Correlations in the sine-Gordon model with finite soliton density

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We study the sine-Gordon (SG) model at finite densities of the topological charge and small SG interaction constant, related to the one-dimensional Hubbard model near half filling. Using the modified Wentzel-Kramers-Brillouin approach, we find that the spectrum of the Gaussian fluctuations around the classical solution reproduces the results of the Bethe ansatz studies. The modification of the collective coordinate method allows us to write down the action, free from infrared divergencies. The behavior of the density-type correlation functions is nontrivial and we demonstrate the existence of leading and subleading asymptotes. A consistent definition of the charge-raising operator is discussed. The superconducting-type correlations are shown to decrease slowly at small soliton densities, while the spectral weight of right (left) moving fermions is spread over neighboring "$4k_F$" harmonics.

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I. INTRODUCTION

In this paper we study the sine-Gordon (SG) model at finite densities of the topological charge and small SG interaction constant. The exactly solvable sine-Gordon model has a large number of applications in different subfields of condensed matter physics and statistical mechanics. The particular case of the SG model of our interest below arose previously in various studies of the one-dimensional Hubbard model close to half filling,[1] commensurate-incommensurate transition,[2] strongly correlated systems in higher dimensions,[3] collective excitations in a multifluxon Josephson junction,[4] superconducting[5] or quantum Hall[6] double-layer systems in parallel magnetic field, and phase excitations in ferroelectric liquid crystals.[7]

Theoretical investigations of the quantum SG model with the finite density of topological excitations (kinks) were undertaken by Haldane,[8] Caux and Tsvelik,[9] and Papa and Tsvelik.[10] These studies, employing the analysis of the Bethe ansatz equations, established the dispersion of low-lying excitations and the so-called Luttinger parameter, defining the asymptotic behavior of the correlation functions.

In the present work, we use an alternative method to investigate this problem, which was introduced by Dashen, Hasslacher, and Neveu,[11] and is usually called the modified Wentzel-Kramers-Brillouin (WKB) or semiclassical approach. In this method, one first finds the classical solution to the equation of motion and then analyzes the quantum fluctuations around this solution.[12] It is known that this method gives excellent agreement with the available exact solutions of several models, and particularly the SG model.

With the use of the modified WKB approach, we find that the spectrum of Gaussian fluctuations in our model gives the spectrum coinciding with the one obtained earlier by the Bethe ansatz method.[8,10] At the same time, applying the standard semiclassical treatment, we encounter strong infrared divergencies in higher orders of perturbation theory. The difficulties arise also with the definition of the so-called dual field associated with the charge-raising operator of the theory. It is known that a troublesome difficulty with the semiclassical approach is the existence of fluctuations bearing zero energy, which corresponds in our case to the translational symmetry of the problem. The treatment of these "zero modes" is best done by the collective coordinate method,[13] whose straightforward application, though, does not cure our problem. It is shown below, however, that it is possible to use a somewhat modified collective coordinate method, to arrive at the action, free from the infrared divergencies. Technically it is done by allowing the fluctuating field to enter the argument of the classical solution. The stable infrared action is achieved at the expense of a more complicated form of the correlations.

Proceeding this way, we obtain the leading and subleading asymptotic terms in the density-density correlation functions in agreement with earlier predictions by Haldane.[8] The definition of the charge-raising operator now becomes free from ambiguities and we can calculate the leading behavior of the superconducting-type correlations. Remarkably, the decay of the latter correlations is slow at small densities of kinks, an observation which confirms earlier results.[8,10]

We demonstrate a nonzero quantum average of the four-fermion umklapp operator in the Hubbard model away from half filling. Particularly, it results in the spectral density of the right-moving fermions being distributed over the neighboring $4k_F$ harmonics of the density.

The remaining part of the paper is organized as follows. We obtain and analyze the classical solution in Sec. II. The fluctuations around it are discussed in Sec. III. The difficulties with the standard application of the semiclassical approach are outlined in Sec. IV. A way of circumventing these difficulties is discussed in Sec. V, we present the form of the correlation functions here. Concluding remarks are found in Sec. VI. The essence of the collective coordinate method is given in the Appendix.

II. CLASSICAL SOLUTION

We consider the Lagrangian density of the form

$$\mathcal{L}=\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}(\partial_x\phi)^2-\frac{m^2}{\beta^2}(1-\cos\beta\phi)+H\frac{\beta}{2\pi}\partial_x\phi \tag{1}$$

on a line of length L. The SG interaction constant is assumed to be small, $\beta \ll 1$, and m is the first breather mass. The term with the chemical potential H does not enter the equation of motion and defines the boundary conditions for the field ϕ. We write the topological charge density $\rho = (1/2\pi)\partial_x(\beta\phi)$ and the average density $\overline{\rho} = Q/L$. Using the well-known correspondence between the fermions and bosons in one spatial dimension,[14] we associate ρ with the fermionic density. Particularly, $\cos\beta\phi$ appears as an umklapp term in the charge sector of the Hubbard model and $\overline{\rho}$ measures the deviation from half filling, as $2\pi\overline{\rho} = 4k_F - 2\pi$. The total charge Q is given by

$$Q = \int_0^L \rho dx = \frac{1}{2\pi}\beta\phi\big|_0^L, \tag{2}$$

which defines the boundary conditions for ϕ.

In what follows, we employ the quasiclassical method of analysis of the quantum system defined by Eq. (1). This method arises most naturally in path integral formulation of the problem and basically corresponds to the steepest descent method. As a first step, one finds the classical configuration of the field, delivering the extremum to the exponentiated action, and then analyzes the spectrum of fluctuations around it. The general criterion for the applicability of the method is the formally large value of the classical action; it was shown, however, that the quasiclassics gives exact results for a particular case of sine-Gordon model with a zero density of the topological charge.[11]

For later convenience we make a shift $\beta\phi \to \beta\phi + \pi$ writing the potential term in the form

$$U(\phi) = \frac{m^2}{\beta^2}(1 + \cos\beta\phi). \tag{3}$$

As a first step we seek a static classical solution ϕ_0 by variating $\mathcal{L}$ in $\delta\phi$. We have

$$\partial_x^2\phi_0 - U'(\phi_0) = 0. \tag{4}$$

Multiplying it by $\partial_x\phi_0$ and integrating over x we obtain

$$(\partial_x\phi_0)^2 = 2U(\phi_0) + C$$

with some constant of integration C, which is integrated again to give

$$x = \int^{\phi_0} \frac{dy}{\sqrt{2U(y) + C}}.$$

Letting

$$C = \frac{4m^2}{\beta^2}\frac{1-k^2}{k^2}$$

with yet undetermined k, we get

$$x = \frac{k}{m}F(\beta\phi_0/2, k) + \text{const},$$

with incomplete elliptic integral F.[15] Inverting the last equality, we have

$$\phi_0 = 2\beta^{-1}\text{am}(xm/k, k), \tag{5}$$

where $\text{am}(x,k)$ the Jacobi amplitude function[15] with the elliptic index k. This index is determined by the above total variation of the field $(\beta\phi/2)$ which results in the equation[16]

$$\frac{m}{2\overline{\rho}} = kK(k) \tag{6}$$

with the complete elliptic integral K. Henceforth we will omit the index k as a second argument of the elliptic functions and use the conventions $k_1 = \sqrt{1-k^2}, K(k) = K$, $K(k_1) = K', E(k) = E$. From Eq. (6) we have

$$k_1 \simeq 4e^{-m/2\overline{\rho}}, \quad \overline{\rho} \ll m, \tag{7}$$

$$\simeq 1, \quad \overline{\rho} \gg m. \tag{8}$$

Introducing the soliton mass $M_s = 8m/\beta^2$ and rescaling the field $H = M_s h$ we write the energy of this classic solution $\mathcal{E}_0 = -\mathcal{L}[\phi_0]$ as follows:

$$\frac{\beta^2}{2m^2}\frac{\mathcal{E}_0}{L} = \frac{2E - k_1^2 K}{k^2 K} - \frac{2h}{kK}. \tag{9}$$

The index k, defining the density (number of kinks), should be chosen in order to minimize the energy $\mathcal{E}_0$. Differentiating Eq. (9) by k, we find that the energy attains its minimum at

$$E/k = h \tag{10}$$

and this minimum has a form

$$\mathcal{E}_0 = -\frac{2m^2 L}{\beta^2}\frac{k_1^2}{k^2}. \tag{11}$$

The last expression coincides, up to the sign, with the pressure $p = -\partial\mathcal{E}_0/\partial L|_Q$. The enthalpy of the system is given by $(2m^2L/\beta^2)(2E/k^2K)$.

A. Number of kinks

At smaller fields, $h < 1$, there are no solutions to Eq. (10). The critical value of the field for the appearance of the "kink condensate" $H_{cr} = M_s$ was found earlier in Ref. 10. At fields slightly exceeding this critical value, we have for the density $(m/\overline{\rho})e^{-m/\overline{\rho}} \simeq (h-1)/4$. Upon further increase of the field, we have approximately linear relation between the field and the density of kinks

$$\overline{\rho} \simeq m\left(\ln\frac{4H_{cr}}{H - H_{cr}}\right)^{-1}, \quad H \simeq H_{cr}, \tag{12}$$

$$\simeq H\beta^2/(4\pi^2), \quad H \gg H_{cr}. \tag{13}$$

Numerically, the asymptotic expression for large kink densities (13) holds with a good accuracy already at $H \geq 2H_{cr}$.

B. Susceptibility

Consider now the susceptibility defined as $\chi = -L^{-1}\partial^2\mathcal{E}_0/\partial H^2$. A simple calculation gives

$$\chi = \frac{\beta^2}{16}\frac{E}{k_1^2K^3} \tag{14}$$

$$\simeq \frac{\beta^2}{8(h-1)}\left(\ln\frac{8}{h-1}\right)^{-2}, \quad h\simeq 1 \tag{15}$$

$$\simeq (\beta/2\pi)^2, \quad h\gg 1. \tag{16}$$

Our expressions for the kink density $\overline{\rho}$ and susceptibility χ are identical with those obtained earlier[10] in the large-field limit, Eqs. (13),(16). At the fields $H\simeq H_{\rm cr}$ our expressions for ρ and χ coincide with their counterparts in Ref. 10 up to an overall factor $\pi/2$.

Near its minimum at $\rho=\overline{\rho}$, the energy can be expanded as follows:

$$\mathcal{E}_0 \simeq \frac{L}{2\chi}(\rho-\overline{\rho})^2 \tag{17}$$

or, $1/\chi = \pi v_N$, in the notations by Haldane.[8]

Another important parameter v_J, according to Haldane, is given by the coefficient in the total Hamiltonian $\mathcal{H} = v_J P^2/(2\pi L\overline{\rho^2})$, at small values of the total field momentum $P=\int_0^L dx\,\dot{\phi}\phi'$. It can be easily shown, that $\mathcal{H} = P^2/2\mathcal{M}_{\rm tot}$ with the mass of the kinks' condensate $\mathcal{M}_{\rm tot} = \int_0^L dx\,(\partial_x\phi_0)^2 = LE/K(2m/\beta k)^2 = QH$. As a result, one finds $v_J = \pi\overline{\rho}/H = \pi\beta^2/(16KE)$. Therefore, one expects[8] the Fermi velocity $v_F = \sqrt{v_N v_J} = k_1 K/E$ and the Luttinger parameter $\mathcal{K} = \sqrt{v_J/v_N} = \pi\beta^2/(16k_1K^2)$, both expectations verified below.

III. FLUCTUATIONS AROUND THE CLASSICAL SOLUTION

A. Lame equation

Now we consider the fluctuations around the classical solution ϕ_0. We write $\phi=\phi_0+\eta$ and expand the Lagrangian into Taylor series[12]

$$\mathcal{L} = \mathcal{L}[\phi_0] + \mathcal{L}_1 \tag{18}$$

$$\mathcal{L}_1 = \frac{1}{2}(\partial_t\eta)^2 - \frac{1}{2}(\partial_x\eta)^2 + \frac{1}{2}m^2(\cos\beta\phi_0)\eta^2 + \text{higher orders in } \eta. \tag{19}$$

The term linear-in-η drops out from this expression and the higher orders in η contain additional powers of β, and we will neglect them for the moment. We look for the solutions η in the form $\eta(x,t) = \Sigma c_i(t)\eta_i(x)$, obeying cyclic boundary conditions. Integrating by parts, we write

$$\mathcal{L}_1 = \frac{1}{2}\eta(-\partial_t^2 + \partial_x^2 + m^2\cos\beta\phi_0)\eta. \tag{20}$$

Next, we seek the "normal modes," satisfying

$$(-\partial_x^2 - m^2\cos\beta\phi_0)\eta_i(x) = \omega_i^2\eta_i(x). \tag{21}$$

The dynamics of these normal modes is simple, $c(t) = e^{\pm i\omega t}c$. Introducing the variable $z = xm/k$ we rewrite Eq. (21) in the form[4]

$$\frac{d^2}{dz^2}\eta = (A + 2k^2 sn^2 z)\eta, \tag{22}$$

with $A = -k^2(\omega^2/m^2+1)$ and $\mathrm{sn}(x)$ Jacobian elliptic function. This is the Jacobi form of the Lamé equation and the solutions to it are described in the literature (Ref. 17, Chap. 23.71). These solutions are parametrized by an index α and are explicitly written as

$$\eta(z,\alpha) = \frac{\mathrm{H}(z-\alpha)}{\Theta(z)}\exp[z\mathrm{Z}(\alpha)] \tag{23}$$

with Jacobi functions $\mathrm{H}(u) = \vartheta_1[\pi u/(2K)]$, $\Theta(u) = \vartheta_4[\pi u/(2K)]$, and $Z(u) = d\ln\Theta(u)/du$; the nome of the ϑ functions $\tilde{q} = \exp(-\pi K'/K)$.[15] The structure of Eq. (23) shows that H/Θ is the modulating Bloch function and Z corresponds to a wave vector. The energy ω is given by

$$\omega = \pm(m/k)dn(\alpha). \tag{24}$$

The second independent solution for η with the same energy (24) is obtained from Eq. (23) by changing $\alpha\to-\alpha$ which corresponds to the symmetry of Eqs. (22),(23) with respect to reflection $z\to -z$.

B. Properties of $\eta(z)$

One can show that the whole family of linearly independent solutions (23), which possess the real-valued energies and are periodic on a length of the chain, is exhausted by the values of α belonging to segments $(K-2iK', K]$ and $[-2iK', 0)$ in the complex plane. We will refer to these segments as I and II, respectively. The energies (24) and the normalized solutions (23) are doubly periodic in α with the periods $2K, 2iK'$.

The solutions (23) are quasiperiodic in $2K$, and

$$\eta(z+2K,\alpha) = -\eta(z,\alpha)\exp[2KZ(\alpha)]. \tag{25}$$

Introducing the Floquet index ν by $e^{-2i\nu K} = -e^{2KZ(\alpha)}$, we have

$$\nu \equiv iZ(\alpha) + \frac{\pi}{2K} = \frac{\pi n}{QK} \tag{26}$$

with integer n. Returning to the original variable $x = zk/m$, we find the allowable wave vectors in the form

$$q = 2\pi\overline{\rho}[i(K/\pi)Z(\alpha) + 1/2] = 2\pi n/L. \tag{27}$$

Consider first the variation of the index α on segment I, the corresponding n in Eq. (27) in the range

$$-Q/2 < n \leq Q/2, \tag{28}$$

which means Q allowed values of n. The energies (24) of solutions $\eta(z,\alpha)$, with α from segment I, lie between

$$\omega=0, \quad q=0 \quad \text{at} \quad \alpha=K\pm iK',$$

and

$$\omega=\omega_1\equiv mk_1/k, \quad q=\pi\bar{\rho} \quad \text{at} \quad \alpha=K. \tag{29}$$

This part of the spectrum is interpreted as a band, originated due to hybridization of the bound states related to individual kinks.

In order to see this, we note that for one kink, $Q=1$, the only value $n=0$ is allowed in segment I. It is the quantized bound state with $\omega=0$ for the single-kink solution of the sine-Gordon equation. In the limit of low density $\bar{\rho}\ll m$, the energies of the lower band are exponentially small,

$$\omega\simeq 4me^{-m/2\bar{\rho}}\sin\frac{|q|}{2\bar{\rho}}, \tag{30}$$

which is the dispersion for a system of weakly coupled harmonic oscillators.

Another branch of the spectrum, parametrized by α from segment II, has one singular point at $\alpha=-iK'$, where both $Z(\alpha)$ and $dn(\alpha)$ have simple poles. This point corresponds to $q\to\pm\infty$ and $\omega\to\infty$.

The energy of this band has the minima

$$\omega=\omega_2\equiv m/k \quad \text{at} \quad |q|=\pi\bar{\rho}, \quad \alpha=-2iK',0. \tag{31}$$

The dispersion at $\bar{\rho}/m=0.26$ is shown in Fig. 1(a) in the extended Brillouin zone scheme. The same dispersion shown in the reduced Brillouin zone scheme is depicted in Fig. 1(b).

We see that there is an energy gap $\omega_2-\omega_1$ between the states in the lower and upper band. It is interesting to note the following relation:

$$\omega_2^2-\omega_1^2=m^2.$$

In the low-density limit $\bar{\rho}\ll m$, we write $\alpha=i\alpha'$, then $q\to -m\tan\alpha'$, $\omega\to m/\cos\alpha'$ so that $\omega=\sqrt{m^2+q^2}$.

At this point it is worthwhile to calculate the Fermi velocity v_F for low-lying excitations in the large-Q limit. Expanding ω and q near the point $\alpha_0=K-iK'$ of sector I, we have

$$\omega\simeq -im\frac{k_1}{k}(\alpha-\alpha_0), \quad q\simeq -i2\bar{\rho}E(\alpha-\alpha_0), \tag{32}$$

and

$$v_F=\omega/q=k_1K/E, \tag{33}$$

$$\simeq\sqrt{(h-1)\ln\left(\frac{8}{h-1}\right)}, \quad h\simeq 1, \tag{34}$$

$$\simeq 1-\left(\frac{\pi}{2h}\right)^4, \quad h\gg 1, \tag{35}$$

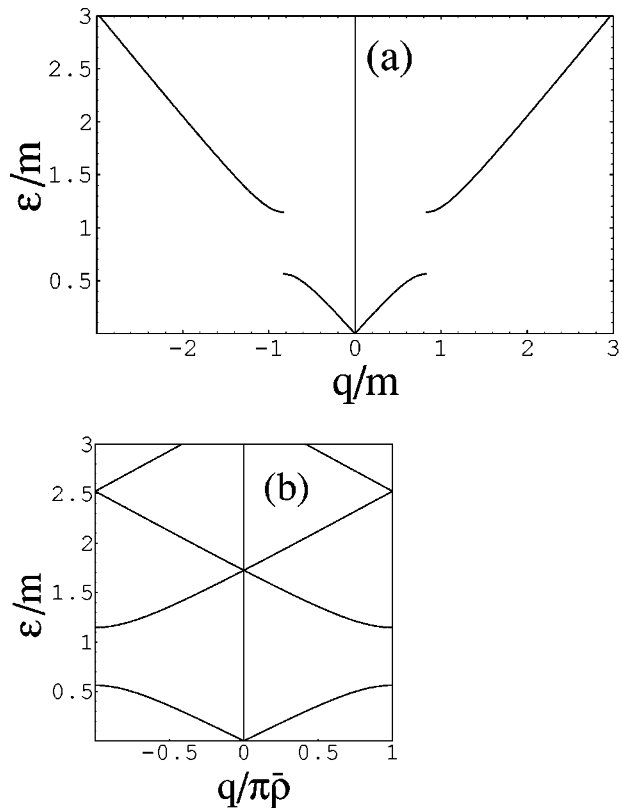

FIG. 1. The dispersion in the (a) extended and (b) reduced Brillouin zone scheme shown at $k=0.87$ and $\bar{\rho}/m=0.26$.

in agreement with the above estimate and the results in Ref. 10. As a result, the Luttinger parameter $\mathcal{K}=\pi\chi v_F$ is found in the form

$$\mathcal{K}=\frac{\pi\beta^2}{16}\frac{1}{k_1K^2}, \tag{36}$$

which shows that $\mathcal{K}$ decreases with increasing the density, and attains its limiting value $\beta^2/(4\pi)$ at $\bar{\rho}\to\infty$.

The form of the eigenfunctions (23) is somewhat simplified at the special points $\alpha=K-iK',K,0$, where one has [see Eqs. (29),(31)]

$$\eta_0(x)=(LE/K)^{-1/2}dn(xm/k), \quad \alpha=K-iK', \tag{37}$$

$$\eta_1(x)\propto cn(xm/k), \quad \alpha=K, \tag{38}$$

$$\eta_2(x)\propto sn(xm/k), \quad \alpha=0 \tag{39}$$

with the normalization factor in $\eta_0(x)$ made explicit. Henceforth, we will call the eigenfunction (37) the zero mode.

Analyzing the lowest-energy states with $\alpha\simeq K-iK'$, one can find a following expression, valid in the leading order in $q/\pi\bar{\rho}$:

$$\eta_q(x)=\eta_0(x)\exp\left(-iqx-i\frac{q\pi}{4KE\bar{\rho}}\frac{\vartheta_3'(\pi x\bar{\rho})}{\vartheta_3(\pi x\bar{\rho})}\right). \tag{40}$$

The second term in the exponential is the phase of the Bloch function. It is $1/\overline{\rho}$ periodic in x, vanishing at $x=0,\pm\overline{\rho}^{-1},\pm 2\overline{\rho}^{-1},\ldots$.

For completeness, we provide here an approximate expression for the eigenfunctions of the upper band in the low-density limit $\overline{\rho}\ll m$ and $k\to 1$. Near the position of nth kink, it is given by

$$\eta(x)_{\omega\simeq m}\propto[\tanh m(x-\overline{\rho}n)-i\tan\alpha]\times\exp[i(x-\overline{\rho}n)m\tan\alpha-inq/\overline{\rho}], \tag{41}$$

$$q=-m\tan\alpha+2\alpha\overline{\rho}+\pi\overline{\rho}, \tag{42}$$

in accordance with Ref. 18. Concluding this subsection, we note that Eqs. (33), (36), (40), and (41) extend the analysis of fluctuations performed by previous authors.[4,7,5]

C. Quantization of fluctuations

The differential operator in Eq. (22) is self-adjoint and, consequently, the eigenfunctions form an orthonormal set

$$\int_0^L dx\,\eta_n(x)\,\eta_m(x)=\delta_{nm}. \tag{43}$$

There is a well-known property that a continuous function $F(x)$ (with the same boundary conditions) can be expanded as a generalized Fourier series

$$F(x)=\sum_n \eta_n(x)\int_0^L dy\,\eta_n(y)F(y) \tag{44}$$

which means

$$\sum_n \eta_n(x)\,\eta_n(y)=\delta(x-y). \tag{45}$$

Now we may introduce the quantized field, defining

$$\Phi(x,t)=\sum_n \frac{1}{\sqrt{2\omega_n}}[\eta_n(x)e^{i\omega_n t}b_n^\dagger+\eta_n^*(x)e^{-i\omega_n t}b_n]. \tag{46}$$

The Bose operators b_n satisfy the commutation relations

$$[b_n,b_m^\dagger]=\delta_{n,m} \tag{47}$$

and therefore for equal-time fields

$$[\Phi(x),\partial_t\Phi(y)]=\frac{i}{2}\sum_n[\eta_n(x)\eta_n^*(y)+\eta_n^*(x)\eta_n(y)] \tag{48}$$

$$=i\delta(x-y). \tag{49}$$

Strictly speaking, the zero mode (37) with $\omega=0$ should be excluded from the sum (46), which leads to the extra term $-i\eta_0(x)\eta_0(y)\sim Q^{-1}\to 0$ in Eq. (48). The reason for it is a nonindependent character of the mode η_0 in the description of the field configuration $\phi_0(x),\{\eta_n(x)\}_{n=0}^\infty$. Indeed, η_0, bearing the zero energy, corresponds to the translation of the classical solution $\phi_0(x)\to\phi(x+X)$, while $\eta_0(x)\sim\partial_x\phi_0(x)$. As a result, the consistent treatment will consist of the functions $\phi_0(x+X),\{\eta_n(x+X)\}_{n=1}^\infty$, with η_0 replaced by the displacement X of the system as whole. The quantity X should be regarded as a dynamical variable in the correct description of the quantum Hamiltonian.[13]

We outline this collective coordinate method in the Appendix and show, that the corrections to the above simpler formula are negligible in the limit $Q\to\infty$. The only thing which we should borrow from the collective coordinate method, is the necessity to integrate over the vacuum location X instead of inclusion of η_0.

It seems that one now has everything to compute various correlation functions, describing the quantum fluctuations of the system. Given a set consisting of the classical solution ϕ_0 and the normal modes around it, one calculates the quantum averages with the simple rules of bosonic algebra and then integrates over the initial position X of the soliton lattice. This program, however, reveals many difficulties described in the next section.

IV. DIFFICULTIES IN THE APPROACH

A. Density-type correlations

Consider the fermionic density $\rho=\partial_x\beta\phi/2\pi$. The classical contribution $\rho_0=\partial_x\beta\phi_0/2\pi=\overline{\rho}w(x)$, where

$$w(x)\equiv\frac{2K}{\pi}dn\frac{m(x+X)}{k}. \tag{50}$$

The averaging over the kinks position gives a simple result here,

$$\langle\rho(x)\rangle_X=\overline{\rho}\langle w(x)\rangle_X=\overline{\rho}.$$

At the same time, the classical contribution to the pairwise density correlator is $\langle\rho(x)\rho(y)\rangle_X=\overline{\rho}^2\langle w(x)w(y)\rangle_X\neq\overline{\rho}^2$. These nontrivial correlations of density at largest distances correspond to Wigner crystallization of the charge Q. This long-range ordering of the charge in one spatial dimension is evidently erroneous and should eventually be removed from the theory.

The quantum corrections to the above correlator in the long-distance limit read as

$$\langle\rho(x)\rho(y)\rangle_q=-\frac{\mathcal{K}}{2\pi^2}\frac{\partial^2}{\partial x\partial y}\langle w(x)w(y)\rangle_X\ln|x-y|, \tag{51}$$

where we have used Eqs. (33), (37), (40), and (46). The CDW-type correlator is similarly estimated as

$$\langle e^{i\beta\phi(x)}e^{-i\beta\phi(y)}\rangle\sim\langle\exp[-2\mathcal{K}\ln(x-y)w(x)w(y)]\rangle_X. \tag{52}$$

Evidently, the modulation $w(x)w(y)$ in Eq. (52), being exponentiated is more pronounced than in Eq. (51).

B. Dual field

The so-called dual field $\theta(x,t)$ used in the construction of the fermion operator, is usually defined as

$$\partial_x\theta=\partial_t\phi,\qquad \partial_t\theta=\partial_x\phi.$$

These relations imply $\partial_t^2\phi=\partial_x^2\phi$, which is not consistent with the linearized theory equation (21) (the discussion of the zero mode contribution to θ is postponed until the next section). Particularly, the usual construction of the charge-raising operator $\mathcal{O}\sim\exp 2\pi i\beta^{-1}\int^{(x,t)}(dt\partial_x+dx\partial_t)\phi$ is inappropriate in the basis $\{\eta_n\}$, being dependent on the contour of integration in $\mathcal{O}$.

One may ignore for a moment this discrepancy, using only a first equation $\partial_x\theta=\partial_t\phi$. This is sufficient for the calculation of the instantaneous correlation function $\langle\exp[4\pi i\beta^{-1}\theta(x)]\exp[-4\pi i\beta^{-1}\theta(y)]\rangle$. Making this calculation, we meet another troublesome property.

The lattice of kinks gives rise to the notion of the Brillouin zone and to the infinite set $\{\eta_n^{(0)}\}$ of the eigenstates with nonzero energy and zero wave vector in the reduced Brillouin zone scheme: $\eta_0^{(0)}\equiv\eta_0$. A constant is not an eigenstate of Eq. (21), and therefore $\overline{\eta_n}\equiv L^{-1}\int_0^L dx\,\eta_n^{(0)}(x)\sim L^{-1/2}$. Further, if the wave vector q_n of the eigenstate η_n is close to zero, a corresponding Fourier transform $\eta_n(q)$ has a pole $\overline{\eta_n}/(q-q_n)$. The quantity $\overline{\eta_n}$ decreases when increasing the energy and this decrease is slower at larger k. In particular, one can show for the above set $\{\eta_n^{(0)}\}$ that $\Sigma_{n=0}^{\infty}(\overline{\eta_n})^2=L^{-1}$, while $\overline{\eta_0}=\pi/(2\sqrt{LKE})$.

The vacuum expectation value of the product of θ fields is then

$$\langle\theta(x+y)\theta(y)\rangle\simeq\sum_{q>0}\frac{v_F}{q}\overline{\eta_0^2}\cos qx+\sum_{q'}\frac{\cos q'x}{2q'^2}\sum_{n\geqslant 1}\omega_{n,q'}\overline{\eta_n^2}. \tag{53}$$

As usual, the constant (divergent) term should be added here in order to analyze the dependence of the correlations on x. The first term in Eq. (53), resulted from the lowest energy modes, contributes to the logarithm. In the second term, with both signs of q' available, we note that linear-in-q' corrections to $\omega_{n,q'}$ vanish and the sum over q' gives a contribution linear in $|x|$. The sum over n in Eq. (53) converges rapidly when elliptic index k is not too close to unity, i.e., at $\bar{\rho}\gtrsim m$. For an estimate, it is sufficient to write $\Sigma_{n\geqslant 1}\omega_{n,q'}\overline{\eta_n^2}\sim\bar{\rho}\Sigma_{n\geqslant 1}\overline{\eta_n^2}=\bar{\rho}(L^{-1}-\overline{\eta_0^2})$. As a result, we have

$$\langle\theta(x+y)\theta(y)\rangle\simeq\frac{\pi k_1}{8E^2}\ln\frac{L}{|x|}-\bar{\rho}|x|[1-\pi^2/(4KE)]O(1). \tag{54}$$

The linear in $|x|$ term is present unless $k\to 0$ $(m/\bar{\rho}\to 0)$, when one recovers the usual expression $\langle\theta(x+y)\theta(y)\rangle\sim-\ln|x|$. As a result, the superconducting correlations $\langle e^{4\pi i\beta^{-1}\theta}e^{-4\pi i\beta^{-1}\theta}\rangle$ decay exponentially at large distances. We see that the above definitions of the dual field are inconsistent in the basis η_n and lead to an exponential decrease of the SC correlations in the gapless situation.

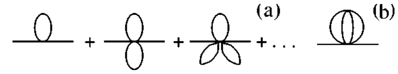

FIG. 2. (a) A tadpole sequence of diagrams, contributing to position-dependent mass renormalization. (b) A stronger divergent diagram not contained in this sequence.

C. Perturbation theory

Consider now the role of the terms of the interaction in Eq. (19), which have been discarded. These terms, containing additional βs, produce divergences in the perturbation theory. Resumming the tadpole sequence of diagrams, Fig. 2(a), one obtains the mass renormalization of the form

$$\eta^2(x)m^2\cos\beta\phi_0(x)\to\eta^2(x)m^2\cos\phi_0(x)e^{-\mathcal{K}w(x)^2\ln\Lambda}$$

with $\Lambda\sim Q$. This renormalization depends on x and, therefore, changes the shape of the potential $\cos\beta\phi_0$. This essential modification is accompanied by stronger than logarithmic infrared divergencies in diagrams, not contained in the tadpole series, e.g., that shown in Fig. 2(b).

V. A PROPOSED SOLUTION

A. Lagrangian

A key to resolve all the above discrepancies is found as follows. We wrote the fluctuations around the classical solution as $\phi=\phi_0+\eta$ and performed the Taylor expansion (19). Instead of that, we can write the fluctuations in the form

$$\phi=2\beta^{-1}\mathrm{am}\left[2K\left(x\bar{\rho}+\frac{\beta\tilde{\eta}}{2\pi}\right)\right]. \tag{55}$$

Formula (55) has the remarkable property that shifting $\tilde{\eta}\to\tilde{\eta}+2\pi/\beta$ increases $\phi\to\phi+2\pi/\beta$. This may serve as a justification of the scaling factor before $\tilde{\eta}$ in Eq. (55). At $\bar{\rho}\geqslant m$ one has $\phi\to 2\pi\bar{\rho}\beta^{-1}x+\tilde{\eta}$, hence Eq. (55) and $\phi=\phi_0+\eta$ coincide in this limiting case. Some calculation shows, that the perturbation theory in $\tilde{\eta}$ can be obtained from the exact form of the full Lagrangian

$$\mathcal{L}=w^2\left[x+\frac{\beta\tilde{\eta}}{2\pi\bar{\rho}}\right]\frac{(\partial_t\tilde{\eta})^2-(\partial_x\tilde{\eta})^2}{2}. \tag{56}$$

The field theory decribed by Eq. (56) is free from infrared divergencies for two reasons. The first is the presence of the field derivatives in each term of the perturbation series. Another reason is a particular form of the formfactors of the interaction, when the nth vertex is obtained by $(n-2)$ differentiations of the restricted function $w^2(x)$.

The Gaussian action is given by

$$\mathcal{L}=\frac{1}{2}w^2(x)[(\partial_t\tilde{\eta})^2-(\partial_x\tilde{\eta})^2], \qquad (57)$$

with the corresponding equation of motion

$$\partial_t^2\tilde{\eta}=w^{-2}\partial_x w^2\partial_x\tilde{\eta}. \qquad (58)$$

Our aim is now to show that the spectrum of Eq. (58) coincides with the initial problem (21), and to find the connection between the fields $\tilde{\eta}$ and η. The equivalence of the spectra stems from the possibility to rewrite the Lamé equation in the form

$$\omega^2\eta(x)=-w^{-1}(x)\partial_x w^2(x)\partial_x w^{-1}(x)\eta(x), \qquad (59)$$

which implies that the Lagrangian is

$$\mathcal{L}=\frac{1}{2}[(\partial_t\eta)^2-(w\partial_x w^{-1}\eta)^2]. \qquad (60)$$

Evidently, Eq. (60) coincides with Eq. (57) after the substitution

$$\tilde{\eta}(x,t)\equiv w^{-1}(x)\eta(x,t). \qquad (61)$$

As a result, the eigenfunctions to Eq. (57) are normalized with a weight

$$\int dx w^2(x)\tilde{\eta}_\alpha(x)\tilde{\eta}_\beta(x)=\delta_{\alpha\beta}, \qquad (62)$$

and Eq. (61) leads to the following form of the correlations at large distances:

$$\beta^2\langle\tilde{\eta}(x,t)\tilde{\eta}(y,0)\rangle\simeq-\mathcal{K}\ln|(x-y)^2-v_F^2t^2|. \qquad (63)$$

Therefore, we find the description of the fluctuations, possessing the same spectrum in the Gaussian approximation, but free from divergencies in higher orders of interaction. This choice of appropriate variables should be hence considered as an effective partial resummation of perturbation series.

B. Density-density correlations

The adoption of the definition (55) makes the calculation of the correlation functions less trivial. Consider first the density-density correlations $\langle\rho(x)\rho(y)\rangle$. Using the series representation[15] for the function $dn(x)$ we write

$$\rho=\left(\bar{\rho}+\frac{\beta}{2\pi}\partial_x\tilde{\eta}\right)\sum_{n=-\infty}^{\infty}\frac{e^{in(2\pi\bar{\rho}x+\beta\tilde{\eta})}}{\cosh n\tau}, \qquad (64)$$

with $\tau=\pi K'/K$. Note that the previous kinks' lattice displacement X corresponds to the new zero mode $\tilde{\eta}_0=$const, Eq. (61). The integration over $\tilde{\eta}_0$ gives evidently $\langle\rho\rangle=\bar{\rho}$, while it selects the terms with equal ns in the pairwise density correlator

$$\langle\rho(x)\rho(y)\rangle_q=\bar{\rho}^2+\frac{\mathcal{K}}{2\pi^2(x-y)^2} \qquad (65)$$

$$+\sum_{n=1}^{\infty}c_n\frac{\cos 2n\pi\bar{\rho}(x-y)}{|x-y|^{2n^2\mathcal{K}}}+\cdots \qquad (66)$$

with omitted faster decaying terms. We see that the weight factors $w(x)$ in Eq. (51) are transformed into the decaying $4k_F=2\pi\bar{\rho}$ harmonics of the density, in line with Ref. 8.

C. CDW-type correlations and umklapp ordering

Keeping in mind the Hubbard model near half filling, we associate the exponent $e^{i\beta\phi/2}$ with the "$2k_F$" density wave operator, whereas $e^{i\beta\phi}$ stands for the "$4k_F$" umklapp process. The "$2k_F$" correlations are estimated as follows. We use Eq. (55) and the series representation

$$e^{i\,\mathrm{am}(2Kz)}=\sum_{n=-\infty}^{\infty}C_n^{(1)}e^{2\pi iz(n+1/2)}, \qquad (67)$$

$$C_n^{(1)}=\frac{\pi e^{\tau(n+1/2)}}{kK\sinh(2n+1)\tau},$$

to obtain

$$\langle e^{i(\beta/2)\phi(x)}e^{-i(\beta/2)\phi(0)}\rangle\sim\sum_{n=-\infty}^{\infty}[C_n^{(1)}]^2\frac{e^{2i\pi\bar{\rho}x(n+1/2)}}{|x|^{2\mathcal{K}(n+1/2)^2}}, \qquad (68)$$

$$\langle e^{i(\beta/2)\phi(x)}e^{i(\beta/2)\phi(0)}\rangle\sim\sum_{n=-\infty}^{\infty}C_n^{(1)}C_{-1-n}^{(1)}\frac{e^{2i\pi\bar{\rho}x(n+1/2)}}{|x|^{2\mathcal{K}(n+1/2)^2}}. \qquad (69)$$

The prefactors of the exponents here are not universal, depending on the cutoff. What matters here, is the ratio of the amplitudes for the forward-going and backward-going waves with the same absolute value of $(n+1/2)$. The leading asymptotes are defined by $n=0,-1$, noting that $C_{-1}^{(1)}=-\tilde{q}C_0^{(1)}$, with $\tilde{q}=e^{-\pi K'/K}$, we write

$$\langle e^{i\beta\phi(x)/2}e^{-i\beta\phi(0)/2}\rangle\sim(e^{i\pi\bar{\rho}x}+\tilde{q}^2e^{-i\pi\bar{\rho}x})|x|^{-\mathcal{K}/2}, \qquad (70)$$

$$\langle e^{i\beta\phi(x)/2}e^{i\beta\phi(0)/2}\rangle\sim-2\tilde{q}\cos(\pi\bar{\rho}x)|x|^{-\mathcal{K}/2}. \qquad (71)$$

Particularly, the above formulas indicate the following. The lowest harmonic of the CDW order parameter field, $\propto\sin(\beta\phi/2)=sn[2K(\bar{\rho}x+\beta\tilde{\eta}/2\pi)]$ has an amplitude $\pi/(kK\sinh\tau/2)$ which remains finite in the total range of variation of $\bar{\rho}$. The similar quantity for the SDW field, $\propto\cos(\beta\phi/2)=cn[2K(\bar{\rho}x+\beta\tilde{\eta}/2\pi)]$, has an amplitude $\pi/(kK\cosh\tau/2)$ which vanishes as $\bar{\rho}/m$ at half filling, $\bar{\rho}=0$.

Using Eq. (67) we write the umklapp order parameter field in the form

$$\cos\beta\phi=\sum_{n=-\infty}^{\infty}C_n^{(2)}e^{in(2\pi\bar{\rho}x+\beta\tilde{\eta})}, \qquad (72)$$

$$2C_n^{(2)} = \sum_{n_1} C_{n-n_1}^{(1)} C_{n_1}^{(1)} + (n \to -n). \tag{73}$$

We observe the appearance of the constant term $C_0^{(2)}$ corresponding to the fact that the average value $\langle e^{i\beta\phi(x)}\rangle = \langle \cos\beta\phi(x)\rangle = C_0^{(2)}$ is not zero at finite density of kinks. Indeed one has

$$C_0^{(2)} = [-2E + K(1+k_1^2)]/(k^2K), \tag{74}$$

$$\simeq 1 - 4\overline{\rho}/m, \quad \overline{\rho}/m \lesssim 0.2, \tag{75}$$

$$\simeq m^2/(8\pi^2\overline{\rho}^2), \quad \overline{\rho}/m \gtrsim 0.2. \tag{76}$$

It should be noted that the nonzero value of $C_0^{(2)}$ arises without "locking" of the field ϕ. We remind that in the absence of kinks the field ϕ is "locked" to one of the discrete values $\phi_n = 2\pi n/\beta$, and it results in the nonzero value of the "$2k_F$" density wave $\langle e^{i\beta\phi/2}\rangle = \pm 1$. In the presence of kinks ϕ connects different vacua ϕ_n, and $\langle e^{i\beta\phi/2}\rangle = 0$, while $\langle e^{i\beta\phi}\rangle \neq 0$ is decreasing function of the kinks density.

The quantum average of two umklapp exponents at long distances has the form

$$\langle \cos\beta\phi(x)\cos\beta\phi(0)\rangle \sim [C_0^{(2)}]^2 + \sum_{n=1}^{\infty} c_n' \frac{\cos 2\pi\overline{\rho}xn)}{|x|^{2\mathcal{K}n^2}}. \tag{77}$$

We see that in the absence of $C_0^{(2)}$ the dimensionality of the operator $\cos\beta\phi$ would be $\mathcal{K}$. As $\mathcal{K}$ increases with decrease of $\overline{\rho}$, one might think that umklapp operator $\cos\beta\phi$ becomes less relevant.[10] However, this increase of $\mathcal{K}$ is accompanied by the increase of $C_0^{(2)}$, and $\langle\cos\beta\phi\rangle \sim 1$ at $\overline{\rho} \sim m$.

D. Dual field

Let us now discuss the modification of the definition of the dual field $\theta(x,t)$. The basic property of the charge raising operator $\mathcal{O} \sim \exp 2\pi i\beta^{-1}\theta$ in the path integral representation is that the contour of integration ("Dirac string") in the definition of θ introduces a discontinuity in the field ϕ, so that the values of the field $\beta\phi$ differ by 2π across this contour.[19] In view of the above property, $\beta\phi \to \beta\phi + 2\pi$ as $\beta\tilde{\eta} \to \beta\tilde{\eta} + 2\pi$, we see that it suffices to require the charge-raising property for the field $\tilde{\eta}$.

The canonical momentum for the field $\tilde{\eta}$ is given by $\tilde{\pi} = \partial\mathcal{L}/\partial(\partial_t\tilde{\eta}) = w^2\partial_t\tilde{\eta}$. The quantization condition reads $[\tilde{\eta}(x), \tilde{\pi}(y)] = i\delta(x-y)$.

Consider the definition

$$\tilde{\theta}(x) = \int^x dx' \,\tilde{\pi}(x') \tag{78}$$

for equal-time fields. Differentiating the last equality by x and then by t we have from Eq. (58) $\partial_t\partial_x\tilde{\theta} = \partial_x w^2 \partial_x \tilde{\eta}$, hence the definitions

$$\partial_x\tilde{\theta} = w^2\partial_t\tilde{\eta}, \quad \partial_t\tilde{\theta} = w^2\partial_x\tilde{\eta}, \tag{79}$$

are consistent with the equation of motion. The charge-raising operator $\mathcal{O} \sim \exp 2\pi i\beta^{-1}\int^{(x,t)} w^2(dt\partial_x + dx\partial_t)\tilde{\eta}$, is now uniquely defined, independent of the contour of integration. Further, one has $[\tilde{\eta}(x), \tilde{\theta}(y)] = i\vartheta(y-x)$ and therefore we have $\exp[2\pi i\beta^{-1}\tilde{\theta}(x)] f[\tilde{\eta}(y)] = f[\tilde{\eta}(y) + 2\pi\beta^{-1}\vartheta(x-y)]\exp[2\pi i\beta^{-1}\tilde{\theta}(x)]$ for any $f(z)$. This property allows one to define the fermion operator, see below.

A counterpart $\tilde{\theta}_0$ of $\tilde{\eta}_0 = \text{const}$ cannot be obtained from Eq. (79), which again shows a particular role of the zero mode. One of the possible ways to introduce the quantized quantity $\tilde{\theta}_0$, relevant to our discussion in the Appendix, is found in Ref. 20. One can write these zero-energy components as follows:

$$\tilde{\eta}_0 = \frac{\pi^2\beta}{8LEK} Jt + (q-Q) W_1(x) \frac{\beta}{2\pi\chi L} + i\frac{2}{\beta}\frac{\partial}{\partial J}, \tag{80}$$

$$\tilde{\theta}_0 = \frac{\pi^2\beta}{8LEK} J W_2(x) + (q-Q) t \frac{\beta}{2\pi\chi L} + i\frac{\beta}{2\pi}\frac{\partial}{\partial q} \tag{81}$$

with $J = P/\pi\overline{\rho}$ the persistent current attaining integer values, the functions $W_1(x) = \int^x w^{-2}(y)dy$ and $W_2(x) = \int^x w^2(y)dy$ are expressed through the elliptic functions $\mathrm{Nd}(x)$ and $\mathrm{Dn}(x)$, respectively. A canonical conjugate to J is $\pi\overline{\rho}X$, so the last term in Eq. (80) is simply the previous shift in the initial kink position. The charge variable q was assumed to be Q in Secs. III, IV and we let it fluctuate now around its equilibrium value. A choice of the factors before J and q variables is dictated by the condition $\int_0^L dx\, w^2(\partial_t\tilde{\eta}_0)^2 + (\partial_x\tilde{\eta}_0)^2 = (q-Q)^2\chi^{-1}L^{-1} + P^2\mathcal{M}_{\text{tot}}^{-1}$, cf. Eq. (17). A particular form of the function $W_1(x)$ in Eq. (80) is not coincidental here. Satisfying the equation $w^{-2}\partial_x w^2 \partial_x W_1(x) = 0$, the function W_1 is another (increasing) zero mode for Eq. (58), which corresponds to the increase of Q in the classical solution $w(x)W_1(x) \propto \partial\phi_0/\partial k$; we have $\beta\tilde{\eta}_0|_{x=0}^{L} = 2\pi(q-Q)$.

Let us discuss now the correlations of the dual field $\tilde{\theta}$. To do it, consider first its equation of motion, as follows from Eq. (79):

$$\partial_t^2\tilde{\theta} = w^2\partial_x w^{-2}\partial_x\tilde{\theta}. \tag{82}$$

Recalling the property of the Jacobi function $\mathrm{dn}(x)$, we write

$$w(x + \overline{\rho}^{-1}/2) = c_1 w^{-1}(x), \quad c_1 = \frac{4K^2k_1}{\pi^2}. \tag{83}$$

Therefore the differential operator in the right-hand side of Eq. (82) coincides with one in Eq. (58), when shifted by its half period $x \to x + \overline{\rho}^{-1}/2$.

For the infrared Gaussian action, given some particular solution $\tilde{\eta}_\alpha$ to Eq. (58), we obtain its dual counterpart $\tilde{\theta}_\alpha$ with the same energy and Floquet index. In view of the completeness of the set $\tilde{\eta}_\alpha$, we have a relation $\tilde{\theta}_\alpha(x) = c_\alpha\tilde{\eta}_\alpha(x + \overline{\rho}^{-1}/2)$. The value of the constant c_α is elucidated from the following sequence of equalities

$$\begin{aligned} c_\alpha \partial_t \tilde{\eta}_\alpha(x+\overline{\rho}^{-1}/2) &= \partial_t \tilde{\theta}_\alpha(x) = w^2(x)\partial_x \tilde{\eta}_\alpha(x) \\ &= e^{2i\nu K} w^2(x) \partial_x \tilde{\eta}_\alpha(x+\overline{\rho}^{-1}) \\ &= e^{2i\nu K} w^2(x) c_\alpha^{-1} \partial_x \tilde{\theta}_\alpha(x+\overline{\rho}^{-1}/2) \\ &= e^{2i\nu K} c_1^2 c_\alpha^{-1} \partial_t \tilde{\eta}_\alpha(x+\overline{\rho}^{-1}/2) \end{aligned} \tag{84}$$

hence

$$c_\alpha = \pm c_1 e^{i\nu K}. \tag{85}$$

One can show, that the sign here is plus for $\alpha \in (K - iK', K)$ and $\alpha \in (-iK', 0)$ and minus for $\alpha \in (K - 2iK', K - iK')$ and $\alpha \in (-2iK', -iK')$. In the large-$\overline{\rho}$ limit the eigenfunctions $\tilde{\eta}_\alpha(x)$ are usual plane waves, $\sim e^{-iq_\alpha x}$, and we recover the usual expression $\tilde{\theta}_\alpha(x) = \mathrm{sgn}(q_\alpha)\tilde{\eta}_\alpha(x)$. The sign in Eq. (85) is irrelevant for calculating the expectation value of the product of two dual fields and we have

$$\langle \tilde{\theta}(x)\tilde{\theta}(y)\rangle = c_1^2 \langle \tilde{\eta}(x)\tilde{\eta}(y)\rangle. \tag{86}$$

Noting from Eqs. (36),(83), that $4\pi c_1 = \beta^2/\mathcal{K}$, we immediately obtain

$$\langle e^{4\pi i \beta^{-1}\tilde{\theta}(x)} e^{-4\pi i\beta^{-1}\tilde{\theta}(y)}\rangle \sim |x-y|^{-2/\mathcal{K}}. \tag{87}$$

The dynamics of the correlations (65),(68),(87) is obtained by replacing $|x-y| \to [(x-y)^2 - v_F^2(t_1-t_2)^2]^{1/2}$. This simple temporal dependence of the correlations is valid as long as we consider the long-time, long-distance behavior $x \sim v_F t \gg \overline{\rho}^{-1}$. At shorter scales, the complicated structure of the wave functions and of the dispersion comes into play and the Lorentz form of the correlations is lost.

E. Fermionic correlations

We now discuss the Hubbard model and regard $\cos\beta\phi$ as a four-fermion umklapp term. In this case, in addition to the "charge" field ϕ discussed insofar, one also considers the spin degree of freedom described by the field ϕ_s and its dual one θ_s. We write the right and left fields as the linear combinations

$$\phi_{R\sigma} = \frac{\beta}{4}\phi - \frac{2\pi}{\beta}\tilde{\theta} + \sigma\sqrt{\frac{\pi}{2}}(\phi_s - \theta_s), \tag{88}$$

$$\phi_{L\sigma} = \frac{\beta}{4}\phi + \frac{2\pi}{\beta}\tilde{\theta} + \sigma\sqrt{\frac{\pi}{2}}(\phi_s + \theta_s) \tag{89}$$

with the spin projection $\sigma = \pm 1$. The operators

$$\psi_{R\sigma} = \chi_\sigma e^{i\phi_{R\sigma}}, \quad \psi_{L\sigma} = \chi_\sigma e^{-i\phi_{L\sigma}} \tag{90}$$

may be identified with the right and left fermion, respectively. Let us discuss this point in more detail.

One can easily check that given the usual relation $[\phi_s(x),\theta_s(y)] = i\vartheta(y-x)$, the operators $\psi^\dagger_{R\sigma}, \psi_{R\sigma}, \psi^\dagger_{L\sigma}, \psi_{L\sigma}$ with the same σ anticommute. The anticommutation of these operators with different projection of spin is ensured by the Majorana fermionic variables χ_σ.[20] These variables obey $\chi_+\chi_+ = \chi_-\chi_- = 1$, while $\chi_+\chi_- = -\chi_-\chi_+$.

A subtler question concerns the relation of the construction (90) to the initial fermionic variables in the Hubbard model. In the bosonization procedure, the initial fermions were written similarly to Eq. (90), with the dual field θ in Eq. (88) instead of our $\tilde{\theta}$. The oscillator representation (46) for the initial right (left) bosons had simple form, which is achieved within our formalism for $\overline{\rho} \neq 0$ at $m=0$. It is not at once clear that the complicated expressions (55),(79),(23) in the presence of umklapp term, $m \neq 0$ are related to right and left moving particles. We state here that the label "right" of the boson field $\phi_{R\sigma}$ simply indicates the species of fermion, which is constructed this way. As we will see shortly, umklapp interaction produces a partial redistribution of the fermionic spectral weight over the $4k_F$ components of the density, hence initially right-moving particle has now left-going components.

In short, our fermions (90) anticommute, have the same initial field ϕ, and explicitly acquire the conventional form at $m=0$. Therefore they are the fermions, which one starts with during the bosonization of the Hubbard model.

The spin fields of the fermions are factorized, and for the charge part we have

$$\psi_{R,L}(x) \propto \exp\left(\pm i\frac{\beta}{4}\phi - i\frac{2\pi}{\beta}\tilde{\theta}\right) \tag{91}$$

$$\sim e^{-2\pi i\beta^{-1}\tilde{\theta}} \sum_{n=-\infty}^{\infty} C_n^{(h)} e^{\pm i(2\pi\overline{\rho}x + \beta\tilde{\eta})(n+1/4)}, \tag{92}$$

$$C_n^{(1)} = \sum_{n_1} C_{n-n_1}^{(h)} C_{n_1}^{(h)}, \tag{93}$$

cf. Eqs. (67),(73). The fermionic correlations are found in the form

$$\langle \psi^\dagger_{R\sigma}(x,t)\psi_{R\sigma}(0)\rangle \sim \frac{1}{(x-v_s t)^{1/2}} \sum_{n=-\infty}^{\infty} \frac{c_n'' e^{2\pi i\overline{\rho}x(n+1/4)}}{(x-v_F t)^{2n+1/2}(x^2-v_F^2t^2)^{\gamma_n}} \tag{94}$$

$$\gamma_n = \frac{[(2n+1/2)\mathcal{K}-1]^2}{4\mathcal{K}}. \tag{95}$$

In the case of free fermions ($m=0, \beta^2=8\pi$) one has $\mathcal{K}=2$, and the anomalous dimension of the fermion vanishes, $\gamma_0 = 0$. We observe that at special values of the Luttinger parameter $\mathcal{K}/2 = 1/3, 1/5, 1/7, \ldots$, certain γ_n vanishes as well, and the fermionic correlations reveal purely chiral (subleading) component. For $\mathcal{K} = 2/(2n+1)$, it is of the form

$$\frac{e^{\pm\pi i\overline{\rho}x(n+1/2)}}{(x \mp v_F t)^{n+1/2}(x-v_s t)^{1/2}} \tag{96}$$

with upper (lower) sign for even (odd) n. The leading asymptote in Eq. (94) is given by a term with $n=0$ and hence the scaling dimension of the fermion operator has its usual value $\dim[\psi_{R\sigma}]=\frac{1}{8}(\sqrt{\mathcal{K}/2}+\sqrt{2/\mathcal{K}})^2$.

F. Low-density limit

We have obtained the critical exponents for the correlation functions which depend on the Luttinger parameter $\mathcal{K}$. Some inspection of the formula (36) reveals that at small densities the Fermi velocity is exponentially small, $v_F \simeq (m/2\overline{\rho})e^{-m/2\overline{\rho}}$, and the parameter $\mathcal{K}\sim\beta^2(\overline{\rho}/m)^2 e^{m/2\overline{\rho}}$ eventually becomes large. Haldane's analysis of the Bethe-ansatz equations has shown the crossover region at $4(\overline{\rho}/m)\ln\beta^{-1}\simeq 1$ and the saturation of $\mathcal{K}\to 1$ at smaller densities. This statement, albeit reasonable, is hard to check in our formalism.

The saturation of $\mathcal{K}$ can be a result of the intervention of some class of diagrams, increasingly important at small densities. As we have shown, the full Lagrangian (56) is free from infrared divergencies. On the other hand, it requires the essential ultraviolet regularization. The situation is reversed when one works with the initial field variables (19). Hence the question of saturation of $\mathcal{K}$ remains open within our formalism.

VI. DISCUSSION AND CONCLUSIONS

We have studied the sine-Gordon model in the presence of a finite density of kinks. This quantum problem is related to the strongly interacting electronic system in one spatial dimension close to commensurability, where the metal-insulator transition occurs. The model was investigated previously by the Bethe ansatz method, which gave a form of the spectrum and the Luttinger parameter, defining the decay of correlations functions at large distances.

In dealing with complicated problems, it is always worth to have alternative methods at hand, suitable for cross checking with existing results and capable for further theoretical predictions. In this paper we used the semiclassical approach and demonstrated that the spectrum and the Luttinger parameter correctly reproduced the results of the Bethe ansatz studies. Among the new features, available due to the quasiclassics, are the explicit form of various correlation functions and the unambigous construction of the charge-raising operator in the theory.

Discussing the correlation functions, we determined their leading and subleading asymptotes, whose amplitudes and critical exponents strongly depend on the kink density. Addressing a more technical issue of the charge-raising operator, or the dual field in the theory, we showed the way it is constructed through the modified collective coordinate method. This allowed us to reestablish the link with the fermionic counterpart of our bosonic problem and to calculate, in the leading order, the behavior of the superconducting-type correlations.

One of the consequences of our approach is the nonzero value of the quantum average, associated with the four-fermion umklapp operator in the Hubbard model away from half filling. As a result, we found that close to commensurability point, the spectral density of the right- (left-) moving fermions is distributed over the neighboring "$4k_F$" harmonics of the fermionic density.

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APPENDIX: COLLECTIVE COORDINATE METHOD

For brevity, we will write $\partial_t f=\dot{f},\partial_x f=f'$ in this section. Following Ref. 13 we write our solution in the form

$$\phi(x,t)=\phi_0(x+X)+\sum_0^\infty c_n(t)\,\eta_n(x+X). \tag{A1}$$

The classical solution ϕ_0 is proportional to β^{-1}, and the fluctuations η are of order of β^0. In terms of the above mass of kinks condensate $\mathcal{M}_{\rm tot}$, the normalized zero-energy solution is

$$\eta_0(x+X)=\mathcal{M}_{\rm tot}^{-1/2}\phi_0'(x+X),$$

in accordance with Eq. (37). In order to pass to the correct form of the quantum Hamiltonian, one has to regard the variable X, which indicates the location of the soliton lattice, as a dynamical variable, replacing the zero mode coordinate c_0. Instead of Eq. (A1) we write

$$\phi(x,t)=\phi_0[x+X(t)]+\sum_1^\infty c_n(t)\,\eta_n[x+X(t)], \tag{A2}$$

and the time derivative

$$\dot{\phi}(x,t)=\left(\phi_0'+\sum_1^\infty c_n\eta_n'\right)\dot{X}+\sum_1^\infty \dot{c}_n\eta_n. \tag{A3}$$

Introducing the new coordinates $u_n, n=0,1,\dots,\infty$ according to $u_0(t)=X(t), u_n(t)=c_n(t)$ for $n>0$ we represent the kinetic energy term in the Lagrangian in the form $\frac{1}{2}\int_0^L dx\,\dot{\phi}^2=\frac{1}{2}\Sigma_{i,j=0}^\infty \dot{u}_i D_{ij}\dot{u}_j$, with a symmetric matrix D_{ij} with elements

$$D_{00}=\int dx\left(\phi_0'+\sum_1^\infty u_n\eta_n'\right)^2,$$

$$D_{0n}=\int dx\,\eta_n\left(\sum_1^\infty u_m\eta_m'\right),\quad n>0,$$

$$D_{nm}=\delta_{nm},\quad n,m>0. \tag{A4}$$

The Hamiltonian is obtained by finding the canonical momenta π_j, conjugate to coordinates u_j, according to $\pi_j=\partial\mathcal{L}/\partial\dot{u}_j=D_{ji}\dot{u}_i$. The classical Hamiltonian then becomes

$$\mathcal{H}=\frac{1}{2}\sum_{i,j=0}^{\infty}\pi_i(D^{-1})_{ij}\pi_j+V(\{u_n\}), \tag{A5}$$

where the potential energy term V is given by $V(\{u_n\}) =\int dx[\frac{1}{2}\phi'^{\,2}+U(\phi)]=\mathcal{M}_{\rm tot}+\frac{1}{2}\Sigma_{n=1}^{\infty}u_n^2\omega_n^2+O(\beta)$.

The elements of the inverse matrix D^{-1} are

$$(D^{-1})_{00}=1/D,$$

$$(D^{-1})_{0n}=-D_{0n}/D, \quad n>0,$$

$$(D^{-1})_{nm}=\delta_{nm}+D_{0n}D_{0m}/D, \quad n,m>0, \tag{A6}$$

and the determinant D of the matrix D_{ij} is given by $D =(\sqrt{\mathcal{M}_{\rm tot}}+a_0)^2$ with $a_0=\int dx\ \eta_0(x)(\Sigma_1^{\infty}u_m\eta_m')$.

Using above formulas, one obtains after some algebra

$$\dot{\phi}(x,t)=\frac{\eta_0}{\sqrt{D}}\pi_0+\sum_{n=1}^{\infty}\left(\eta_n+\frac{D_{0n}}{\sqrt{D}}\eta_0\right)\pi_n. \tag{A7}$$

Requiring now $[u_n(t),\pi_m(t)]=i\delta_{nm}$ and using Eq. (A7), one verifies the exactness of the equation $[\phi(x,t),\dot{\phi}(y,t)] =i\delta(x-y)$, which result improves the previous formula (48). Among the new features, brought about by the described method is the explicit absence of the variable $X(t)$ in the Hamiltonian (A5). As a result, its conjugate momentum π_0 is independent of time and equals to the conserved total field momentum $P\equiv\int_0^L dx\ \dot{\phi}\phi'$, which equality is easily checked.

If we consider the soliton lattice in its rest frame, $P=0$, then the Hamiltonian is simplified and becomes

$$\mathcal{H}_{P=0}=\frac{1}{2}\sum_{n,m=1}^{\infty}\left(\delta_{nm}+\frac{D_{0n}D_{0m}}{D}\right)\pi_n\pi_m+V(\{u_n\}). \tag{A8}$$

At first glance, Eqs. (A7),(A8) contain nontrivial admixtures $\propto D_{0n}/\sqrt{D}$, which should interfere into the subsequent consideration. This is true, when one deals with a vacuum, consisting of a finite number of kinks Q. However, in the limit $Q\sim L\to\infty$, one has $\eta_n\propto L^{-1/2}$ and therefore $D_{0n}=O(1)$, whereas $D=O(Q\beta^{-2})$. As a result, the terms $D_{0n}/\sqrt{D} \sim\beta Q^{-1/2}$ in Eqs. (A7), (A8) should be neglected in this limit and we arrive at simpler equations $\mathcal{H}=\mathcal{E}_0+\frac{1}{2}\Sigma_{n=1}^{\infty}(\pi_n^2 +\omega_n^2u_n^2)+O(\beta)$ and $\dot{\phi}(x,t)_{P=0}=\Sigma_{n=1}^{\infty}\eta_n(x+X)\pi_n$.

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[1] See, e.g., T. Giamarchi, Phys. Rev. B **44**, 2905 (1991).
[2] V.L. Pokrovsky and A. Talapov, Phys. Rev. Lett. **42**, 65 (1979).
[3] A. Luther, Phys. Rev. B **50**, 11 446 (1994).
[4] P. Lebwohl and M.J. Stephen, Phys. Rev. **163**, 376 (1967); A. Shnirman, Z. Hermon, A.V. Ustinov, B.A. Malomed, and E. Ben-Jacob, Phys. Rev. B **50**, 12 793 (1994).
[5] J.P. Rodriguez, J. Phys.: Condens. Matter **9**, 5117 (1997).
[6] C.B. Hanna, A.H. MacDonald, and S.M. Girvin, Phys. Rev. B **63**, 125305 (2001).
[7] B. Kutnjak-Urbanc and B. Zeks, Phys. Rev. E **52**, 3892 (1995).
[8] F.D.M. Haldane, J. Phys. A **15**, 507 (1982).
[9] J.-S. Caux and A.M. Tsvelik, Nucl. Phys. B **474**, 715 (1996).
[10] E. Papa and A.M. Tsvelik, Phys. Rev. B **60**, 12 752 (1999); **63**, 085109 (2001).
[11] R.F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).
[12] R. Rajaraman, *Solitons and Instantons* (North Holland, Amsterdam, 1982).
[13] N.H. Christ and T.D. Lee, Phys. Rev. D **12**, 1606 (1975); Chap. 8 of Ref. 12.
[14] J. Voit, Rep. Prog. Phys. **58**, 977 (1994); A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998).
[15] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (National Bureau of Standards, New York, 1964); W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Berlin, 1966).
[16] F.C. Frank and J.H. van der Merwe, Proc. R. Soc. London, Ser. A **198**, 205 (1949).
[17] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).
[18] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 2.
[19] S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B **607**, 437 (2001).
[20] T. Banks, D. Horn, and H. Neuberger, Nucl. Phys. B **108**, 119 (1976).