted by means of ARPES experiments on quasi-1D insulator.[7]

The branch cut corresponds to the lack of integrity of the hole excitation, which breaks up into a spinon and a holon. Here we will show that, in the thermodymanic limit, the probability enhancement $p_{mn}(1)$ $[p'_{mn}(1)]$ turns into the square-root singularity at threshold for a spinon-holon pair. As a consequence, we prove that the square-root singularity in the hole spectral function is a direct consequence of the interaction between spinons and holons. Therefore, it can be directly experimentally measured.

We begin with the calculation of $A_h^{\text{sp, ho}}(\omega,q)$ for a finite lattice. In Lehman representation we obtain

$$A_h^{\text{sp, ho}}(\omega,q)=\text{Im}\left\{\sum_X \frac{\left|\langle X|\sum_{h_0}(h_0)^{-k}c_{h_0\uparrow}|\Psi_{GS}\rangle\right|^2}{\pi N\langle X|X\rangle\langle\Psi_{GS}|\Psi_{GS}\rangle}\times\frac{1}{\omega+i\eta-(E_X-E_{GS})}\right\}, \tag{85}$$

where $|X\rangle$ is an exact one-spinon one-holon eigenstate of $H_{KY},|\Phi_{mn}\rangle$ with energy $E_X=E_{mn}$, and $q=2\pi k/N$ [below we discuss why only forward-propagating states contribute to Eq. (85)].

Using $A_h^{\text{sp,ho}}(\omega,q)$ instead of $A_h(\omega,q)$ is equivalent to approximating

$$c_{h_0\uparrow}|\Psi_{GS}\rangle\approx\Psi_{h_0h_0}=\sum_{n=1}^{M+2}\sum_{m=0}^{n-2}h_0^{m-n}p_{mn}(1)\Phi_{mn}+\sum_{n=1}^{M+2}\sum_{m=n-1}^{M}h_0^{m-n}p'_{mn}(1)\Phi_{mn}. \tag{86}$$

[Equation (86) basically amounts to neglecting contributions to $c_{h_0\uparrow}|\Psi_{GS}\rangle$ coming from multi-spinon one-holon states.]

Since $\mathcal{H}_{KY}$ contains the Gutzwiller projector P, its matrix elements between states with at least a doubly occupied site are zero. Therefore, at half-filling, $A_h(q,\omega)$ takes contributions only from forward-propagating hole states. Hence, using Eqs. (85) and (86) we obtain

$$A_h^{\text{sp, ho}}(\omega,q)=\text{Im}\frac{1}{\pi}\left\{\sum_{l=2}^{M+2}\sum_{m=0}^{l-2}\frac{\delta_{k-m+l}p_{ml}^2(1)}{\omega+i\eta-(E_{mn}-E_{GS})}+\sum_{l=1}^{M+2}\sum_{m=l-1}^{M}\frac{\delta_{k-m+l}(p'_{ml})^2(1)}{\omega+i\eta-(E_{mn}-E_{GS})}\right\}\times\frac{\langle\Phi_{ml}|\Phi_{ml}\rangle}{\langle\Psi_{GS}|\Psi_{GS}\rangle}. \tag{87}$$

Equation (87) shows that only the p_{mn}'s at $z=1$ determine the spinon-holon contribution to the hole spectral function. Therefore, the contribution is completely determined by the spinon-holon interaction.

Let us now analyze the thermodynamic limit of Eq. (87). In the thermodynamic limit, the Γ functions can be approximated by using Stirling's formula

$$\Gamma[z]\approx\sqrt{\pi}(z-1)^{z-1/2}e^{-(z-1)}. \tag{88}$$

From Eqs. (87) and (88) we get, in the thermodynamic limit,

$$A_h^{\mathrm{sp,ho}}(q,\omega)\approx\frac{2}{\pi(M+1)}\left\{\sum_{l=2}^{M+2}\sum_{m=0}^{l-2}\frac{\delta_{r-m+l}}{\omega+i\eta-(E_{mn}-E_{GS})}\right.$$
$$\times\sqrt{\frac{M-m-\frac{1}{2}}{m}}+\sum_{l=2}^{M+2}\sum_{m=l-1}^{M}$$
$$\left.\times\frac{\delta_{r-m+l}}{\omega+i\eta-(E_{mn}-E_{GS})}\sqrt{\frac{m+\frac{1}{2}}{M-m}}\right\}. \qquad (89)$$

(Notice that, in order to stabilize the hole occupation at 1, we had to introduce a chemical potential $J\pi^2/4$, which is added to the energies E_{mn}.)

By defining the auxiliary variables

$$q_{\mathrm{sp}}=\frac{2\pi}{N}m,\quad q_{\mathrm{ho}}=\frac{2\pi}{N}l,$$

Eq. (89), in the thermodynamic limit, may be written in the form already obtained in Ref. 14.

$$A_h^{\mathrm{sp\ ho}}(\omega,q)=2\,\mathrm{Im}\int_0^{\pi}\frac{dq_{\mathrm{ho}}}{\pi}\left\{\int_0^{q_{\mathrm{ho}}}\frac{dq_{\mathrm{sp}}}{\pi}\sqrt{\frac{\pi-q_{\mathrm{sp}}}{q_{\mathrm{sp}}}}\right.$$
$$\left.+\int_{q_{\mathrm{ho}}}^{\pi}\frac{dq_{\mathrm{sp}}}{\pi}\sqrt{\frac{q_{\mathrm{sp}}}{\pi-q_{\mathrm{sp}}}}\right\}$$
$$\times\frac{\delta(q-q_{\mathrm{sp}}+q_{\mathrm{ho}})}{\omega-\mu+i\eta-E(q_{\mathrm{sp}},q_{\mathrm{ho}})}. \qquad (90)$$

In the region $0\leqslant q\leqslant\pi$ the integration of Eq. (90) gives

$$A_h^{\mathrm{sp\ ho}}(\omega,q)=\frac{1}{\pi^2Jq}\sqrt{\frac{J\left[q+\frac{\pi}{2}\right]^2-\omega}{\omega-J\left[q-\frac{\pi}{2}\right]^2}}\Theta\left[\omega-J\left(q\right.\right.$$
$$\left.\left.-\frac{\pi}{2}\right)^2\right]\Theta\left[J\left\{\frac{\pi^2}{4}+q(\pi-q)\right\}-\omega\right]. \qquad (91)$$

This formula shows that the spinon-holon probability enhancement $p_{mn}^2(1)$ turned into a square-root singularity in $A_h^{\mathrm{sp,\,ho}}(\omega,q)$ at the threshold energy for creation of a spinon-holon pair. Because the spinon-holon joint density of states is uniform, the main conclusion we trace from our calculation is that the sharp nonanalytic threshold in $A_h^{\mathrm{sp,ho}}(\omega,q)$ is the direct consequence of spinon-holon interaction.

IX. CONCLUSIONS

In this paper, we have extended the formalism introduced in Ref. 9 to analyze spinon interaction in the Haldane-Shastry model, to the case where also charge degrees of freedom are involved. Our formalism allows us to define a quantum-mechanical real-space representation of the one-spinon one-holon wave function. We construct a Dirac-like equation, whose solution is the spinon-holon wave function in real space coordinates. By means of a careful study of the real-space one-spinon one-holon wave function, we show the existence of the spinon-holon interaction and its survival in the thermodynamic limit. Spinon-holon interaction generates a short-range enhancement in the probability for a spinon and a holon to be on the same site. The attraction, however, is not strong enough to form a spinon-holon bound state, which would correspond to a Landau quasihole resonance. This makes, in the thermodynamic limit, the hole excitation fully unstable against decay in one-holon multi-spinon states and the quasiparticle peak disappear. Correspondingly, in the thermodynamic limit, the probability enhancement develops into a square-root singularity followed by a branch cut, which reflects the full instability of the hole excitation. Hence, by means of a sequence of exact, straightforward steps, we prove that spinon-holon attraction is what makes Landau's Fermi-liquid theory break down in $1D$ strongly correlated electron systems.

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APPENDIX A: FUNCTIONS EXPRESSED IN TERMS OF ↓-SPIN COORDINATES

In this section we prove the formulas that express the states of the KY model in terms of the ↓-spin coordinates, once the expression in terms of the ↑-spin coordinates is known. The starting point is the following identity (z_α, z_β are Nth roots of the identity):

$$\prod_{\alpha:z_\alpha\neq z_\beta}(z_\alpha-z_\beta)=\lim_{z\to z_\beta}\frac{z^N-1}{z-z_\beta}=\frac{N}{z_\beta}. \qquad (A1)$$

The ground-state wave function at half-filling expressed in terms of the ↑-spin coordinates is given by

$$\Psi_{GS}(z_1,\ldots,z_M)=\prod_{i<j}^{M}(z_i-z_j)^2\prod_{t}^{M}z_t, \qquad (A2)$$

where N is even and $M=N/2$. Let $\eta_1,\ldots,\eta_M$ be the ↓-spin coordinates. Upon applying Eq. (A1), we get

$$\prod_{i<j}^{M}(z_i-z_j)^2\prod_t^M z_t=(-1)^{M(M-1)/2}\prod_{i\neq j}^{M}(z_i-z_j)\prod_t^M z_t$$
$$=(-1)^{M(M+1)/2}\frac{\prod_t^M z_t\prod_t^M\left(\frac{N}{z_t}\right)}{\prod_{z_j,\eta_i}(z_j-\eta_i)}$$
$$=\prod_{i<j}^{M}(\eta_i-\eta_j)^2\prod_t^M\eta_t, \tag{A3}$$

which proves that $\Psi_{GS}(z_1,\ldots,z_M)=\Psi_{GS}(\eta_1,\ldots,\eta_M)$.

The one-holon wave function is given by

$$\Psi_n^{\mathrm{ho}}(z_1,\ldots,z_M|h)=h^n\prod_j^M(z_j-h)\prod_{i<j}^M(z_i-z_j)^2\prod_t^M z_t, \tag{A4}$$

where now N is odd, $M=(N-1)/2$ and h is the coordinate of the empty site.

The same steps as for Ψ_{GS} apply to the one-holon wave function. We have

$$h^n\prod_j^M(z_j-h)\prod_{i<j}^M(z_i-z_j)^2\prod_t^M z_t$$
$$=h^n(-1)^{M(M+1)/2}\frac{\prod_t^M z_t\prod_t^M\left(\frac{N}{z_t}\right)}{\prod_{\eta_i,z_j}(z_j-\eta_i)}$$
$$=h^n\prod_i^M(\eta_i-h)\prod_{i<j}^M(\eta_i-\eta_j)^2\prod_t^M\eta_t; \tag{A5}$$

this proves that $\Psi_h^{\mathrm{ho}}(z_1,\ldots,z_M|h)=\Psi_h^{\mathrm{ho}}(\eta_1,\ldots,\eta_M|h)$.

The one-spinon one-holon state $\Psi_s^n(z_1,\ldots,z_M|h)$ is given by

$$\Psi_s^n(z_1,\ldots,z_M|h)=h^n\prod_j^M(z_j-s)(z_j-h)$$
$$\times\prod_{i<j}^M(z_i-z_j)^2\prod_t^M z_t, \tag{A6}$$

where N is even, $M=N/2-1$, s is the coordinate of the $\downarrow$-spin, and h is the location of the empty site. As in the previous cases, we have

$$h^n\prod_j^M(z_j-s)(z_j-h)\prod_{i<j}^M(z_i-z_j)^2\prod_t^M z_t$$
$$=h^n(-1)^{M(M+1)/2}\frac{\prod_t^M z_t\prod_t^M\left(\frac{N}{z_t}\right)}{\prod_{\eta_i,z_j}(z_j-\eta_i)}h^n$$
$$\times\prod_i^M(\eta_i-s)(\eta_i-h)\prod_{i<j}^M(\eta_i-\eta_j)^2\prod_t^M\eta_t. \tag{A7}$$

Equation (A7) provides the proof that $\Psi_s^n(z_1,\ldots,z_M|h)=\Psi_s^n(\eta_1,\ldots,\eta_M|h)$.

The last identity we need refers to the case where the spinon and holon are at the same site, which we had to consider in deriving Eq. (56). We have

$$\Psi_s^n(z_1,\ldots,z_M|s)=s^n\prod_j^M(z_j-s)^2\prod_{i<j}^M(z_i-z_j)^2\prod_t^M z_t$$
$$=s^n(-1)^{M(M-1)/2}\prod_j^M\frac{(z_j-s)}{(z_j-h)}\frac{\left(\prod_t^M z_t\right)\left[\prod_t^M\left(\frac{N}{z_t}\right)\right]}{\prod_{z_j,\eta_i}(z_j-\eta_i)}$$
$$=s^n\prod_j^M\frac{(z_j-s)}{(z_j-h)}\frac{\left(\prod_t^M z_t\right)\left[\prod_t^M\left(\frac{N}{z_t}\right)\right]}{\prod_i^M\left(\frac{N}{\eta_i}\right)}\prod_{i<j}^M(\eta_i-\eta_j)^2\prod_i^M(\eta_i-s)(\eta_i-h)$$
$$=-\left(\frac{s}{h}\right)^{n-1}h^n\prod_i^M(\eta_i-h)^2\prod_{i<j}(\eta_i-\eta_j)^2\prod_t^M\eta_t. \tag{A8}$$

Equation (A8) proves the identity

$$\Psi_s^n(z_1,\ldots,z_M|s)=-\left(\frac{s}{h}\right)^{n-1}\Psi_h^n(\eta_1,\ldots,\eta_M|h).$$

APPENDIX B: THE NORM OF ONE-HOLON WAVE FUNCTION

In this appendix we discuss in detail the calculation of the norm of the negative-energy one-holon states, $\langle\Psi_n^{\mathrm{ho}}|\Psi_n^{\mathrm{ho}}\rangle$,

$$\langle\Psi_n^{\rm ho}|\Psi_n^{\rm ho}\rangle=\sum_{z_1,\ldots,z_M,h}|\Psi_n^{\rm ho}(z_1,\ldots,z_M|h)|^2. \quad \text{(B1)}$$

Let $P_m(z_1,\ldots,z_M)$ be the mth degree symmetric polynomial in $z_1,\ldots,z_M$. One obtains[9]

$$\Psi_n^{\rm ho}(z_1,\ldots,z_M|h)=\sum_{m=0}^{M}h^{m+n}P_m(z_1,\ldots,z_M)\times\prod_{i<j}^{M}(z_i-z_j)^2. \quad \text{(B2)}$$

From Eqs. (B1) and (B2), we derive

$$\langle\Psi_n^{\rm ho}|\Psi_n^{\rm ho}\rangle=N\sum_{m=0}^{M}\sum_{z_1,\ldots,z_M}|P_m(z_1,\ldots,z_M)|^2\prod_{i<j}^{M}|z_i-z_j|^4 =N\sum_{m=0}^{M}\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle. \quad \text{(B3)}$$

The norm of one-spinon wave functions, $\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle$, has been derived in Ref. 9, where it has been shown that

$$\langle\Psi_m^{\rm sp}|\Psi_m^{\rm sp}\rangle=\frac{\Gamma[M+1]}{\Gamma\left[M+\frac{1}{2}\right]}\frac{\Gamma\left[m+\frac{1}{2}\right]\Gamma\left[M-m+\frac{1}{2}\right]}{\Gamma[m+1]\Gamma[M-m+1]}N^M\frac{(2M)!}{2^M}. \quad \text{(B4)}$$

Therefore, we can rewrite Eq. (B3) as

$$\langle\Psi_n^{\rm ho}|\Psi_n^{\rm ho}\rangle=\Gamma\left[\frac{1}{2}\right]\frac{(2M)!}{2^M}g_{M0}(1), \quad \text{(B5)}$$

where the polynomial $g_{mn}(z)$ is the two-spinon relative wave function, denoted by $p_{mn}(z)$ in Ref. 9,

$$g_{mn}(z)=\frac{\Gamma[m-n+1]}{\Gamma\left[\frac{1}{2}\right]\Gamma\left[m-n+\frac{1}{2}\right]}\times\sum_{k=0}^{m-n}\frac{\Gamma\left[k+\frac{1}{2}\right]\Gamma\left[m-n-k+\frac{1}{2}\right]}{\Gamma[k+1]\Gamma[m-n-k+1]}z^k. \quad \text{(B6)}$$

By means of generic properties of hypergeometric functions[20] one gets

$$g_{mn}(1)=\frac{\Gamma\left[\frac{1}{2}\right]\Gamma[m-n+1]}{\Gamma\left[m-n+\frac{1}{2}\right]}. \quad \text{(B7)}$$

Therefore, we obtain

$$\langle\Psi_n^{\rm ho}|\Psi_n^{\rm ho}\rangle=N^{M+1}\frac{(2M)!}{2^M}\frac{\Gamma\left[\frac{1}{2}\right]\Gamma[M+1]}{\Gamma\left[M+\frac{1}{2}\right]}\quad\forall n. \quad \text{(B8)}$$

APPENDIX C: THE NORM OF ONE-SPINON ONE-HOLON ENERGY EIGENSTATES

In this section we generalize the recursion procedure introduced in Ref. 8 and 9 to calculate the norm of one-spinon and two-spinon wave function to the calculation of the norm of one-spinon one-holon energy eigenstates. As in the calculation in Refs. 8 and 9, the key operator is given by $e_1(z_1,\ldots,z_M)$, defined as

$$e_1(z_1,\ldots,z_M)=z_1+\cdots+z_M. \quad \text{(C1)}$$

The state for one holon and one spinon localized at s is given by

$$\Psi_s^n(z_1,\ldots,z_M)=\Phi_s^n(z_1,\ldots,z_M|h)\Psi_{GS}(z_1,\ldots,z_M), \quad \text{(C2)}$$

where

$$\Phi_s^n(z_1,\ldots,z_M|h)=h^n\prod_j^M(z_j-s)(z_j-h).$$

On $\Psi_s^n(z_1,\ldots,z_M)$, e_1 acts as a ladder operator, as we are going to show next.

In order to work out the action of H_{KY} on holon eigenstate, we split it into five terms: $H_{KY}/(J/2)(2\pi/N)^2=h_S^T+h_S^V+h_N+h_Q^{\downarrow}+h_Q^{\uparrow}$. Among those five terms, the only ones that do not commute with e_1 are the spin-exchange operator h_S^T and the $\uparrow$-spin charge propagation operator $h_Q^{\uparrow}$. On the state Ψ_s^n, h_S^T is realized as

$$h_S^T\Psi_s^n=\Psi_{GS}\left\{\text{const}+\frac{1}{2}\sum_i^M z_i^2\frac{\partial^2}{\partial z_i^2}+2\sum_{i\neq j}^M\frac{z_i^2}{z_i-z_j}\frac{\partial}{\partial z_i}-\frac{N-3}{2}\sum_i^M z_i^2\frac{\partial}{\partial z_i}\right\}\Phi_s^n. \quad \text{(C3)}$$

From Eq. (C3), we derive that

$$[h_S^Te_1\Phi_s^n\Psi_{GS}]-e_1\Phi_s^n[h_S^T\Psi_{GS}] =\Psi_{GS}\left\{\left[M-\frac{3}{2}\right]e_1\Phi_s^n+\sum_i^M z_i^2\frac{\partial}{\partial z_i}\Phi_s^n\right\} =\Psi_{GS}\left\{\left(M+\frac{1}{2}\right)e_1\Phi_s^n+M(s+h)\Phi_s^n -s^2\frac{\partial}{\partial s}\Phi_s^n-h^{n+2}\frac{\partial}{\partial h}\left(\frac{\Phi_s^n}{h^n}\right)\right\}. \quad \text{(C4)}$$

In the sense clarified by Eq. (C4), e_1 commutes with h_S^V, h_N, and $h_Q^{\downarrow}$. On the other hand, it does not commute with

$h_Q^{\uparrow}$. Indeed, since, in order to derive the action of $h_Q^{\uparrow}$ on Ψ_s^n we have to express the state in terms of the $\downarrow$-spin coordinates, we must do the same with e_1. In order to do so, we notice that $\{z_j\}$, $\{\eta_j\}$, h, and s taken all together are the set of the Nnth roots of 1. Therefore, we obtain

$$z_1+\cdots+z_M+\eta_1+\cdots+\eta_M+h+s=0,$$

which implies

$$e_1(z_1,\ldots,z_M)=-e_1(\eta_1,\ldots,\eta_M)-h-s. \tag{C5}$$

The action of $h_Q^{\uparrow}$ on $e_1\Psi_s^n$ gives

$$\begin{aligned}[h_Q^{\uparrow}e_1]\Psi_s^n=\Psi_{GS}\Bigg\{&\left[\frac{N^2-1}{12}+\frac{n(n-N)}{2}\right]e_1\Phi_s^n(\{\eta\}|h)-\left[\frac{N-1}{2}-n\right]h^{n+1}\frac{\partial}{\partial h}\left[\left(-h-s-\sum_j\eta_j\right)\frac{\Phi_s^n(\{\eta\}|h)}{h^n}\right]\\&+\frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\left[\left(-h-s-\sum_j^M\eta_j\right)\frac{\Phi_s^n(\{\eta\}|h)}{h^n}\right]+\frac{sh}{(s-h)^2}\left[-\sum_j\eta_j-2s\right]\Phi_s^n(\{\eta\}|s)\Bigg\}\\&=\Psi_{GS}e_1[h_Q^{\uparrow}\Phi_s^n](\eta_1,\ldots,\eta_M|h)+\left[\frac{N-1}{2}-n\right]h\Phi_n^s(\eta_1,\ldots,\eta_M|h)-h^{n+2}\frac{\partial}{\partial h}\left(\frac{\Phi_s^n(\eta_1,\ldots,\eta_M|h)}{h^n}\right)\\&+\frac{h}{h-s}\Phi_s^{n+1}(\eta_1,\ldots,\eta_M|s).\end{aligned} \tag{C6}$$

Equation (C6) yields the result

$$\begin{aligned}[h_Q^{\uparrow},e_1]\Psi_s^n=\Psi_{GS}\Bigg\{&h\frac{h}{h-s}\left[\Phi_s^n(z_1,\ldots,z_M|h)\right.\\&\left.-\left(\frac{s}{h}\right)^n\Phi_h^n(z_1,\ldots,z_M|h)\right]\\&+h^{n+2}\frac{\partial}{\partial h}\left(\frac{\Phi_s^n(z_1,\ldots,z_M|h)}{h^n}\right)\\&-nh\Phi_s^n(z_1,\ldots,z_M|h)\Bigg\}.\end{aligned} \tag{C7}$$

Upon summing Eqs. (C4) and (C7), we derive the basic relation we need in order to work out the recursion relations,

$$\begin{aligned}\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Psi_s^n=&\left(M+\frac{1}{2}\right)e_1\Psi_s^n+M(s+h)\Psi_s^n\\&-s^2\frac{\partial}{\partial s}\Psi_s^n+h\frac{h}{h-s}\sum_{m=0}^M\\&\times\left[s^m-\left(\frac{s}{h}\right)^nh^m\right]\Psi_{mn}-nh\Psi_s^n,\end{aligned} \tag{C8}$$

where we have introduced the one-spinon one-holon plane waves Ψ_{mn} defined in Sec. VI B.

In order to further manipulate terms in Eq. (C8), let us consider, now, the identity

$$\begin{aligned}&h\frac{h}{h-s}\sum_s\frac{s^{-k}}{N}\sum_{m=0}^M\left[s^m-\left(\frac{s}{h}\right)^nh^m\right]\Psi_{mn}\\&=\sum_{m=0}^M\Psi_{mn}\frac{h^{1+m-k}}{N}\sum_{\{s/h\}}\left[\frac{\left(\frac{s}{h}\right)^{n-k}}{\frac{s}{h}-1}-\frac{\left(\frac{s}{h}\right)^{m-k}}{\frac{s}{h}-1}\right]\\&=\frac{1}{N}\sum_{m=0}^{n-1}\Psi_{mn}h^{1+m-k}\sum_{\{s/h\}}\sum_{r=0}^{n-m-1}\left(\frac{s}{h}\right)^{m+r-k}\\&\quad-\frac{1}{N}\sum_{m=n+1}^M\Psi_{mn}h^{1+m-k}\sum_{\{s/h\}}\sum_{r=0}^{m-n-1}\left(\frac{s}{h}\right)^{n+r-k}.\end{aligned} \tag{C9}$$

(The symbol $\Sigma_{\{s/h\}}$ means that we have to sum over $s/h\in S^N$).

Suppose, now, $n-k+1<0$. In this case, the sum in Eq. (C9) over m from $n+1$ to M will give 0, while the second sum will be reduced to

$$\sum_{m=0}^k\Psi_{mn}h^{1+m-k}\sum_{\{s/h\}}\sum_{r=0}^{n-m-1}\left(\frac{s}{h}\right)^{m+r-k}.$$

On the other hand, as $k-n+1\geqslant 0$, the only nonzero term in the sum will be given by

$$-\sum_{m=k+1}^M\Psi_{mn}h^{1+m-k}\sum_{\{s/h\}}\sum_{r=0}^{m-n-1}\left(\frac{s}{h}\right)^{n-k+r}.$$

In order to write down the action of $[H_{KY},e_1]$ on the plane-wave states Ψ_{kn}, we multiply both members of Eq. (C7) by s^{-k} and sum over s. By using Eq. (C9), it is straightforward to derive the following equations.

If $k-n+1<0$,

$$\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Psi_{kn} = \left(M+\frac{1}{2}\right)e_1\Psi_{kn}+(M-k+1)\Psi_{k-1,n} + (M-n+1)h\Psi_{kn}+\sum_{m=0}^{k-1}\Psi_{mn}\frac{h^{1+m-k}}{N} \times\sum_{\{s/h\}}\sum_{r=0}^{n-m-1}\left(\frac{s}{h}\right)^{m-k+r}. \tag{C10}$$

If $k-n+1\geqslant 0$

$$\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Psi_{kn} = \left(M+\frac{1}{2}\right)e_1\Psi_{kn}+(M-k+1)\Psi_{k-1,n}+(M-n)h\Psi_{kn} - \sum_{m=k+1}^{M}\Psi_{mn}\frac{h^{1+m-k}}{N}\sum_{\{s/h\}}\sum_{r=0}^{m-n-1}\left(\frac{s}{h}\right)^{n-k+r}. \tag{C11}$$

Let us now work out the basic recursion relation in both cases.

Case $k-n+1<0$.

In this case energy eigenstates are given by

$$\Phi_{kn}=\sum_{l=0}^{k}a_l\Phi_{k-l,n-l}.$$

(see Sec. VI B for the definition of the coefficients a_j).

Therefore, we have

$$\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Phi_{kn}=\sum_{l=0}^{k}a_l\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Psi_{k-l,n-l}$$
$$=\sum_{l=0}^{k}a_l\left(M+\frac{1}{2}\right)e_1\Psi_{k-l,n-l}+\sum_{l=0}^{k-1}a_l(M-k+l+1)\Psi_{k-1-l,n-l}+\sum_{l=0}^{k}a_l(M-n+l+1)\Psi_{k-l,n-l+1} + \sum_{l=0}^{k-1}a_l\sum_{m=0}^{k-l-1}\Psi_{m,n+1+m-k}\left[\frac{1}{N}\sum_{\{s/h\}}\sum_{r=0}^{n-m-1}\left(\frac{s}{h}\right)^{m-k+r}\right]. \tag{C12}$$

From Eq. (C12), we get

$$(E_{k-1,n}-E_{kn}-M-\tfrac{1}{2})\langle\Phi_{k-1,n}|e_1|\Phi_{kn}\rangle$$
$$=\sum_{l=0}^{k-1}a_l(M-k+l+1)\langle\Phi_{k-1,n}|\Psi_{k-1-l,n-l}\rangle+\sum_{l=0}^{k}a_l(M-n+l+1)\langle\Phi_{k-1,n}|\Psi_{k-l,n+1-l}\rangle+\langle\Phi_{k-1,n}|\Psi_{k-1,n}\rangle$$
$$=[M-k+2+b_1(M-n+1)+a_1(M-n+2)]\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle=(M-k+\tfrac{3}{2})\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle, \tag{C13}$$

(notice that, from their definition, we have $a_1=-1/2,b_1=1/2$). Equation (C13) may be recast in the following compact form:

$$\frac{\langle\Phi_{k-1,n}|e_1|\Phi_{kn}\rangle}{\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle}=-\frac{M-k+\frac{3}{2}}{2(M-k+1)}. \tag{C14}$$

Case $k-n+1\geqslant 0$.

In this case we have

$$\Phi_{kn}=\sum_{l=0}^{M-k}a_l\Phi_{k+l,n+l}.$$

Equation (C12) now takes the form

$$\left[\frac{H_{KY}}{\frac{J}{2}\left(\frac{2\pi}{N}\right)^2},e_1\right]\Phi_{kn}=\sum_{l=0}^{M-k}a_l\left(M+\frac{1}{2}\right)e_1\Psi_{k+l,n+l} + \sum_{l=0}^{M-k}a_l(M-k-l+1)\Psi_{k+l-1,n+l} + \sum_{l=0}^{M-k}a_l(M-n-l)\Psi_{k+l,n+1+l} - \sum_{l=0}^{M-k}a_l\sum_{m=k+l+1}^{M}\Psi_{m,n+1+m-k} \times\frac{1}{N}\sum_{\{s/h\}}\left[\sum_{r=0}^{m-n-1}\left(\frac{s}{h}\right)^{n-k+r}\right]. \tag{C15}$$

From Eq. (C15) and by working exactly as in the previous case, we get the identity

$$(E_{k-1,n}-E_{kn}-M-\tfrac{1}{2})\langle\Phi_{k-1,n}|e_1|\Phi_{kn}\rangle =(M-k+1)\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle, \tag{C16}$$

which yields the relation

$$\frac{\langle\Phi_{k-1,n}|e_1|\Phi_{kn}\rangle}{\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle}=-\frac{M-k+1}{2\left(M-k+\frac{1}{2}\right)}. \tag{C17}$$

In order to complete the recursion procedure, we need one more induction relation that is derived by inserting in the product $\langle\Phi_{ab}|e_1|\Phi_{cd}\rangle$ the product $e_M e_M^*=1$, where

$$e_M(z_1,\ldots,z_M)=z_1\cdots z_M .$$

From the definition of scalar product, Eq. (25), it is straightforward to show that

$$\langle\Phi_{ab}|e_1|\Phi_{cd}\rangle=\langle[(e_M)^2\Phi_{ab}]|e_1|[(e_M)^2\Phi_{cd}]\rangle =\langle\Phi_{M-c,M-d}|e_1|\Phi_{M-a,M-b}\rangle. \tag{C18}$$

By applying Eqs. (C18) to (C13), and by using Eq. (C17), we obtain

$$\langle\Phi_{k-1,n}|e_1|\Phi_{kn}\rangle=\langle\Phi_{M-k,M-n}|e_1|\Phi_{M-k+1,M-n}\rangle =-\frac{k}{2\left(k-\frac{1}{2}\right)}\langle\Phi_{kn}|\Phi_{kn}\rangle. \tag{C19}$$

By putting together Eqs. (C15) and (C19), one finally obtains

$$\frac{\langle\Phi_{kn}|\Phi_{kn}\rangle}{\langle\Phi_{k-1,n}|\Phi_{k-1,n}\rangle}=\frac{\left(k-\frac{1}{2}\right)\left(M-k+\frac{3}{2}\right)}{k(M-k+1)}, \tag{C20}$$

which provides the formula for the norm of the one-spinon one-holon energy eigenstates in the case $k-n+1<0$,

$$\langle\Phi_{kn}|\Phi_{kn}\rangle=N^{M+1}\frac{(2M)!}{2^M}\left(M+\frac{1}{2}\right)\frac{\Gamma[M-k+1]\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[M-k+\frac{3}{2}\right]\Gamma[k+1]}. \tag{C21}$$

In the complementary case, $k-n+1\geq 0$, we may follow the same step to prove that

$$\langle\Phi_{kn}|\Phi_{kn}\rangle=\langle\Phi_{M-k,M-n}|\Phi_{M-k,M-n}\rangle =N^{M+1}\frac{(2M)!}{2^M}\left(M+\frac{1}{2}\right)\frac{\Gamma\left[M-k+\frac{1}{2}\right]\Gamma[k+1]}{\Gamma[M-k+1]\Gamma\left[k+\frac{3}{2}\right]}. \tag{C22}$$

APPENDIX D: SOLUTION OF THE EQUATION OF MOTION FOR THE ONE-SPINON ONE-HOLON WAVE FUNCTION

In this appendix we will derive the solution to the equation of motion for the relative coordinate part of the spinon-holon wave functions, $p_{mn}(z),p'_{mn}(z)$. In order to do so, let us consider first the case $m-n+1<0$, in which we express the solution to Eq. (79) as a power series of z,

$$p_{mn}(z)=\sum_k a_k z^k. \tag{D1}$$

From Eq. (D1), we get the following equation for the coefficients a_k:

$$-2\sum_k k a_{k+1}z^k+\sum_k (2k-1)a_k z^k - z^{M-m}\sum_k a_k=0. \tag{D2}$$

As $k\leq M-m$, the following recursion relation between the a_k's holds:

$$\frac{a_{k+1}}{a_k}=\frac{k+\frac{1}{2}}{k+1}. \tag{D3}$$

Equation (D3) is satisfied by

$$a_k=\frac{\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[k+1]}. \tag{D4}$$

To calculate a_{M-m+1}, we need the following identity, valid for any positive integer R (C_0 is a closed path centered at $z=0$):

$$\begin{aligned}\sum_{k=0}^{R}\frac{\Gamma\left[k+\frac{1}{2}\right]}{\Gamma[k+1]}&=\sum_{k=0}^{R}\oint_{C_0}\frac{dz}{2\pi i}\frac{1}{z^{k+1}}\frac{1}{\sqrt{1-z}}\\&=\oint_{C_0}\frac{dz}{2\pi i z^{R+1}}\frac{z^{R+1}-1}{z-1}\frac{1}{\sqrt{1-z}}\\&=\oint_{C_0}\frac{dz}{z^{R+1}2\pi i}\frac{1}{(1-z)^{3/2}}\\&=2\frac{\Gamma\left[R+\frac{1}{2}\right]}{\Gamma[R+1]}\left(R+\frac{1}{2}\right).\end{aligned} \tag{D5}$$

Since the recursion relation for a_{M-m} is

$$-2(M-m)a_{M-m}+2(M-m-\tfrac{1}{2})a_{M-m-1}-\sum_k a_k=0, \tag{D6}$$

equation (D5) implies $a_{M-m}=0$ and

$$p_{mn}(z)=\sum_{k=0}^{M-m-1}\frac{\Gamma\left[k+\frac{1}{2}\right]}{\Gamma\left[\frac{1}{2}\right]\Gamma[k+1]}z^k. \tag{D7}$$

Equation (D7) provides the formula for $p_{mn}(z)$ used throughout the paper.

The same steps followed in the case $m-n+1<0$ allow us to find the spinon-holon wave function in the case $m-n+1\geq 0$, provided the coordinate z is substituted with $\xi=1/z$.

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