Quantum orders and symmetric spin liquids

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A concept—quantum order—is introduced to describe a new kind of orders that generally appear in quantum states at zero temperature. Quantum orders that characterize the universality classes of quantum states (described by *complex* ground-state wave functions) are much richer than classical orders that characterize the universality classes of finite-temperature classical states (described by *positive* probability distribution functions). Landau's theory for orders and phase transitions does not apply to quantum orders since they cannot be described by broken symmetries and the associated order parameters. We introduced a mathematical object—projective symmetry group—to characterize quantum orders. With the help of quantum orders and projective symmetry groups, we construct hundreds of symmetric spin liquids, which have SU(2), U(1), or Z_2 gauge structures at low energies. We found that various spin liquids can be divided into four classes: (a) Rigid spin liquid—spinons (and all other excitations) are fully gapped and may have bosonic, fermionic, or fractional statistics. (b) Fermi spin liquid—spinons are gapless and are described by a Fermi liquid theory. (c) Algebraic spin liquid—spinons are gapless, but they are not described by free fermionic-bosonic quasiparticles. (d) Bose spin liquid—low-lying gapless excitations are described by a free-boson theory. The stability of those spin liquids is discussed in detail. We find that stable two-dimensional spin liquids exist in the first three classes (a)–(c). Those stable spin liquids occupy a finite region in phase space and represent quantum phases. Remarkably, some of the stable quantum phases support gapless excitations even without any spontaneous symmetry breaking. In particular, the gapless excitations in algebraic spin liquids interact down to zero energy and the interaction does not open any energy gap. We propose that it is the quantum orders (instead of symmetries) that protect the gapless excitations and make algebraic spin liquids and Fermi spin liquids stable. Since high-T_c superconductors are likely to be described by a gapless spin liquid, the quantum orders and their projective symmetry group descriptions lay the foundation for a spin liquid approach to high-T_c superconductors.

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I. INTRODUCTION

Due to its long length, we would like to first outline the structure of the paper so readers can choose to read the parts of interest. Section X summarizes the main results of the paper, which also serves as a guide of the whole paper. The concept of quantum order is introduced in Sec. I A. A concrete mathematical description of quantum order is described in Secs. IV A and IV B. Readers who are interested in the background and motivation of quantum orders may choose to read Sec. I A. Readers who are familiar with the slave-boson approach and just want a quick introduction to quantum orders may choose to read Secs. IV A and IV B. Readers who are not familiar with the slave-boson approach may find the review in Secs. II and III useful. Readers who do not care about the slave-boson approach but are interested in applications to high-T_c superconductors and experimental measurements of quantum orders may choose to read Secs. I A, I B, and VII and consult Figs. 1–15 to gain some intuitive picture of the spinon dispersion and neutron scattering behavior of various spin liquids.

A. Topological orders and quantum orders

Matter can have many different states, such as gas, liquid, and solid. Understanding states of matter is the first step in understanding matter. Physicists find matter can have much more different states than just gas, liquid, and solid. Even solids and liquids can appear in many different forms and states. With so many different states of matter, a general theory is needed to gain a deeper understanding of the states of matter.

All the states of matter are distinguished by their internal structures or orders. The key step in developing a general theory for states of matter is the realization that all orders are associated with symmetries (or rather the breaking of symmetries). Based on the relation between orders and symmetries, Landau developed a general theory of orders and the transitions between different orders.[1,2] Landau's theory is so successful and one starts to have a feeling that we understand, in principle, all kinds of orders that matter can have.

However, nature never stops surprising us. In 1982, Tsui, Stormer, and Gossard[3] discovered a new kind of state—fractional quantum-Hall (FQH) liquid.[4] Quantum-Hall liquids have many amazing properties. A quantum-Hall liquid is more ''rigid'' than a solid (a crystal), in the sense that a quantum Hall liquid cannot be compressed. Thus a quantum Hall liquid has a fixed and well-defined density. When we measure the electron density in terms of filling factor ν, we find that all discovered quantum Hall states have such densities that the filling factors are exactly given by some rational numbers, such as $\nu=1,1/3,2/3,2/5,\ldots$ Knowing that FQH liquids exist only at certain magical filling factors, one cannot help to guess that FQH liquids should have some internal orders or ''patterns.'' Different magical filling factors should be due to those different internal ''patterns.'' How-

ever, the hypothesis of internal ''patterns'' appears to have one difficulty—FQH states are liquids, and how can liquids have any internal ''patterns''?

In 1989, it was realized that the internal orders in FQH liquids (as well as the internal orders in chiral spin liquids[5,6]) are different from any other known orders and cannot be observed and characterized in any conventional ways.[7,8] What is really new (and strange) about the orders in chiral spin liquids and FQH liquids is that they are not associated with any symmetries (or the breaking of symmetries) and cannot be described by Landau's theory using physical order parameters.[9] This kind of order is called *topological order*. Topological order is a new concept and a whole new theory was developed to describe it.[9,10]

Knowing FQH liquids contain a new kind of order—topological order—we would like to ask why FQH liquids are so special. What is missed in Landau's theory for states of matter so that the theory fails to capture the topological order in FQH liquids?

When we talk about orders in FQH liquids, we are really talking about the internal structure of FQH liquids at *zero* temperature. In other words, we are talking about the internal structure of the quantum ground state of FQH systems. So the topological order is a property of the ground-state wave function. Landau's theory is developed for systems at finite temperatures where quantum effects can be ignored. Thus one should not be surprised that Landau's theory does not apply to states at zero temperature where quantum effects are important. The very existence of topological orders suggests that finite-temperature orders and zero-temperature orders are different, and zero-temperature orders contain richer structures. We see that what is missed by Landau's theory is simply the quantum effect. Thus FQH liquids are not that special. Landau's theory and symmetry characterization can fail for any quantum states at zero temperature. As a consequence, new kind of orders with no broken symmetries and local order parameters (such as topological orders) can exist for any quantum states at zero temperature. Because the orders in quantum states at zero temperature and the orders in classical states at finite temperatures are very different, here we would like to introduce two concepts to stress their differences:[11] (a) *quantum orders*,[93] which describe the universality classes of quantum ground states (i.e., the universality classes of *complex* ground-state wave functions with infinity variables), and (b) *classical orders*, which describe the universality classes of classical statistical states (i.e., the universality classes of *positive* probability distribution functions with infinity variables).

From the above definition, it is clear that the quantum orders associated with complex functions are richer than the classical orders associated with positive functions. Landau's theory is a theory for classical orders, which suggests that classical orders may be characterized by broken symmetries and local order parameters.[94] The existence of topological order indicates that quantum orders cannot be completely characterized by broken symmetries and order parameters. Thus we need to develop a new theory to describe quantum orders.

In a sense, the classical world described by positive probabilities is a world with only ''black and white.'' Landau's theory and the symmetry principle for classical orders are color-blind which can only describe different ''shades of gray'' in the classical world. The quantum world described by complex wave functions is a ''colorful'' world. We need to use new theories, such as the theory of topological order and the theory developed in this paper, to describe the rich ''color'' of the quantum world.

The quantum orders in FQH liquids have a special property that all excitations above ground state have finite energy gaps. This kind of quantum orders is called topological orders. In general, a topological order is defined as a quantum order where all the excitations above ground state have finite energy gapes.

Topological orders and quantum orders are general properties of any states at zero temperature. Nontrivial topological orders not only appear in FQH liquids; they also appear in spin liquids at zero temperature. In fact, the concept of topological order was first introduced in a study of spin liquids.[9] FQH liquid is not even the first experimentally observed state with nontrivial topological orders. That honor goes to the superconducting state discovered in 1911.[12] In contrast to a common point of view, a superconducting state cannot be characterized by broken symmetries. It contains nontrivial topological orders[13] and is fundamentally different from a superfluid state.

After a long introduction, now we can state the main subject of this paper. In this paper, we will study a new class of quantum orders where the excitations above the ground state are gapless. We believe that the gapless quantum orders are important in understanding high-T_c superconductors. To connect to high-T_c superconductors, we will study quantum orders in quantum spin liquids on a two-dimensional (2D) square lattice. We will concentrate on how to characterize and classify quantum spin liquids with different quantum orders. We introduce projective symmetry groups to help us to achieve this goal. The projective symmetry group can be viewed as a generalization of symmetry group that characterize different classical orders.

B. Spin-liquid approach to high-T_c superconductors

There are many different approaches to the high-T_c superconductors. Different physicists have different points of view on what are the key experimental facts for the high-T_c superconductors. The different choice of key experimental facts leads to many different approaches and theories. The spin-liquid approach is based on a point of view that the high-T_c superconductors are doped Mott insulators.[14–16] (Here by Mott insulator we means a insulator with an odd number of electrons per unit cell.) We believe that the most important property of the high-T_c superconductors is that the materials are insulators when the conduction band is *half* filled. The charge gap, obtained by the optical conductance experiments, is about 2 eV, which is much larger than the antiferromagnetic (AF) transition temperature $T_{\rm AF}\sim 250$ K, the superconducting transition temperature $T_c\sim 100$ K, and the spin pseudogap scale $\Delta\sim 40$ meV.[17–19] The insulating

property is completely due to the strong correlations present in the high-T_c materials. Thus the strong correlations are expected to play a very important role in understanding high-T_c superconductors. Many important properties of high-T_c superconductors can be directly linked to the Mott insulator at half filling, such as (a) the low charge density[20] and superfluid density,[21] (b) T_c being proportional to doping $T_c \propto x$,[22–24] (c) the positive charge carried by the charge carrier,[20] etc.

In the spin-liquid approach, the strategy is to try to understand the properties of the high-T_c superconductors from the low-doping limit. We first study the spin-liquid state at half filling and try to understand the parent Mott insulator. (In this paper, by spin liquid, we mean a spin state with translation and spin rotation symmetry.) At half filling, the charge excitations can be ignored due to the huge charge gap. Thus we can use a pure spin model to describe the half filled system. After understand the spin liquid, we try to understand the dynamics of a few doped holes in the spin-liquid states and to obtain the properties of the high-T_c superconductors at low doping. One advantage of the spin-liquid approach is that experiments (such as angle resolved photoemission,[17,18,25,26] NMR,[27] neutron scattering,[28–30] etc.) suggest that underdoped cuperates have many striking and qualitatively new properties which are very different from the well-known Fermi liquids. It is thus easier to approve or disapprove a new theory in the underdoped regime by studying those qualitatively new properties.

Since the properties of the doped holes (such as their statistics, spin, effective mass, etc.) are completely determined by the spin correlation in the parent spin liquids, thus in the spin-liquid approach, each possible spin liquid leads to a possible theory for high-T_c superconductors. Using the concept of quantum orders, we can say that possible theories for high-T_c superconductors in the low-doping limits are classified by possible quantum orders in spin liquids on 2D square lattices. Thus one way to study high-T_c superconductors is to construct all possible spin liquids that have the same symmetries as those observed in high-T_c superconductors and then analyze the physical properties of those spin liquids with dopings to see which one actually describes the high-T_c superconductor. Although we cannot say that we have constructed all symmetric spin liquids, in this paper we have found a way to construct a large class of symmetric spin liquids. (Here by symmetric spin liquids we mean spin liquids with all the lattice symmetries: translation, rotation, parity, and the time reversal symmetries.) We also find a way to characterize the quantum orders in those spin liquids via projective symmetry groups. This gives us a global picture of possible high-T_c theories. We would like to mention that a particular spin liquid—the staggered-flux/*d*-wave state[31,32]—may be important for high-T_c superconductors. Such a state can explain[33,34] the highly unusual pseudogap metallic state found in underdoped cuperates,[17,18,25,26] as well as the *d*-wave superconducting state.[32]

The spin liquids constructed in this paper can be divided into four classes: (a) Rigid spin liquid—spinons are fully gapped and may have bosonic, fermionic, or fractional statistics. (b) Fermi spin liquid—spinons are gapless and are described by a Fermi liquid theory. (c) Algebraic spin liquid—spinons are gapless, but they are not described by free fermionic-bosonic quasiparticles. (d) Bose spin liquid—low-lying gapless excitations are described by a free-boson theory. We find that some of the constructed spin liquids are stable and represent stable quantum phases, while others are unstable at low energies due to long-range interactions caused by gauge fluctuations. The algebraic spin liquids and Fermi spin liquids are interesting since they can be stable despite their gapless excitations. Those gapless excitations are not protected by symmetries. This is particularly striking for algebraic spin liquids since their gapless excitations interact down to zero energy and the states are still stable. We propose that it is the quantum orders that protect the gapless excitations and ensure the stability of the algebraic spin liquids and Fermi spin liquids.

We would like to point out that both stable and unstable spin liquids may be important for understanding high-T_c superconductors. Although at zero temperature high-T_c superconductors are always described stable quantum states, some important states of high-T_c superconductors, such as the pseudogap metallic state for underdoped samples, are observed only at finite temperatures. Such finite-temperature states may correspond to (doped) unstable spin liquids, such as the staggered flux state. Thus even unstable spin liquids can be useful in understanding finite-temperature metallic states.

There are many different approaches to spin liquids. In addition to the slave-boson approach,[6,15,16,31–33,35–40] spin liquids has been studied using the slave-fermion/σ-model approach,[41–46] quantum dimer model,[47–51] and various numerical methods.[52–55] In particular, the numerical results and recent experimental results[56] strongly support the existence of quantum spin liquids in some frustrated systems. A 3D quantum orbital liquid was also proposed to exist in $L_a T_i O_3$.[57]

However, I must point out that there is no generally accepted numerical results yet that prove the existence of spin liquids with an odd number of electrons per unit cell for spin-1/2 systems, despite intensive search in the last ten years. But it is my faith that spin liquids exist in spin-1/2 systems. For more general systems, spin liquids do exist. Read and Sachdev[43] found stable spin liquids in a Sp(N) model in the large-N limit. The spin-1/2 model studied in this paper can be easily generalized to the SU(N) model with $N/2$ fermions per site.[31,58] In the large-N limit, one can easily construct various Hamiltonians[58,59] whose ground states realize the various U(1) and Z_2 spin liquids constructed in this paper. The quantum orders in those large-N spin liquids can be described by the methods introduced in this paper.[58] Thus, despite the uncertainly about the existence of spin-1/2 spin liquids, the methods and the results presented in this paper are not about (possibly) nonexisting ''ghost states.'' Those methods and results apply, at least, to certain large-N systems. In short, nontrivial quantum orders exist in theory. We just need to find them in nature. (In fact, our vacuum is likely to be a state with a nontrivial quantum order, due to the fact that light exists.[58]) Knowing the existence of spin liquids

in large-N systems, it is not such a big leap to go one step further to speculate that spin liquids exist for spin-1/2 systems.

C. Spin-charge separation in (doped) spin liquids

Spin-charge separation and the associated gauge theory in spin liquids (and in doped spin liquids) are very important concepts in our attempt to understand the properties of high-T_c superconductors.[14–16,39,60] However, the exact meaning of spin-charge separation is different for different researchers. The term ''spin-charge separation'' has at least two different interpretations. In the first interpretation, the term means that it is better to introduce separate spinons (a neutral spin-1/2 excitation) and holons (a spinless excitation with unit charge) to understand the dynamical properties of high-T_c superconductors, instead of using the original electrons. However, there may be long-range interactions (possibly, even confining interactions at long distance) between the spinons and holons, and the spinons and holons may not be well-defined quasiparticles. We will call this interpretation pseudo spin-charge separation. The algebraic spin liquids have the pseudo spin-charge separation. The essence of the pseudo spin-charge separation is not that spin and charge separate. *The pseudo spin-charge separation is simply another way to say that the gapless excitations cannot be described by free fermions or bosons*. In the second interpretation, the term ''spin-charge separation'' means that there are only *short-ranged* interactions between the spinons and holons. The spinons and holons are well-defined quasiparticles at least in the dilute limit or at low energies. We will call the second interpretation the true spin-charge separation. The rigid spin liquids and the Fermi spin liquids have true spin-charge separation.

The electron operator is not a good starting point to describe states with pseudo spin-charge separation or true spin-charge separation. To study those states, we usually rewrite the electron operator as a product of several other operators. Those operators are called parton operators. (The spinon operator and the holon operator are examples of parton operators.) We then construct the mean-field state in the enlarged Hilbert space of partons. The gauge structure can be determined as the most general transformations between the partons that leave the electron operator unchanged.[61] After identifying the gauge structure, we can project the mean-field state onto the physical (i.e., the gauge invariant) Hilbert space and obtain a strongly correlated electron state. This procedure in its general form is called projective construction. It is a generalization of the slave-boson approach.[15,16,33,36–38,40] The general projective construction and the related gauge structure have been discussed in detail for quantum Hall states.[61] Now we see a third (but technical) meaning of spin-charge separation: to construct a strongly correlated electron state, we need to use partons and projective construction. The resulting effective theory naturally contains a gauge structure.

Although it is not clear which interpretation of spin-charge separation actually applies to high-T_c superconductors, the possibility of true spin-charge separation in an electron system is very interesting. The first concrete example of true spin-charge separation in 2D is given by the chiral spin-liquid state,[5,6] where the gauge interaction between the spinons and holons becomes short-ranged due to a Chern-Simons term. The Chern-Simons term breaks time reversal symmetry and gives the spinons and holons a fractional statistics. Later in 1991, it was realized that there is another way to make the gauge interaction short ranged through the Anderson-Higgs mechanism.[38,43] This led to a mean-field theory[38,40] of the short-ranged resonating valence bond (RVB) state[47,48] conjectured earlier. We will call such a state a Z_2 spin-liquid state, to stress the *unconfined* Z_2 gauge field that appears in the *low-energy* effective theory of those spin liquids. (See the remarks at the end of this section. We also note that the Z_2 spin liquids studied in Ref. 43 all break the 90° rotation symmetry and are different from the short-ranged RVB state studied Refs. 38, 40, 47, and 48.) Since the Z_2 gauge fluctuations are weak and are not confining, the spinons and holons have only short-ranged interactions in the Z_2 spin-liquid state. The Z_2 spin-liquid state also contains a Z_2 vortex-like excitation.[38,62] The spinons and holons can be bosons or fermions depending on if they are bound with the Z_2 vortex.

Recently, true spin-charge separation, the Z_2 gauge structure, and the Z_2 vortex excitations were also proposed in a study of quantum-disordered superconducting states in a continuum model[63] and in a Z_2 slave-boson approach.[64] The resulting liquid state (which was named a nodal liquid) has all the novel properties of a Z_2 spin-liquid state such as the Z_2 gauge structure and the Z_2 vortex excitation (which was named vison). From the point of view of the universality class, the nodal liquid is one kind of Z_2 spin liquids. However, the particular Z_2 spin liquid studied in Refs. 38 and 40 and the nodal liquid are two different Z_2 spin liquids, despite having the same symmetry. The spinons in the first Z_2 spin liquid have a finite energy gap while the spinons in the nodal liquid are gapless and have a Dirac-like dispersion. In this paper, we will use the projective construction to obtain more general spin liquids. We find that one can construct hundreds of different Z_2 spin liquids. Some Z_2 spin liquids have finite energy gaps, while others are gapless. Among those gapless Z_2 spin liquids, some have finite Fermi surfaces while others have only Fermi points. The spinons near the Fermi points can have linear $E(k)\propto|k|$ or quadratic $E(k)\propto k^2$ dispersions. We find that there are more than one Z_2 spin liquid whose spinons have a massless Dirac-like dispersion. Those Z_2 spin liquids have the same symmetry but different quantum orders. Their *Ansätze* are given by Eq. (42), Eq. (39), Eq. (106), etc.

Both chiral spin-liquid and Z_2 spin-liquid states are Mott insulators with one electron per unit cell if not doped. Their internal structures are characterized by a new kind of order—topological order—if they are gapped or if the gapless sector decouples. Topological order is not related to any symmetries and has no (local) order parameters. Thus, the topological order is robust against all perturbations that can break any symmetries (including random perturbations that break translation symmetry).[9,10] (This point was also emphasized in Ref. 65 recently.) Even though there are no order parameters

to characterize them, the topological orders can be characterized by other measurable quantum numbers, such as ground-state degeneracy in compact space as proposed in Refs. 9 and 10. Recently, Ref. 65 introduced a very clever experiment to test the ground-state degeneracy associated with the nontrivial topological orders. In addition to the ground-state degeneracy, there are other practical ways to detect topological orders. For example, the excitations on top of a topologically ordered state can be defects of the underlying topological order, which usually leads to unusual statistics for those excitations. Measuring the statistics of those excitations also allows us to measure topological orders.

The concepts of topological order and quantum order are very important in understanding quantum spin liquids (or any other strongly correlated quantum liquids). In this paper we are going to construct hundreds of different spin liquids. Those spin liquids all have the same symmetry. To understand those spin liquids, we need to first learn how to characterize those spin liquids. Those states break no symmetries and hence have no order parameters. One would get into a wrong track if trying to find an order parameter to characterize the spin liquids. We need to use a completely new way, such as topological orders and quantum orders, to characterize those states.

In addition to the above Z_2 spin liquids, in this paper we will also study many other spin liquids with different low-energy gauge structures, such as U(1) and SU(2) gauge structures. We will use the terms Z_2 spin liquids, U(1) spin liquids, and SU(2) spin liquids to describe them. We would like to stress that Z_2, U(1), and SU(2) here are gauge groups that appear in the low-energy effective theories of those spin liquids. They should not be confused with the Z_2, U(1), and SU(2) gauge group in the slave-boson approach or other theories of projective construction. The latter are high-energy gauge groups. The high-energy gauge groups have nothing to do with the low-energy gauge groups. A high-energy Z_2 gauge theory (or a Z_2 slave-boson approach) can have a low-energy effective theory that contains SU(2), U(1), or Z_2 gauge fluctuations. Even the t-J model, which has no gauge structure at lattice scale, can have a low-energy effective theory that contains SU(2), U(1), or Z_2 gauge fluctuations. The spin liquids studied in this paper all contain some kind of low-energy gauge fluctuations. Despite their different low-energy gauge groups, all those spin liquids can be constructed from any one of SU(2), U(1), or Z_2 slave-boson approaches. After all, all those slave-boson approaches describe the same t-J model and are equivalent to each other. In short, the high-energy gauge group is related to the way in which we write down the Hamiltonian, while the low-energy gauge group is a property of the ground state. Thus we should not regard Z_2 spin liquids as the spin liquids constructed using Z_2 slave-boson approach. A Z_2 spin liquid can be constructed from the U(1) or SU(2) slave-boson approaches as well. A precise mathematical definition of the low-energy gauge group will be given in Sec. IV A.

D. Organization

In this paper we will use the method outlined in Refs. 38 and 40 to study gauge structures in various spin-liquid states. In Sec. II we review SU(2) mean-field theory of spin liquids. In Sec. III, we construct simple symmetric spin liquids using translationally invariant *Ansätze*. In Sec. IV, the projective symmetry group is introduced to characterize quantum orders in spin liquids. In Sec. V, we study the transition between different symmetric spin liquids, using the results obtained in Ref. 66, where we find a way to construct all symmetric spin liquids in the neighborhood of some well-known spin liquids. We also study the spinon spectrum to gain some intuitive understanding of the dynamical properties of the spin liquids. Using the relation between the two-spinon spectrum and quantum order, we propose, in Sec. VII, a practical way to use neutron scattering to measure quantum orders. We study the stability of Fermi spin liquids and algebraic spin liquids in Sec. VIII. We find that both Fermi spin liquids and algebraic spin liquids can exist as zero-temperature phases. This is particularly striking for algebraic spin liquids since their gapless excitations interacts even at lowest energies and there are no free fermionic-bosonic quasiparticle excitations at low energies. We show how quantum order can protect gapless excitations. The Appendix contains an algebraic description of projective symmetry groups, which can be used to classify projective symmetry groups.[66] Section X summarizes the main results of the paper.

II. PROJECTIVE CONSTRUCTION OF 2D SPIN LIQUIDS: A REVIEW OF THE SU(2) SLAVE-BOSON APPROACH

In this section, we are going to use projective construction to construct 2D spin liquids. We are going to review a particular projective construction: namely, the SU(2) slave-boson approach.[15,16,33,36–38,40] The gauge structure discovered by Baskaran and Anderson[16] in the slave-boson approach plays a crucial role in our understanding of strongly correlated spin liquids.

We will concentrate on the spin-liquid states of a pure spin-1/2 model on a 2D square lattice:

$$H_{\rm spin}=\sum_{\langle ij\rangle} J_{ij}S_i\cdot S_j+\cdots, \tag{1}$$

where the summation is over different links (i.e., $\langle ij\rangle$ and $\langle ji\rangle$ are regarded as the same) and the ellipsis represents possible terms which contain three or more spin operators. Those terms are needed in order for many exotic spin-liquid states introduced in this paper to become the ground state. To obtain the mean-field ground state of the spin liquids, we introduce fermionic parton operator $f_{i\alpha}$, $\alpha=1,2$, which carries spin 1/2 and no charge. The spin operator $\boldsymbol{S}_i$ is represented as

$$\boldsymbol{S}_i=\frac{1}{2}f^\dagger_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}f_{i\beta}. \tag{2}$$

In terms of the fermion operators the Hamiltonian, Eq. (1), can be rewritten as

$$H=\sum_{\langle ij\rangle} -\frac{1}{2}J_{ij}\left(f^\dagger_{i\alpha}f_{j\alpha}f^\dagger_{j\beta}f_{i\beta}+\frac{1}{2}f^\dagger_{i\alpha}f_{i\alpha}f^\dagger_{j\beta}f_{j\beta}\right). \tag{3}$$

Here we have used $\sigma_{\alpha\beta}\cdot\sigma_{\alpha'\beta'}=2\delta_{\alpha\beta'}\delta_{\alpha'\beta}-\delta_{\alpha\beta}\delta_{\alpha'\beta'}$. We also added proper constant terms $\Sigma_i f^\dagger_{i\alpha}f_{i\alpha}$ and $\Sigma_{\langle ij\rangle}f^\dagger_{i\alpha}f_{i\alpha}f^\dagger_{j\beta}f_{j\beta}$ to get the above form. Notice that the Hilbert space of Eq. (3) is generated by the parton operators f_α and is larger than that of Eq. (1). The equivalence between Eq. (1) and Eq. (3) is valid only in the subspace where there is exactly one fermion per site. Therefore to use Eq. (3) to describe the spin state we need to impose the constraint[15,16]

$$f^\dagger_{i\alpha}f_{i\alpha}=1, \quad f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta}=0. \tag{4}$$

The second constraint is actually a consequence of the first one.

A mean-field ground state at ''zeroth'' order is obtained by making the following approximations. First we replace the constraint, Eq. (4), by its ground-state average

$$\langle f^\dagger_{i\alpha}f_{i\alpha}\rangle=1, \quad \langle f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta}\rangle=0. \tag{5}$$

Such a constraint can be enforced by including a *site-dependent* and time-independent Lagrangian multiplier $a^l_0(i)$, $l=1,2,3$, in the Hamiltonian. At zeroth order we ignore the fluctuations (i.e., the time dependence) of a^l_0. If we included the fluctuations of a^l_0, the constraint, Eq. (5), would become the original constraint, Eq. (4).[15,16,36,37] Second we replace the operators $f^\dagger_{i\alpha}f_{j\beta}$ and $f_{i\alpha}f_{i\beta}$ by their ground-state expectations value

$$\eta_{ij}\epsilon_{\alpha\beta}=-2\langle f_{i\alpha}f_{j\beta}\rangle, \quad \eta_{ij}=\eta_{ji},$$
$$\chi_{ij}\delta_{\alpha\beta}=2\langle f^\dagger_{i\alpha}f_{j\beta}\rangle, \quad \chi_{ij}=\chi^\dagger_{ji}, \tag{6}$$

again ignoring their fluctuations. In this way we obtain the zeroth-order mean-field Hamiltonian

$$H_{\rm mean}=\sum_{\langle ij\rangle}-\frac{3}{8}J_{ij}[(\chi_{ji}f^\dagger_{i\alpha}f_{j\alpha}+\eta_{ij}f^\dagger_{i\alpha}f^\dagger_{j\beta}\epsilon_{\alpha\beta}+\text{H.c.}-|\chi_{ij}|^2-|\eta_{ij}|^2]+\sum_i\{a^3_0(f^\dagger_{i\alpha}f_{i\alpha}-1)+[(a^1_0+ia^2_0)f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta}+\text{H.c.}]\}. \tag{7}$$

χ_{ij} and η_{ij} in Eq. (7) must satisfy the self-consistency condition, Eq. (6), and the site-dependent fields $a^l_0(i)$ are chosen such that Eq. (5) is satisfied by the mean-field ground state. Such χ_{ij}, η_{ij}, and a^l_0 give us a mean-field solution. The fluctuations in χ_{ij}, η_{ij}, and $a^l_0(i)$ describe the collective excitations above the mean-field ground state.

The Hamiltonian, Eq. (7), and the constraints, Eq. (4), have a local SU(2) symmetry.[36,37] The local SU(2) symmetry becomes explicit if we introduce the doublet

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}f_\uparrow\\ f^\dagger_\downarrow\end{pmatrix} \tag{8}$$

and matrix

$$U_{ij}=\begin{pmatrix}\chi^\dagger_{ij} & \eta_{ij}\\ \eta^\dagger_{ij} & -\chi_{ij}\end{pmatrix}=U^\dagger_{ji}. \tag{9}$$

Using Eq. (8) and Eq. (9) we can rewrite Eq. (5) and Eq. (7) as

$$\langle\psi^\dagger_i\tau^l\psi_i\rangle=0, \tag{10}$$

$$H_{\rm mean}=\sum_{\langle ij\rangle}\frac{3}{8}J_{ij}\left[\frac{1}{2}\text{Tr}(U^\dagger_{ij}U_{ij})-(\psi^\dagger_iU_{ij}\psi_j+\text{H.c.})\right]+\sum_i a^l_0\psi^\dagger_i\tau^l\psi_i, \tag{11}$$

where τ^l, $l=1,2,3$, are the Pauli matrices. From Eq. (11) we can see clearly that the Hamiltonian is invariant under a local SU(2) transformation $W(i)$:

$$\psi_i\to W(i)\psi_i,$$
$$U_{ij}\to W(i)U_{ij}W^\dagger(j). \tag{12}$$

The SU(2) gauge structure originates from Eq. (2). The SU(2) is the most general transformation between the partons that leave the physical spin operator unchanged. Thus once we write down the parton expression of the spin operator, Eq. (2), the gauge structure of the theory is determined.[61] [The SU(2) gauge structure discussed here is a high-energy gauge structure.]

We note that both components of ψ carry spin up. Thus the spin-rotation symmetry is not explicit in our formalism and it is hard to tell if Eq. (11) describes a spin-rotation-invariant state or not. In fact, for a general U_{ij} satisfying $U_{ij}=U^\dagger_{ji}$, Eq. (11) may not describe a spin-rotation-invariant state. However, if U_{ij} has a form

$$U_{ij}=i\rho_{ij}W_{ij},$$
$$\rho_{ij}=\text{real number},$$
$$W_{ij}\in\text{SU}(2), \tag{13}$$

then Eq. (11) will describe a spin-rotation-invariant state. This is because the above U_{ij} can be rewritten in a form of Eq. (9). In this case Eq. (11) can be rewritten as Eq. (7) where the spin-rotation invariance is explicit.

To obtain the mean-field theory, we have enlarged the Hilbert space. Because of this, the mean-field theory is not even qualitatively correct. Let $|\Psi^{(U_{ij})}_{\rm mean}\rangle$ be the ground state of the Hamiltonian, Eq. (11), with energy $E(U_{ij},a^l_i\tau^l)$. It is clear that the mean-field ground state is not even a valid wave function for the spin system since it may not have one fermion per site. Thus it is very important to include fluctuations of a^l_0 to enforce one-fermion-per-site constraint. With this understanding, we may obtain a valid wave function of the spin system $\Psi_{\rm spin}(\{\alpha_i\})$ by projecting the mean-field state to the subspace of one fermion per site:

$$\Psi_{\rm spin}(\{\alpha_i\})=\langle 0|\prod_i f_{i\alpha_i}|\Psi^{(U_{ij})}_{\rm mean}\rangle. \tag{14}$$

Now the local SU(2) transformation, Eq. (12), can have a very physical meaning: $|\Psi_{\rm mean}^{(U_{ij})}\rangle$ and $|\Psi_{\rm mean}^{(W(i)U_{ij}W^\dagger(j))}\rangle$ give rise to the same spin wave function after projection:

$$\langle 0|\prod_i f_{i\alpha_i}|\Psi_{\rm mean}^{(U_{ij})}\rangle=\langle 0|\prod_i f_{i\alpha_i}|\Psi_{\rm mean}^{(W(i)U_{ij}W^\dagger(j))}\rangle. \quad (15)$$

Thus U_{ij} and $U'_{ij}=W(i)U_{ij}W^\dagger(j)$ are just two different labels which label the *same physical state*. Within the mean-field theory, a local SU(2) transformation changes a mean-field state $|\Psi_{\rm mean}^{(U_{ij})}\rangle$ to a different mean-field state $|\Psi_{\rm mean}^{(U'_{ij})}\rangle$. If the two mean-field states always have the same physical properties, the system has a local SU(2) *symmetry*. However, after projection, the physical spin quantum state described by wave function $\Psi_{\rm spin}(\{\alpha_i\})$ is invariant under the local SU(2) transformation. A local SU(2) transformation just transforms one label U_{ij} of a physical spin state to another label U'_{ij} which labels exactly the same physical state. Thus after projection, local SU(2) transformations become gauge transformations. The fact that U_{ij} and U'_{ij} label the same physical spin state creates an interesting situation when we consider the fluctuations of U_{ij} around a mean-field solution—some fluctuations of U_{ij} do not change the physical state and are unphysical. Those fluctuations are called pure gauge fluctuations.

The above discussion also indicates that in order for the mean-field theory to make any sense, we must at least include the SU(2) gauge (or other gauge) fluctuations described by a_0^l and W_{ij} in Eq. (13), so that the SU(2) gauge structure of the mean-field theory is revealed and the physical spin state is obtained. We will include the gauge fluctuations to the zeroth-order mean-field theory. The new theory will be called the first-order mean-field theory. It is this first-order mean-field theory that represents a proper low-energy effective theory of the spin liquid.

Here, we would like to make a remark about ''gauge symmetry'' and ''gauge symmetry breaking.'' We see that two *Ansätze* U_{ij} and $U'_{ij}=W(i)U_{ij}W^\dagger(j)$ have the same physical properties. This property is usually called the ''gauge symmetry.'' However, from the above discussion, we see that the ''gauge symmetry'' is *not* a symmetry. A symmetry is about two *different* states having the same properties. U_{ij} and U'_{ij} are just two labels that label the same state, and the same states always have the same properties. We do not usually call the same state having the same properties a symmetry. Because the same states always have the same properties, the ''gauge symmetry'' can never be broken. It is very misleading to call the Anderson-Higgs mechanism ''gauge symmetry breaking.'' With this understanding, we see that a superconductor is fundamentally different from a superfluid. A superfluid is characterized by U(1) symmetry breaking, while a superconductor has no symmetry breaking once we include the dynamical electromagnetic gauge fluctuations. A superconductor is actually the first topologically ordered state observed in experiments,[13] which has no symmetry breaking, no long-range order, and no (local) order parameter. However, when the speed of light $c=\infty$, a superconductor becomes similar to a superfluid and is characterized by U(1) symmetry breaking.

The relation between the mean-field state and the physical spin-wave function, Eq. (14), allows us to construct transformation of the physical spin wave function from the mean-field *Ansätze*. For example the mean-field state $|\Psi_{\rm mean}^{(U'_{ij})}\rangle$ with $U'_{ij}=U_{i-l,j-l}$ produces a physical spin wave function which is translated by a distance l from the physical spin wave function produced by $|\Psi_{\rm mean}^{(U_{ij})}\rangle$. The physical state is translationally symmetric if and only if the translated *Ansatz* U'_{ij} and the original *Ansatz* U_{ij} are gauge equivalent (it does not require $U'_{ij}=U_{ij}$). We see that the gauge structure can complicate our analysis of symmetries, since the physical spin wave function $\Psi_{\rm spin}(\{\alpha_i\})$ may have more symmetries than the mean-field state $|\Psi_{\rm mean}^{(U_{ij})}\rangle$ before projection.

Let us discuss time reversal symmetry in more detail. A quantum system described by

$$i\hbar\partial_t\Psi(t)=H\Psi(t) \quad (16)$$

has a time reversal symmetry if $\Psi(t)$ satisfying the equation of motion implies that $\Psi^*(-t)$ also satisfying the equation of motion. This requires that $H=H^*$. We see that, for a time-reversal-symmetric system, if Ψ is an eigenstate, then Ψ^* will be an eigenstate with the same energy.

For our system, the time reversal symmetry means that if the mean-field wave function $\Psi_{\rm mean}^{(U_{ij},a_i^l\tau^l)}$ is a mean-field ground-state wave function for *Ansatz* $(U_{ij},a_i^l\tau^l)$, then $(\Psi_{\rm mean}^{(U_{ij},a_i^l\tau^l)})^*$ will be the mean-field ground-state wave function for *Ansatz* $(U_{ij}^*,a_i^l(\tau^l)^*)$. That is,

$$(\Psi_{\rm mean}^{(U_{ij},a_i^l\tau^l)})^*=\Psi_{\rm mean}^{(U_{ij}^*,a_i^l(\tau^l)^*)}. \quad (17)$$

For a system with time reversal symmetry, the mean-field energy $E(U_{ij},a_i^l\tau^l)$ satisfies

$$E(U_{ij},a_i^l\tau^l)=E(U_{ij}^*,a_i^l(\tau^l)^*). \quad (18)$$

Thus, if an *Ansatz* $(U_{ij},a_i^l\tau^l)$ is a mean-field solution, then $(U_{ij}^*,a_i^l(\tau^l)^*)$ is also a mean-field solution with the same mean-field energy.

From the above discussion, we see that under the time reversal transformation, the *Ansätz* transforms as

$$U_{ij}\to U'_{ij}=(-i\tau^2)U_{ij}^*(i\tau^2)=-U_{ij},$$

$$a_i^l\tau^l\to a_i'^l\tau^l=(-i\tau^2)(a_i^l\tau^l)^*(i\tau^2)=-a_i^l\tau^l. \quad (19)$$

Note here we have included an additional SU(2) gauge transformation $W_i=-i\tau^2$. We also note that under the time reversal transformation, the loop operator transforms as $P_C=e^{i\theta+i\theta^l\tau^l}\to(-i\tau^2)P_C^*(i\tau^2)=e^{-i\theta+i\theta^l\tau^l}$. We see that the U(1) flux changes the sign while the SU(2) flux is not changed.

Before ending this review section, we would like to point out that the mean-field *Ansätze* of the spin liquids U_{ij} can be divided into two classes: unfrustrated *Ansätze* where U_{ij}

only link an even lattice site to an odd lattice site and frustrated *Ansätze* where U_{ij} are nonzero between two even sites and/or two odd sites. An unfrustrated *Ansatz* has only pure SU(2) flux through each plaquette, while an frustrated *Ansatz* has U(1) flux of multiple of $\pi/2$ through some plaquettes in addition to the SU(2) flux.

III. SPIN LIQUIDS FROM TRANSLATIONALLY INVARIANT *ANSATZ*

In this section, we will study many simple examples of spin liquids and their *Ansatz*. Through those simple examples, we gain some understanding as to what kind of spin liquids are possible. This understanding helps us to develop the characterization and classification of spin liquids using projective symmetry groups.

Using the above SU(2) projective construction, one can construct many spin-liquid states. To limit ourselves, we will concentrate on spin liquids with translation and 90° rotation symmetries. Although a mean-field *Ansatz* with translation and rotation invariance always generates a spin liquid with translation and rotation symmetries, a mean-field *Ansatz* without those invariances can also generate a spin liquid with those symmetries.[95] Because of this, it is quite difficult to construct all the translation- and rotation-symmetric spin liquids. In this section we will consider a simpler problem. We will limit ourselves to spin liquids generated from translationally invariant *Ansätze*:

$$U_{i+l,j+l}=U_{ij}, \quad a_0^l(i)=a_0^l. \tag{20}$$

In this case, we only need to find the conditions under which the above *Ansätze* can give rise to a rotationally symmetric spin liquid. First let us introduce u_{ij}:

$$\frac{3}{8}J_{ij}U_{ij}=u_{ij}. \tag{21}$$

For translationally invariant *Ansätze*, we can introduce a shorthand notation

$$u_{ij}=u^{\mu}_{-i+j}\tau^{\mu}\equiv u_{-i+j}, \tag{22}$$

where $u_l^{1,2,3}$ are real, u_l^0 is imaginary, τ^0 is the identity matrix, and $\tau^{1,2,3}$ are the Pauli matrices. The fermion spectrum is determined by the Hamiltonian

$$H=-\sum_{\langle ij\rangle}(\psi_i^{\dagger}u_{j-i}\psi_j+\text{H.c.})+\sum_i \psi_i^{\dagger}a_0^l\tau^l\psi_i. \tag{23}$$

In k space we have

$$H=-\sum_k \psi_k^{\dagger}[u^{\mu}(k)-a_0^{\mu}]\tau^{\nu}\psi_k, \tag{24}$$

where $\mu=0,1,2,3$,

$$u^{\mu}(k)=\sum_l u_l^{\mu}e^{il\cdot k}, \tag{25}$$

$a_0^0=0$, and N is the total number of sites. The fermion spectrum has two branches and is given by

$$E_{\pm}(k)=u^0(k)\pm E_0(k),$$

$$E_0(k)=\sqrt{\sum_l [u^l(k)-a_0^l]^2}. \tag{26}$$

The constraints can be obtained from $\partial E_{\text{ground}}/\partial a_0^l=0$ and have the form

$$N\langle\psi_i^{\dagger}\tau^l\psi_i\rangle=\sum_{k,E_-(k)<0}\frac{u^l(k)-a_0^l}{E_0(k)}-\sum_{k,E_+(k)<0}\frac{u^l(k)-a_0^l}{E_0(k)}=0, \tag{27}$$

which allows us to determine a_0^l, $l=1,2,3$. It is interesting to see that if $u_i^0=0$ and the *Ansatz* is unfrustrated, then we can simply choose $a_0^l=0$ to satisfy the mean-field constraints (since $u^{\mu}(k)=-u^{\mu}[k+(\pi,\pi)]$ for unfrustrated *Ansatze*). Such *Ansätze* always have time reversal symmetry. This is because U_{ij} and $-U_{ij}$ are gauge equivalent for unfrustrated *Ansätze*.

Now let us study some simple examples. First let us assume that only the nearest-neighbor couplings $u_{\hat{x}}$ and $u_{\hat{y}}$ are nonzero. In order for the *Ansatz* to describe a rotationally symmetric state, the rotated *Ansatz* must be gauge equivalent to the original *Ansatz*. One can easily check that the following *Ansatz* has rotation symmetry:

$$a_0^l=0,$$

$$u_{\hat{x}}=\chi\tau^3+\eta\tau^1,$$

$$u_{\hat{y}}=\chi\tau^3-\eta\tau^1, \tag{28}$$

since the 90° rotation followed by a gauge transformation $W_i=i\tau^3$ leaves the *Ansatz* unchanged. The above *Ansatz* also has time reversal symmetry, since the time reversal transformation $u_{ij}\rightarrow -u_{ij}$ followed by a gauge transformation $W_i=i\tau^2$ leaves the *Ansatz* unchanged.

To understand the gauge fluctuations around the above mean-field state, we note that the mean-field *Ansatz* may generate nontrivial SU(2) flux through plaquettes. That flux may break the SU(2) gauge structure down to U(1) or Z_2 gauge structures as discussed in Refs. 38 and 40. In particular, the dynamics of the gauge fluctuations in the breakdown from SU(2) to Z_2 has been discussed in detail in Ref. 40. According to Refs. 38 and 40, the SU(2) flux plays the role of Higgs fields. A nontrivial SU(2) flux corresponds to a condensation of Higgs fields which can break the gauge structure and give the SU(2) and/or U(1) gauge boson a mass. Thus to understand the dynamics of the gauge fluctuations, we need to find the SU(2) flux.

The SU(2) flux is defined for loops with a base point. The loop starts and ends at the base point. For example, we can consider the following two loops $C_{1,2}$ with the same base point i: $C_1=i\rightarrow i+\hat{x}\rightarrow i+\hat{x}+\hat{y}\rightarrow i+\hat{y}\rightarrow i$ and C_2 is the 90° rotation of C_1: $C_2=i\rightarrow i+\hat{y}\rightarrow i-\hat{x}+\hat{y}\rightarrow i-\hat{x}\rightarrow i$. The SU(2) flux for the two loops is defined as

$$P_{C_1} \equiv u_{i,i+\hat{y}} u_{i+\hat{y},i+\hat{x}+\hat{y}} u_{i+\hat{x}+\hat{y},i+\hat{x}} u_{i+\hat{x},i} = u_{\hat{y}}^\dagger u_{\hat{x}}^\dagger u_{\hat{y}} u_{\hat{x}},$$

$$P_{C_2} \equiv u_{i,i-\hat{x}} u_{i-\hat{x},i-\hat{x}+\hat{y}} u_{i-\hat{x}+\hat{y},i+\hat{y}} u_{i+\hat{y},i} = u_{\hat{x}} u_{\hat{y}}^\dagger u_{\hat{x}} u_{\hat{y}}. \tag{29}$$

As discussed in Refs. 38 and 40, if the SU(2) flux P_C for all loops is trivial $P_C \propto \tau^0$, then the SU(2) gauge structure is unbroken. This is the case when $\chi = \eta$ or when $\eta = 0$ in the above *Ansatz*, Eq. (28). The spinon in the spin liquid described by $\eta = 0$ has a large Fermi surface. We will call this state the SU(2)-gapless state. (This state is called the uniform RVB state in the literature.) The state with $\chi = \eta$ has gapless spinons only at isolated k points. We will call such a state a SU(2)-linear state to stress the linear dispersion $E \propto |k|$ near the Fermi points. (Such a state was called the π-flux state in the literature.) The low-energy effective theory for the SU(2)-linear state is described by massless Dirac fermions (the spinons) coupled to a SU(2) gauge field.

After proper gauge transformations, the SU(2)-gapless *Ansatz* can be rewritten as

$$u_{\hat{x}} = i\chi,$$

$$u_{\hat{y}} = i\chi \tag{30}$$

and the SU(2)-linear *Ansatz* as

$$u_{i,i+\hat{x}} = i\chi,$$

$$u_{i,i+\hat{y}} = i(-)^{i_x}\chi. \tag{31}$$

In these forms, the SU(2) gauge structure is explicit since $u_{ij} \propto i\tau^0$. Here we would also like to mention that under the projective-symmetry-group classification, the SU(2)-gapless *Ansatz* Eq. (30), is labeled by SU2An0 and the SU(2)-linear *Ansätze* Eq. (31), by SU2Bn0 [see Eq. (100)].

When $\chi \neq \eta \neq 0$, the flux P_C is nontrivial. However, P_C commutes with $P_{C'}$ as long as the two loops C and C' have the same base point. In this case the SU(2) gauge structure is broken down to a U(1) gauge structure.[38,40] The gapless spinon still only appears at isolated **k** points. We will call such a state a U(1)-linear state. (This state was called the staggered flux state and/or d-wave pairing state in the literature.) After a proper gauge transformation, the U(1)-linear state can also be described by the *Ansatz*

$$u_{i,i+\hat{x}} = i\chi - (-)^i \eta\tau^3,$$

$$u_{i,i+\hat{y}} = i\chi + (-)^i \eta\tau^3, \tag{32}$$

where the U(1) gauge structure is explicit. Under the projective-symmetry-group classification, such a state is labeled by U1Cn01n (see Sec. IV C). The low-energy effective theory is described by massless Dirac fermions (the spinons) coupled to a U(1) gauge field.

The above results are all known. In the following we are going to study a new class of translation and rotation symmetric *Ansatz*, which has the form

$$a_0^l = 0,$$

$$u_{\hat{x}} = i\eta\tau^0 - \chi(\tau^3 - \tau^l),$$

$$u_{\hat{y}} = i\eta\tau^0 - \chi(\tau^3 + \tau^1), \tag{33}$$

with χ and η nonzero. The above *Ansatz* describes the SU(2)-gapless spin liquid if $\chi = 0$, and the SU(2)-linear spin liquid if $\eta = 0$.

After a 90° rotation R_{90}, the above *Ansatz* becomes

$$u_{\hat{x}} = -i\eta\tau^0 - \chi(\tau^3 + \tau^1),$$

$$u_{\hat{y}} = i\eta\tau^0 - \chi(\tau^3 - \tau^1). \tag{34}$$

The rotated *Ansatz* is gauge equivalent to the original *Ansatz* under the gauge transformation $G_{R_{90}}(i) = (-)^{i_x}(1 - i\tau^2)/\sqrt{2}$. After a parity $x \to -x$ transformation P_x, Eq. (33) becomes

$$u_{\hat{x}} = -i\eta\tau^0 - \chi(\tau^3 - \tau^1),$$

$$u_{\hat{y}} = i\eta\tau^0 - \chi(\tau^3 + \tau^1), \tag{35}$$

which is gauge equivalent to the original *Ansatz* under the gauge transformation $G_{P_x}(i) = (-)^{i_x} i(\tau^3 + \tau^1)/\sqrt{2}$. Under time reversal transformation T, Eq. (33) is changed to

$$u_{\hat{x}} = -i\eta\tau^0 + \chi(\tau^3 - \tau^1),$$

$$u_{\hat{y}} = -i\eta\tau^0 + \chi(\tau^3 + \tau^1), \tag{36}$$

which is again gauge equivalent to the original *Ansatz* under the gauge transformation $G_T(i) = (-)^i$. [In fact any *Ansatz* which only has links between two nonoverlapping sublattices (i.e., the unfrustrated *Ansatz*) is time reversal symmetric if $a_0^l = 0$.] To summarize, the *Ansatz*, Eq. (33), is invariant under the rotation R_{90}, parity P_x, and time reversal transformation T, *followed by the following gauge transformations*:

$$G_{R_{90}}(i) = (-)^{i_x}(1 - i\tau^2)/\sqrt{2},$$

$$G_{P_x}(i) = (-)^{i_x} i(\tau^3 + \tau^1)/\sqrt{2},$$

$$G_T(i) = (-)^i. \tag{37}$$

Thus the *Ansatz*, Eq. (33), describes a spin liquid with translation, rotation, parity, and time reversal symmetries.

Using time reversal symmetry we can show that the vanishing a_0^l in our *Ansatz*, Eq. (33), indeed satisfy the constraint, Eq. (27). This is because $a_0^l \to -a_0^l$ under the time reversal transformation. Thus $\partial E_{\text{mean}}/\partial a_0^l = 0$ when $a_0^l = 0$ for any time-reversal-symmetric *Ansatz*, including the *Ansatz*, Eq. (33).

The spinon spectrum is given by [see Fig. 5(a)]

$$E_\pm = 2\eta[\sin(k_x) + \sin(k_y)] \pm 2|\chi|\sqrt{2\cos^2(k_x) + 2\cos^2(k_y)}. \tag{38}$$

The spinons have two Fermi points and two small Fermi pockets (for small η). The SU(2) flux is nontrivial. Furthermore, P_{C_1} and P_{C_2} do not commute. Thus the SU(2) gauge

structure is broken down to a Z_2 gauge structure by the SU(2) flux P_{C_1} and P_{C_2}.[38,40] We will call the spin liquid described by Eq. (33) a $Z2$-gapless spin liquid. The low-energy effective theory is described by massless Dirac fermions and fermions with small Fermi surfaces, coupled to a Z_2 gauge field. Since the Z_2 gauge interaction is irrelevant at low energies, the spinons are *free* fermions at low energies and we have a true spin-charge separation in the Z_2-gapless spin liquid. The Z_2-gapless spin liquid is one of the Z_2 spin liquids classified in Ref. 66. Its projective symmetry group is labeled by $\mathrm{Z2A}\tau_-^{13}\tau_+^{13}\tau^3\tau_-^0$ or equivalently by $\mathrm{Z2A}x2(12)n$ [see Sec. IV B and Eq. (85)].

Now let us include longer links. First we still limit ourselves to unfrustrated *Ansätze*. An interesting *Ansatz* is given by

$$
\begin{aligned}
a_0^l &= 0,\\
u_{\hat{x}} &= \chi\tau^3 + \eta\tau^1,\\
u_{\hat{y}} &= \chi\tau^3 - \eta\tau^1,\\
u_{2\hat{x}+\hat{y}} &= \lambda\tau^2,\\
u_{-\hat{x}+2\hat{y}} &= -\lambda\tau^2,\\
u_{2\hat{x}-\hat{y}} &= \lambda\tau^2,\\
u_{\hat{x}+2\hat{y}} &= -\lambda\tau^2.
\end{aligned}
\tag{39}
$$

By definition, the *Ansatz* is invariant under translation and parity $x\to -x$. After a 90° rotation, the *Ansatz* is changed to

$$
\begin{aligned}
u_{\hat{x}} &= -\chi\tau^3 - \eta\tau^1,\\
u_{\hat{y}} &= -\chi\tau^3 + \eta\tau^1,\\
u_{2\hat{x}+\hat{y}} &= -\lambda\tau^2,\\
u_{-\hat{x}+2\hat{y}} &= +\lambda\tau^2,\\
u_{2\hat{x}-\hat{y}} &= -\lambda\tau^2,\\
u_{\hat{x}+2\hat{y}} &= +\lambda\tau^2,
\end{aligned}
\tag{40}
$$

which is gauge equivalent to Eq. (39) under the gauge transformation $G_{R_{90}}(i) = i\tau_3$. Thus the *Ansätze* describe a spin liquid with translation, rotation, parity, and the time reversal symmetries. The spinon spectrum is given by [see Fig. 1(c)]

$$
\begin{aligned}
E_\pm &= \pm\sqrt{\epsilon_1(k)^2+\epsilon_2(k)^2+\epsilon_3(k)^2},\\
\epsilon_1 &= -2\chi[\cos(k_x)+\cos(k_y)],\\
\epsilon_2 &= -2\eta[\cos(k_x)-\cos(k_y)],\\
\epsilon_3 &= -2\lambda[\cos(2k_x+k_y)+\cos(2k_x-k_y)\\
&\quad -\cos(k_x-2k_y)-\cos(k_x+2k_y)].
\end{aligned}
\tag{41}
$$

Thus the spinons are gapless only at four $\mathbf{k}$ points $(\pm\pi/2, \pm\pi/2)$. We also find that P_{C_3} and P_{C_4} do not commute, where the loops $C_3 = i\to i+\hat{x}\to i+2\hat{x}\to i+2\hat{x}+\hat{y}\to i$ and $C_4 = i\to i+\hat{y}\to i+2\hat{y}\to i+2\hat{y}-\hat{x}\to i$. Thus the SU(2) fluxes P_{C_3} and P_{C_4} break the SU(2) gauge structure down to a Z_2 gauge structure. The spin liquid described by Eq. (39) will be called the Z_2-linear spin liquid. The low-energy effective theory is described by massless Dirac fermions coupled to a Z_2 gauge field. Again the Z_2 coupling is irrelevant and the spinons are free fermions at low energies. We have a true spin-charge separation. According to the classification scheme summarized in Sec. IV B, the above Z_2-linear spin liquid is labeled by $\mathrm{Z2A}003n$.

Next let us discuss frustrated *Ansätze*. A simple Z_2 spin liquid can be obtained from the following frustrated *Ansätze*:

$$
\begin{aligned}
a_0^3 &\neq, \quad a_0^{1,2} = 0,\\
u_{\hat{x}} &= \chi\tau^3 + \eta\tau^1,\\
u_{\hat{y}} &= \chi\tau^3 - \eta\tau^1,\\
u_{\hat{x}+\hat{y}} &= \gamma\tau^3,\\
u_{-\hat{x}+\hat{y}} &= \gamma\tau^3.
\end{aligned}
\tag{42}
$$

The *Ansatz* has translation, rotation, parity, and the time reversal symmetries. When $a_0^3 \neq 0$, $\chi \neq \pm\eta$, and $\chi\eta \neq 0$, $a_0^l\tau^l$ does not commute with the loop operators. Thus the *Ansatz* breaks the SU(2) gauge structure to a Z_2 gauge structure. The spinon spectrum is given by [see Fig. 1(a)]

$$
\begin{aligned}
E_\pm &= \pm\sqrt{\epsilon^2(\mathbf{k})+\Delta^2(\mathbf{k})},\\
\epsilon(k) &= 2\chi[\cos(k_x)+\cos(k_y)]\\
&\quad + a_0^3 2\gamma[\cos(k_x+k_y)+\cos(k_x-k_y)],\\
\Delta(k') &= 2\eta[\cos(k_x)-\cos(k_y)] + a_0^3,
\end{aligned}
\tag{43}
$$

which is gapless only at four $\mathbf{k}$ points with a linear dispersion. Thus the spin liquid described by Eq. (42) is a Z_2-linear spin liquid, which has a true spin-charge separation. The Z_2-linear spin liquid is described by the projective symmetry group Z2A0032 or equivalently Z2A0013 (see Sec. IV B). From the above two examples of Z_2-linear spin liquids, we find that it is possible to obtain true spin-charge separation with massless Dirac points (or nodes) within a pure spin model without charge fluctuations. We also find that there is more than one way to do it.

A well-known frustrated *Ansatz* is the *Ansatz* for the chiral spin liquid:[6]

$$
\begin{aligned}
u_{\hat{x}} &= -\chi\tau^3 - \chi\tau^1,\\
u_{\hat{y}} &= -\chi\tau^3 + \chi\tau^1,\\
u_{\hat{x}+\hat{y}} &= \eta\tau^2,\\
u_{-\hat{x}+\hat{y}} &= -\eta\tau^2,\\
a_0^l &= 0.
\end{aligned}
\tag{44}
$$

The chiral spin liquid breaks the time reversal and parity symmetries. The SU(2) gauge structure is unbroken.[38] The low-energy effective theory is an SU(2) Chern-Simons theory (of level 1). The spinons are gapped and have a semionic statistics.[5,6] The third interesting frustrated *Ansatz* is given in Refs. 38 and 40:

$$u_{\hat{x}} = u_{\hat{y}} = -\chi\tau^3,$$

$$u_{\hat{x}+\hat{y}} = \eta\tau^1 + \lambda\tau^2,$$

$$u_{-\hat{x}+\hat{y}} = \eta\tau^1 - \lambda\tau^2,$$

$$a_0^{2,3} = 0, \quad a_0^1 \neq 0. \tag{45}$$

This *Ansatz* has translation, rotation, parity, and time reversal symmetries. The spinons are fully gapped and the SU(2) gauge structure is broken down to a Z_2 gauge structure. We may call such a state a Z_2-gapped spin liquid (it was called the RVB state in Refs. 38 and 40). It is described by the projective symmetry group Z2A$xx0z$. Both the chiral spin liquid and the Z_2-gapped spin liquid have true spin-charge separation.

IV. QUANTUM ORDERS IN SYMMETRIC SPIN LIQUIDS

A. Quantum orders and projective symmetry groups

We have seen that there can be many different spin liquids with the *same* symmetries. The stability analysis in Sec. VIII shows that many of those spin liquids occupy a finite region in phase space and represent stable quantum phases. So here we are facing a similar situation as in the quantum Hall effect: there are many distinct quantum phases not separated by symmetries and order parameters. The quantum Hall liquids have finite energy gaps and are rigid states. The concept of topological order was introduced to describe the internal order of those rigid states. Here we can also use the topological order to describe the internal orders of rigid spin liquids. However, we also have many other stable quantum spin liquids that have gapless excitations.

To describe internal orders in gapless quantum spin liquids (as well as gapped spin liquids), we have introduced a new concept—quantum order—which describes the internal orders in any quantum phases. The key point in introducing quantum orders is that quantum phases, in general, cannot be completely characterized by broken symmetries and local order parameters. This point is illustrated by the quantum Hall states and by the stable spin liquids constructed in this paper. However, to make the concept of quantum order useful, we need to find concrete mathematical characterizations of the quantum orders. Since quantum orders are not described by symmetries and order parameters, we need to find a completely new way to characterize them. Here we would like to propose to use the projective symmetry group to characterize quantum (or topological) orders in quantum spin liquids. The projective symmetry group is motivated from the following observation. Although *Ansätze* for different symmetric spin liquids all have the same symmetry, the *Ansätze* are invariant under symmetry transformations followed by *different* gauge transformations. We can use those different gauge transformations to distinguish different spin liquids with the same symmetry. In the following, we will introduce the projective symmetry group in a general and formal setting.

We know that to find quantum numbers that characterize a phase is to find the universal properties of the phase. For classical systems, we know that symmetry is a universal property of a phase and we can use symmetry to characterize different classical phases. To find universal properties of quantum phases we need to find universal properties of many-body wave functions. This is too hard. Here we want to simplify the problem by limiting ourselves to a subclass of many-body wave functions which can be described by *Ansatz* $(u_{ij}, a_0^l \tau^l)$ via Eq. (14). Instead of looking for the universal properties of many-body wave functions, we try to find the universal properties of *Ansatz* $(u_{ij}, a_0^l \tau^l)$. Certainly, one may object that the universal properties of the *Ansatz* (or the subclass of wave functions) may not be the universal properties of the spin quantum phase. This is indeed the case for some *Ansätze*. However, if the mean-field state described by *Ansatz* $(u_{ij}, a_0^l \tau^l)$ is stable against fluctuations (i.e., the fluctuations around the mean-field state do not cause any infrared divergence), then the mean-field state faithfully describes a spin quantum state and the universal properties of the *Ansatz* will be the universal properties of the correspond spin quantum phase. This completes the link between the properties of *Ansätze* and properties of physical spin liquids. Motivated by Landau's theory for classical orders, here we whould like to propose that the invariance group (or the ''symmetry'' group) of an *Ansatz* is a universal property of the *Ansatz*. Such a group will be called the projective symmetry group (PSG). We will show that the PSG can be used to characterize quantum orders in quantum spin liquids.

Let us give a detailed definition of the PSG. A PSG is a property of an *Anstaz*. It is formed by all the transformations that keep the *Ansatz* unchanged. Each transformation (or each element in the PSG) can be written as a combination of a symmetry transformation U (such as translation) and a gauge transformation G_U. The invariance of the *Ansatz* under its PSG can be expressed as

$$G_U U(u_{ij}) = u_{ij},$$

$$U(u_{ij}) \equiv u_{U(i),U(j)},$$

$$G_U(u_{ij}) \equiv G_U(i) u_{ij} G_U^\dagger(j),$$

$$G_U(i) \in \mathrm{SU}(2), \tag{46}$$

for each $G_U U \in \mathrm{PSG}$.

Every PSG contains a special subgroup, which will be called invariant gauge group (IGG). The IGG (denoted by $\mathcal{G}$) for an *Ansatz* is formed by all the gauge transformations that leave the *Ansatz* unchanged:

$$\mathcal{G} = \{W_i \,|\, W_i u_{ij} W_j^\dagger = u_{ij}, W_i \in \mathrm{SU}(2)\}. \tag{47}$$

If we want to relate the IGG to a symmetry transformation, then the associated transformation is simply an identity transformation.

If the IGG is nontrivial, then for a fixed symmetry transformation U there can be many gauge transformations G_U that leave the *Ansatz* unchanged. If $G_U U$ is in the PSG of u_{ij}, $GG_U U$ will also be in the PSG iff $G \in \mathcal{G}$. Thus, for each symmetry transformation U, the different choices of G_U have a one-to-one correspondence with the elements in the IGG. From the above definition, we see that the PSG, the IGG, and the symmetry group (SG) of an *Ansatz* are related:

$$\mathrm{SG} = \mathrm{PSG}/\mathrm{IGG}. \tag{48}$$

This relation tells us that a PSG is a projective representation or an extension of the symmetry group.[96] (In the Appendix we will introduce a closely related but different definition of the PSG. To distinguish the two definitions, we will call the PSG defined above the invariant PSG and the PSG defined in the Appendix the algebraic PSG.)

Certainly the PSG's for two gauge-equivalent *Ansätze* u_{ij} and $W(i)u_{ij}W^\dagger(j)$ are related. From $WG_U U(u_{ij}) = W(u_{ij})$, where $W(u_{ij}) \equiv W(i)u_{ij}W^\dagger(j)$, we find $WG_U UW^{-1}W(u_{ij}) = WG_U W_U^{-1} UW(u_{ij}) = W(u_{ij})$, where $W_U \equiv UWU^{-1}$ is given by $W_U(i) = W(U(i))$. Thus if $G_U U$ is in the PSG of *Ansatz* u_{ij}, then $(WG_U W_U)U$ is in the PSG of gauge-transformed *Ansatz* $W(i)u_{ij}W^\dagger(j)$. We see that the gauge transformation G_U associated with the symmetry transformation U is changed in the following way:

$$G_U(i) \to W(i)G_U(i)W^\dagger(U(i)) \tag{49}$$

after a gauge transformation $W(i)$.

Since the PSG is a property of an *Ansatz*, we can group all the *Ansätze* sharing the same PSG together to form a class. We claim that such a class is formed by one or several universality classes that correspond to quantum phases. (A more detailed discussion of this important point is given in Sec. VIII E.) It is in this sense that we say that quantum orders are characterized by PSG's.

We know that a classical order can be described by its symmetry properties. Mathematically, we say that a classical order is characterized by its symmetry group. Using the projective symmetry group to describe a quantum order, conceptually, is similar to using the symmetry group to describe a classical order. The symmetry description of a classical order is very useful since it allows us to obtain many universal properties, such as the number of Nambu-Goldstone modes, without knowing the details of the system. Similarly, knowing the PSG of a quantum order also allows us to obtain the low-energy properties of a quantum system without knowing its details. As an example, we will discuss a particular kind of low-energy fluctuations—the gauge fluctuations—in a quantum state. We will show that the low-energy gauge fluctuations can be determined completely from the PSG. In fact the gauge group of the low-energy gauge fluctuations is nothing but the IGG of the *Ansatz*.

To see this, let us assume that, as an example, an IGG $\mathcal{G}$ contains a U(1) subgroup which is formed by the following constant gauge transformations:

$$\{W_i = e^{i\theta\tau^3} \,|\, \theta \in [0, 2\pi)\} \subset \mathcal{G}. \tag{50}$$

Now we consider the following type of fluctuations around the mean-field solution $\overline{u}_{ij}$: $u_{ij} = \overline{u}_{ij} e^{ia^3_{ij}\tau^3}$. Since $\overline{u}_{ij}$ is invariant under the constant gauge transformation $e^{i\theta\tau^3}$, a spatial-dependent gauge transformation $e^{i\theta_i\tau^3}$ will transform the fluctuation a^3_{ij} to $\tilde{a}^3_{ij} = a^3_{ij} + \theta_i - \theta_j$. This means that a^3_{ij} and $\tilde{a}^3_{ij}$ label the same physical state and a^3_{ij} correspond to gauge fluctuations. The energy of the fluctuations has a gauge invariance $E(\{a^3_{ij}\}) = E(\{\tilde{a}^3_{ij}\})$. We see that the mass term of the gauge field, $(a^3_{ij})^2$, is not allowed and the U(1) gauge fluctuations described by a^3_{ij} will appear at low energies.

If the U(1) subgroup of $\mathcal{G}$ is formed by spatial-dependent gauge transformations

$$\{W_i = e^{i\theta n_i \cdot \tau} \,|\, \theta \in [0, 2\pi), |n_i| = 1\} \subset \mathcal{G}, \tag{51}$$

we can always use an SU(2) gauge transformation to rotate n_i to the $\hat{z}$ direction on every site and reduce the problem to the one discussed above. Thus, regardless if the gauge transformations in the IGG have spatial dependence or not, the gauge group for low-energy gauge fluctuations is always given by $\mathcal{G}$.

We would like to remark that sometimes low-energy gauge fluctuations not only appear near $\mathbf{k} = 0$, but also appear near some other $\mathbf{k}$ points. In this case, we will have several low-energy gauge fields, one for each $\mathbf{k}$ point. Examples of this phenomenon are given by some *Ansätze* of SU(2) slave-boson theory discussed in Sec. VI, which have an SU(2) $\times$ SU(2) gauge structures at low energies. We see that the low-energy gauge structure SU(2) $\times$ SU(2) can even be larger than the high-energy gauge structure SU(2). Even for this complicated case where low-energy gauge fluctuations appear around different k points, the IGG still correctly describes the low-energy gauge structure of the corresponding *Ansatz*. If the IGG contains gauge transformations that are independent of spatial coordinates, then such transformations correspond to the gauge group for gapless gauge fluctuations near $\mathbf{k} = 0$. If the IGG contains gauge transformations that depend on spatial coordinates, then those transformations correspond to the gauge group for gapless gauge fluctuations near nonzero $\mathbf{k}$. Thus the IGG gives us a unified treatment of all low-energy gauge fluctuations, regardless of their momenta.

In this paper, we have used the terms Z_2 spin liquids, U(1) spin liquids, SU(2) spin liquids, and SU(2) $\times$ SU(2) spin liquids in many places. Now we can have a precise definition of those low-energy Z_2, U(1), SU(2), and SU(2) $\times$ SU(2) gauge groups. Those low-energy gauge groups are nothing but the IGG of the corresponding *Ansätze*. They have nothing to do with the high-energy gauge groups that appear in the SU(2), U(1), or Z_2 slave-boson approaches. We also used the terms Z_2 gauge structure, U(1) gauge structure, and SU(2) gauge structure of a mean-field state. Their precise mathematical meaning is again the IGG of the corresponding *Anastz*. When we say a U(1) gauge structure is broken down to a Z_2 gauge structure, we mean that an *Ansatz* is changed in such a way that its IGG is changed from the U(1) to the Z_2 group.

B. Classification of symmetric Z_2 spin liquids

As an application of the PSG characterization of quantum orders in spin liquids, we would like to classify the PSG's associated with translation transformations assuming the IGG $\mathcal{G}=Z_2$. Such a classification leads to a classification of translation-symmetric Z_2 spin liquids.

When $\mathcal{G}=Z_2$, it contains two elements—gauge transformations G_1 and G_2:

$$\mathcal{G}=\{G_1,G_2\},$$

$$G_1(i)=\tau^0, \quad G_2(i)=-\tau^0. \tag{52}$$

Let us assume that a Z_2 spin liquid has a translation symmetry. The PSG associated with the translation group is generated by four elements $\pm G_xT_x, \pm G_yT_x$ where

$$T_x(u_{ij})=u_{i-\hat{x},j-\hat{x}}, \quad T_y(u_{ij})=u_{i-\hat{y},j-\hat{y}}. \tag{53}$$

Due to the translation symmetry of the *Ansatz*, we can choose a gauge in which all the loop operators of the *Ansatz* are translation invariant. That is, $P_{C_1}=P_{C_2}$ if the two loops C_1 and C_2 are related by a translation. We will call such a gauge a uniform gauge.

Under transformation G_xT_x, a loop operator P_C based at i transforms as $P_C\rightarrow G_x(i')P_{T_xC}G_x^\dagger(i')=G_x(i')P_CG_x^\dagger(i')$ where $i'=T_xi$ is the base point of the translated loop $T_x(C)$. We see that translation invariance of P_C in the uniform gauge requires

$$G_x(i)=\pm\tau^0, \quad G_y(i)=\pm\tau^0, \tag{54}$$

since different loop operators based at the same base point do not commute for Z_2 spin liquids. We note that the gauge transformations of the form $W(i)=\pm\tau^0$ do not change the translation-invariant property of the loop operators. Thus we can use such gauge transformations to further simplify $G_{x,y}$ through Eq. (49). First we can choose a gauge to make

$$G_y(i)=\tau^0. \tag{55}$$

We note that a gauge transformation satisfying $W(i)=W(i_x)$ does not change the condition $G_y(i)=\tau^0$. We can use such a kind of gauge transformations to make

$$G_x(i_x,i_y=0)=\tau^0. \tag{56}$$

Since the translations in the x and y directions commute, $G_{x,y}$ must satisfy (for any *Ansatz*, Z_2 or not Z_2)

$$\begin{aligned}&G_xT_xG_yT_y(G_xT_x)^{-1}(G_yT_y)^{-1}\\&\quad=G_xT_xG_yT_yT_x^{-1}G_x^{-1}T_y^{-1}G_y^{-1}\in\mathcal{G}.\end{aligned} \tag{57}$$

That means

$$G_x(i)G_y(i-\hat{x})G_x^{-1}(i-\hat{y})G_y(i)^{-1}\in\mathcal{G}. \tag{58}$$

For Z_2 spin liquids, Eq. (58) reduces to

$$G_x(i)G_x^{-1}(i-\hat{y})=+\tau^0 \tag{59}$$

or

$$G_x(i)G_x^{-1}(i-\hat{y})=-\tau^0. \tag{60}$$

When combined with Eq. (55) and Eq. (56), we find that there are only two gauge-inequivalent extensions of the translation group when IGG is $\mathcal{G}=Z_2$. The two PSG's are given by

$$G_x(i)=\tau^0, \quad G_y(i)=\tau^0 \tag{61}$$

and

$$G_x(i)=(-)^{i_y}\tau^0, \quad G_y(i)=\tau^0. \tag{62}$$

Thus, under PSG classification, there are only two types of Z_2 spin liquids if they have only the translation symmetry and no other symmetries. The *Ansätze* that satisfy Eq. (61) have a form

$$u_{i,i+m}=u_m \tag{63}$$

and the ones that satisfy Eq. (62) have a form

$$u_{i,i+m}=(-)^{m_yi_x}u_m. \tag{64}$$

Through the above example, we see that the PSG is a very powerful tool. It can lead to a complete classification of (mean-field) spin liquids with prescribed symmetries and low-energy gauge structures.

In the above, we have studied Z_2 spin liquids which have only the translation symmetry and no other symmetries. We find that there are only two types of such spin liquids. However, if spin liquids have more symmetries, then they can have much more types. In Ref. 66, we give a classification of symmetric Z_2 spin liquids using the PSG. Here we use the term "symmetric spin liquid" to refer to a spin liquid with translation symmetry $T_{x,y}$, time reversal symmetry T: $u_{ij}\rightarrow -u_{ij}$, and three parity symmetries P_x: $(i_x,i_y)\rightarrow(-i_x,i_y)$, P_y: $(i_x,i_y)\rightarrow(i_x,-i_y)$, and P_{xy}: $(i_x,i_y)\rightarrow(i_y,i_x)$. The three parity symmetries also imply a 90° rotation symmetry. The classification is obtained by noticing that the gauge transformations G_{xy}, $G_{P_x,P_y,P_{xy}}$, and G_T must satisfy certain algebraic relations (see the Appendix A). Solving those algebraic relations and factoring out gauge-equivalent solutions,[66] we find that there are 272 different extensions of the symmetry group $\{T_{x,y},P_{x,y,xy},T\}$ if IGG $\mathcal{G}=Z_2$. Those PSG's are generated by $(G_xT_x,G_yT_y,G_TT,G_{P_x}P_x,G_{P_y}P_y,G_{P_{xy}}P_{xy})$. The PSG's can be divided into two classes. The first class is given by

$$\begin{aligned}&G_x(i)=\tau^0, \quad G_y(i)=\tau^0,\\&G_{P_x}(i)=\eta_{xpx}^{i_x}\eta_{xpy}^{i_y}g_{P_x}, \quad G_{P_y}(i)=\eta_{xpy}^{i_x}\eta_{xpx}^{i_y}g_{P_y},\\&G_{P_{xy}}(i)=g_{P_{xy}}, \quad G_T(i)=\eta_t^ig_T\end{aligned} \tag{65}$$

and the second class by

$$\begin{aligned}&G_x(i)=(-)^{i_y}\tau^0, \quad G_y(i)=\tau^0,\\&G_{P_x}(i)=\eta_{xpx}^{i_x}\eta_{xpy}^{i_y}g_{P_x}, \quad G_{P_y}(i)=\eta_{xpy}^{i_x}\eta_{xpx}^{i_y}g_{P_y},\\&G_{P_{xy}}(i)=(-)^{i_xi_y}g_{P_{xy}}, \quad G_T(i)=\eta_t^ig_T.\end{aligned} \tag{66}$$

Here the three η's can independently take two values ± 1. g's have 17 different choices which are given by (see Ref. 66)

$$g_{Pxy}=\tau^0,\quad g_{P_x}=\tau^0,\quad g_{P_y}=\tau^0,\quad g_T=\tau^0; \tag{67}$$

$$g_{Pxy}=\tau^0,\quad g_{P_x}=i\tau^3,\quad g_{P_y}=i\tau^3,\quad g_T=\tau^0; \tag{68}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=\tau^0,\quad g_{P_y}=\tau^0,\quad g_T=\tau^0; \tag{69}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=i\tau^3,\quad g_{P_y}=\tau^3,\quad g_T=\tau^0; \tag{70}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^1,\quad g_T=\tau^0; \tag{71}$$

$$g_{Pxy}=\tau^0,\quad g_{P_x}=\tau^0,\quad g_{P_y}=\tau^0,\quad g_T=i\tau^3; \tag{72}$$

$$g_{Pxy}=\tau^0,\quad g_{P_x}=i\tau^3,\quad g_{P_y}=i\tau^3,\quad g_T=i\tau^3; \tag{73}$$

$$g_{Pxy}=\tau^0,\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^1,\quad g_T=i\tau^3; \tag{74}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=\tau^0,\quad g_{P_y}=\tau^0,\quad g_T=i\tau^3; \tag{75}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=i\tau^3,\quad g_{P_y}=i\tau^3,\quad g_T=i\tau^3; \tag{76}$$

$$g_{Pxy}=i\tau^3,\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^1,\quad g_T=i\tau^3; \tag{77}$$

$$g_{Pxy}=i\tau^1,\quad g_{P_x}=\tau^0,\quad g_{P_y}=\tau^0,\quad g_T=i\tau^3; \tag{78}$$

$$g_{Pxy}=i\tau^1,\quad g_{P_x}=i\tau^3,\quad g_{P_y}=i\tau^3,\quad g_T=i\tau^3; \tag{79}$$

$$g_{Pxy}=i\tau^1,\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^1,\quad g_T=i\tau^3; \tag{80}$$

$$g_{Pxy}=i\tau^1,\quad g_{P_x}=i\tau^2,\quad g_{P_y}=i\tau^2,\quad g_T=i\tau^3; \tag{81}$$

$$g_{Pxy}=i\tau^{12},\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^2,\quad g_T=i\tau^0; \tag{82}$$

$$g_{Pxy}=i\tau^{12},\quad g_{P_x}=i\tau^1,\quad g_{P_y}=i\tau^2,\quad g_T=i\tau^3; \tag{83}$$

where

$$\tau^{ab}=\frac{\tau^a+\tau^b}{\sqrt{2}},\quad \tau^{\bar{ab}}=\frac{\tau^a-\tau^b}{\sqrt{2}}. \tag{84}$$

Thus there are $2\times 17\times 2^3=272$ different PSG's. They can potentially lead to 272 different types of symmetric Z_2 spin liquids on a 2D square lattice.

To label the 272 PSG's, we propose the following scheme:

$$\mathrm{Z2A}(g_{px})_{\eta_{xpx}}(g_{py})_{\eta_{xpy}}g_{pzy}(g_t)_{\eta_t}, \tag{85}$$

$$\mathrm{Z2B}(g_{px})_{\eta_{xpx}}(g_{py})_{\eta_{xpy}}g_{pxy}(g_t)_{\eta_t}. \tag{86}$$

The label Z2A... corresponds to the case of Eq. (65), and the label Z2B... corresponds to the case of Eq. (66). A typical label will look like $Z2A\tau^1_+\tau^2_-\tau^{12}\tau^3_-$. We will also use an abbreviated notation. An abbreviated notation is obtained by replacing $(\tau^0,\tau^1,\tau^2,\tau^3)$ or $(\tau^0_+,\tau^1_+,\tau^2_+,\tau^3_+)$ by (0,1,2,3) and $(\tau^0_-,\tau^1_-,\tau^2_-,\tau^3_-)$ by (n,x,y,z). For example, $\mathrm{Z2A}\tau^1_+\tau^0_-\tau^{12}\tau^3_-$ can be abbreviated as $\mathrm{Z2A1}n(12)z$.

Those 272 different Z_2 PSG's, strictly speaking, are the so-called algebraic PSG's. The algebraic PSG's are defined as extensions of the symmetry group. They can be calculated through the algebraic relations listed in the Appendix. The algebraic PSG's are different from the invariant PSG's which are defined as a collection of all transformations that leave an *Ansatz* u_{ij} invariant. Although an invariant PSG must be an algebraic PSG, an algebraic PSG may not be an invariant PSG. This is because certain algebraic PSG's have the following properties: any *Ansatz* u_{ij} that is invariant under an algebraic PSG may actually be invariant under a larger PSG. In this case the original algebraic PSG cannot be an invariant PSG of the *Ansatz*. The invariant PSG of the *Ansatz* is really given by the larger PSG. If we limit ourselves to the spin liquids constructed through the *Ansatz* u_{ij}, then we should drop the algebraic PSG's that are not invariant PSG's. This is because those algebraic PSG's do not characterize mean-field spin liquids.

We find that among the 272 algebraic Z_2 PSG's, at least 76 of them are not invariant PSG's. Thus the 272 algebraic Z_2 PSG's can at most lead to 196 possible Z_2 spin liquids. Since some of the mean-field spin liquid states may not survive the quantum fluctuations, the number of physical Z_2 spin liquids is even smaller. However, for the physical spin liquids that can be obtained through the mean-field states, the PSG's do offer a characterization of the quantum orders in those spin liquids.

C. Classification of symmetric U(1) and SU(2) spin liquids

In addition to the Z_2-symmetric spin liquids studied above, there can be symmetric spin liquids whose low-energy gauge structure is U(1) or SU(2). Such U(1)- and SU(2)-symmetric spin liquids (at the mean-field level) are classified by U(1)- and SU(2)-symmetric PSG's. The U(1)- and SU(2)-symmetric PSG's are calculated in Ref. 66. In the following we just summarize the results.

We find that the PSG's that characterize mean-field symmetric U(1) spin liquids can be divided into four types: U1A, U1B, U1C, and $\mathrm{U1}^m_n$. There are 24 type-U1A PSG's:

$$\begin{aligned} &G_x=g_3(\theta_x),\quad G_y=g_3(\theta_y),\\ &G_{P_x}=\eta^{i_y}_{ypx}g_3(\theta_{px}),\quad G_{P_y}=\eta^{i_x}_{ypx}g_3(\theta_{py}),\\ &G_{P_{xy}}=g_3(\theta_{pxy}),\quad g_3(\theta_{pxy})i\tau^1,\\ &G_T=\eta^i_t g_3(\theta_t)|_{\eta_t=-1},\quad \eta^i_t g_3(\theta_t)i\tau^1 \end{aligned} \tag{87}$$

and

$$\begin{aligned} &G_x=g_3(\theta_x),\quad G_y=g_3(\theta_y),\\ &G_{P_x}=\eta^{i_x}_{xpx}g_3(\theta_{px})i\tau^1,\quad G_{P_y}=\eta^{i_y}_{xpx}g_3(\theta_{py})i\tau^1,\\ &G_{P_{xy}}=g_3(\theta_{pxy}),g_3(\theta_{pxy})i\tau^1,\\ &G_T=\eta^i_t g_3(\theta_t)|_{\eta_t=1},\quad \eta^i_t g_3(\theta_t)i\tau^1, \end{aligned} \tag{88}$$

where

$$g_a(\theta)\equiv e^{i\theta\tau^a}. \tag{89}$$

We will use U1A$a_{\eta_{xpx}} b_{\eta_{ypx}} c d_{\eta_t}$ to label the 24 PSG's. a, b, c, and d are associated with G_{P_x}, G_{P_y}, $G_{P_{xy}}$, and G_T, respectively. They are equal to τ^1 if the corresponding G contains a τ^1 and equal to τ^0 otherwise. A typical notation looks like U1A$\tau^1_-\tau^1\tau^0\tau^1_-$ which can be abbreviated as U1A$x10x$.

There are also 24 type-U1B PSG's:

$$G_x=(-)^{i_y}g_3(\theta_x),\quad G_y=g_3(\theta_y),$$

$$G_{P_x}=\eta_{ypx}^{i_y}g_3(\theta_{px}),\quad G_{P_y}=\eta_{ypx}^{i_x}g_3(\theta_{py}),$$

$$(-)^{i_x i_y}G_{P_{xy}}=g_3(\theta_{pxy}),\quad g_3(\theta_{pxy})i\tau^1,$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1},\quad \eta_t^i g_3(\theta_t)i\tau^1 \tag{90}$$

and

$$G_x=(-)^{i_y}g_3(\theta_x),\quad G_y=g_3(\theta_y),$$

$$G_{P_x}=\eta_{xpx}^{i_x}g_3(\theta_{px})i\tau^1,\quad G_{P_y}=\eta_{xpx}^{i_y}g_3(\theta_{py})i\tau^1,$$

$$(-)^{i_x i_y}G_{P_{xy}}=g_3(\theta_{pxy}),\quad g_3(\theta_{pxy})i\tau^1,$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1},\quad \eta_t^i g_3(\theta_t)i\tau^1. \tag{91}$$

We will use U1B$a_{\eta_{xpx}} b_{\eta_{ypx}} c d_{\eta_t}$ to label the 24 PSG's.

The 60 type-U1C PSG's are given by

$$G_x=g_3(\theta_x)i\tau^1,\quad G_y=g_3(\theta_y)i\tau^1,$$

$$G_{P_x}=\eta_{xpx}^{i_x}\eta_{ypx}^{i_y}g_3(\theta_{px}),\quad G_{P_y}=\eta_{ypx}^{i_x}\eta_{px}^{i_y}(\theta_{py}),$$

$$G_{P_{xy}}=\eta_{pxy}^{i_x}g_3\left(\eta_{pxy}^i\frac{\pi}{4}+\theta_{pxy}\right),$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1},\quad \eta_{pxy}^{i_x}g_3(\theta_t)i\tau^1, \tag{92}$$

$$G_x=g_3(\theta_x)i\tau^1,\quad G_y=g_3(\theta_y)i\tau^1,$$

$$G_{P_x}=\eta_{xpx}^{i_x}g_3(\theta_{px})i\tau^1,\quad G_{P_y}=\eta_{xpx}^{i_y}\eta_{pxy}^i g_3(\theta_{py})i\tau^1,$$

$$G_{P_{xy}}=\eta_{pxy}^{i_x}g_3\left(\eta_{pxy}^i\frac{\pi}{4}+\theta_{pxy}\right),$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1},\quad \eta_{pxy}^{i_x}\eta_t^i g_3(\theta_t)i\tau^1, \tag{93}$$

$$G_x=g_3(\theta_x)i\tau^1,\quad G_y=g_3(\theta_y)i\tau^1,$$

$$G_{P_x}=\eta_{xpx}^{i_x}\eta_{ypx}^{i_y}g_3(\theta_{px}),\quad G_{P_y}=\eta_{ypx}^{i_x}\eta_{xpx}^{i_y}g_3(\theta_{py}),$$

$$G_{P_{xy}}=g_3(\theta_{pxy})i\tau^1,$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1}, \tag{94}$$

$$G_x=g_3(\theta_x)i\tau^1,\quad G_y=g_3(\theta_y)i\tau^1,$$

$$G_{P_x}=\eta_{xpx}^{i_x}\eta_{ypx}^{i_y}g_3(\theta_{px}),\quad G_{P_y}=\eta_{ypx}^{i_x}\eta_{xpx}^{i_y}g_3(\theta_{py}),$$

$$G_{P_{xy}}=g_3\left(\eta_{pxy}^i\frac{\pi}{4}+\theta_{pxy}\right)i\tau^1,$$

$$G_T=\eta_{pxy}^{i_x}\eta_t^i g_3(\theta_t)i\tau^1, \tag{95}$$

$$G_x=g_3(\theta_x)i\tau^1,\quad G_y=g_3(\theta_y)i\tau^1,$$

$$G_{P_x}=\eta_{xpx}^{i_x}g_3(\theta_{px})i\tau^1,\quad G_{P_y}=\eta_{xpx}^{i_y}\eta_{pxy}^i g_3(\theta_{py})i\tau^1,$$

$$G_{P_{xy}}=g_3\left(\eta_{pxy}^i\frac{\pi}{4}+\theta_{pxy}\right)i\tau^1,$$

$$G_T=\eta_t^i g_3(\theta_t)|_{\eta_t=-1},\quad \eta_t^i\eta_{pxy}^{i_x}g_3(\theta_t)i\tau^1, \tag{96}$$

which will be labeled by U1C$a_{\eta_{xpx}} b_{\eta_{ypx}} c_{\eta_{pxy}} d_{\eta_t}$.

The type-U1$_n^m$ PSG's have not been classified. However, we do know that for each rational number $m/n\in(0,1)$, there exists at least one mean-field symmetric spin liquid, which is described by the *Ansatz*

$$u_{i,i+\hat{x}}=\chi\tau^3,\quad u_{i,i+\hat{y}}=\chi g_3\left(\frac{m\pi}{n}i_x\right)\tau^3. \tag{97}$$

It has $\pi m/n$ flux per plaquette. Thus there are infinite many type-$U1_n^m$ spin liquids.

We would like to point out that the above 108 U1[A,B,C] PSG's are algebraic PSG's. They are only a subset of all possible algebraic U(1) PSG's. However, they do contain all the invariant U(1) PSG's of type U1A, U1B, and U1C. We find that 46 of the 108 PSG's are also invariant PSG's. Thus there are 46 different mean-field U(1) spin liquids of type U1A, U1B, and U1C. Their *Ansatz* and labels are given in Ref. 66.

To classify symmetric SU(2) spin liquids, we find eight different SU(2) PSG's which are given by

$$G_x(i)=g_x,\quad G_y(i)=g_y,$$

$$G_{P_x}(i)=\eta_{xpx}^{i_x}\eta_{xpy}^{i_y}g_{P_x},\quad G_{P_y}(i)=\eta_{xpy}^{i_x}\eta_{xpx}^{i_y}g_{P_y},$$

$$G_{P_{xy}}(i)=g_{P_{xy}},\quad G_T(i)=(-)^i g_T \tag{98}$$

and

$$G_x(i)=(-)^{i_y}g_x,\quad G_y(i)=g_y,$$

$$G_{P_x}(i)=\eta_{xpx}^{i_x}\eta_{xpy}^{i_y}g_{P_x},\quad G_{P_y}(i)=\eta_{xpy}^{i_x}\eta_{xpx}^{i_y}g_{P_y},$$

$$G_{P_{xy}}(i)=(-)^{i_x i_y}g_{P_{xy}},\quad G_T(i)=(-)^i g_T, \tag{99}$$

where g's are in SU(2). We would like to use the notation

$$\mathrm{SU2A}\tau^0_{\eta_{xpx}}\tau^0_{\eta_{xpy}},$$

$$\mathrm{SU2B}\tau^0_{\eta_{xpx}}\tau^0_{\eta_{xpy}} \tag{100}$$

to denote the above eight PSG's. $\mathrm{SU2A}\tau^0_{\eta_{xpx}}\tau^0_{\eta_{xpy}}$ is for Eq. (98) and $\mathrm{SU2B}\tau^0_{\eta_{xpx}}\tau^0_{\eta_{xpy}}$ for Eq. (99). We find that only four of the eight SU(2) PSG's, SU2A$[n0,0n]$ and SU2B$[n0,0n]$, lead to SU(2) symmetric spin liquids. The SU2A$n0$ state is the uniform RVB state and the SU2B$n0$ state is the π-flux state. The other two SU(2) spin liquids are given by SU2A$0n$,

$$\begin{aligned} u_{i,i+2\hat{x}+\hat{y}} &= +i\chi\tau^0, \\ u_{i,i-2\hat{x}+\hat{y}} &= -i\chi\tau^0, \\ u_{i,i+\hat{x}+2\hat{y}} &= +i\chi\tau^0, \\ u_{i,i-\hat{x}+2\hat{y}} &= +i\chi\tau^0, \end{aligned} \tag{101}$$

and SU2B$0n$,

$$\begin{aligned} u_{i,i+2\hat{x}+\hat{y}} &= +i(-)^{i_x}\chi\tau^0, \\ u_{i,i-2\hat{x}+\hat{y}} &= -i(-)^{i_x}\chi\tau^0, \\ u_{i,i+\hat{x}+2\hat{y}} &= +i\chi\tau^0, \\ u_{i,i-\hat{x}+2\hat{y}} &= +i\chi\tau^0. \end{aligned} \tag{102}$$

The above results give us a classification of symmetric U(1) and SU(2) spin liquids at the mean-field level. If a mean-field state is stable against fluctuations, it will correspond to a physical U(1)- or SU(2)-symmetric spin liquids. In this way the U(1) and SU(2) PSG's also provide a description of some physical spin liquids.

V. CONTINUOUS TRANSITIONS AND SPINON SPECTRA IN SYMMETRIC SPIN LIQUIDS

A. Continuous phase transitions without symmetry breaking

After classifying mean-field symmetric spin liquids, we would like to know how those symmetric spin liquids are related to each other. In particular, we would like to know which spin liquids can change into each other through a *continuous* phase transition. This problem is studied in detail in Ref. 66, where the symmetric spin liquids in the neighborhood of some important symmetric spin liquids were obtained. After lengthy calculations, we found all the mean-field symmetric spin liquids around the Z_2-linear state Z2A001n in Eq. (39), the U(1)-linear state U1Cn01n in Eq. (32), the SU(2)-gapless state SU2A$n0$ in Eq. (30), and the SU(2)-linear state SU2B$n0$ in Eq. (31). We find that, at the mean-field level, the U(1)-linear spin liquid U1Cn01n can continuously change into 8 different Z_2 spin liquids, the SU(2)-gapless spin liquid SU2A$n0$ can continuously change into 12 U(1) spin liquids and 52 Z_2 spin liquids, and the SU(2)-linear spin liquid SU2B$n0$ can continuously change into 12 U(1) spin liquids and 58 Z_2 spin liquids.

We would like to stress that the above results on the continuous transitions are valid only at the mean-field level. Some of the mean-field results survive the quantum fluctuations while others do not. One needs to do a case-by-case study to see which mean-field results can be valid beyond the mean-field theory. In Ref. 40, a mean-field transition between a $\mathrm{SU}(2)\times\mathrm{SU}(2)$-linear spin liquid and a Z_2-gapped spin liquid was studied. In particular the effects of quantum fluctuations were discussed.

We would also like to point out that all the above spin liquids have the same symmetry. Thus the continuous transitions between them, if they exist, represent a new class of continuous transitions which do not change any symmetries.[67]

B. Symmetric spin liquids around the U(1)-linear spin liquid U1Cn01n

The SU(2)-linear state SU2B$n0$ (the π-flux state), the U(1)-linear state U1Cn01n (the staggered-flux/d-wave state), and the SU(2)-gapless state SU2A$n0$ (the uniform RVB state) are closely related to high-T_c superconductors. They reproduce the observed electron spectra function for undoped, underdoped, and overdoped samples, respectively. However, theoretically, those spin liquids are unstable at low energies due to the U(1) or SU(2) gauge fluctuations. Those states may change into more stable spin liquids in their neighborhood. In the next few subsections, we are going to study those more stable spin liquids. Since there are still many different spin liquids involved, we will only present some simplified results by limiting the length of nonzero links. Those spin liquids with short links should be more stable for simple spin Hamiltonians. The length of a link between i and j is defined as $|i_x-j_x|+|i_y-j_y|$. By studying the spinon dispersion in those mean-field states, we can understand some basic physical properties of those spin liquids, such as their stability against the gauge fluctuations and the qualitative behaviors of spin correlations which can be measured by neutron scattering. Those results allow us to identify them if those spin liquids exist in certain samples or appear in numerical calculations. We would like to point out that we will only study symmetric spin liquids here. The above three unstable spin liquids may also change into some other states that break certain symmetries. Such symmetry breaking transitions actually have been observed in high-T_c superconductors (such as the transitions to antiferromagnetic state, d-wave superconducting state, and stripe state).

First, let us consider the spin liquids around the U(1)-linear state U1Cn01n. In the neighborhood of the U1Cn01n *Ansatz*, Eq. (32), there are eight different spin liquids that break the U(1) gauge structure down to a Z_2 gauge structure. Those eight spin liquids are labeled by different PSG's despite all having the same symmetry. In the following, we will study those eight Z_2 spin liquids in more detail. In particular, we would like to find out the spinon spectra in them.

The first one is labeled by Z2A0013 and takes the following form:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+\hat{y}}=\chi\tau^1+\eta\tau^2,$$
$$u_{i,i+\hat{x}+\hat{y}}=+\gamma_1\tau^1,$$
$$u_{i,i-\hat{x}+\hat{y}}=+\gamma_1\tau^1,$$
$$u_{i,i+2\hat{x}}=\gamma_2\tau^1+\lambda_2\tau^2,$$
$$u_{i,i+2\hat{y}}=\gamma_2\tau^1-\lambda_2\tau^2,$$
$$a_0^1\neq 0,\quad a_0^{2,3}=0. \tag{103}$$

It has the same quantum order as that in the *Ansatz* Eq. (42). The label Z2A0013 tells us the PSG that characterizes the spin liquid. The second *Ansatz* is labeled by Z2A*zz*13:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+\hat{y}}=\chi\tau^1+\eta\tau^2,$$
$$u_{i,i+\hat{x}+\hat{y}}=-\gamma_1\tau^1,$$
$$u_{i,i-\hat{x}+\hat{y}}=+\gamma_1\tau^1,$$
$$u_{i,i+2\hat{x}}=u_{i,i+2\hat{y}}=0,$$
$$a_0^{1,2,3}=0. \tag{104}$$

The third one is labeled by Z2A001*n* (or equivalently Z2A003*n*):

$$a_0^l=0,$$
$$u_{i,i+\hat{x}}=\chi\tau^1+\eta\tau^2,$$
$$u_{i,i+\hat{y}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+2\hat{x}+\hat{y}}=\lambda\tau^3,$$
$$u_{i,i-\hat{x}+2\hat{y}}=-\lambda\tau^3,$$
$$u_{i,i+2\hat{x}-\hat{y}}=\lambda\tau^3,$$
$$u_{i,i+\hat{x}+2\hat{y}}=-\lambda\tau^3. \tag{105}$$

Such a spin liquid has the same quantum order as Eq. (39). The fourth one is labeled by Z2A*zz*1*n*:

$$a_0^l=0,$$
$$u_{i,i+\hat{x}}=\chi\tau^1+\eta\tau^2,$$
$$u_{i,i+\hat{y}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+2\hat{x}+\hat{y}}=\chi_1\tau^1+\eta_1\tau^2+\lambda\tau^3,$$
$$u_{i,i-\hat{x}+2\hat{y}}=\chi_1\tau^1-\eta_1\tau^2+\lambda\tau^3,$$
$$u_{i,i+2\hat{x}-\hat{y}}=\chi_1\tau^1+\eta_1\tau^2-\lambda\tau^3,$$
$$u_{i,i+\hat{x}+2\hat{y}}=\chi_1\tau^1-\eta_1\tau^2-\lambda\tau^3. \tag{106}$$

The above four *Ansätze* have translation invariance. The next four Z_2 *Ansätze* do not have translation invariance. (But they still describe translation symmetric spin liquids after the projection.) Those Z_2 spin liquids are the following.

Z2B0013:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+\hat{y}}=(-)^{i_x}(\chi\tau^1+\eta\tau^2),$$
$$u_{i,i+2\hat{x}}=-\gamma_2\tau^1+\lambda_2\tau^2,$$
$$u_{i,i+2\hat{y}}=-\gamma_2\tau^1-\lambda_2\tau^2,$$
$$a_0^1\neq 0,\quad a_0^{2,3}=0. \tag{107}$$

Z2B*zz*13:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,$$
$$u_{i,i+\hat{y}}=(-)^{i_x}(\chi\tau^1+\eta\tau^2),$$
$$u_{i,i+2\hat{x}+2\hat{y}}=-\gamma_1\tau^1,$$
$$u_{i,i-2\hat{x}+2\hat{y}}=\gamma_1\tau^1,$$
$$a_0^{1,2,3}=0. \tag{108}$$

Z2B001*n*:

$$u_{i,\hat{i}+x}=\chi\tau^1+\eta\tau^2,$$
$$u_{i,\hat{i}+y}=(-)^{i_x}(\chi\tau^1-\eta\tau^2),$$
$$u_{i,i+\hat{2}x+y}=(-)^{i_x}\lambda\tau^3,$$
$$u_{i,i-\hat{x}+2y}=-\lambda\tau^3,$$
$$u_{i,i+\hat{2}x-y}=(-)^{i_x}\lambda\tau^3,$$
$$u_{i,i+\hat{x}+2y}=-\lambda\tau^3,$$
$$a_0^l=0. \tag{109}$$

Z2B*zz*1*n*:

$$u_{\hat{x}}=\chi\tau^1+\eta\tau^2,$$
$$u_{\hat{y}}=(-)^{i_x}(\chi\tau^1-\eta\tau^2),$$
$$u_{2\hat{x}+y}=(-)^{i_x}(\chi_1\tau^1+\eta_1\tau^2+\lambda\tau^3),$$
$$u_{-\hat{x}+2y}=\chi_1\tau^1-\eta_1\tau^2+\lambda\tau^3,$$
$$u_{2\hat{x}-y}=(-)^{i_x}(\chi_1\tau^1+\eta_1\tau^2-\lambda\tau^3),$$
$$u_{\hat{x}+2y}=\chi_1\tau^1-\eta_1\tau^2-\lambda\tau^3,$$
$$a_0^l=0. \tag{110}$$

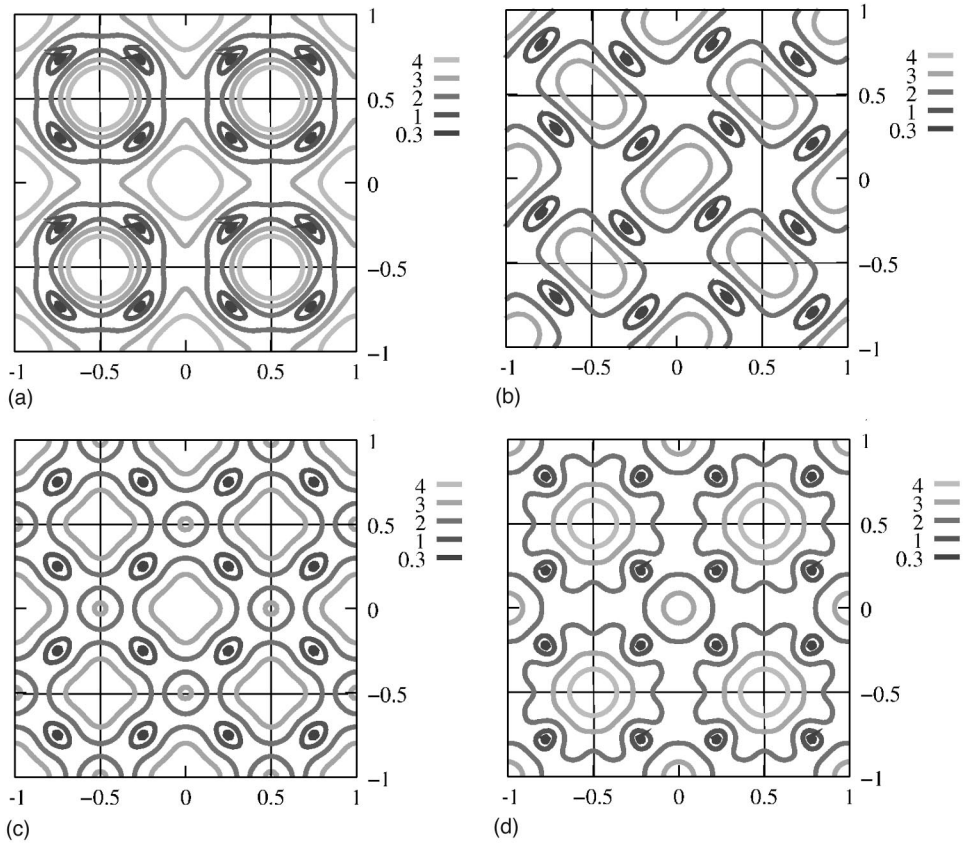

FIG. 1. Contour plot of the spinon dispersion $E_+(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for the Z_2-linear spin liquids. (a) is for the Z2A0013 state in Eq. (103), (b) for the Z2Azz13 state in Eq. (104), (c) for the Z2A001n state in Eq. (105), and (d) for the Z2Azz1n state in Eq. (106).

The spinons are gapless at four isolated points with a linear dispersion for the first four Z_2 spin liquids Eq. (103), Eq. (104), Eq. (105), and Eq. (106) (see Fig. 1). Therefore the four *Ansätze* describe symmetric Z_2-linear spin liquids. The single-spinon dispersion for the second Z_2 spin liquid Z2Azz13 is quite interesting. It has 90° rotation symmetry around $k=(0,\pi)$ and parity symmetry about $k=(0,0)$. One very important thing to notice is that the spinon dispersions for the four Z_2-linear spin liquids, Eq. (103), Eq. (104), Eq. (105), and Eq. (106), have some qualitative differences between them. Those differences can be used to physically measure quantum orders (see Sec. VII).

Next let us consider the *Ansatz* Z2B0013 in Eq. (107). The spinon spectrum for the *Ansatz*, Eq. (107), is determined by

$$H=-2\chi\cos(k_x)\Gamma_0-2\eta\cos(k_x)\Gamma_2-2\chi\cos(k_y)\Gamma_1 +2\eta\cos(k_y)\Gamma_3+\lambda\Gamma_4, \tag{111}$$

where $k_x\in(0,\pi)$, $k_y\in(-\pi,\pi)$, and

$$\Gamma_0=\tau^1\otimes\tau^3,\quad \Gamma_1=\tau^1\otimes\tau^1,$$

$$\Gamma_2=\tau^2\otimes\tau^3,\quad \Gamma_3=\tau^2\otimes\tau^1,$$

$$\Gamma_4=\tau^1\otimes\tau^0, \tag{112}$$

assuming $\gamma_{1,2}=\lambda_2=0$. The four bands of spinon dispersion have a form $\pm E_1(k),\pm E_2(k)$. We find that the spinon spectrum vanishes at eight isolated points near $k=(\pi/2,\pm\pi/2)$ [see Fig. 2(a)]. Thus the state Z2B0013 is a Z_2-linear spin liquid.

Knowing the translation symmetry of the above Z_2-linear spin liquid, it seems strange to find that the spinon spectrum is defined only on half of the lattice Brillouin zone. However, this is not inconsistent with translation symmetry since the single-spinon excitation is not physical. Only two-spinon excitations correspond to physical excitations and their spectrum should be defined on the full Brillouin zone. Now the problem is that how to obtain the two-spinon spectrum defined on the full Brillouin zone from the single-spinon spectrum defined on half of the Brillouin zone. Let $|k,1\rangle$ and $|k,2\rangle$ be the two eigenstates of the single spinon with positive energies $E_1(k)$ and $E_2(k)$ [here $k_x\in(-\pi/2,\pi/2)$ and $k_y\in(-\pi,\pi)$]. The translation by $\hat{x}$ (followed by a gauge transformation) changes $|k,1\rangle$ and $|k,2\rangle$ to the other two eigenstates with the same energies:

$$|k,1\rangle\rightarrow|k+\pi\hat{y},1\rangle,$$

$$|k,2\rangle\rightarrow|k+\pi\hat{y},2\rangle. \tag{113}$$

Now we see that the momentum and energy of the two-spinon states $|k_1,\alpha_1\rangle|k_2,\alpha_2\rangle\pm|k_1+\pi\hat{y},\alpha_1\rangle|k_2+\pi\hat{y},\alpha_2\rangle$ are given by

$$E_{2\text{-spinon}}=E_{\alpha_1}(k_1)+E_{\alpha_2}(k_2),$$

$$k=k_1+k_2,\quad k_1+k_2+\pi\hat{x}. \tag{114}$$

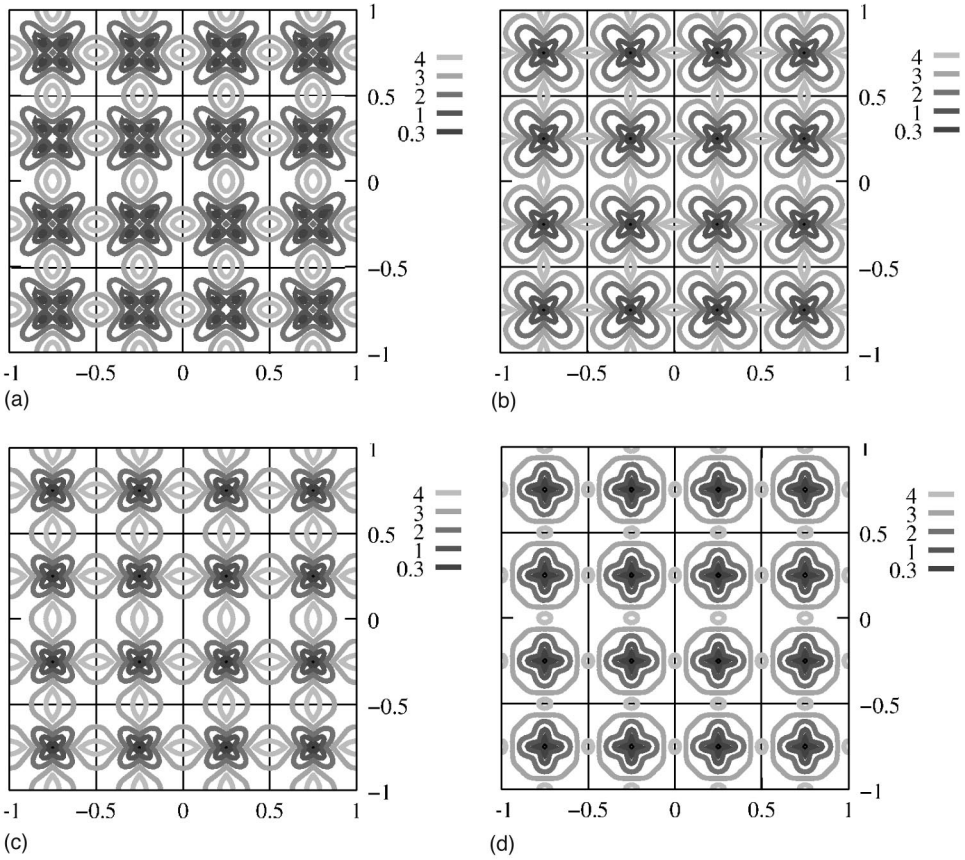

FIG. 2. Contour plot of the spinon dispersion $\min(E_1(k),E_2(k))$ as a function of $(k_x/2\pi,k_y/2\pi)$ for the Z_2-linear states. (a) is for the Z2B0013 state in Eq. (107), (b) for the Z2Bzz13 state in Eq. (108), (c) for the Z2B001n state in Eq. (109), and (d) for the Z2B$zz1n$ state in Eq. (110).

Equation (114) allows us to construct the two-spinon spectrum from the single-spinon spectrum.

Now let us consider the *Ansatz* Z2Bzz13 in Eq. (108). The spinon spectrum for the *Ansatz*, Eq. (108), is determined by

$$H=-2\chi\cos(k_x)\Gamma_0-2\eta\cos(k_x)\Gamma_2-2\chi\cos(k_y)\Gamma_1+2\eta\cos(k_y)\Gamma_3-2\gamma_1\cos(2k_x+2k_y)\Gamma_4+2\gamma_1\cos(2k_x-2k_y)\Gamma_4, \quad (115)$$

where $k_x\in(0,\pi)$, $k_y\in(-\pi,\pi)$, and

$$\Gamma_0=\tau^1\otimes\tau^3,\quad \Gamma_1=\tau^1\otimes\tau^1,$$
$$\Gamma_2=\tau^2\otimes\tau^3,\quad \Gamma_3=\tau^2\otimes\tau^1,$$
$$\Gamma_4=\tau^1\otimes\tau^0. \quad (116)$$

We find the spinon spectrum to vanish at two isolated points $k=(\pi/2,\pm\pi/2)$ [see Fig. 2(b)]. The state Z2Bzz13 is a Z_2-linear spin liquid.

The spinon spectrum for the *Ansatz* Z2B001n in Eq. (109) is determined by

$$H=-2\chi\cos(k_x)\Gamma_0-2\eta\cos(k_x)\Gamma_2-2\chi\cos(k_y)\Gamma_1+2\eta\cos(k_y)\Gamma_3+2\lambda[\cos(k_x+2k_y)+\cos(-k_x+2k_y)]\Gamma_4-2\lambda[\cos(2k_x+k_y)+\cos(2k_x-k_y)]\Gamma_5, \quad (117)$$

where $k_x\in(0,\pi)$, $k_y\in(-\pi,\pi)$, and

$$\Gamma_0=\tau^1\otimes\tau^3,\quad \Gamma_1=\tau^1\otimes\tau^1,$$
$$\Gamma_2=\tau^2\otimes\tau^3,\quad \Gamma_3=\tau^2\otimes\tau^1,$$
$$\Gamma_4=\tau^3\otimes\tau^3,\quad \Gamma_5=\tau^3\otimes\tau^1. \quad (118)$$

The spinon spectrum vanishes at two isolated points $k=(\pi/2,\pm\pi/2)$ [see Fig. 2(c)]. The state Z2B001n is also a Z_2-linear spin liquid.

The spinon spectrum for the *Ansatz* Z2B$zz1n$ in Eq. (110) can be obtained from

$$H=-2\chi\cos(k_x)\Gamma_0-2\eta\cos(k_x)\Gamma_2-2\chi\cos(k_y)\Gamma_1+2\eta\cos(k_y)\Gamma_3+2\lambda[\cos(k_x+2k_y)-\cos(-k_x+2k_y)]\Gamma_4-2\lambda[\cos(2k_x+k_y)-\cos(2k_x-k_y)]\Gamma_5, \quad (119)$$

where $k_x\in(0,\pi)$, $k_y\in(-\pi,\pi)$, and

$$\Gamma_0=\tau^1\otimes\tau^3,\quad \Gamma_1=\tau^1\otimes\tau^1,$$
$$\Gamma_2=\tau^2\otimes\tau^3,\quad \Gamma_3=\tau^2\otimes\tau^1,$$
$$\Gamma_4=\tau^3\otimes\tau^3,\quad \Gamma_5=\tau^3\otimes\tau^1. \quad (120)$$

We have also assumed that $\chi_1=\eta_1=0$. The spinon spectrum vanishes at two isolated points $k=(\pi/2,\pm\pi/2)$ [see Fig. 2(d)]. The state Z2B$zz1n$ is again a Z_2-linear spin liquid.

C. Symmetric spin liquids around the SU(2)-gapless spin liquid SU2An0

There are many types of symmetric *Ansätze* in the neighborhood of the SU(2)-gapless state, Eq. (30). Let us first consider the 12 classes of symmetric U(1) spin liquids around the SU(2)-gapless state. Here we just present the simple cases where u_{ij} are nonzero only for links with length $\leqslant 2$. Among the 12 classes of symmetric *Ansätze*, we find that 5 classes actually give us the SU(2)-gapless spin liquid when the link length is limited to $\leqslant 2$. The other 7 symmetric U(1) spin liquids are given below.

The U1Cn01n state:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,\quad u_{i,i+\hat{y}}=\chi\tau^1+\eta\tau^2,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=i\tau^1,\quad G_T=(-)^i\tau^0. \tag{121}$$

In the above, we have also listed the gauge transformations $G_{x,y}$, $G_{P_x,P_y,P_{xy}}$, and G_T associated translation, parity, and time reversal transformations. Those gauge transformations define the PSG that characterizes the U(1) spin liquid. In Sec. IV C, we have introduced a notation U1Cn01n to label the above PSG. We will use the same notation to label the *Ansatz*.

U1Cn00x state:

$$u_{i,i+\hat{x}}=\chi\tau^1,\quad u_{i,i+\hat{y}}=\chi\tau^1,$$
$$u_{i,i+\hat{x}+\hat{y}}=\eta_1\tau^3,\quad u_{i,i-\hat{x}+\hat{y}}=\eta_1\tau^3,$$
$$u_{i,i+2\hat{x}}=\eta_2\tau^3,\quad u_{i,i+2\hat{y}}=\eta_2\tau^3,$$
$$a_0^3=\eta_3,\quad a_0^{1,2}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=\tau^0,\quad G_T=i\tau^2. \tag{122}$$

U1Cn01x state:

$$u_{i,i+\hat{x}}=\chi\tau^1,\quad u_{i,i+\hat{y}}=\chi\tau^1,$$
$$u_{i,i+2\hat{x}}=-\eta_2\tau^3,\quad u_{i,i+2\hat{y}}=\eta_2\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=i\tau^1,\quad G_T=i\tau^2. \tag{123}$$

U1Cx10x state:

$$u_{i,i+\hat{x}}=\chi\tau^1,\quad u_{i,i+\hat{y}}=\chi\tau^1,$$
$$u_{i,i+\hat{x}+\hat{y}}=-\eta\tau^3,\quad u_{i,i-\hat{x}+\hat{y}}=\eta\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_y}=i\tau^1,$$
$$G_{P_{xy}}=\tau^0,\quad G_T=i\tau^2. \tag{124}$$

U1A0001 state:

$$u_{i,i+\hat{x}}=i\chi\tau^0,\quad u_{i,i+\hat{y}}=i\chi\tau^0,$$
$$u_{i,i+\hat{x}+\hat{y}}=-\eta_1\tau^3,\quad u_{i,i-\hat{x}+\hat{y}}=\eta_1\tau^3,$$
$$u_{i,i+2\hat{x}}=\eta_2\tau^3,\quad u_{i,i+2\hat{y}}=\eta_2\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad (-)^{i_x}G_{P_x}=(-)^{i_x}G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=\tau^0,\quad G_T=i(-)^i\tau^1. \tag{125}$$

U1A0011 state:

$$u_{i,i+\hat{x}}=i\chi\tau^0,\quad u_{i,i+\hat{y}}=i\chi\tau^0,$$
$$u_{i,i+2\hat{x}}=-\eta_2\tau^3,\quad u_{i,i+2\hat{y}}=\eta_2\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad (-)^{i_x}G_{P_x}=(-)^{i_x}G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=i\tau^1,\quad G_T=i(-)^i\tau^1. \tag{126}$$

U1Ax10x state:

$$u_{i,i+\hat{x}}=i\chi\tau^0,\quad u_{i,i+\hat{y}}=i\chi\tau^0,$$
$$u_{i,i+\hat{x}+\hat{y}}=\eta\tau^3,\quad u_{i,i-\hat{x}+\hat{y}}=\eta\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad (-)^{i_x}G_{P_x}=(-)^{i_x}G_{P_y}=i\tau^1,$$
$$G_{P_{xy}}=\tau^0,\quad G_T=i(-)^i\tau^1. \tag{127}$$

In addition to the labels, we also explicitly list the gauge transformations $G_{x,y}$, $G_{P_x,P_y,P_{xy}}$, and G_T for each *Ansatz*. Note that when we define the labels of the U(1) PSG from the gauge transformations $G_{x,y}$, $G_{P_x,P_y,P_{xy}}$, and G_T, we have chosen a particular gauge called the canonical gauge. In the canonical gauge, the IGG is generated by a constant gauge transformation $e^{i\theta\tau^3}$. Some of the above *Ansätze* are given in the canonical gauge while others are not. For the

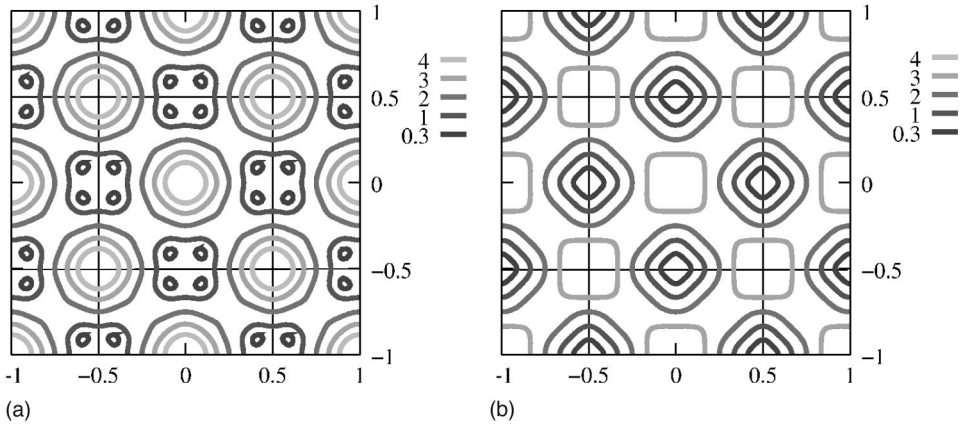

FIG. 3. Contour plot of the spinon dispersion $E_+(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for (a) the U(1)-linear state U1Cn00x in Eq. (122) and (b) the U(1)-quadratic state U1Cx10x in Eq. (124). In the U(1)-quadratic state, the spinon energy vanishes as Δk^2 near two points $k=(\pi,0),(0,\pi)$.

latter *Ansätze*, the listed gauge transformations $G_{x,y}$, $G_{P_x,P_y,P_{xy}}$, and G_T are different from those in the canonical gauge.

Equation (121) is the U1Cn01n U(1)-linear state (the staggered flux state) studied before. After examining the spinon dispersion, we find that the U1Cn00x state in Eq. (122) can be a U(1)-linear or a U(1)-gapped state depending on the value of a_0^3. If it is a U(1)-linear state, it will have eight isolated Fermi points [see Fig. 3(a)]. The U1Cn01x state in Eq. (123) is a U(1)-gapless state [see Fig. 4(a)]. The U1Cx10x state in Eq. (124) has two Fermi points at $k_1=(\pi,0)$ and $k_2=(0,\pi)$ [see Fig. 3(b)]. However, the spinon energy has a quadratic form $E(k)\propto(k-k_{1,2})^2$ near k_1 and k_2. Thus we call the U1Cx10x spin liquid, Eq. (124), a U(1)-quadratic state. The U1A0001 state in Eq. (125), the U1A0011 state in Eq. (126), and the U1Ax10x state in Eq. (127) are U(1)-gapless states (see Fig. 4). Again the spinon dispersions for the U(1) spin liquids have some qualitative differences between each other, which can be used to detect different quantum orders in those U(1) spin liquids.

We next consider the 52 classes of symmetric Z_2 spin liquids around the SU(2)-gapless state. Here we just present the simplest case where u_{ij} are nonzero only for links with length $\leqslant 1$. We find that 48 out of the 52 classes of *Ansätze* reduce to U(1) or SU(2) spin liquids when the link's length is $\leqslant 1$. In the following we discuss the four remaining Z_2 *Ansätze*.

The first one is Z_2 spin liquid Z2Ax2(12)n described by Eq. (33). The second one is Z_2 spin liquid Z2A0013 described by Eq. (103) or Eq. (42). The third one is Z_2 spin

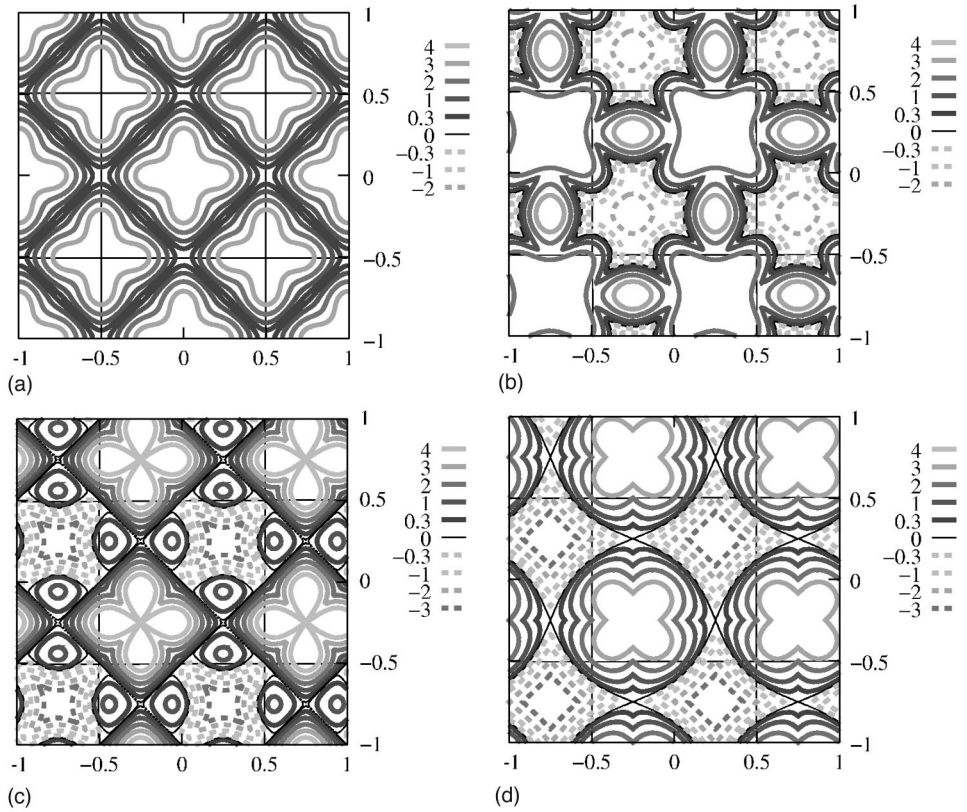

FIG. 4. Contour plot of the spinon dispersion $E_+(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for the U(1)-gapless states. (a) is for the U1Cn01x state in Eq. (123), (b) for the U1A0001 state in Eq. (125), (c) for the U1A0011 state in Eq. (126), and (d) for the U1Ax10x state in Eq. (127).

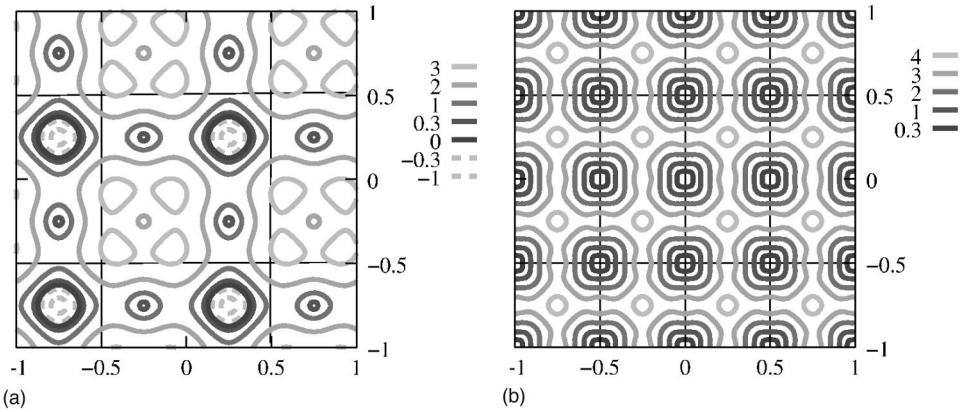

FIG. 5. Contour plot of the spinon dispersion $E_+(k)$ as a function of $(k_x/2\pi,k_y/2\pi)$ for the Z_2 spin liquids. (a) is for Z_2-gapless state Z2Ax2(12)n in Eq. (33), and (b) is for Z_2-quadratic state Z2Bx2(12)n in Eq. (128). Despite the lack of rotation and parity symmetries in the single-spinon dispersion in (a), the two-spinon spectrum does have those symmetries.

liquid Z2By1(12)n [note that Z2By1(12)n is gauge equivalent to Z2Bx2(12)n]:

$$u_{i,i+\hat{x}} = i\chi\tau^0 + \eta_1\tau^1,$$

$$u_{i,i+\hat{y}} = (-)^{i_x}(i\chi\tau^0 + \eta_1\tau^2),$$

$$a_0^{1,2,3} = 0. \tag{128}$$

The fourth one is Z_2 spin liquid Z2B0013, which is described by Eq. (107).

The *Ansatz* Z2Bx2(12)n in Eq. (128) is a new Z_2 spin liquid. The spinon spectrum for the *Ansatz*, Eq. (128), is determined by

$$H = -2\chi\sin(k_x)\Gamma_0 + 2\eta\cos(k_x)\Gamma_2 - 2\chi\sin(k_y)\Gamma_1 + 2\eta\cos(k_y)\Gamma_3, \tag{129}$$

where $k_x \in (-\pi/2,\pi/2)$, $k_y \in (-\pi,\pi)$, and

$$\Gamma_0 = \tau^0\otimes\tau^3, \quad \Gamma_2 = \tau^1\otimes\tau^3,$$

$$\Gamma_1 = \tau^0\otimes\tau^1, \quad \Gamma_3 = \tau^2\otimes\tau^1. \tag{130}$$

The spinon spectrum can be calculated exactly and its four branches take a form $\pm E_1(k)$ and $\pm E_2(k)$. The spinon energy vanishes at two isolated points $k=(0,0),(0,\pi)$. Near $k=0$ the low-energy spectrum is given by [see Fig. 5(b)]

$$E = \pm\eta^{-1}\sqrt{(\chi^2+\eta^2)^2(k_x^2-k_y^2)^2 + 4\chi^4 k_x^2 k_y^2}. \tag{131}$$

It is interesting to see that the energy does not vanish linearly as $k\to 0$: instead it vanishes like k^2.

We find that the loop operators for the loops $i\to i+\hat{x}\to i+\hat{x}+\hat{y}\to i+\hat{y}\to i$ and $i\to i+\hat{y}\to i-\hat{x}+\hat{y}\to i-\hat{x}\to i$ do not commute as long as both χ and η are nonzero. Thus the spin liquid described by Eq. (128) indeed has a Z_2 gauge structure. We will call such a state Z_2-quadratic spin liquid to stress the quadratic $E\propto k^2$ dispersion. Such a state cannot be constructed from translation-invariant *Ansätze*. The two-spinon spectrum is still related to the one-spinon spectrum through Eq. (114) (see Fig. 6).

D. Symmetric spin liquids around the SU(2)-linear spin liquid SU2Bn0

Last, we consider symmetric states in the neighborhood of the SU(2)-linear state, Eq. (31). We would like to use the following result proved in Ref. 66. Given a PSG generated by $G_{x,y,T}$ and $G_{P_x,P_y,P_{xy}}$, the generators

$$\tilde{G}_x(i) = (-)^{i_y}G_x(i), \quad \tilde{G}_y(i) = G_y(i),$$

$$\tilde{G}_{P_x}(i) = G_{P_x}(i), \quad \tilde{G}_{P_y}(i) = G_{P_y}(i),$$

$$\tilde{G}_{P_{xy}}(i) = (-)^{i_x i_y}G_{P_{xy}}(i), \quad \tilde{G}_T(i) = G_T(i) \tag{132}$$

generate a new PSG. The new PSG has the same IGG and is an extension of the same symmetry group as the original PSG. The PSG's for states in the neighborhood of the SU(2)-

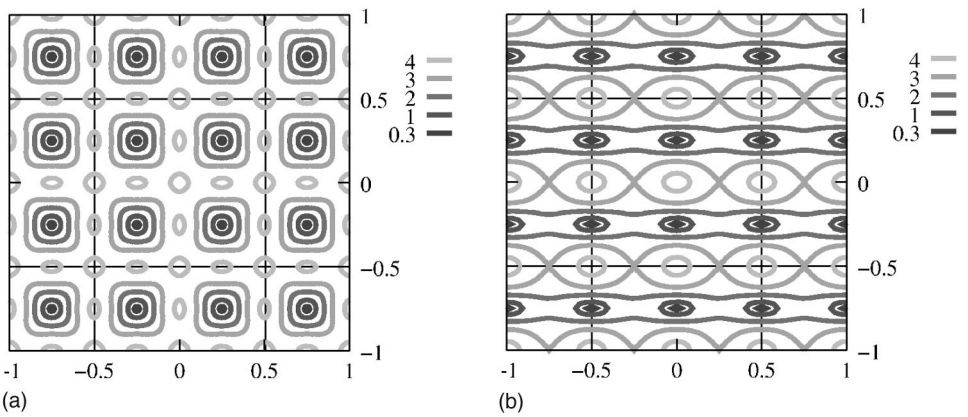

FIG. 6. Contour plot of the spinon dispersion $E_+(k)$ as a function of $(k_x/2\pi,k_y/2\pi)$ for (a) the U(1)-linear state U1Cn0x1 in Eq. (134) and (b) the U(1)-linear state Eq. (142).

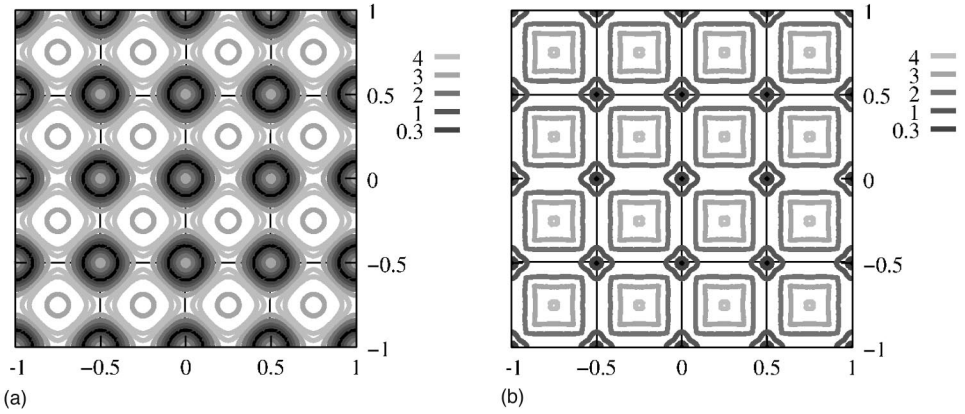

FIG. 7. Contour plot of the spinon dispersion $\min(E_1(k), E_2(k))$ as a function of $(k_x/2\pi, k_y/2\pi)$ for the U(1) spin-liquid states. (a) is for the U(1)-gapless state U1B0001 in Eq. (136) and (b) is for the U(1)-linear state U1B0011 in Eq. (137).

linear state can be obtained through above mapping from the PSG's of symmetric spin liquids around the SU(2)-gapless spin liquid.

Here we will only consider the 12 classes of symmetric U(1) spin liquids around the SU(2)-linear state. We will just present the simple cases where u_{ij} are nonzero only for links with length $\leqslant 2$. We find that 5 of 12 classes of *Ansätze* reduce SU(2)-gapless spin liquids when the link length is $\leqslant 2$. Thus we only obtain the following 5 symmetric U(1) spin liquids.

U1Cn01n *Ansatz*:

$$u_{i,i+\hat{x}}=\chi\tau^1-\eta\tau^2,\quad u_{i,i+\hat{y}}=\chi\tau^1+\eta\tau^2,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_x}=\tau^0,$$
$$G_{P_{xy}}=i\tau^1,\quad G_T=(-)^i\tau^0, \tag{133}$$

which has the same quantum order as in the U(1)-linear state, Eq. (32).

U1Cn0x1 *Ansatz*:

$$u_{i,i+\hat{x}}=\chi\tau^2,\quad u_{i,i+\hat{y}}=\chi\tau^1,$$
$$u_{i,i+2\hat{x}}=-\eta\tau^3,\quad u_{i,i+2\hat{y}}=\eta\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$G_x=G_y=\tau^0,\quad G_{P_x}=G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=i\tau^{12},\quad G_T=(-)^{i_y}i\tau^1. \tag{134}$$

U1Cn0n1 *Ansatz*:

$$u_{i,i+\hat{x}}=\chi\tau^2,\quad u_{i,i+\hat{y}}=\chi\tau^1,$$
$$u_{i,i+2\hat{x}}=\eta\tau^3,\quad u_{i,i+2\hat{y}}=\eta\tau^3,$$
$$a_0^3=\eta_1,\quad a_0^{1,2}=0,$$
$$G_x=G_y=\tau^0,\quad G_T=(-)^{i_y}i\tau^1,$$
$$G_{P_x}=G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=(-)^{i_x i_y}g_3\{[(-)^{i_x}-(-)^{i_y}]\pi/4\}. \tag{135}$$

U1B0001 *Ansatz*:

$$u_{i,i+\hat{x}}=i\chi\tau^0,\quad u_{i,i+\hat{y}}=i(-)^{i_x}\chi\tau^0,$$
$$u_{i,i+2\hat{x}}=\eta\tau^3,\quad u_{i,i+2\hat{y}}=\eta\tau^3,$$
$$a_0^3=\eta_1,\quad a_0^{1,2,}=0,$$
$$(-)^{i_y}G_x=G_y=\tau^0,\quad (-)^{i_x}G_{P_x}=(-)^{i_x}G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=(-)^{i_x i_y}\tau^0,\quad G_T=i(-)^i\tau^1. \tag{136}$$

U1B0011 *Ansatz*:

$$u_{i,i+\hat{x}}=i\chi\tau^0,\quad u_{i,i+\hat{y}}=i(-)^{i_x}\chi\tau^0,$$
$$u_{i,i+2\hat{x}}=-\eta\tau^3,\quad u_{i,i+2\hat{y}}=\eta\tau^3,$$
$$a_0^{1,2,3}=0,$$
$$(-)^{i_y}G_x=G_y=\tau^0,\quad (-)^{i_x}G_{P_x}=(-)^{i_x}G_{P_y}=\tau^0,$$
$$G_{P_{xy}}=i(-)^{i_x i_y}\tau^1,\quad G_T=i(-)^i\tau^1. \tag{137}$$

Now let us discuss spinon dispersions in the above U(1) spin liquids. The spinon in the U1Cn0x1 state Eq. (134) has four linear nodes at $(\pm\pi/2,\pm\pi/2)$. Thus the U1Cn0x1 state is a U(1)-linear spin liquid. The U1Cn0n1 state Eq. (135) has fully gapped spinons and is a U(1)-gapped spin liquid.

The four spinon bands in the U1B0001 state Eq. (136) are given by [see Fig. 7(a)]

$$\pm 2\chi\sqrt{\sin^2(k_x)+\sin^2(k_y)}\pm[2\eta\cos(2k_x)+2\eta\cos(2k_y)+\eta_1]. \tag{138}$$

We find that the U1B0001 state is a U(1)-gapless spin liquid. The four spinon bands in the U1B0011 state Eq. (137) are given by [see Fig. 7(b)]

$$\pm 2\chi\sqrt{\sin^2(k_x)+\sin^2(k_y)}\pm 2\eta[\cos(2k_x)-\cos(2k_y)]. \tag{139}$$

Hence, the U1B0011 state is a U(1)-linear spin liquid.

TABLE I. Spin liquids.

Z_2-gapped	Z2A$xx0z$
Z_2-linear	Z2A0013, Z2A$zz13$, Z2A001n Z2A$zz1n$, Z2B0013, Z2B$zz13$ Z2B001n, Z2B$zz1n$
Z_2-quadratic	Z2B$x2(12)n$
Z_2-gapless	Z2A$x2(12)n$
U(1)-gapped	U1C$n00x$
U(1)-linear	U1B0011, U1C$n00x$, U1C$n01n$ U1C$n0x1$
U(1)-quadratic	U1C$x10x$
U(1)-gapless	U1A0001, U1A0011, U1A$x10x$ U1B0001, U1C$n01x$
SU(2)-linear	SU2B$n0$
SU(2)-gapless	SU2A$n0$

To summarize we list all the spin liquids discussed so far in Table I.

VI. MEAN-FIELD PHASE DIAGRAM OF THE J_1-J_2 MODEL

To see which of the Z_2, U(1), and SU(2) spin liquids discussed in the last section have low ground energies and may appear in real high-T_c superconductors, we calculate the mean-field energy of a large class of *translation-invariant Ansätze*. In Fig. 8, we present the resulting mean-field phase diagram for a J_1-J_2 spin system. Here J_1 is the nearest-neighbor spin coupling and J_2 is the next-nearest-neighbor spin coupling. We have fixed $J_1+J_2=1$. The y axis is the mean-field energy per site (multiplied by a factor 8/3). The phase (A) is the π-flux state [the SU2B$n0$ SU(2)-linear state], Eq. (31). The phase (B) is a state with two independent uniform RVB states on the diagonal links. It has SU(2)×SU(2) gauge fluctuations at low energies and will be called an SU(2)×SU(2)-gapless state. Its *Ansatz* is given by

$$u_{i,i+\hat{x}+\hat{y}}=\chi\tau^3,$$

$$u_{i,i+\hat{x}-\hat{y}}=\chi\tau^3,$$

$$a_0^l=0. \tag{140}$$

The phase (C) is a state with two independent π-flux states on the diagonal links. It has SU(2)×SU(2) gauge fluctuations at low energies and will be called an SU(2)×SU(2)-linear state. Its *Ansatz* is given by

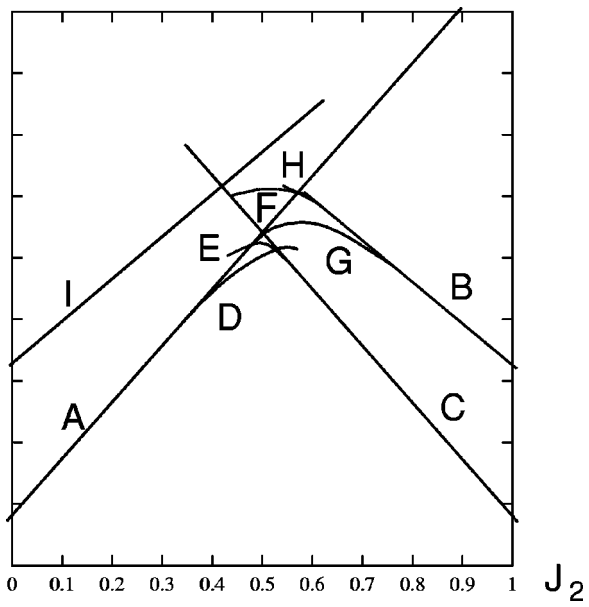

FIG. 8. The mean-field energies for various phases in a J_1-J_2 spin system. (A) The π-flux state [the SU(2)-linear state SU2B$n0$]. (B) The SU(2)×SU(2)-gapless state in Eq. (140). (C) The SU(2)×SU(2)-linear state in Eq. (141). (D) The chiral spin state [an SU(2)-gapped state]. (E) The U(1)-linear state Eq. (142) which breaks 90° rotation symmetry. (F) The U(1)-gapped state U1C$n00x$ in Eq. (122). (G) The Z_2-linear state Z2A$zz13$ in Eq. (104). (H) The Z_2-linear state Z2A0013 in Eq. (103). (I) The uniform RVB state [the SU(2)-gapless state SU2A$n0$].

$$u_{i,i+\hat{x}+\hat{y}}=\chi(\tau^3+\tau^1),$$

$$u_{i,i+\hat{x}-\hat{y}}=\chi(\tau^3-\tau^1),$$

$$a_0^l=0. \tag{141}$$

The phase (D) is the chiral spin state, Eq. (44). The phase (E) is described by an *Ansatz*

$$u_{i,i+\hat{x}+\hat{y}}=\chi_1\tau^1+\chi_2\tau^2,$$

$$u_{i,i+\hat{x}-\hat{y}}=\chi_1\tau^1-\chi_2\tau^2,$$

$$u_{i,i+\hat{y}}=\eta\tau^3,$$

$$a_0^l=0, \tag{142}$$

which breaks the 90° rotation symmetry and is a U(1)-linear state [see Fig. 6(b)]. The phase (F) is described by the U1C$n00x$ *Ansatz* in Eq. (122). The U1C$n00x$ state can be a U(1)-linear or a U(1)-gapped state. The state for phase (F) turns out to be a U(1)-gapped state. The phase (G) is described by the Z2A$zz13$ *Ansatz* in Eq. (104) which is a Z_2-linear state. The phase (H) is described by the Z2A0013 *Ansatz* in Eq. (103) and is also a Z_2-linear state. The phase (I) is the uniform RVB state [the SU(2)-gapless state SU2A$n0$, Eq. (30)].

From Fig. 8, we see continuous phase transitions (at mean-field level) between the following pairs of phases: (A,D), (A,G), (B,G), (C,E), and (B,H). The three continuous transitions (B,G), (B,H), and (A,G) do not change any sym-

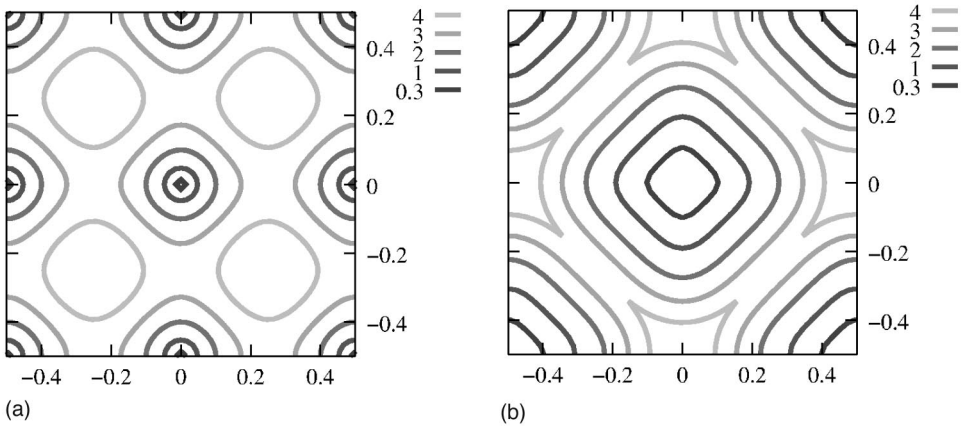

FIG. 9. Contour plot of the dispersion for spin-1 excitation, $E_{2s}(k)$, as a function of $(k_x/2\pi, k_y/2\pi)$ for (a) the SU(2)-linear spin liquid SU2Bn0 in Eq. (31) (the π-flux phase) and (b) the U(1)-quadratic spin liquid U1Cx10x in Eq. (124).

metries. We also note that the SU(2) gauge structure in the phase (A) breaks down to Z_2 in the continuous transition from the phase (A) to the phase (G). The $SU(2)\times SU(2)$ gauge structure in the phase (B) breaks down to Z_2 in the two transitions (B,G) and (B,H).

VII. PHYSICAL MEASUREMENTS OF QUANTUM ORDERS

After characterizing the quantum orders using the PSG mathematically, we would like to ask how to measure quantum orders in experiments. The quantum orders in gapped states are related to the topological orders. The measurement of topological orders are discussed in Refs. 9, 10 and 65. The quantum order in a state with gapless excitations can be measured, in general, by the dynamical properties of gapless excitation. However, not all dynamical properties are universal. Thus we need to identify the universal properties of gapless excitations before using them to characterize and measure quantum orders. The PSG characterization of quantum orders allows us obtain those universal properties. We simply need to identify the common properties of gapless excitations that are shared by all the *Ansatz* with the same PSG.

To demonstrate the above idea, we would like to study the spectrum of two-spinon excitations. We note that spinons can only be created in pairs. Thus the one-spinon spectrum is not physical. We also note that the two-spinon spectrum include spin-1 excitations which can be measured in experiments. At a given momentum, the two-spinon spectrum is distributed in one or several ranges of energy. Let $E_{2s}(k)$ be the lower edge of the two-spinon spectrum at momentum k. In the mean-field theory, the two-spinon spectrum can be constructed from the one-spinon dispersion

$$E_{2\text{-spinon}}(k)=E_{1\text{-spinon}}(q)+E_{1\text{-spinon}}(k-q). \quad (143)$$

In Figs. 9–15 we present the mean field E_{2s} for some simple spin liquids. If the mean-field state is stable against the gauge fluctuations, we expect that the mean field E_{2s} should qualitatively agree with the real E_{2s}.

Among our examples, there are eight Z_2-linear spin liquids (see Fig. 10 and Fig. 11). We see that some of those eight different Z_2-linear spin liquids (or eight different quantum orders) have a different number of gapless points. The gapless points of some spin liquids are pinned at position $k=(\pi,\pi)$ and/or $k=(\pi,0),(0,\pi)$. By measuring the low-energy spin excitations (say, using neutron scattering), we can distinguish those Z_2 spin liquids. We note that all two-spinon spectra have rotation and parity symmetries around $k=0$. This is expected. Since the two-spinon spectra are physical, they should have all the symmetries the spin liquids have.

We also have four U(1)-linear spin liquids. Some of them can be distinguished by their different numbers of gapless points. It is interesting to note that all the U(1) spin liquids discussed here have a gapless point in the two-spinon spectrum pinned at position $k=(\pi,\pi)$. The U(1)-linear spin liquids are also different from the Z_2-linear spin liquids in that the spin-spin correlations have different decay exponents once the U(1) gauge fluctuations are included. We also see that E_{2s} has a quadratic form $E_{2s}\propto k^2$ for the U(1)-quadratic spin liquid. E_{2s} vanishes in two finite regions in k space for the Z_2-gapless spin liquids.

Neutron scattering experiments probe the two-spinon sector. Thus low-energy neutron scattering allows us to measure quantum orders in high-T_c superconductors.

Let us discuss the U(1) linear state U1Cn01n (the staggered-flux state) in more detail. The U1Cn01n state is proposed to describe the pseudogap metallic state in underdoped high-T_c superconductors.[33,34] The U1Cn01n state naturally explains the spin pseudogap in the underdoped metallic state. As an algebraic spin liquid, the U1Cn01n state also explain the Luttinger-like electron spectral function[34] and the enhancement of the (π,π) spin fluctuations[68] in the pseudogap state. From Fig. 13(a), we see that gapless points of the spin-1 excitations in the U1Cn01n state are always at $k=(\pi,\pi)$, (0,0), $(\pi,0)$, and $(0,\pi)$. The equal energy contour for the edge of the spin-1 continuum has a shape of two overlapped ellipses at all four k points. Also the energy contours are not perpendicular to the zone boundary. All those are the universal properties of the U1Cn01n state. Measuring those properties in neutron scattering experiments will allow us to determine if the pseudogap metallic state is described by the U1Cn01n (the staggered-flux) state or not.

We have seen that at low energies, the U1Cn01n state is unstable due to the instanton effect. Thus the U1Cn01n state has to change into some other states, such as the eight Z_2

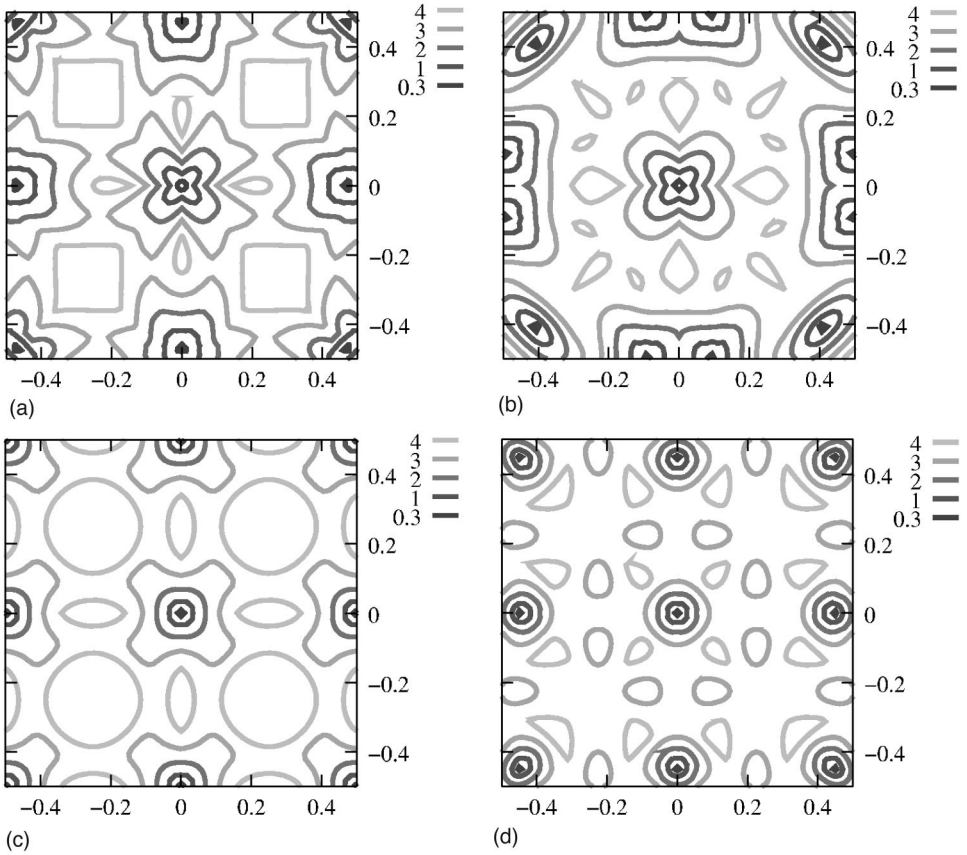

FIG. 10. Contour plot of $E_{2s}(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for the Z_2-linear spin liquids. (a) is for the Z2A0013 state in Eq. (103), (b) for the Z2A*zz*13 state in Eq. (104), (c) for the Z2A001*n* state in Eq. (105), and (d) for the Z2A*zz*1*n* state in Eq. (106).

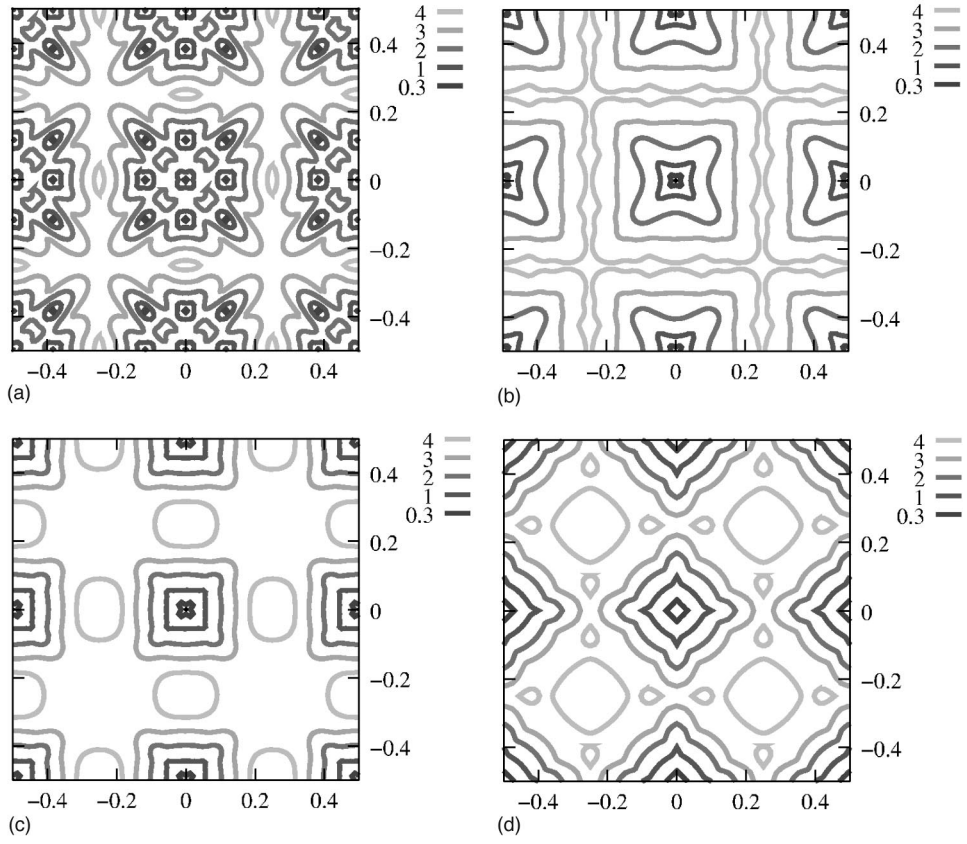

FIG. 11. Contour plot of $E_{2s}(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for the Z_2-linear spin liquids. (a) is for the Z2B0013 state in Eq. (107), (b) for the Z2B*zz*13 state in Eq. (108), (c) for the Z2B001*n* state in Eq. (109), and (d) for the Z2B*zz*1*n* state in Eq. (110).

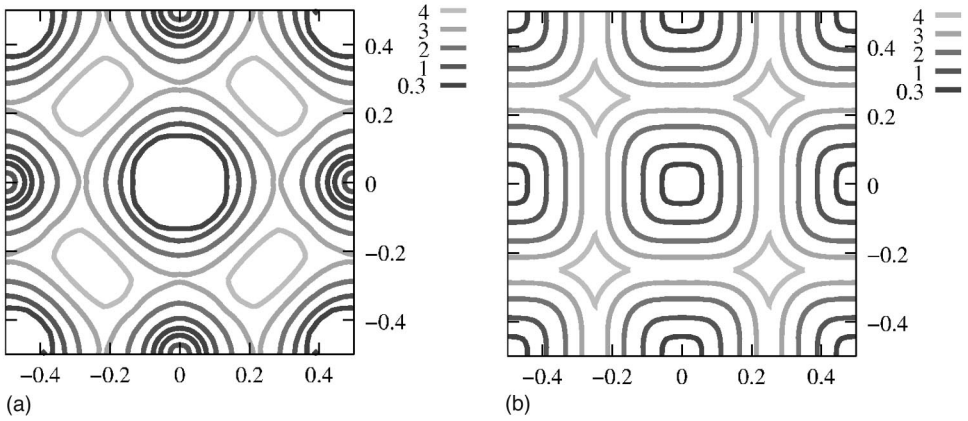

FIG. 12. Contour plot of $E_{2s}(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for (a) the Z_2-gapless state Z2Ax2(12)n in Eq. (33), and (b) the Z_2-quadratic state Z2Bx2(12)n in Eq. (128).

spin liquids discussed in Sec. V or some other states not discussed in this paper. From Fig. 10(a), we see that the transition from the U1Cn01n state to the Z_2-linear state Z2A0013 can be detected by neutron scattering if one observes the splitting of the node at (π,π) into four nodes at $(\pi\pm\delta,\pi\pm\delta)$ and the splitting of the nodes at $(\pi, 0)$ and $(0,\pi)$ into two nodes at $(\pi\pm\delta,0)$ and $(0,\pi\pm\delta)$. From Fig. 10(b), we see that, for the transition from the U1Cn01n state to the Z_2-linear state Z2Azz13, the node at (π,π) still splits into four nodes at $(\pi\pm\delta,\pi\pm\delta)$. However, the nodes at $(\pi,0)$ and $(0,\pi)$ split differently into two nodes at $(\pi,\pm\delta)$ and $(\pm\delta,\pi)$. We can also study the transition from the U1Cn01n state to other six Z_2 spin liquids. We find that the spectra of spin-1 excitations all change in certain characteristic ways. Thus, by measuring the spin-1 excitation spectrum and its evolution, we not only can detect a quantum transition that does not change any symmetries, we can also tell which transition is happening.

The neutron scattering on high-T_c superconductors indeed showed a splitting of the scattering peak at (π,π) into four peaks at $(\pi\pm\delta,\pi),(\pi,\pi\pm\delta)$ (Refs. 30 and 69–75) or into two peaks at $(\pi,\pi)\rightarrow(\pi+\delta,\pi-\delta),(\pi-\delta,\pi+\delta)$ (Refs. 28 and 76) as we lower the energy. This is consistent with our belief that the U1Cn01n state is unstable at low energies. However, it is still unclear if we can identify the position of the neutron scattering peak as the position of the node in the spin-1 spectrum. If we do identify the scattering peak as the node, then none of the eight Z_2 spin liquids in the neighborhood of the U1Cn01n state can explain the splitting pattern $(\pi\pm\delta,\pi),(\pi,\pi\pm\delta)$. This will imply that the U1Cn01n state changes into another state not studied in this paper. This example illustrates that detailed neutron scattering experiments are powerful tools in detecting quantum orders and studying new transitions between quantum orders that may not change any symmetries.

VIII. FOUR CLASSES OF SPIN LIQUIDS AND THEIR STABILITY

We have concentrated on the mean-field states of spin liquids and presented many examples of mean-field *Ansätze* for symmetric spin liquids. In order for those mean-field states to represent real physical spin liquids, we need to include the gauge fluctuations. We also need to show that the inclusion of the gauge fluctuations does not destabilize the mean-field states at low energies. This requires that (a) the gauge interaction be not too strong and (b) the gauge interaction be not a relevant perturbation. (The gauge interaction, however, can be a marginal perturbation.) The requirement (a) can be satisfied through the large-N limit and/or adjustment of short-range spin couplings in the spin Hamiltonian, if necessary. Here we will mainly consider the requirement (b). We find that, at least in certain large-N limits, many (but not all) mean-field states do correspond to real quantum spin liquids which are stable at low energies. In this case, the characterization of the mean-field states by PSG's corre-

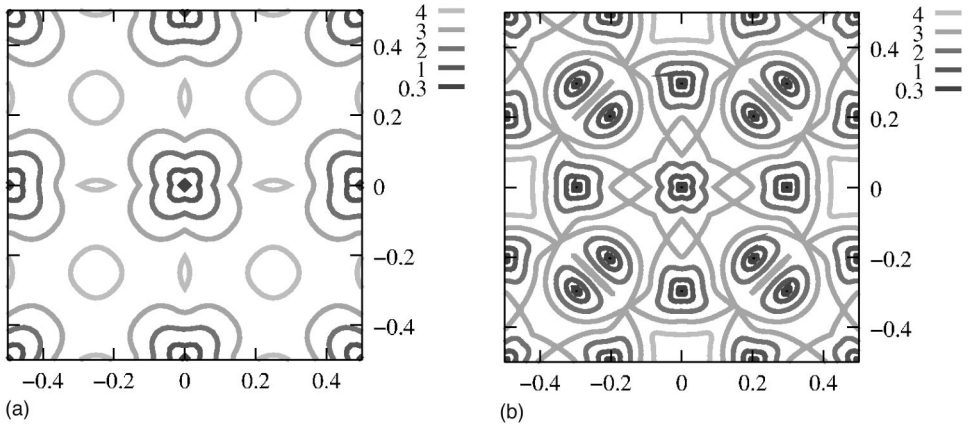

FIG. 13. Contour plot of $E_{2s}(k)$ as a function of $(k_x/2\pi, k_y/2\pi)$ for two U(1)-linear spin liquids. (a) is for the U1Cn01n state Eq. (32) (the staggered flux phase) and (b) for the U1Cn00x state Eq. (122) in the gapless phase.

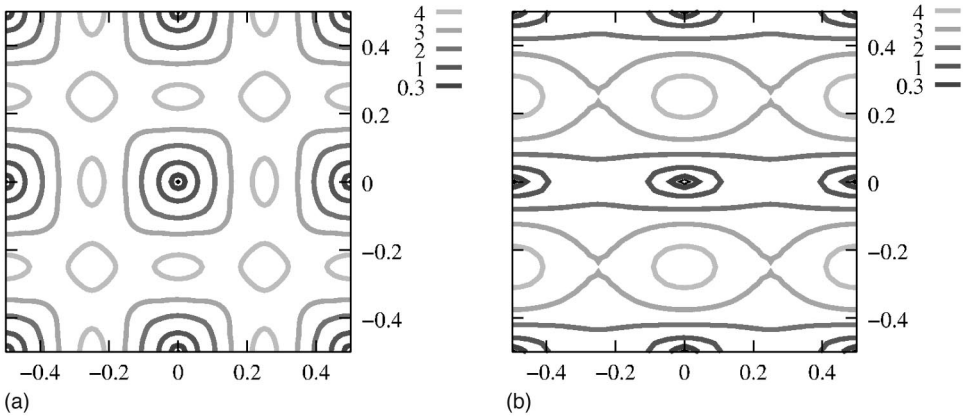

FIG. 14. Contour plot of the two-spinon dispersion $E_{2s}(k)$ as a function of $(k_x/2\pi,k_y/2\pi)$ for (a) the U(1)-linear spin liquid state U1Cn0x1 in Eq. (134) and (b) the U(1)-linear spin liquid Eq. (142).

sponds to the characterization of real quantum spin liquids.

All spin liquids (with an odd number of electrons per unit cell) studied so far can be divided into four classes. In the following we will study each class in turn.

A. Rigid spin liquid

In rigid spin liquids, by definition, the spinons and all other excitations are fully gapped. The gapped gauge field only induces short-range interactions between spinons due to Chern-Simons terms or the Anderson-Higgs mechanism. By definition, the rigid spin liquids are locally stable and self-consistent. The rigid spin liquids are characterized by topological orders and they have true spin-charge separation. The low-energy effective theories for rigid spin liquids are topological field theories. The Z_2-gapped spin liquid and chiral spin liquid are examples of rigid spin liquids.

B. Bose spin liquid

The U(1)-gapped spin liquid discussed in the last section is not a rigid spin liquid. It is a Bose spin liquid. Although the spinon excitations are gapped, the U(1) gauge fluctuations are gapless in the U(1)-gapped spin liquid. The dynamics of the gapless U(1) gauge fluctuations is described by low-energy effective theory

$$\mathcal{L}=\frac{1}{2g}(f_{\mu\nu})^2, \tag{144}$$

where $f_{\mu\nu}$ is the field strength of the U(1) gauge field. However, in $1+2$ dimensions and after including the instanton effect, the U(1) gauge fluctuations will gain an energy gap.[77] The properties of the resulting quantum state remain to be an open problem.

C. Fermi spin liquid

The Fermi spin liquids have gapless excitations that are described by spin-1/2 fermions. Those gapless excitations have only short-range interactions between them. The Z_2-linear, Z_2-quadratic, and Z_2-gapless spin liquids discussed above are examples of Fermi spin liquids.

The spinons have a massless Dirac dispersion in Z_2-linear spin liquids. Thus Z_2-linear spin liquids are locally stable since short-range interactions between massless Dirac fermions are irrelevant in $1+2$ dimensions. We would like to point out that the massless Dirac dispersion of the Z_2-linear spin liquids is protected by the PSG (or the quantum order). That is, any perturbations around, for example, the Z_2-linear *Ansatz* Z2A003z in Eq. (39) cannot destroy the massless Dirac dispersion as long as the PSG's are not changed by the perturbations. To understand this result, we start with the most general form of symmetric perturbations (see Ref. 66):

$$u_{i,i+m}=u_m^l\tau^l|_{l=1,2,3},$$

$$u^{1,2}_{P_{xy}(m)}=-u_m^{1,2},$$

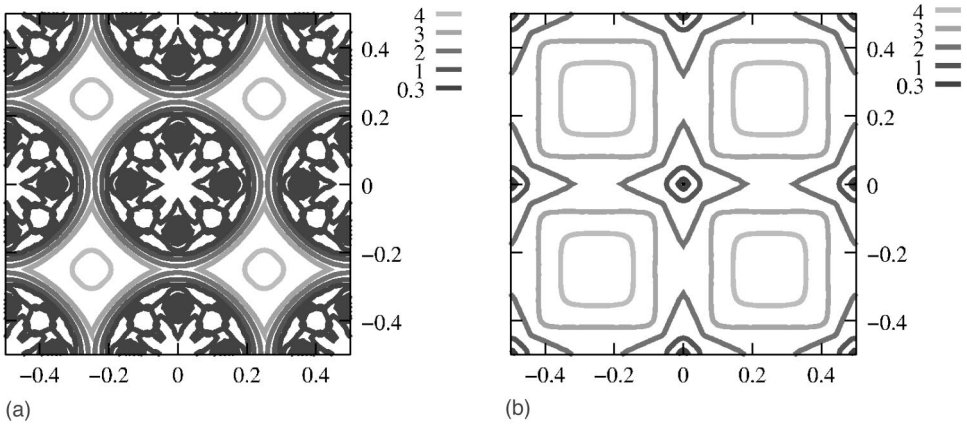

FIG. 15. Contour plot of the two-spinon dispersion $E_{2s}(k)$ as a function of $(k_x/2\pi,k_y/2\pi)$ for the U(1) spin liquid states. (a) is for the U(1)-gapless state U1B0001 in Eq. (136) and (b) is for the U(1)-linear state U1B0011 in Eq. (137).

$$u^3_{P_{xy}(m)} = u^3_m,$$

$$u^{1,2,3}_{P_x(m)} = u^{1,2,3}_m,$$

$$u^{1,2,3}_{P_y(m)} = u^{1,2,3}_m,$$

$$u_m = 0, \quad \text{for} \quad m = \text{even}, \tag{145}$$

around the Z_2-linear *Ansatz*, Eq. (39). We find that such perturbations vanish in momentum space at $k = (\pm\pi/2, \pm\pi/2)$. The translation, parity, and the time reversal symmetries do not allow any mass terms or chemical potential terms. Thus the Z_2-linear spin liquid is a phase that occupies a finite region in the phase space (at $T=0$). *One does not need any fine-tuning of the coupling constants and* u_{ij} *to get a massless Dirac spectrum.*

Now let us consider the stability of the Z_2-quadratic spin liquid Z2Bx2(12)n in Eq. (128). The spinons have a gapless quadratic dispersion in the Z_2-quadratic spin liquid. The gapless quadratic dispersion of the Z_2-quadratic spin liquid is also protected by the symmetries. The most general form of symmetric perturbations around the Z_2-quadratic *Ansatz*, Eq. (128), is given by (see Ref. 66)

$$u_{i,i+m} = (-)^{m_y i_x}(u^0_m \tau^0 + u^1_m \tau^1 + u^2_m \tau^2),$$

$$u^{0,1,2}_m = 0, \quad \text{for} \quad m = \text{even}. \tag{146}$$

In momentum space, the most general symmetric Z_2-quadratic *Ansätze* give rise to the following Hamiltonian (after considering the 90 ° rotation symmetry):

$$\begin{aligned} H = &-2\sum \chi_{mn}[\sin(nk_x - mk_y)\Gamma_0 + \sin(mk_x + nk_y)\Gamma_1] \\ &+2\sum \eta_{mn}[\cos(nk_x - mk_y)\Gamma_2 + \cos(mk_x + nk_y)\Gamma_3] \\ &+2\sum \lambda_{mn}[\cos(nk_x - mk_y)\Gamma_4 + \cos(mk_x + nk_y)\Gamma_6], \end{aligned} \tag{147}$$

where

$$\Gamma_0 = \tau^0 \otimes \tau^3, \quad \Gamma_2 = \tau^1 \otimes \tau^3,$$

$$\Gamma_1 = \tau^0 \otimes \tau^1, \quad \Gamma_3 = \tau^2 \otimes \tau^1,$$

$$\Gamma_4 = -\tau^2 \otimes \tau^3, \quad \Gamma_5 = \tau^1 \otimes \tau^1, \tag{148}$$

and the summation is over $m=$ even, $n=$ odd. We find that the spinon dispersion still vanishes at $k=(0,0),0,\pi)$ and the energy still satisfies $E \propto k^2$. The translation, parity, and the time reversal symmetric perturbations do not change the qualitative behavior of the low-energy spinon dispersion. Thus, at the mean-field level, the Z_2-quadratic spin liquid is a phase that occupies a finite region in phase space (at $T=0$). One does not need any fine-tuning of the coupling constants to get a gapless quadratic dispersion of the spinons. However, unlike the Z_2-linear spin liquid, the short-range four-fermion interactions between the gapless spinons in the Z_2-quadratic state are marginal in $1+2$ dimensions. Further studies are needed to understand the dynamical stability of the Z_2-quadratic spin liquid beyond the mean-field level.

The Z_2-gapless spin liquid is as stable as a Fermi liquid in $1+2$ dimensions. Again we expect the Z_2-gapless spin liquid to be a phase that occupies a finite region in phase space, at least at the mean-field level.

D. Algebraic spin liquid

U(1)-linear spin liquids are examples of algebraic spin liquids. Their low-lying excitations are described by massless Dirac fermions coupled to a U(1) gauge field. Although the massless Dirac fermions are protected by quantum orders, the gauge couplings remain large at low energies. Thus the low-lying excitations in the U(1)-linear spin liquids are not described by free fermions. This makes a discussion of the stability of those states much more difficult.

Here we would like to concentrate on the U(1)-linear spin liquid U1Cn01n in Eq. (32). The spinons have a massless Dirac dispersion in the U(1)-linear spin liquid. First we would like to know if the massless Dirac dispersion is generic property of the U(1)-linear spin liquid, i.e., if the massless Dirac dispersion is a property shared by all the spin liquids that have the same quantum order as that in Eq. (32). The most general perturbations around the U(1)-linear *Ansatz*, Eq. (32), are given by

$$\delta u_{i,i+m} = \delta u^0_m \tau^0 + (-)^i \delta u^3_m \tau^3, \tag{149}$$

$$u_{i,i+m} = u^0_m \tau^0 + (-)^i u^3_m \tau^3,$$

$$u^{0,3}_m = 0, \quad \text{for} \quad m = \text{even},$$

$$u^0_{P_{xy}(m)} = u^0_m,$$

$$u^3_{P_{xy}(m)} = -u^3_m,$$

$$u^{0,3}_{P_x(m)} = (-)^{m_x} u^{0,3}_m,$$

$$u^{0,3}_{P_y(m)} = (-)^{m_y} u^{0,3}_m, \tag{150}$$

if the perturbations respect translation, parity, and time reversal symmetries, and if the perturbations do not break the U(1) gauge structure. Since $\delta u^3_m = \delta u^0_m = 0$ for $m=$ even, their contributions in momentum space vanish at $k=(0,0)$ and $k=(0,\pi)$. The spinon energy also vanishes at those points for the *Ansatz*, Eq. (32). Thus the massless Dirac dispersion is protected by the symmetries and the U(1) gauge structure in the U(1)-linear spin liquid, Eq. (32). In other words, the massless Dirac dispersion is protected by the quantum order in the U(1)-linear spin liquid.

Next we consider if the symmetries and the U(1) gauge structure in the U(1)-linear spin liquid can be broken spontaneously due to interactions and fluctuations at low energy. The low-energy effective theory is described by the Lagrangian (in imaginary time)

$$\mathcal{L}=\sum_{a,\mu}\psi_a^\dagger\gamma^0[v_{\mu,a}\gamma^\mu(\partial_\mu+ia_\mu)]\psi_a, \tag{151}$$

where $\mu=0,1,2$, $a=1,2$, γ^μ are 4×4 γ matrices, $v_a^0=1$, and $(v_{1,a},v_{2,a})$ are velocities for a^θ fermions in the x and y directions. We make a large-N generalization of the above effective theory and allow $a=1,2,\ldots,N$. Our first concern is about whether the self-energy from the gauge interaction can generate any mass/chemical-potential term, due to infrared divergence. It turns out that, in the $1/N$ expansion, the gauge fluctuations represent an exact marginal perturbation that does not generate any mass/chemical=potential term.[78] Instead the gauge interaction changes the quantum fixed point described by free massless Dirac fermions to a new quantum fixed point which has no free fermionic excitations at low energies.[34,78] The new quantum fixed point has gapless excitations and correlation functions all have algebraic decay. Such a quantum fixed point was called an algebraic spin liquid.[34] Actually, it is easy to understand why the gauge fluctuations represent an exact marginal perturbation. This is because the conserved current that couples to the gauge potential cannot have any anomalous dimensions. Thus, if the gauge interaction is marginal at first order, then it is marginal at all orders. The gauge interaction as an exact marginal perturbation is also supported by the following results. The gauge-invariant Green's function of ψ is found to be gapless after coupling to the gauge field, to all orders in the $1/N$ expansion.[78] Recently it was argued that the U(1) gauge interaction does not generate any mass perturbatively even when N is as small as 2.[79]

Now let us discuss other possible instabilities. First we would like consider a possible instability that change the U(1)-linear state to the Z_2-linear state. To study such an instability we add a charge-2 Higgs field to our effective theory:

$$\begin{aligned}\mathcal{L}=&\psi_a^\dagger\gamma^0[\gamma^\mu(\partial_\mu+ia_\mu)]\psi_a+|(\partial_0-2ia_0)\phi|^2\\&+v^2|(\partial_i-2ia_1)\phi|^2+V(\phi),\end{aligned} \tag{152}$$

where $V(\phi)$ has its minimum at $\phi=0$ and we have assumed $v_{1,a}=v_{2,a}=1$ for simplicity. [Note that ϕ corresponds to λ in Eq. (39). It is a nonzero λ that breaks the U(1) gauge structure down to the Z_2 gauge structure.] If after integrating out ψ and a_μ the resulting effective potential $V_{\rm eff}(|\phi|)$ has its minimum at a nonzero ϕ, then the U(1)-linear state has an instability towards the Z_2-linear state.

To calculate $V_{\rm eff}(|\phi|)$, we first integrate out ψ and get

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}a_\mu\pi_{\mu\nu}a_\nu+|[(\partial_0-2ia_0)\phi]^2\\&+v^2|(\partial_i-2ia_i)\phi|^2+V(\phi),\end{aligned} \tag{153}$$

where

$$\pi_{\mu\nu}=\frac{N}{8}(p^2)^{-1/2}(p^2\delta_{\mu\nu}-p_\mu p_\nu). \tag{154}$$

Now the effective potential $V_{\rm eff}(|\phi|)$ can be obtained by integrating out a_μ (in the $a_0=0$ gauge) and the phase θ of the ϕ field, $\phi=\rho e^{i\theta}$ (Ref. 97):

$$\begin{aligned}V_{\rm eff}(\phi)-V(\phi)=&\int_0^\infty\frac{d\omega}{\pi}\int\frac{d^2k}{(2\pi)^2}\frac{1}{2}\,\mathrm{Im}\ln\{\ln[-\mathcal{K}^\perp(i\omega)]\\&+\ln[-\mathcal{K}^\parallel(i\omega)]\}\\=&\int_0^\infty\frac{d\omega}{\pi}\int\frac{d^2k}{(2\pi)^2}\\&\times\mathrm{Im}\ln\left(-\frac{N}{8}(k^2-\omega^2-0^+)^{1/2}-4|\phi|^2\right),\end{aligned} \tag{155}$$

where

$$\mathcal{K}^\perp=\frac{N}{8}(\omega^2+k^2)^{1/2}+4|\phi|^2,$$

$$\begin{aligned}\mathcal{K}^\parallel=&\frac{N}{8}(\omega^2+k^2)^{-1/2}\omega^2+4|\phi|^2\frac{\omega^2}{\omega^2+v^2k^2}\\=&\frac{\omega^2}{\omega^2+v^2k^2}\left(\frac{N}{8}(\omega^2+k^2)^{1/2}+4|\phi|^2\right).\end{aligned} \tag{156}$$

We find that $V_{\rm eff}=V-C_1|\phi|^6\ln|\phi|$ where C_1 is a constant. Now it is clear that the gapless gauge fluctuations cannot shift the minimum of V from $\phi=0$ and the U(1)-linear state is stable against spontaneously changing into the Z_2-linear state.

So far we only considered the effects of perturbative fluctuations. The nonperturbative instanton effects can also cause instability of the algebraic spin liquid. The instanton effects have been discussed in Ref. 60 for the case $v_a^1=v_a^2$. It was found that the instanton effects represent a relevant perturbation which can destabilize the algebraic spin liquid when $N<24$. In the following, we will generalize the analysis of Ref. 60 to the $v_a^1\neq v_a^2$ case. First we rewrite

$$\begin{aligned}S=&\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}a_\mu(-k)\pi_{\mu\nu}a_\nu(k)\\=&\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}f_\mu(-k)K_{\mu\nu}f_\nu(k),\end{aligned} \tag{157}$$

where

$$f_\mu=\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{158}$$

When $\pi_{\mu\nu}=k^2\delta_{\mu\nu}-k_\mu k_\nu$, we find $K_{\mu\nu}=\delta_{\mu\nu}$. When $\pi_{\mu\nu}=(k^2\delta_{\mu\nu}-k_\mu k_\nu)/\sqrt{k^2}$, we may assume $K_{\mu\nu}=\delta_{\mu\nu}/\sqrt{k^2}$. When $v_{1,a}\neq v_{2,a}$ we have

$$(K_{\mu\nu})=\sum_a \frac{1}{8}(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2} \times\begin{pmatrix} v_{1,a}v_{2,a} & 0 & 0 \\ 0 & v_{2,a}/v_{1,a} & 0 \\ 0 & 0 & v_{1,a}/v_{2,a}\end{pmatrix}. \quad (159)$$

The instanton field f_μ minimizes the action, Eq. (157), and satisfies

$$K_{\mu\nu}f_\nu=c(k)k_\mu, \quad (160)$$

where $c(k)$ is chosen such that $k_\mu f_\mu=2i\pi$. We find that

$$f_0=\frac{8c\omega}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{1,a}v_{2,a}},$$

$$f_1=\frac{8ck_1}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{2,a}/v_{1,a}},$$

$$f_2=\frac{8ck_2}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{1,a}/v_{2,a}}, \quad (161)$$

and

$$c=2i\pi\left(\frac{8\omega^2}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{1,a}v_{2,a}} + \frac{8k_1^2}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{2,a}/v_{1,a}} + \frac{8k_2^2}{\Sigma_a(\omega^2+v_{1,a}^2k_1^2+v_{2,a}^2k_2^2)^{-1/2}v_{1,a}/v_{2,a}}\right)^{-1}. \quad (162)$$

Using the above solution, we can calculate the action for a single instanton, which has the form

$$S_{inst}=\frac{N}{2}\alpha(v_2/v_1)\ln(L), \quad (163)$$

where L is the size of the system and we have assumed that $N/2$ fermions have velocity $(v_x,v_y)=(v_1,v_2)$ and the other $N/2$ fermions have velocity $(v_x,v_y)=(v_2,v_1)$. We find $\alpha(1)=1/4+O(1/N)$ and $\alpha(0.003)=3+O(1/N)$. When $(N/2)\alpha(v_2/v_1)>3$, the instanton effect is irrelevant. We see that even for the case $N=2$, the instanton effect can be irrelevant for small enough v_2/v_1. Therefore, the algebraic spin liquid exists and can be stable.

It has been proposed that the pseudogap metallic state in underdoped high-T_c superconductors is described by the (doped) staggered flux state [the U(1)-linear state U1Cn01n] *which contains a long-range* U(1) *gauge interaction*.[33,34] From the above result, we see that, for realistic $v_2/v_1\sim 0.1$ in high-T_c superconductors, the U1Cn01n spin liquid is unstable at low energies. However, this does not mean that we cannot not use the algebraic spin liquid U1Cn01n to describe the pseudogap metallic state. It simply means that, at low temperatures, the algebraic spin liquid will change into other stable quantum states, such as superconducting state or antiferromagnetic state[80] as observed in experiments.

The unstable algebraic spin liquid can be viewed as an unstable quantum fixed point. Thus the algebraic-spin-liquid approach to the pseudogap metallic state in underdoped samples looks similar to the quantum-critical-point approach.[81,82] However, there is an important distinction between the two approaches. The quantum-critical-point approach assumes a nearby continuous phase transition that changes symmetries and strong fluctuations of local order parameters that cause the criticality. The algebraic-spin-liquid approach does not require any nearby symmetry breaking state and there is no local order parameter to fluctuate.

E. Quantum order and the stability of spin liquids

After introducing quantum orders and the PSG, we can have a deeper discussion of the stability of mean-field states. The existence of the algebraic spin liquid is a very striking phenomenon, since gapless excitations interact down to zero energy and cannot be described by free fermions or free bosons. According to a conventional wisdom, if bosons and fermions interact at low energies, the interaction will open an energy gap for those low-lying excitations. This implies that a system either has free bosonic and fermionic excitations at low energies or has no low-energy excitations at all. According to the discussion in Sec. VIII, such conventional wisdom is incorrect. But it nevertheless raises an important question: what protects gapless excitations (in particular when they interact at all energy scales). There should be a ''reason'' or ''principle'' for the existence of the gapless excitations. Here we would like to propose that *it is the quantum order that protects the gapless excitations*. We would like to stress that gapless excitations in the Fermi spin liquids and in the algebraic spin liquids exist even without any spontaneous symmetry breaking and they are not protected by symmetries. The existence of gapless excitations without symmetry breaking is a truly remarkable feature of quantum-ordered states. In addition to the gapless Nambu-Goldstone modes from spontaneous continuous symmetry breaking, quantum orders offer another origin for gapless excitations.

We have seen from several examples discussed in Sec. V that the quantum order (or the PSG) not only protects the zero-energy gap, it also protects certain qualitative properties of the low-energy excitations. Those properties include the linear, quadratic, or gapless dispersions, the k locations where the two-spinon energy $E_{2s}(k)$ vanishes, etc.

Since quantum order is a generic property for any quantum state at zero temperature, we expect that the existence of interacting gapless excitations is also a generic property of quantum states. We see that the algebraic state is a norm. It is the Fermi liquid state that is special.

In the following, we would like to argue that the PSG can be a stable (or universal) property of a quantum state. It is robust against perturbative fluctuations. Thus, the PSG, as a universal property, can be used to characterize a quantum phase. From the examples discussed in Secs. VIII C and

VIII D, we see that the PSG protects gapless excitations. Thus, the stability of the PSG also implies the stability of gapless excitations.

We know that a mean-field spin-liquid state is characterized by $U_{ij}=\langle \psi_i \psi_j^\dagger \rangle$. If we include perturbative fluctuations around the mean-field state, we expect U_{ij} to receive perturbative corrections δU_{ij}. Here we would like to argue that the perturbative fluctuations can only change U_{ij} in such a way that U_{ij} and $U_{ij}+\delta U_{ij}$ have the same PSG.

First we would like to note the following well known facts: the perturbative fluctuations cannot change the symmetries and the gauge structures. For example, if U_{ij} and the Hamiltonian have a symmetry, then δU_{ij} generated by perturbative fluctuations will have the same symmetry. Similarly, the perturbative fluctuations cannot generate δU_{ij} that, for example, break a U(1) gauge structure down to a Z_2 gauge structure.

Since both the gauge structure (described by the IGG) and the symmetry are part of the PSG, it is reasonable to generalize the above observation by saying that not only the IGG and the symmetry in the PSG cannot be changed: the whole PSG cannot be changed by the perturbative fluctuations. In fact, the mean-field Hamiltonian and the mean-field ground state are invariant under the transformations in the PSG. Thus in a perturbative calculation around a mean-field state, the transformations in the PSG behave just like symmetry transformations. Therefore, the perturbative fluctuations can only generate δU_{ij} that are invariant under the transformations in the PSG.

Since the perturbative fluctuations (by definition) do not change the phase, U_{ij} and $U_{ij}+\delta U_{ij}$ describe the same phase. In other words, we can group U_{ij} into classes (which are called universality classes) such that U_{ij} in each class are connected by the the perturbative fluctuations and describe the same phase. We see that if the above argument is true, then the universality classes are classified by the PSG's (or quantum orders).

We would like to point out that we have assumed the perturbative fluctuations to have no infrared divergence in the above discussion. The infrared divergence implies the perturbative fluctuations to be relevant perturbations, which cause phase transitions.

IX. RELATION TO PREVIOUSLY CONSTRUCTED SPIN LIQUIDS

Since the discovery of high-T_c superconductor in 1987, many spin liquids were constructed. After classifying and constructing a large class of spin liquids, we would like to understand the relation between the previously constructed spin liquids and the spin liquids constructed in this paper.

Anderson, Baskaran, and Zou[14–16] first used the slave-boson approach to construct a uniform RVB state. The uniform RVB state is a symmetric spin liquid which has all the symmetries of the lattice. It is a SU(2)-gapless state characterized by the PSG SU2An0. Later two more spin liquids were constructed using the same U(1) slave-boson approach. One is the π-flux phase and the other is the staggered-flux/d-wave state.[31,32,83] The π-flux phase is a SU(2)-linear symmetric spin liquid characterized by PSG SU2Bn0. The staggered-flux/d-wave state is a U(1)-linear symmetric spin liquid characterized by PSG U1Cn01n. The U1Cn01n state was found to be the mean-field ground for underdoped samples. Upon doping the U1Cn01n state becomes a metal with a pseudogap at high temperatures and a d-wave superconductor at low temperatures.

It is amazing to see that the slave-boson approach, which is regarded as a very unreliable approach, predicted the d-wave superconducting state 5 years before its experimental confirmation.[21,84–86] Maybe predicting the d-wave superconductor is not a big deal. After all, the d-wave superconductor is a commonly known state and the paramagnon approach[87,88] predicted the d-wave superconductor before the slave-boson approach. However, what is really a big deal is that the slave-boson approach also predicted the pseudogap metal which is a completely new state of matter. It is very rare in condensed matter physics to predict a new state of matter before experiments.

The above U(1) and SU(2) spin liquids are likely to be unstable at low energies and may not appear as the ground states of spin systems. The first known stable spin liquid is the chiral spin liquid.[5,6] It has a true spin-charge separation. The spinons and holons carry fractional statistics. Such a state breaks the time reversal and parity symmetries and is a SU(2)-gapped state. The SU(2) gauge fluctuations in the chiral spin state does not cause any instability since the gauge fluctuations are suppressed and become massive due to the Chern-Simons term. Due to the broken time reversal and parity symmetries, the chiral spin state does not fit within our classification scheme.

Spin liquids can also be constructed using the slave-fermion/σ-model approach.[41,42] Some gapped spin liquids were constructed using this approach.[43,44] Those states turn out to be Z_2 spin liquids. But they are not symmetric spin liquids since the 90° rotation symmetry is broken. Thus they do not fit within our classification scheme. Later, a Z_2-gapped symmetric spin liquid was constructed using the SU(2) slave-boson approach [or the SU(2) projective construction].[38] The PSG for such a state is Z2Axx0z. Recently, another Z_2 state was constructed using slave-boson approach.[63,64] It is a Z_2-linear-symmetric spin liquid. Its PSG is given by Z2A0013. New Z_2 spin liquids were also obtained recently using the slave-fermion/σ-model approach.[46] It appears that most of those states break certain symmetries and are not symmetric spin liquids. We would like to mention that Z_2 spin liquids have a nice property that they are stable at low energies and can appear as the ground states of spin systems.

Many spin liquids were also obtained in quantum dimer model[47–51] and in various numerical approaches.[52–55] It is hard to compare those states with the spin liquids constructed here. This is because either the spectrum of spin-1 excitations was not calculated or the model has a very different symmetry than the model discussed here. We need to generalize our classification to models with different symmetries so that we can have a direct comparison with those interesting results and with the nonsymmetric spin liquids obtained in the slave-fermion/σ-model approach. In the quantum

dimer model and in numerical approaches, we usually know the explicit form of the ground-state wave function. However, at this moment, we do not know how to obtain the PSG from the ground-wave function. Thus, knowing the explicit ground-state wave function does not help us to obtain the PSG. We see that it is important to understand the relation between the ground-state wave function and PSG so that we can understand quantum order in the states obtained in numerical calculations.

X. SUMMARY OF THE MAIN RESULTS

In the following we will list the main results obtained in this paper. The summary also serves as a guide of the whole paper.

(1) A concept of quantum order is introduced. The quantum order describes the orders in zero-temperature quantum states. The opposite of quantum order—classical order describes the orders in finite-temperature classical states. Mathematically, the quantum order characterizes universality classes of complex ground-state wave functions. It is richer than the classical order that characterizes the universality classes of positive distribution functions. Quantum orders cannot be completely described by symmetries and order parameters. Landau's theory of orders and phase transitions does not apply to quantum orders (see Sec. I A).

(2) Projective symmetry group is introduced to describe different quantum orders. It is argued that the PSG is a universal property of a quantum phase. The PSG extends the symmetry group description of classical orders and can distinguish different quantum orders with the same symmetries (see Secs. IV A and VIII E).

(3) As an application of the PSG description of quantum phases, we propose the following principle that governs the continuous phase transition between quantum phases. Let PSG_1 and PSG_2 be the PSG's of the two quantum phases on the two sides of a transition, and PSG_{cr} be the PSG that describes the quantum critical state. Then $\mathrm{PSG}_1 \subseteq \mathrm{PSG}_{cr}$ and $\mathrm{PSG}_2 \subseteq \mathrm{PSG}_{cr}$. We note that the two quantum phases may have the same symmetry and continuous quantum phase transitions are possible between quantum phases with same symmetry.[67] The continuous transitions between different mean-field symmetric spin liquids are discussed in Sec. V and in Ref. 66 which demonstrate the above principle. However, for continuous transitions between mean-field states, we have an additional condition $\mathrm{PSG}_1 = \mathrm{PSG}_{cr}$ or $\mathrm{PSG}_2 = \mathrm{PSG}_{cr}$.

(4) With the help of the PSG, we find that, within the SU(2) mean-field slave-boson approach, there are 4 symmetric SU(2) spin liquids and infinite many symmetric U(1) spin liquids. There are at least 103 and at most 196 symmetric Z_2 spin liquids. Those symmetric spin liquids have translation, rotation, parity, and time reversal symmetries. Although the classifications are done for the mean-field states, they apply to real physical spin liquids if the corresponding mean-field states turn out to be stable against fluctuations (see Sec. IV).

(5) The stability of mean-field spin liquid states is discussed in detail. We find many gapless mean-field spin liquids to be stable against quantum fluctuations. They can be stable even in the presence of long-range gauge interactions. In that case the mean-field spin liquid states become algebraic spin liquids where the gapless excitations interact down to zero energy (see Sec. VIII).

(6) The existence of algebraic spin liquids is a striking phenomenon since there is no spontaneous broken symmetry to protect the gapless excitations. There should be a ''principle'' that prevents the interacting gapless excitations from opening an energy gap and makes the algebraic spin liquids stable. We propose that quantum order is such a principle. To support our idea, we showed that just like the symmetry group of a classical state determines the gapless Nambu-Goldstone modes, the PSG of a quantum state determines the structure of gapless excitations. The gauge group of the low-energy gauge fluctuations is given by the IGG, a subgroup of the PSG. The PSG also protects massless Dirac fermions from gaining a mass due to radiative corrections. We see that the stabilities of algebraic spin liquids and Fermi spin liquids are protected by their PSG's. The existence of gapless excitations (the gauge bosons and gapless fermions) without symmetry breaking is a truly remarkable feature of quantum-ordered states. The gapless gauge and fermion excitations originate from the quantum orders, just like the phonons originate from translation symmetry breaking [see Secs. VIII C, VIII D, and VIII E and discussions below Eq. (49)].

(7) Many Z_2 spin liquids are constructed. Their low-energy excitations are described by free fermions. Some Z_2 spin liquids have gapless excitations and others have a finite energy gap. For those gapless Z_2 spin liquids some have Fermi surface while others have only Fermi points. The spinon dispersion near the Fermi points can be linear $E \propto |k|$ (which gives us Z_2-linear spin liquids) or quadratic $E \propto k^2$ (which gives us Z_2-quadratic spin liquids). In particular, we find there can be many Z_2-linear spin liquids with different quantum orders. All those different Z_2-linear spin liquids have nodal spinon excitations (see Secs. III and V).

(8) Many U(1) spin liquids are constructed. Some U(1) spin liquids have gapless excitations near the isolated Fermi point with a linear dispersion. Those U(1) linear states can be stable against quantum fluctuations. Due to long-range U(1) gauge fluctuations, the gapless excitations interact at low energies. The U(1)-linear spin liquids can be concrete realizations of algebraic spin liquids[34,68] (see Secs. III and V).

(9) Spin liquids with the same symmetry and different quantum orders can have continuous phase transitions between them. Those phase transitions are very similar to the continuous topological phase transitions between quantum Hall states.[67,89–91] We find that, at the mean-field level, the U1Cn01n spin liquid in Eq. (32) (the staggered flux phase) can continuously change into 8 different symmetric Z_2 spin liquids. The SU2An0 spin liquid in Eq. (30) (the uniform RVB state) can continuously change into 12 symmetric U(1) spin liquids and 52 symmetric Z_2 spin liquids. The SU2Bn0 spin liquid in Eq. (31) (the π-flux phase) can continuously change into 12 symmetric U(1) spin liquids and 58 symmetric Z_2 spin liquids (see Ref. 66).

(10) We show that spectrum of spin-1 excitations (i.e., the two-spinon spectrum), which can be probed in neutron scattering experiments, can be used to measure quantum orders.

The gapless points of the spin-1 excitations in the U1Cn01n (the staggered-flux) state are always at $k=(\pi,\pi)$, (0,0), $(\pi,0)$, and $(0,\pi)$. In the pseudogap metallic phase of underdoped high-T_c superconductors, the observed splitting of the neutron scattering peak $(\pi,\pi)\rightarrow(\pi\pm\delta,\pi),(\pi,\pi\pm\delta)$ (Refs. 30 and 69–75) or $(\pi,\pi)\rightarrow(\pi+\delta,\pi-\delta),(\pi-\delta,\pi+\delta)$ (Refs. 28 and 76) at low energies indicates a transition of the U1Cn01n state into a state with a different quantum order, *if we can indeed identify the scattering peak as the gapless node*. None of the eight symmetric Z_2 spin liquids in the neighborhood of the U1Cn01n state can explain the splitting pattern. Thus we might need to construct a new low-energy state to explain the splitting. This illustrates that detailed neutron scattering experiments are powerful tools in detecting quantum orders and studying transitions between quantum orders (see Sec. VII).

(11) The mean-field phase diagram, Fig. 8, for a J_1-J_2 spin system is calculated. (Only translation symmetric states are considered.) We find four mean-field ground states as we change J_2/J_1: the π-flux state (the SU2An0 state), the chiral spin state [an SU(2)-gapped state], the U(1)-linear state in Eq. (142) which breaks 90° rotation symmetry, and the $SU(2)\times SU(2)$-linear state in Eq. (141). We also find several locally stable mean-field states: the U(1)-gapped state U1Cn00x in Eq. (122) and two Z_2-linear states Z2Azz13 in Eq. (104) and Z2A0013 in Eq. (103). Those spin liquids have a better chance to appear in underdoped high-T_c superconductors. The Z2A0013 Z_2-linear state has a spinon dispersion very similar to the electron dispersion observed in underdoped samples. The spinon dispersion in the Z2Azz13 Z_2-linear state may also be consistent with electron dispersion in underdoped samples. We note that the two-spinon spectrum for the two Z_2-linear states have some qualitative differences [see Figs. 10(a) and 10(b) and note the positions of the nodes]. Thus we can use neutron scattering to distinguish the two states (see Sec. VI).

Next we list some remarks and comments that may clarify certain confusing points and help to avoid possible misunderstanding.

(a) Gauge structure is simply a redundant labeling of quantum states. The ''gauge symmetries'' (referring different labels of same physical state give rise to the same result) are not symmetries and can never be broken [see the discussion below Eq. (15)].

(b) The gauge structures referred to in this paper [such as in Z_2, U(1), or SU(2) spin liquids] are ''low-energy'' gauge structures. They are different from the ''high-energy'' gauge structures that appear in Z_2, U(1), and SU(2) slave-boson approaches. The ''low-energy'' gauge structures are properties of the quantum orders in the ground state of a spin system. The ''high-energy'' gauge structure is a particular way of writing down the Hamiltonian of spin systems. The two kinds of gauge structures have nothing to do with each other (see discussions at the end of Sec. I C and at the end of Sec. IV A).

(c) There are (at least) two different interpretations of spin-charge separation. The first interpretation (pseudo spin-charge separation) simply means that the low-energy excitations cannot be described by electronlike quasiparticles. The second interpretation (true spin-charge separation) means the existence of free spin-1/2 neutral quasiparticles and spin-0 charged quasiparticles. In this paper both interpretations are used. The algebraic spin liquids have a pseudo spin-charge separation. The Z_2 and chiral spin liquids have a true spin-charge separation (see Sec. I C).

(d) Although in this paper we stress that quantum orders can be characterized by the PSG's, we need to point out that the PSG's do not completely characterize quantum orders. Two different quantum orders may be characterized by the same PSG. As an example, we have seen that the *Ansatz*, Eq. (122), can be a U(1)-linear state or a U(1)-gapped state depending on the values of parameters in the *Ansatz*. Both states are described by the same PSG U1Cn00x. Thus the PSG cannot distinguish the different quantum orders carried by the U(1)-linear state and the U(1)-gapped state.

(e) The unstable spin liquids can be important in understanding the finite-temperature states in high-T_c superconductors. The pseudogap metallic state in underdoped samples is likely to be described by the unstable U1Cn01n algebraic spin liquid (the staggered flux state) which contains a long-range U(1) gauge interaction[33,34] (see discussions at the end of Sec. VIII).

(f) Although we have been concentrated on the characterization of stable quantum states, quantum order and the PSG characterization can also be used to describe the internal order of quantum critical states. Here we define ''quantum critical states'' as states that appear at the continuous phase transition points between two states with different symmetries or between two states with different quantum orders (but the same symmetry). We would like to point out that ''quantum critical states'' thus defined are more general than ''quantum critical points.'' ''Quantum critical points,'' by definition, are the continuous phase transition points between two states with different symmetries. The distinction is important. ''Quantum critical points'' are associated with broken symmetries and order parameters. Thus the low-energy excitations at ''quantum critical points'' come from the strong fluctuations of order parameters. While ''quantum critical states'' may not be related to broken symmetries and order parameters. In that case it is impossible to relate the gapless fluctuations in a ''quantum critical state'' to fluctuations of an order parameter. The unstable spin liquids mentioned in (e) can be more general quantum critical states. Since some finite temperature phases in high-T_c superconductors may be described by quantum critical states or stable algebraic spin liquids, their characterization through quantum order and PSG's is useful for describing those finite-temperature phases.

(g) In this paper, we only studied quantum orders and topological orders at zero temperature. However, we would like to point out that topological orders and quantum orders may also apply to finite-temperature systems. The quantum effect can be important even at finite temperatures. In Ref. 13, a dimension index (DI) is introduced to characterize the robustness of the ground-state degeneracy of a topologically ordered state. We find that if $\mathrm{DI}\leqslant 1$, topological orders cannot exist at finite temperature. However, if $\mathrm{DI}>1$, topologi-

cal order can exist at finite temperatures and one expect a finite-temperature phase transition without any change of symmetry. Topological orders in FQH states have DI=1, and they cannot exist at finite temperatures. The topological order in 3D superconductors has DI=2. Such a topological order can exist at finite temperatures, and we have a continuous finite-temperature superconductor-metal transition that does not change any symmetry.

(h) If we regard a down-spin as an empty site and an up-spin as a site occupied by a boson, then our spin system can be viewed as an interacting boson system. The gapless spin liquids studied here are examples of boson metals that exist at zero temperature.

Although we mainly discussed quantum orders in 2D spin systems, the concept of quantum order is not limited to 2D spin systems. The concept applies to any quantum systems in any dimensions. Actually, a superconductor is the simplest example of a state with nontrivial quantum order if the dynamical electromagnetic fluctuations are included. A superconductor breaks no symmetries and cannot be characterized by order parameters. An *s*-wave and a *d*-wave superconductor, having the same symmetry, are distinguished only by their different quantum orders. The gapless excitations in a *d*-wave superconductor are not produced by broken symmetries, but by quantum orders. We see that a superconductor has many properties characteristic of quantum-ordered states, and it is a quantum-ordered state. The quantum orders in the superconducting states can also be characterized using PSG's. The IGG $\mathcal{G}=Z_2$ if the superconducting state is caused by electron-pair condensation, and the IGG $\mathcal{G}=Z_4$ if the superconducting state is caused by four-electron-cluster condensation. The different quantum orders in an *s*-wave and a *d*-wave superconductor can be distinguished by their different PSG's. The *Ansatz* of the *s*-wave superconductor is invariant under the 90° rotation, while the *Ansatz* of the *d*-wave superconductor is invariant under the 90° rotation followed by gauge transformations $c_i \to \pm e^{i\pi/2} c_i$.

It would be interesting to study quantum orders in 3D systems. In particular, it is interesting to find out the quantum order that describes the physical vacuum that we all live in. The existence of light—a massless excitation—without any sign of spontaneous symmetry breaking suggests that our vacuum contains a nontrivial quantum order that protect the massless photons. Thus quantum order provides an origin of light.[58]

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APPENDIX A: GENERAL CONDITIONS ON PROJECTIVE SYMMETRY GROUPS

The transformations in a symmetry group satisfy various algebraic relations so that they form a group. Those algebraic relations lead to conditions on the elements of the PSG. Solving those conditions for a given symmetry group and a given IGG allows us to find possible extensions of the symmetry group or, in another word, to find possible PSG's associated with the symmetry group. In Sec. IV A, we have seen that the relation $T_x T_y T_x^{-1} T_y^{-1} = 1$ between translations in the x and y directions leads to the condition

$$\begin{aligned} &G_x T_x G_y T_y (G_x T_x)^{-1} (G_y T_y)^{-1} \\ &\quad = G_x T_x G_y T_y T_x^{-1} G_x^{-1} T_y^{-1} G_y^{-1} \in \mathcal{G}, \end{aligned} \tag{A1}$$

or

$$G_x(i) G_y(i-\hat{x}) G_x^{-1}(i-\hat{y}) G_y(i)^{-1} \in \mathcal{G} \tag{A2}$$

on elements $G_x T_x$ and $G_y T_y$ of the PSG. Here $\mathcal{G}$ is the IGG. This condition allows us to determine that there are only two different extensions [given by Eq. (61) and Eq. (62)] for the translation group generated by T_x and T_y, if $\mathcal{G}=Z_2$.

However, a bigger symmetry group can have many more extensions. In the following we are going to consider PSG's for the symmetry group generated by two translations $T_{x,y}$, three parity transformations $P_{x,y,xy}$, and the time reversal transformation T. Since translations and the time reversal transformation commute, we have

$$\begin{aligned} &(G_x T_x)^{-1} (G_T T)^{-1} G_x T_x G_T T \in \mathcal{G}, \\ &(G_y T_y)^{-1} (G_T T)^{-1} G_y T_y G_T T \in \mathcal{G}, \end{aligned} \tag{A3}$$

which reduces to the following two conditions on $G_{x,y}(i)$ and $G_T(i)$:

$$\begin{aligned} &G_x^{-1}(i) G_T^{-1}(i) G_x(i) G_T(i-\hat{x}) \in \mathcal{G}, \\ &G_y^{-1}(i) G_T^{-1}(i) G_y(i) G_T(i-\hat{y}) \in \mathcal{G}. \end{aligned} \tag{A4}$$

Since $T^{-1} P_x^{-1} T P_x = 1$, $T^{-1} P_y^{-1} T P_y = 1$, and $T^{-1} P_{xy}^{-1} T P_{xy} = 1$, one can also show that

$$\begin{aligned} &G_T^{-1}(P_x(i)) G_{P_x}^{-1}(i) G_T(i) G_{P_x}(i) \in \mathcal{G}, \\ &G_T^{-1}(P_y(i)) G_{P_y}^{-1}(i) G_T(i) G_{P_y}(i) \in \mathcal{G}, \\ &G_T^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) G_T(i) G_{P_{xy}}(i) \in \mathcal{G}. \end{aligned} \tag{A5}$$

From the relation between the translations and the parity transformations, $T_x P_x^{-1} T_x P_x = T_y^{-1} P_x^{-1} T_y P_x = T_y P_y^{-1} T_y P_y = T_x^{-1} P_y^{-1} T_x P_y = T_y^{-1} P_{xy}^{-1} T_x P_{xy} = T_x^{-1} P_{xy}^{-1} T_y P_{xy} = 1$, we find that

$$\begin{aligned} &(G_x T_x)(G_{P_x} P_x)^{-1} G_x T_x G_{P_x} P_x \in \mathcal{G}, \\ &(G_y T_y)^{-1} (G_{P_x} P_x)^{-1} G_y T_y G_{P_x} P_x \in \mathcal{G}, \\ &(G_y T_y)(G_{P_y} P_y)^{-1} G_y T_y G_{P_y} P_y \in \mathcal{G}, \\ &(G_x T_x)^{-1} (G_{P_y} P_y)^{-1} G_x T_x G_{P_y} P_y \in \mathcal{G}, \end{aligned} \tag{A6, A7}$$

$$(G_y T_y)^{-1}(G_{P_{xy}} P_{xy})^{-1} G_x T_x G_{P_{xy}} P_{xy} \in \mathcal{G},$$

$$(G_x T_x)^{-1}(G_{P_{xy}} P_{xy})^{-1} G_y T_y G_{P_{xy}} P_{xy} \in \mathcal{G}, \qquad \text{(A8)}$$

or

$$G_x(P_x(i)) G_{P_x}^{-1}(i+\hat{x}) G_x(i+\hat{x}) G_{P_x}(i) \in \mathcal{G},$$

$$G_y^{-1}(P_x(i)) G_{P_x}^{-1}(i) G_y(i) G_{P_x}(i-\hat{y}) \in \mathcal{G}, \qquad \text{(A9)}$$

$$G_y(P_y(i)) G_{P_y}^{-1}(i+\hat{y}) G_y(i+\hat{y}) G_{P_y}(i) \in \mathcal{G},$$

$$G_x^{-1}(P_y(i)) G_{P_y}^{-1}(i) G_x(i) G_{P_y}(i-\hat{x}) \in \mathcal{G}, \qquad \text{(A10)}$$

$$G_y^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) G_x(i) G_{P_{xy}}(i-\hat{x}) \in \mathcal{G},$$

$$G_y^{-1}(P_{xy}(i)) G_{P_{xy}}^{-1}(i) G_x(i) G_{P_{xy}}(i-\hat{y}) \in \mathcal{G}, \qquad \text{(A11)}$$

We also have $P_{xy} P_x P_{xy} P_y^{-1} = P_y P_x P_y^{-1} P_x^{-1} = 1$. Thus

$$G_{P_{xy}} P_{xy} G_{P_x} P_x G_{P_{xy}} P_{xy} (G_{P_y} P_y)^{-1} \in \mathcal{G},$$

$$G_{P_y} P_y G_{P_x} P_x (G_{P_x} P_y)^{-1} (G_{P_x} P_x)^{-1} \in \mathcal{G}, \qquad \text{(A12)}$$

which implies

$$G_{P_{xy}}(i) G_{P_x}(P_{xy}(i)) G_{P_{xy}}(P_{xy} P_x(i)) G_{P_y}^{-1}(i) \in \mathcal{G},$$

$$G_{P_y}(i) G_{P_x}(P_y(i)) G_{P_y}^{-1}(P_x(i)) G_{P_x}^{-1}(i) \in \mathcal{G}. \qquad \text{(A13)}$$

The fact that $T^2 = 1$ leads to the condition

$$G_T^2(i) \in \mathcal{G}, \qquad \text{(A14)}$$

and $P_x^2 = P_y^2 = P_{xy}^2 = 1$ leads to

$$G_{P_x}(i) G_{P_x}(P_x(i)) \in \mathcal{G},$$

$$G_{P_y}(i) G_{P_y}(P_y(i)) \in \mathcal{G},$$

$$G_{P_{xy}}(i) G_{P_{xy}}(P_{xy}(i)) \in \mathcal{G}. \qquad \text{(A15)}$$

The above conditions completely determine the PSG's. The solutions of the above equations for $\mathcal{G} = Z_2$, U(1), and SU(2) allow us to obtain PSG's for Z_2, U(1), and SU(2) spin liquids. However, we would like to point out that the above conditions define the so-called algebraic PSG's, which are somewhat different from the invariant PSG defined in Sec. IV A. More precisely, an algebraic PSG is defined for a given IGG and a given symmetry group SG. It is a group equipped with a projection P and satisfies the following conditions:

$$\text{IGG} \subset \text{PSG}, \quad P(\text{PSG}) = \text{SG},$$

$$P(gu) = P(u), \quad \text{for any } u \in \text{PSG} \text{ and } g \in \text{IGG}. \qquad \text{(A16)}$$

It is clear that an invariant PSG is always an algebraic PSG. However, some algebraic PSG's are not invariant PSG's. This is because a generic *Ansatz* u_{ij} that is invariant under an algebraic PSG may be invariant under a larger invariant PSG. If we limit ourselves to spin liquids constructed using u_{ij}, then an algebraic PSG characterizes a mean-field spin liquid only when it is also an invariant PSG at the same time.

We would like to remark that the definition of invariant PSG can be generalized. In Sec. IV A, the invariant PSG is defined as a collection of transformations that leave an *Ansatz* u_{ij} invariant. More generally, a spin liquid is not only characterized by the two-point correlation $(U_{ij})_{\alpha\beta} = \langle \psi_{\alpha i} \psi^\dagger_{\beta j} \rangle$ but also by many-point correlations such as $(U_{ijmn})_{\alpha\beta\gamma\lambda} = \langle \psi_{\alpha i} \psi_{\beta j} \psi^\dagger_{\gamma m} \psi^\dagger_{\lambda n} \rangle$. We may define the generalized invariant PSG as a collection of transformations that leave the many-point correlation invariant. It would be very interesting to see if the generalized invariant PSG coincides with the algebraic PSG.

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[93] A more precise definition of quantum order is given in Ref. 11.
[94] Landau's theory may not even be able to describe all the classical orders. Some classical phase transitions, such as the Kosterliz-Thouless transition, do not change any symmetries.
[95] In this paper we will distinguish the invariance of an *Ansatz* and the symmetry of an *Ansatz*. We say an *Ansatz* has a translation invariance when the *Ansatz* itself does not change under translation. We say an *Ansatz* has a translation symmetry when the physical spin wave function obtained from the *Ansatz* has a translation symmetry.
[96] In his unpublished study of quantum antiferromagnetism with a symmetry group of large rank, Wiegmenn (Ref. 92) constructed a gauge theory which realizes a double-valued magnetic space group. The double-valued magnetic space group extends the space group and is a special case of the protective symmetry group.
[97] We need to integrate out the phase of the ϕ field to get a gauge-invariant result.