Calculus of sea-displacement operators

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Sea-displacement operators for fermions are defined in terms of the Fermi fields in a one-component Fermi system. The main conclusions of this article fully corroborate the conjectures made in our earlier works, and provide a mathematically rigorous foundation for these earlier works. These ideas are generalized to electron-hole systems where we are able to clearly explore the nature of exciton-exciton interactions. We find that exciton-exciton interactions in an ideal model of GaAs are not adequately treated simply as of the two-body type; rather the interactions are mediated by the exchange of other bosons that are present in this system. These bosons are identified explicitly and the exciton Green function is calculated. This exercise is also intended to be a precursor to a systematic nonperturbative treatment of gauge theories.

I. INTRODUCTION

This article is meant to address some technical issues that prevented our earlier work[1] from being universally embraced, and to place some of the conjectures that appeared there on a firm mathematical foundation. Indeed as Cune and Apostol[2] pointed out in their very pertinent critique, there were significant technical drawbacks relating to the meaning of the square-root of the number operator in the denominator that cast doubt on the soundness of the physical conclusions. This article lays to rest once and for all such doubts, and as an added benefit, we are able to study the precise nature of exciton-exciton interactions. It is interesting to note that the rigorous formulation presented here is indispensable for a correct treatment of charge-conserving two-component Fermi systems such as excitons and relativistic electrons and positrons. This is somewhat surprising since in our earlier work[1] we showed that the lack of a proper definition of the sea-displacement operator does not invalidate the physical conclusions in the one-component Fermi system. Here we find that a naive approach underestimates the nature and strength of interactions between excitons but a more careful treatment brings out many subtle features that are easy to overlook. Many authors who study exciton-exciton interactions assume that excitons are bosons that interact via two-body interactions where the exciton number is conserved. Kavoulakis and Baym[3] have pointed out the need to include Auger-like processes where the exciton number is not conserved. Although assuming that excitons are bosons is totally acceptable (it is a matter of definition), there are some differences between a system such as a hydrogen atom (which interacts via two-body forces with other hydrogen atoms) and an exciton. The main difference is that in a hydrogen atom the proton and electron do not recombine leaving behind photons, where as an electron can recombine with a hole. In fact, when two excitons scatter off each other it is conceivable that nonradiative recombination processes take place in addition to radiative recombination. But in place of that, we find that an exciton can recombine with a special kind of electron-hole pair, which we call a *solitron* (to be made clear in the main text) and can create an intraband

electron-hole pair (in other words, the usual kind of sea-displacement bosons found in one-component Fermi systems) called a ''valeron'' or ''conductron'' depending upon the nature of the band. Thus we find that the two-component charge-conserving undoped Fermi system may be thought of as consisting of several kinds of elementary excitations (which are postulated to be exact bosons). The usual excitons are just one class of such excitations. Excitons can be in a bound state or in a scattering state such as that in a hydrogen atom. Furthermore, the excitons can possess a net center-of-mass momentum. These excitons possess an energy dispersion that is slightly different from the zero-center of mass excitons and are somewhat important in the analysis. Next in order of importance in our system is a solitron. A *solitron* is an electron-hole pair that resides at the bottom of the conduction and valence bands. This pair is unbound and its existence is needed in order that excitons interact with one another, especially when we are dealing with Auger-like processes where exciton number is not conserved. In addition to the solitron, we find the need to invoke two other kinds of bosons, valerons and conductrons. Valerons are intravalence band electron-hole excitations. They are analogs of the usual sea-bosons[1] in one-component Fermi systems. Conductrons similarly are particle-hole excitations in the conduction band. All these bosons interact with one another and the resulting system is completely equivalent to the interacting Fermi system. Only the exciton couples to external radiation fields. Therefore, we have to consider excitons as the primary objects of interest and the other bosons in the system are similar to gauge bosons. Material particles (excitons) interact by exchanging these other bosons. Furthermore, these gauge bosons interact amongst themselves, suggesting an analogy with nonabelian gauge theories. In future publications, we intend applying these ideas to the study of relativistic gauge theories. In this article, we compute the exciton Green function using the interaction terms that correspond to inelastic scattering of the excitons off the other bosons. In future publications, we intend to investigate more thoroughly the practical aspects of this formalism—specifically, the biexciton Green function and nonlinear optical susceptibilities.

II. CONDENSATE DISPLACEMENT OPERATORS

In this section, we provide some details regarding the correspondence between the canonical bosons and their condensate displacement analogs. These details are not found in our earlier works.[1] They are important since they provide a springboard from which we may write down the analogous statements for fermions. The correspondence between the sea-displacement operators and the canonical fermions was not made sufficiently clear in our earlier works. Indeed, there were inconsistencies in the technical definition, that even though had no impact on the physical conclusions (as we shall see in the present article once and for all) they did leave some room for doubting the soundness of the framework. Let us therefore start off with the familiar Bose systems. Let $b_{\mathbf{k}}^{\dagger}$ and $b_{\mathbf{k}}$ be canonical boson creation and annihilation operators, respectively. They obey the commutation rules $[b_{\mathbf{k}}, b_{\mathbf{k}'}] = 0$ and $[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}, \mathbf{k}'}$. Let us now introduce the following new objects known as condensate displacement operators

$$d_{\mathbf{q}/2}(\mathbf{q}) = \frac{1}{\sqrt{\hat{N}_0}} b_{\mathbf{0}}^{\dagger} b_{\mathbf{q}}, \quad \mathbf{q} \neq 0, \tag{1}$$

$$d_{\mathbf{0}}(\mathbf{0}) = 0, \tag{2}$$

where $\hat{N}_0 = b_{\mathbf{0}}^{\dagger} b_{\mathbf{0}}$ is the number operator for the zero-momentum state. A word of caution regarding notation. The symbol N^0 without a hat refers to a positive integer (a c number) that corresponds to the total number of particles in the system, in both Bose as well as Fermi systems. The symbol $\hat{N}$ refers to the operator that corresponds to the total number of particles in the system, be it Bose or Fermi. The object $\hat{\mathbf{1}} = \hat{N}/N^0$ is therefore the unit operator. The symbol $\hat{N}_0$ refers to the operator that corresponds to the total number of bosons in the zero-momentum state. This object does not appear in the next section where we deal with fermions. The square root of the operator in the denominator of Eq. (2) deserves special attention. In particular, when we act $d_{\mathbf{q}/2}^{\dagger}(\mathbf{q})$ on a state containing no particle in the zero-momentum state, we get an infinity multiplied by the same state, and when this further is acted on by $b_{\mathbf{0}}$, we get a factor of zero. Zero multiplied by infinity is indeterminate. This tells us that the condensate displacement operator is an ill-defined operator on the Fock space of bosons. This is a technical problem that cannot be wished away. We will mitigate the severity of this problem by postulating that all states of the interacting system (both ground state and excited states) may be expressed as linear combinations of states from a restricted Hilbert space that contains states of the noninteracting system with a fixed total number of particles, but excludes those that contain no particles in the zero-momentum state. Although we are unable to say precisely when this assumption breaks down, it is reasonable to assert that even in the case when the interactions are strong, either because of the intrinsic nature of the interaction or apparently strong due to the dimensionality of the system, the zero-momentum state of the interacting system will have at least one boson in it, if not a macro-

scopically large number of them. (The number operator is no longer a good quantum number in such systems but perhaps this is true in some average sense with small fluctuations around the average.) The subsequent development will justify this. Our assumption enables us to write down a polar decomposition of the operator $b_{\mathbf{0}}$ as $b_{\mathbf{0}} = \exp(-iX_0^r)\sqrt{\hat{N}_0}$ where $X_0^r = (1/2)(X_0 + X_0^{\dagger})$ is an operator that is strictly self-adjoint but it is not exactly but almost a canonical conjugate to the number operator $\hat{N}_0 = b_{\mathbf{0}}^{\dagger} b_{\mathbf{0}} \geqslant 0$ (in Appendix A we see that an appropriate interpretation of the definition of the conjugate allows us to circumvent these issues, for fermions we have to be more careful). On the other hand, X_0 is almost self-adjoint but is strictly a conjugate to the number operator $\hat{N}_0$ (however, in the next section when we deal with fermions, X_0 refers to the canonical conjugate of the *total* number of *fermions*). If a definition of X_0^r is desired, we may claim that it is given by the manifestly self-adjoint formula

$$X_0^r = \frac{i}{2} \ln\left(b_{\mathbf{0}} \frac{1}{\sqrt{\hat{N}_0}} \right) - \frac{i}{2} \ln\left(\frac{1}{\sqrt{\hat{N}_0}} b_{\mathbf{0}}^{\dagger} \right). \tag{3}$$

In addition to the square root of the number operator in the denominator that we have already made sense of, we have to make sense of the logarithm. It is defined to be the power-series expansion suggested by rewriting the above formula as

$$X_0^r = \frac{i}{2} \ln\left[1 + (b_{\mathbf{0}} - \sqrt{\hat{N}_0}) \frac{1}{\sqrt{\hat{N}_0}} \right]$$

$$- \frac{i}{2} \ln\left[1 + \frac{1}{\sqrt{\hat{N}_0}} (b_{\mathbf{0}}^{\dagger} - \sqrt{\hat{N}_0}) \right]. \tag{4}$$

The proof of the fact that X_0 is canonically conjugate to the number operator $\hat{N}_0$ in the restricted Hilbert space is relegated to Appendix A. Therefore we may write $[X_0, \hat{N}_0] = i$. Armed with these facts we now are able to prove that $d_{\mathbf{q}/2}(\mathbf{q})$ is a canonical boson annihilation operator. We may now write $d_{\mathbf{q}/2}(\mathbf{q}) = \exp(iX_0^r) b_{\mathbf{q}}$. Since from the definition, $\exp(iX_0^r)$ depends on neither $b_{\mathbf{q}}$ nor $b_{\mathbf{q}}^{\dagger}$, we see that as far as commutation rules of $d_{\mathbf{q}/2}(\mathbf{q})$ go, they are identical to those of $b_{\mathbf{q}}$ since the two differ by a trivial phase that commutes with both these objects and their Hermitian conjugates. Next, we reproduce some facts that have been proved satisfactorily[1] elsewhere, and these will be used as the point of departure for a rigorous treatment of fermions. Earlier we proved[1] (for both $\mathbf{q} = 0$ and $\mathbf{q} \neq 0$), the following combined formula:

$$b_{\mathbf{k}+\mathbf{q}/2}^{\dagger} b_{\mathbf{k}-\mathbf{q}/2} = \hat{N}_0 \delta_{\mathbf{k},0} \delta_{\mathbf{q},0} + \sqrt{\hat{N}_0} d_{-\mathbf{q}/2}(-\mathbf{q}) \delta_{\mathbf{k}+\mathbf{q}/2,0}$$

$$+ d_{\mathbf{q}/2}^{\dagger}(\mathbf{q}) \sqrt{\hat{N}_0} \delta_{\mathbf{k}-\mathbf{q}/2,0} + d_{(1/2)(\mathbf{k}+\mathbf{q}/2)}^{\dagger}$$

$$\times (\mathbf{k}+\mathbf{q}/2) d_{(1/2)(\mathbf{k}-\mathbf{q}/2)}(\mathbf{k}-\mathbf{q}/2), \tag{5}$$

$$\hat{N}_0 = \hat{N} - \sum_{\mathbf{q} \neq 0} d_{\mathbf{q}/2}^{\dagger}(\mathbf{q}) d_{\mathbf{q}/2}(\mathbf{q}). \tag{6}$$

This equation may be rewritten more elaborately and in a manner that is conducive to generalization as ($\mathbf{q} \neq 0$)

$$b^\dagger_{\mathbf{k}+\mathbf{q}/2} b_{\mathbf{k}-\mathbf{q}/2} = \sqrt{n_{\mathbf{k}+\mathbf{q}/2}} A_{\mathbf{k}}(-\mathbf{q}) + A^\dagger_{\mathbf{k}}(\mathbf{q}) \sqrt{n_{\mathbf{k}-\mathbf{q}/2}}$$

$$+ \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A^\dagger_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$- \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A^\dagger_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1),$$

$$(7)$$

$$n_{\mathbf{k}} = b^\dagger_{\mathbf{k}} b_{\mathbf{k}} = n_B(\mathbf{k}) \frac{\hat{N}}{N^0} + \sum_{\mathbf{q}_1 \neq 0} A^\dagger_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1)$$

$$- \sum_{\mathbf{q}_1 \neq 0} A^\dagger_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1).$$

$$(8)$$

Here $n_B(\mathbf{k})$ is the momentum distribution of noninteracting bosons at zero temperature, $n_B(\mathbf{k}) = \delta_{\mathbf{k},0} N^0$. Also, $A_{\mathbf{k}}(\mathbf{q}) = \delta_{\mathbf{k}-\mathbf{q}/2,0} d_{\mathbf{q}/2}(\mathbf{q})$. It is worthwhile to consider an alternative scheme for making sense of the definition in Eq. (2). It involves writing the number operator as

$$\hat{N}_0 = N^0 \left(\hat{\mathbf{1}} - \frac{1}{N^0} \sum_{\mathbf{q} \neq 0} d^\dagger_{\mathbf{q}/2}(\mathbf{q}) d_{\mathbf{q}/2}(\mathbf{q}) \right). \quad (9)$$

Furthermore, we may write for the occupation number not in the zero-momentum state ($\mathbf{k} \neq 0$) as $n_{\mathbf{k}} = d^\dagger_{(1/2)\mathbf{k}}(\mathbf{k}) d_{(1/2)\mathbf{k}}(\mathbf{k})$. With these identifications, we may rewrite Eq. (2) as ($\mathbf{q} \neq 0$),

$$d_{\mathbf{q}/2}(\mathbf{q}) = \frac{1}{\sqrt{N^0}} \left(\hat{\mathbf{1}} - \frac{1}{N^0} \sum_{\mathbf{q} \neq 0} d^\dagger_{\mathbf{q}/2}(\mathbf{q}) d_{\mathbf{q}/2}(\mathbf{q}) \right)^{-1/2} b^\dagger_{\mathbf{0}} b_{\mathbf{q}}. \quad (10)$$

If Eq. (10) is interpreted as a power-series expansion around unity, we can construct an iterative procedure to solve for $d_{\mathbf{q}/2}(\mathbf{q})$. This may seem redundant given the fact that we have already made an elegant argument that pins down the meaning of $d_{\mathbf{q}/2}(\mathbf{q})$ in terms of X_0. The reason for this new approach is that in the case of fermions we will not have the luxury of introducing an object similar to X_0 for reasons that will become clear in the next section. Thus we are forced to seek alternatives that are more fermion friendly. Unfortunately, these alternatives do not allow us to venture very far from the noninteracting case, as we shall soon see. Nevertheless, this exercise is very instructive since it tells us that the correspondence that we write down for fermions in the next section has exactly the same features and are therefore correct, pending the resolution of the interpretation of the ubiquitous square root in the denominator. Retaining only the lowest order in the series expansion gives us

$$d_{\mathbf{q}/2}(\mathbf{q}) = \frac{1}{\sqrt{N^0}} b^\dagger_{\mathbf{0}} b_{\mathbf{q}}, \quad (11)$$

$$\hat{N}_0 = N^0 \hat{\mathbf{1}}. \quad (12)$$

Further we have ($\mathbf{k} \neq 0$) $n_{\mathbf{k}} = d^\dagger_{(1/2)\mathbf{k}}(\mathbf{k}) d_{(1/2)\mathbf{k}}(\mathbf{k}) = (1/N^0)(\hat{\mathbf{1}} + \hat{N}_0) b^\dagger_{\mathbf{k}} b_{\mathbf{k}}$. But we also know that $n_{\mathbf{k}} = b^\dagger_{\mathbf{k}} b_{\mathbf{k}}$. This seems to suggest one of two things, namely, $n_{\mathbf{k}} = 0$ or we should restrict ourselves to cases when $\hat{N}_0 = N^0 \hat{\mathbf{1}} \gg \hat{\mathbf{1}}$. The latter possibility seems the most attractive until one realizes that the restricted Hilbert space is not so restrictive as to exclude such systems where the number of particles is small and finite. This will become clear in Appendix A where we need to only assume that the restricted Hilbert space contains no states that have zero particles in the zero-momentum state. Therefore we are left with the other possibility $n_{\mathbf{k}} = 0$. This result is not as alarming as it seems since Eq. (12) is consistent only with the state where all the bosons are in the zero-momentum state and no bosons have higher momenta. Further iterations do not change this picture. That is, if we interpret the square root as a power-series expansion, we obtain the first order correction

$$d_{\mathbf{q}/2}(\mathbf{q}) \approx \left(\hat{\mathbf{1}} + \frac{1}{2N^0} \sum_{\mathbf{q}_1 \neq 0} d^\dagger_{\mathbf{q}_1/2}(\mathbf{q}_1) d_{\mathbf{q}_1/2}(\mathbf{q}_1) \right) \frac{1}{\sqrt{N^0}} b^\dagger_{\mathbf{0}} b_{\mathbf{q}}.$$

$$(13)$$

In order to conform to the iterative scheme, we are obliged to replace the the d's on the right side by the zeroth-order d's. But we know that in the zeroth order $d^\dagger_{\mathbf{q}_1/2}(\mathbf{q}_1) d_{\mathbf{q}_1/2}(\mathbf{q}_1) = 0$ and $\hat{N} = N^0 \hat{\mathbf{1}}$. Therefore the first-order d is the same as in the zeroth order, $d_{\mathbf{q}/2}(\mathbf{q}) = (1/\sqrt{N^0}) b^\dagger_{\mathbf{0}} b_{\mathbf{q}}$. All this points to the futility of interpreting the square root as a power-series expansion around the noninteracting ground state. All further iterations lead to precisely the same result. This should tell us that one should solve the system self-consistently. Perhaps we should expand around an expectation value [i.e., $\hat{N}_0 = \langle \hat{N}_0 \rangle \hat{\mathbf{1}} + (\hat{N}_0 - \langle \hat{N}_0 \rangle \hat{\mathbf{1}})$] with the expectation value determined self-consistently. This expectation value will be different from N^0 even when interactions are absent, for example, if we consider finite temperature. Let us now try and compute the finite-temperature momentum distribution of noninteracting bosons. From elementary considerations we know that $\langle \hat{N}_0 \rangle = N^0 - \Sigma \mathbf{k} \neq 0 \langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \rangle$ and $\langle n_{\mathbf{k}} \rangle = \langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \rangle$. The thermodynamic expectation values involve the chemical potential μ:

$$\langle n_{\mathbf{k}} \rangle = \frac{1}{\exp[\beta(\epsilon_{\mathbf{k}} - \mu)] - 1}. \quad (14)$$

The reason this appears is that in the grand canonical ensemble we have to compute the trace with the Boltzmann weight $\langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \rangle = (1/Z) \text{Tr} \{\exp[-\beta(H-\mu N)] b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\}$, where $Z = \text{Tr} \{\exp[-\beta(H-\mu N)]\}$ is the grand partition function. Using the cyclic property of the trace, we may write

$$\langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \rangle = \frac{1}{Z} \text{Tr}(e^{-\beta(H-\mu N)} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}) = \frac{1}{Z} \text{Tr}(b_{\mathbf{k}} e^{-\beta(H-\mu N)} b^\dagger_{\mathbf{k}})$$

$$= \frac{1}{Z} \text{Tr}(e^{-\beta(H-\mu N)} e^{\beta(H-\mu N)} b_{\mathbf{k}} e^{-\beta(H-\mu N)} b^\dagger_{\mathbf{k}})$$

$$= e^{-\beta(\epsilon_{\mathbf{k}} - \mu)} \langle b_{\mathbf{k}} b^\dagger_{\mathbf{k}} \rangle = e^{-\beta(\epsilon_{\mathbf{k}} - \mu)} (1 + \langle b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \rangle).$$

In other words,

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \rangle = \frac{1}{\exp[\beta(\epsilon_{\mathbf{k}} - \mu)] - 1}.$$

Let us now try to evaluate this quantity using the condensate-displacement language

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} \rangle = \langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle = \frac{1}{Z} \mathrm{Tr}[e^{-\beta(H-\mu N)} d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k})],$$

$$H = \sum_{\mathbf{k} \neq 0} \epsilon_{\mathbf{k}} d_{(1/2)\mathbf{k}}^{\dagger}(\mathbf{k}) d_{(1/2)\mathbf{k}}(\mathbf{k}).$$

We also know that $[\hat{N}, d_{(1/2)\mathbf{k}}(\mathbf{k})] = 0$:

$$\langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle = \frac{1}{Z} \mathrm{Tr}[e^{-\beta(H-\mu N)} d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k})]$$

$$= \frac{1}{Z} \mathrm{Tr}[e^{-\beta(H-\mu N)} e^{\beta(H-\mu N)}$$
$$\times d_{\mathbf{k}/2}(\mathbf{k}) e^{-\beta(H-\mu N)} d_{\mathbf{k}/2}^{\dagger}(\mathbf{k})],$$

$$\langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle = e^{-\beta \epsilon_{\mathbf{k}}} \frac{1}{Z} \mathrm{Tr}[e^{-\beta(H-\mu N)} d_{\mathbf{k}/2}(\mathbf{k}) d_{\mathbf{k}/2}^{\dagger}(\mathbf{k})]$$

$$= e^{-\beta \epsilon_{\mathbf{k}}} [1 + \langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle],$$

$$\langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle = \frac{1}{\exp(\beta \epsilon_{\mathbf{k}}) - 1}.$$

The crucial μ seems to be missing. The reason is somewhat subtle. It has to do with the fact that the trace in the original case using the parent bosons spans all states including those with $\hat{N}_0 = 0$. However, in the condensate-displacement language, the trace is over all states *except* those that have $\hat{N}_0 = 0$. There are quite a number of states that have $\hat{N}_0 = 0$ and an arbitrary total number of particles. It is therefore not surprising that we have encountered a discrepancy. The best way to resolve this is to introduce a Lagrange multiplier that allows us to control how many bosons there are in states with zero momentum. Thus when it comes to taking the trace over states in the condensate-displacement language, we have to be careful to include a new chemical potential that couples to $\hat{N}_0$ and not just to $\hat{N}$. When this is done we can easily show

$$\langle d_{\mathbf{k}/2}^{\dagger}(\mathbf{k}) d_{\mathbf{k}/2}(\mathbf{k}) \rangle = \frac{1}{\exp[\beta(\epsilon_{\mathbf{k}} - \mu)] - 1}.$$

These considerations carry over to fermions where the difficulties are understandably far more severe. After all, trying to describe fermions using Bose-like objects, and to do it exactly, is a daunting task. We shall now use this insight to write down a correspondence for fermions.

III. THE NATURE OF SEA-DISPLACEMENT OPERATORS

In this section, we provide details of the correspondence between the sea-displacement operators and canonical fermi-

ons. The sea-displacement operators have been introduced elsewhere[1] but an explicit formula for the sea-displacement operator in terms of the Fermi fields was lacking. In this section, we once and for all pin down the definition of the sea-displacement operator and show that they obey commutation rules that are somewhat complicated but in the limit of a generalized random-phase approximation (RPA) (to be made clear in Appendix B) they obey canonical boson commutation rules. This exercise hopefully is the final word as far as the technicalities of the correspondence goes. Furthermore, as we shall see in the conclusion, the rigorous formulation reinforces the rather startling claim[1] that in one dimension (1D), so long as the interaction between the fermions is purely repulsive and possesses an appropriate Fourier transform, and is sufficiently weak so that all states of the interacting system (both ground state as well as excited states) may be expressed as linear combinations of low lying excited states of the noninteracting system and its ground state, then the momentum distribution of the interacting system possesses a sharp Fermi surface at precisely the same place as the noninteracting system. Let us now proceed to the main task at hand.

Let $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{\dagger}$ be canonical fermion annihilation and creation operators. We may write $\{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = 0$ and $\{c_{\mathbf{k}}, c_{\mathbf{k}'}^{\dagger}\} = \delta_{\mathbf{k},\mathbf{k}'}$. Let $n_{\mathbf{k}} = c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}}$ denote the number operator. The sea-displacement operators for fermions are defined as ($\mathbf{q} \neq 0$)

$$A_{\mathbf{k}}(\mathbf{q}) = n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)] \frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}} c_{\mathbf{k}-\mathbf{q}/2}^{\dagger} c_{\mathbf{k}+\mathbf{q}/2}, \tag{15}$$

$$A_{\mathbf{k}}(\mathbf{0}) = 0, \tag{16}$$

where $n_F(\mathbf{k}) = \theta(k_F - |\mathbf{k}|)$ is the zero-temperature Fermi distribution. As before, the square root of the number operator in the denominator in the definition requires clarification, a point noted by Cune and Apostol.[2] Their approach for dealing with this problem is unfortunately, not adequate.[4] Neither it seems is the approach used for dealing with this problem in the case of bosons, namely that we be able to interpret the object $\hat{U}(\mathbf{k}) = (1/\sqrt{n_{\mathbf{k}}}) c_{\mathbf{k}}^{\dagger}$ as being a unitary operator. This reason for this additional complication probably stems from the fact that in order for this object to be unitary, we should be able to find a self-adjoint canonical conjugate $P_{\mathbf{k}}$ of the number operator such that $[P_{\mathbf{k}}, n_{\mathbf{k}'}] = i \delta_{\mathbf{k},\mathbf{k}'}$. Such an object is not likely to exist not only because of the positivity of the number operator $n_{\mathbf{k}}$, but also because of idempotence.[5] In fact, it is a simple matter to convince ourselves that idempotence is inconsistent with the existence of $P_{\mathbf{k}}$. On the one hand, we have $[P_{\mathbf{k}}, n_{\mathbf{k}}^2] = [P_{\mathbf{k}}, n_{\mathbf{k}}] n_{\mathbf{k}} + n_{\mathbf{k}}[P_{\mathbf{k}}, n_{\mathbf{k}}] = 2i n_{\mathbf{k}}$. On the other hand, $[P_{\mathbf{k}}, n_{\mathbf{k}}^2] = [P_{\mathbf{k}}, n_{\mathbf{k}}] = i$. This suggests that $n_{\mathbf{k}} = (1/2)\hat{\mathbf{1}}$—a state of affairs seldom realized if at all. Some readers of this work familiar with the more traditional bosonization approaches may point to the importance of "point splitting"—a procedure that does not allow us to write $n_{\mathbf{k}} = c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}}$. They may wish to suggest this as the main reason for all these difficulties. It is likely that point splitting is a necessary technical consideration only in systems that

have $k_F \to \infty$ and $m \to \infty$ but $v_F = k_F/m < \infty$, in other words, a system where both the mass of the particle and the density are infinite such that the Fermi velocity is finite.[6] We deal with systems that are more physical (finite k_F and m) and are, by and large, immune to these considerations. While the metaphorical definition of Eq. (15) is quite adequate for most practical purposes, it is desirable to have a better understanding of the meaning of the square root.

Let us divert our attention for some time to the more mechanical aspects of this program in the hope that soon we will be able to address this delicate technical problem of the square root in the denominator. In a manner entirely analogous to that of bosons we may write

$$c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} = \sqrt{n_{\mathbf{k}+\mathbf{q}/2}} A_{\mathbf{k}}(-\mathbf{q}) + A_{\mathbf{k}}^{\dagger}(\mathbf{q}) \sqrt{n_{\mathbf{k}-\mathbf{q}/2}}$$

$$+ \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

(17)

$$- \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1).$$

(18)

Furthermore,

$$n_{\mathbf{k}} = c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} = n_F(\mathbf{k}) \frac{\hat{N}}{N^0} + \lambda \sum_{\mathbf{q}_1 \neq \mathbf{0}} A_{\mathbf{k}-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1)$$

$$- \lambda \sum_{\mathbf{q}_1 \neq \mathbf{0}} A_{\mathbf{k}+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1),$$

(19)

where $n_F(\mathbf{k}) = \theta(k_F - |\mathbf{k}|)$ is the zero-temperature Fermi distribution function. The parameter λ is a book-keeping device, familiar from perturbation theory, where in the end we set $\lambda = 1$. This is a very useful device as we shall soon find out, one that partially addresses the problem of the square root in the denominator. The proof of Eqs. (17) and (19) rests crucially on the meaning of the definition of $A_{\mathbf{k}}(\mathbf{q})$ shown in Eq. (15). Let us now consider the problem of ascertaining the meaning of $A_{\mathbf{k}}(\mathbf{q})$ in a "perturbative" manner. That is, we shall interpret the square root of the number operator as being the square root of the Eq. (19) where we find a natural unit operator about which we may expand the square root. Thus if we consider the lowest order in λ we have $n_{\mathbf{k}} = n_F(\mathbf{k})\hat{\mathbf{1}}$. Since $n_{\mathbf{k}-\mathbf{q}/2} = \mathbf{1}$ when $n_F(\mathbf{k}-\mathbf{q}/2) = 1$, we may write

$$A_{\mathbf{k}}(\mathbf{q}) = n_F(\mathbf{k}-\mathbf{q}/2)[1 - n_F(\mathbf{k}+\mathbf{q}/2)] c_{\mathbf{k}-\mathbf{q}/2}^{\dagger} c_{\mathbf{k}+\mathbf{q}/2}.$$

(20)

These equations obviously solve Eqs. (17) and (19) in the lowest order $\lambda^0 = 1$ except that we seem to have the additional curiosity $c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} = 0$ if $n_F(\mathbf{k}+\mathbf{q}/2) = 1$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 1$ or if $n_F(\mathbf{k}+\mathbf{q}/2) = 0$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 0$. This is hardly surprising given the fact that $n_{\mathbf{k}} = n_F(\mathbf{k})\hat{\mathbf{1}}$ is consistent with our restricted Hilbert space having exactly one element, namely, the ground state of the noninteracting Fermi sea $|\text{FS}\rangle$. This means $c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}|\text{FS}\rangle = 0$ when $n_F(\mathbf{k}+\mathbf{q}/2)$

$= 1$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 1$ or if $n_F(\mathbf{k}+\mathbf{q}/2) = 0$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 0$, a fact easily verified. Parenthetically, we note that Eq. (20) was anticipated in the comment by Cune and Apostol.[2] Just as in the case of bosons, further iterations do not affect these conclusions. Therefore, if one starts with the noninteracting system and expands around the noninteracting ground state, one is able to treat only the noninteracting system. Just for the sake of completeness, let us see where the first order terms lead us. When $n_F(\mathbf{k}+\mathbf{q}/2) = 1$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 0$, we have from the definition Eq. (15) a result that is independent of λ and therefore valid to all orders in λ, $\sqrt{n_{\mathbf{k}+\mathbf{q}/2}} A_{\mathbf{k}}(-\mathbf{q}) = c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}$. In addition, we have $A_{\mathbf{k}}^{\dagger}(\mathbf{q}) \sqrt{n_{\mathbf{k}-\mathbf{q}/2}} = c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}$ when $n_F(\mathbf{k}+\mathbf{q}/2) = 0$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 1$. These two identifications are entirely consistent with Eq. (17) to all orders in λ. The other cases, namely, $n_F(\mathbf{k}\pm\mathbf{q}/2) = 0$ and $n_F(\mathbf{k}\pm\mathbf{q}/2) = 1$ are given below. They have to be proven iteratively. When $n_F(\mathbf{k}+\mathbf{q}/2) = 0$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 0$, we can see from Eq. (17) that $c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} = \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$. We now show that this result is identically zero. This is consistent with our earlier claim that further iterations do not change the zeroth order results. Since, now if we replace the A's by the zeroth order ones, we get a first order result for $c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}$ equal to zero:

$$c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} = \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$

$$= \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} n_F(\mathbf{k}+\mathbf{q}/2$$

$$- \mathbf{q}_1) c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1} c_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}$$

$$= \lambda \sum_{\mathbf{q}_1 \neq \mathbf{q}, \mathbf{0}} n_F(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1)$$

$$\times (\hat{\mathbf{1}} - n_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1}) c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}.$$

The number operator $n_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1} = n_F(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1)\hat{\mathbf{1}}$, and therefore the first order result is zero as well. Similarly, we find the same answer when $n_F(\mathbf{k}+\mathbf{q}/2) = 1$ and $n_F(\mathbf{k}-\mathbf{q}/2) = 1$. To put it another way, iterations around $n_{\mathbf{k}} = n_F(\mathbf{k})\hat{\mathbf{1}}$ do not change the form of $n_{\mathbf{k}}$. That is, $n_{\mathbf{k}}$ remains frozen at the value $n_F(\mathbf{k})\hat{\mathbf{1}}$. This means that the only state consistent with such an identity is the noninteracting ground state. Using this, we can convince ourselves that the program is consistent, just as the program was consistent in the Bose case which has been proved rigorously by other means.

In order to do better than just remain at the noninteracting ground state, one must, just as in the Bose case replace $n_{\mathbf{k}} = \langle n_{\mathbf{k}} \rangle \hat{\mathbf{1}} + (n_{\mathbf{k}} - \langle n_{\mathbf{k}} \rangle \hat{\mathbf{1}})$, and expand around an expectation value $\langle n_{\mathbf{k}} \rangle \neq 0$ for all $\mathbf{k}$. Thus we are obliged to consider a system with interactions and at a finite temperature in order for this scheme to be of practical significance. The philosophy is that we solve for $\langle n_{\mathbf{k}} \rangle$ self-consistently and then pass to the limit of weak interactions or low temperature if one wants to study the ideal case. Therefore, we may write

$$\frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}} = \frac{1}{\sqrt{\langle n_{\mathbf{k}-\mathbf{q}/2}\rangle}}\left[\hat{\mathbf{1}} + \frac{n_{\mathbf{k}-\mathbf{q}/2} - \langle n_{\mathbf{k}-\mathbf{q}/2}\rangle\hat{\mathbf{1}}}{\langle n_{\mathbf{k}-\mathbf{q}/2}\rangle}\right]^{-1/2}.$$

$$(21)$$

Let us now use the insight obtained in the case of bosons to write down the correct commutation rules obeyed by the sea-displacement operators. This is a nontrivial task given the subtleties involved. The Bose case was soluble exactly via a polar decomposition which we do not have here. We write down here the final answers, details of which may be found in Appendix B. The exact commutation rules obeyed by the sea-displacement operators it seems are rather complicated. The sea-displacement operators do, however, obey *exact closed* commutation rules as we shall see in Appendix B. There is a natural and simple regime where the *approximate* commutation rules are those of canonical bosons. It is the regime of the RPA of Bohm and Pines.[7] They are

$$[A_{\mathbf{k}}(\mathbf{q}), A_{\mathbf{k}'}(\mathbf{q}')]_{\mathrm{RPA}} = 0, \qquad (22)$$

$$[A_{\mathbf{k}}(\mathbf{q}), A_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]_{\mathrm{RPA}}$$
$$= n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)]\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}.$$
$$(23)$$

We saw in the Bose case that the RPA-commutation rules may be taken to be the exact rules the only care that we had to exercise was to couple the condensate bosons to their own chemical potential. Here too we have to couple the sea-bosons to a momentum dependent chemical potential as we will soon show. We find that it is quite appropriate to view the objects $A_{\mathbf{k}}(\mathbf{q})$ as being exact bosons, although for special values of the indices they behave rather strangely. These simple-looking results belie the notoriously difficult and technical problem of the square root in the denominator, which incidentally, we have only partially resolved [see Eq. (21)]. One potential criticism of this is to claim that a scheme such as Eq. (21) is inconsistent with the exact commutation rules presented in Appendix B. The rebuttal to such a critique is that even Eq. (21) is approximate, no matter how many orders are summed—the reason being that this assumes that fluctuations in the number operator are small compared with the mean. This is inconsistent since we may show, for example, that $\langle n_{\mathbf{k}}^2\rangle - \langle n_{\mathbf{k}}\rangle^2 = \langle n_{\mathbf{k}}\rangle(1 - \langle n_{\mathbf{k}}\rangle)$. Therefore, the more noideal the momentum distribution is, the more it fluctuates. In fact, a momentum distribution that is highly non-ideal $\langle n_{\mathbf{k}}\rangle \approx 1/2$ for most momenta, has the largest fluctuation equal to the mean itself. There is another important point that should be mentioned. The definition in Eq. (15) closely resembles the definition of condensate displacement operators for bosons. This is certainly a plus. Further, the factor $n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)]$ serves as a cutoff function. It fulfills a very important role. It ensures that the kinetic energy operator is positive definite. The authors initially tried a number of alternatives that attempted to include terms beyond the RPA-like term, for instance, one with $[1 - n_F(\mathbf{k} + \mathbf{q}/2)][1 - n_F(\mathbf{k} - \mathbf{q}/2)]$. Such attempts are always unsuccessful for the simple reason that they lead to nonpositive kinetic energy

operators. With the cutoff function we have introduced, we may write the kinetic energy operator in the sea-displacement language as $K = k_0\hat{N} + \Sigma\mathbf{k}, \mathbf{q} \neq 0\,\omega_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})$. It can be seen that $\omega_{\mathbf{k}}(\mathbf{q}) = (\mathbf{k} \cdot \mathbf{q}/m)n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)] \geq 0$. Here $k_0 = (1/N^0)\Sigma_{\mathbf{k}}\epsilon_{\mathbf{k}}n_F(\mathbf{k})$ is the kinetic energy per particle. In order not to disappoint the attentive reader, we collect here some facts that are true in the absolute sense. From the exact definition in Eq. (15) it is fairly obvious that (independent of the meaning of the square root)

$$[A_{\mathbf{k}}(\mathbf{q}), n_{\mathbf{p}}] = A_{\mathbf{k}}(\mathbf{q})(\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}). \qquad (24)$$

More generally,

$$F([\{n_{\mathbf{p}} + \delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2} - \delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}\}])A_{\mathbf{k}}(\mathbf{q}) = A_{\mathbf{k}}(\mathbf{q})F([n]).$$
$$(25)$$

From the definition in Eq. (15) we may write

$$A_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}^{\dagger}(\mathbf{q}) = n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)](1 - n_{\mathbf{k}+\mathbf{q}/2}).$$
$$(26)$$

The other commutation rules are collected in Appendix B. They are important only for special values of the indices $\mathbf{k}, \mathbf{k}', \mathbf{q}$ and $\mathbf{q}'$. In the RPA sense and whenever we are willing to overlook these nuances (which is always, practically speaking) the RPA commutation rules of Eqs. (22) and (23) are quite sufficient and easy to use. Let us now try and compute the properties of the free Fermi theory using the machinery that we have just laid down. For this we need to first ask, how do we express $A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})$? We expect the answer to depend crucially on the meaning of the square root. We do not have any more insight into the nature of this object, but we will be needing its expectation value when we try to compute the finite temperature properties of noninteracting Fermi system in the sea-displacement language. Fortunately, the expectation value is obtained by a reasonably straightforward method and one that is very reminiscent of the Bose case. We shall have occasion to introduce a chemical potential that is momentum dependent and scales as the logarithm of the volume of the system so that one may venture into regions where the square root in the denominator vanishes (as does the numerator) where the sea-displacement method breaks down. The introduction of the chemical potential restores the meaning of the trace that in the Fermi language was intended to span over all the states. Let us now evaluate the finite-temperature properties, specifically, the finite-temperature momentum distribution of noninteracting fermions in the sea-displacement language. To this end, let us now compute the following correlation function:

$$\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = \frac{1}{Z(\mathbf{k} - \mathbf{q}/2)}\mathrm{Tr}(e^{-\beta(H - \mu N - \mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}$$

$$\times A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})), \qquad (27)$$

$$Z(\mathbf{k} - \mathbf{q}/2) = \mathrm{Tr}(e^{-\beta(H - \mu N - \mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}). \qquad (28)$$

Using the cyclic property of the trace, we have

$$\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = \frac{1}{Z(\mathbf{k}-\mathbf{q}/2)}\mathrm{Tr}[A_{\mathbf{k}}(\mathbf{q})e^{-\beta(H-\mu N-\mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}A_{\mathbf{k}}^{\dagger}(\mathbf{q})]$$

$$= \frac{1}{Z(\mathbf{k}-\mathbf{q}/2)}\mathrm{Tr}[e^{-\beta(H-\mu N-\mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}e^{\beta(H-\mu N-\mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}A_{\mathbf{k}}(\mathbf{q})e^{-\beta(H-\mu N-\mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}A_{\mathbf{k}}^{\dagger}(\mathbf{q})]$$

$$= e^{-\beta\frac{\mathbf{k}\cdot\mathbf{q}}{m}}e^{-\beta\mu_{\mathbf{k}-\mathbf{q}/2}}\frac{1}{Z(\mathbf{k}-\mathbf{q}/2)}\mathrm{Tr}[e^{-\beta(H-\mu N-\mu_{\mathbf{k}-\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2})}A_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}^{\dagger}(\mathbf{q})]$$

$$= e^{-\beta\mathbf{k}\cdot\mathbf{q}/m}e^{-\beta\mu_{\mathbf{k}-\mathbf{q}/2}}\langle A_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}^{\dagger}(\mathbf{q})\rangle. \tag{29}$$

But we know from Eq. (26) that

$$\langle A_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}^{\dagger}(\mathbf{q})\rangle = n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]$$
$$\times[1-n^{\beta}(\mathbf{k}+\mathbf{q}/2)]$$
$$\approx n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)] \tag{30}$$

at low temperatures, since the thermodynamic expectation value $n^{\beta}(\mathbf{k}+\mathbf{q}/2)$ vanishes exponentially when $n_F(\mathbf{k}+\mathbf{q}/2)$ $=0$. Therefore we may write $\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]\exp(-\beta\mathbf{k}\cdot\mathbf{q}/m)\exp(-\beta\mu_{\mathbf{k}-\mathbf{q}/2})$. The chemical potential $\mu_{\mathbf{k}}$ has to be chosen so that the correct thermodynamic expectation values are recovered. It is clear that in the absence of the chemical potential, we are bound to obtain answers that are incorrect, unless one is at absolute zero. It is also clear that in order for Eq. (31) to make sense, one must choose a chemical potential that scales as the logarithm of the volume of the system. Let us set $\exp(\beta\mu_{\mathbf{k}})=V\exp(\lambda_{\mathbf{k}})$ for some $\lambda_{\mathbf{k}}$. This may be determined by demanding the self-consistency of

$$n_{F,\beta}(\mathbf{k})=n_F(\mathbf{k})+\sum_{\mathbf{q}\neq0}n_F(\mathbf{k}-\mathbf{q})$$
$$\times[1-n_F(\mathbf{k})]e^{-\beta\mathbf{k}\cdot\mathbf{q}/m}e^{\beta\epsilon_{\mathbf{q}}}e^{-\beta\mu_{\mathbf{k}-\mathbf{q}}}$$
$$-\sum_{\mathbf{q}\neq0}n_F(\mathbf{k})[1-n_F(\mathbf{k}+\mathbf{q})]e^{-\beta\mathbf{k}\cdot\mathbf{q}/m}e^{-\beta\epsilon_{\mathbf{q}}}e^{-\beta\mu_{\mathbf{k}}},$$
$$\tag{31}$$

$$n_{F,\beta}(k)=n_F(k)\left(1-\frac{e^{-\lambda_k}}{V}e^{\beta(k^2/2m)}\right.$$
$$\times\sum_{\mathbf{q}\neq\mathbf{k}}[1-n_F(q)]e^{-\beta(q^2/2m)}\Bigg)+[1-n_F(k)]$$
$$\times e^{-\beta(k^2/2m)}\frac{1}{V}\sum_{\mathbf{q}}n_F(q)e^{\beta(q^2/2m)}e^{-\lambda_q}. \tag{32}$$

Let us now focus on a system in two space dimensions, in which case

$$n_F(k)e^{-\lambda_k}=\left(\frac{4\pi\beta}{2m}\right)n_F(k)[1-n_{F,\beta}(k)]e^{-\beta(k^2/2m)}e^{\beta(k_F^2/2m)}$$
$$\approx\left(\frac{2\pi\beta}{m}\right)n_F(k),$$

$$\frac{1}{V}\sum_{\mathbf{q}}n_F(q)e^{\beta(q^2/2m)}e^{-\lambda_q}$$
$$=\frac{1}{(2\pi)^2}\int_0^{k_F}2\pi q\,dq\left(\frac{4\pi\beta}{2m}\right)[1-n_{F,\beta}(q)]e^{\beta(k_F^2/2m)}$$
$$=e^{\beta(k_F^2/2m)}.$$

The last result follows if we assume, as we shall, that we are working at low enough temperatures $k_BT\ll E_F$. For this program to be self-consistent, we have to make this assumption. Perhaps there is a way out of it, but for now the authors are unable to find it. In any event, it is this regime that is of considerable physical significance and the fact that the sea-displacement scheme is consistent in this regime is very comforting indeed. Moving on, we write

$$\frac{1}{V}\sum_{\mathbf{q}}[1-n_F(\mathbf{q})]e^{-\beta(q^2/2m)}=\left(\frac{m}{2\pi\beta}\right)e^{-\beta(k_F^2/2m)}. \tag{33}$$

Substituting these back into Eq. (32) we find $n_{F,\beta}(\mathbf{k})$ $=n_F(\mathbf{k})n_{F,\beta}(\mathbf{k})+[1-n_F(\mathbf{k})]\exp[-\beta(k^2-k_F^2)/2m]$. At low temperatures when $|\mathbf{k}|>k_F$, we know that $n_{F,\beta}(\mathbf{k})$ $\approx\exp[-\beta(k^2-k_F^2)/2m]$. Therefore, the sea-displacement language gives the exact same result as the original Fermi language. Having successfully fixed the chemical potential of the sea-displacements, we now turn to the problem of computing the correlation functions of the noninteracting system at finite temperature. For instance, we have the four-point function $F(\mathbf{k}\cdot\mathbf{q};\mathbf{k}'\cdot\mathbf{q}')=\langle c_{\mathbf{k}+\mathbf{q}/2}^{\dagger}c_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}'/2}^{\dagger}c_{\mathbf{k}'+\mathbf{q}'/2}\rangle$. In the Fermi language, it is simply given by

$$F(\mathbf{k}\cdot\mathbf{q};\mathbf{k}'\cdot\mathbf{q}')=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}n_{F,\beta}(\mathbf{k}+\mathbf{q}/2)[1-n_{F,\beta}(\mathbf{k}-\mathbf{q}/2)]. \tag{34}$$

It may be shown that in the sea-displacement language, the same quantity may be evaluated as

$F(\mathbf{k}\cdot\mathbf{q};\mathbf{k}'\cdot\mathbf{q}')$

$$= \delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}[\langle\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})A_{\mathbf{k}'}^{\dagger}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}\rangle$$

$$+\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')\rangle]+\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}$$

$$\times\sum_{\mathbf{q}_1\neq\mathbf{q},0}\langle A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\rangle$$

$$\times n_F(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1)[1-n_F(\mathbf{k}-\mathbf{q}/2)]$$

$$+\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}\sum_{\mathbf{q}_1\neq\mathbf{q},0}\langle A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\rangle$$

$$\times n_F(\mathbf{k}+\mathbf{q}/2)[1-n_F(\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1)]. \tag{35}$$

Equation (35) is obtained by placing two expressions in Eq. (17) next to each other, contracting the indices and finally using Eq. (30). Insofar as the scheme for computing the momentum distribution is satisfactory, it may shown that formula in Eq. (35) is in fact identical to Eq. (34). To show this rigorously, we have to properly interpret the average $\langle\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})A_{\mathbf{k}'}^{\dagger}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}\rangle$. In fact it is a simple matter to evaluate this quantity exactly in the Fermi language. From Eq. (15) we have $\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})$ $=c_{\mathbf{k}+\mathbf{q}/2}^{\dagger}c_{\mathbf{k}-\mathbf{q}/2}n_F(\mathbf{k}+\mathbf{q}/2)[1-n_F(\mathbf{k}-\mathbf{q}/2)]$. Therefore we may write quite unambiguously

$$\langle\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})A_{\mathbf{k}'}^{\dagger}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}\rangle$$

$$=n_{F,\beta}(\mathbf{k}+\mathbf{q}/2)[1-n_{F,\beta}(\mathbf{k}-\mathbf{q}/2)]n_F(\mathbf{k}+\mathbf{q}/2)$$

$$\times[1-n_F(\mathbf{k}-\mathbf{q}/2)]\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}. \tag{36}$$

Similarly we may write,

$$\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')\rangle$$

$$=n_{F,\beta}(\mathbf{k}+\mathbf{q}/2)[1-n_{F,\beta}(\mathbf{k}-\mathbf{q}/2)]n_F(\mathbf{k}-\mathbf{q}/2)$$

$$\times[1-n_F(\mathbf{k}+\mathbf{q}/2)]\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}. \tag{37}$$

The main conclusion of these arguments is that one must be careful when evaluating these expectation values. In particular, it is wrong to make the approximation $n_{\mathbf{k}+\mathbf{q}/2}$ $\approx\langle n_{\mathbf{k}+\mathbf{q}/2}\rangle\hat{\mathbf{1}}$, since this implies

$$\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')\rangle$$

$$=_?n_F(\mathbf{k}-\mathbf{q}/2)n_F(\mathbf{k}'-\mathbf{q}'/2)\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}'}(\mathbf{q}')\rangle$$

$$=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\mathbf{q},\mathbf{q}'}$$

$$\times\left(\frac{1}{V}\right)e^{-\beta(\mathbf{k}\cdot\mathbf{q}/m)}\left(\frac{2\pi\beta}{m}\right). \tag{38}$$

This result is inconsistent with the correct result in Eq. (37) that does not vanish in the thermodynamic limit. Having said this, it is still not a poor approximation at low temperatures $(k_BT\ll E_F)$. The reason being that although the approximation on the right-hand side of Eq. (38) vanishes in the thermodynamic limit at any finite temperature, the exact answer for this correlation function as derived above also vanishes

for a different reason. At low temperatures, due to the presence of a product such as $n_{F,\beta}(\mathbf{k}+\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]$, the correlation function vanishes exponentially fast as $\exp[-(\epsilon_{\mathbf{k}+\mathbf{q}/2}-E_F)/k_BT]$. Therefore, while it is certainly advisable to be cautious, we are allowed some leeway at low temperatures. Having done all this, it is now a simple matter to convince ourselves that Eq. (35) is identical to Eq. (34). If $n_F(\mathbf{k}+\mathbf{q}/2)=1$ and $n_F(\mathbf{k}-\mathbf{q}/2)=0$, then all but the first term in Eq. (35) vanish and the identity is trivially satisfied. If $n_F(\mathbf{k}+\mathbf{q}/2)=0$ and $n_F(\mathbf{k}-\mathbf{q}/2)=1$, then all but the second term vanish and the identity is again trivially satisfied. What remains now is to compute the terms when $n_F(\mathbf{k}+\mathbf{q}/2)=0$ and $n_F(\mathbf{k}-\mathbf{q}/2)=0$ and when $n_F(\mathbf{k}+\mathbf{q}/2)=1$ and $n_F(\mathbf{k}-\mathbf{q}/2)=1$. In the first instance, we have to evaluate the sum $S_1=\Sigma_{\mathbf{q}_1\neq\mathbf{q},0}\langle A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\rangle n_F(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1)$. For this we have to make use of the simplified expression in 2D $\langle A_{\mathbf{k}}^{\dagger}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]\exp(-\beta\mathbf{k}\cdot\mathbf{q}/m)2\pi\beta/(mV)$. Therefore, we may write S_1 as

$$S_1=\frac{1}{V}\sum_{\mathbf{Q}_1}n_F(\mathbf{Q}_1)e^{\beta(Q_1^2/2m)}\left(\frac{2\pi\beta}{m}\right)e^{-\beta[(\mathbf{k}+\mathbf{q}/2)^2/2m]}$$

$$=e^{\beta[k_F^2-(\mathbf{k}+\mathbf{q}/2)^2/2m]}. \tag{39}$$

At low temperatures we may approximate Eq. (34) as $F(\mathbf{k}\cdot\mathbf{q};\mathbf{k}\cdot\mathbf{q})=\exp\{\beta[k_F^2-(\mathbf{k}+\mathbf{q}/2)^2]/2m\}$ since when $n_F(\mathbf{k}-\mathbf{q}/2)=0$ and $E_F\gg kT$, and $n_F(\mathbf{k}+\mathbf{q}/2)=0$ implies $|\mathbf{k}+\mathbf{q}/2|>k_F$. Therefore we have $1-n_{F,\beta}(\mathbf{k}-\mathbf{q}/2)\approx1$ and $n_{F,\beta}(\mathbf{k}+\mathbf{q}/2)=\exp\{\beta[k_F^2-(\mathbf{k}+\mathbf{q}/2)^2]/2m\}$, and hence the result follows. Similarly, when $n_F(\mathbf{k}+\mathbf{q}/2)=1$ and $n_F(\mathbf{k}-\mathbf{q}/2)=1$ we have

$$\sum_{\mathbf{q}_1\neq\mathbf{q},0}\langle A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\rangle$$

$$\times[1-n_F(\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1)]$$

$$=\frac{1}{V}\sum_{\mathbf{Q}_1}[1-n_F(\mathbf{Q}_1)]e^{-\beta(Q_1^2/2m)}\left(\frac{2\pi\beta}{m}\right)e^{\beta[(\mathbf{k}-\mathbf{q}/2)^2/2m]}$$

$$=e^{\beta[(\mathbf{k}-\mathbf{q}/2)^2-k_F^2/2m]}. \tag{40}$$

Analogous to the earlier case, we may approximate Eq. (34) as $F(\mathbf{k}\cdot\mathbf{q};\mathbf{k}\cdot\mathbf{q})=\exp\{\beta[(\mathbf{k}-\mathbf{q}/2)^2-k_F^2]/2m\}$. Therefore, this scheme gives the right answers at finite temperature for both the momentum distribution and the four-point functions.

A. Role of partial isometries

The term partial isometry[8] refers to, intuitively, an almost unitary operator. The absolute unitary nature is spoiled by an object that is by now ubiquitous in this article, namely the canonical conjugate X_0 of the total number operator. Thus we would like to interpret the object

$$U(\mathbf{k})=n_F(\mathbf{k})\frac{1}{\sqrt{n_{\mathbf{k}}}}c_{\mathbf{k}}^{\dagger} \tag{41}$$

as an almost unitary operator when $n_F(\mathbf{k})=1$. The presence of the square root of the number operator in the denominator led others[2] to conclude that our theory contains divergences. A closer examination tells us that this is not the case. The creation operator $c_\mathbf{k}^\dagger$ scales as the square root of the number operator. In other words, if we write

$$c_\mathbf{k}^\dagger = \sqrt{n_\mathbf{k}}\exp\{-i\Theta([n];\mathbf{k})\}\exp(iP_\mathbf{k}^\dagger), \qquad (42)$$

the creation operator becomes a nontrivial complex operator of unit modulus (ideally) times the square root of the number operator. The creation operator differs from the square root of the number operator by a phase factor. Analogous to what we found in our earlier works,[1] this phase factor consists of a canonical conjugate to the number operator denoted by $P_\mathbf{k}$ and a real functional of the number operator $\Theta([n];\mathbf{k})$. The sea-displacement operator may be written in the following suggestive form:

$$A_\mathbf{k}(\mathbf{q})=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]$$
$$\times e^{-i\Theta([n];\mathbf{k}-\mathbf{q}/2)}e^{i\,P_{\mathbf{k}-\mathbf{q}/2}^\dagger}c_{\mathbf{k}+\mathbf{q}/2}. \qquad (43)$$

This form, although more manageable than the definition in Eq. (15), is ambiguous since we have yet to pin down the meaning of $P_\mathbf{k}$ and $\Theta([n];\mathbf{k})$. We shall adopt an approach similar to the one we outlined in our earlier works[1] where we transform to the Fourier space and write down a formula for $P_\mathbf{k}$ in terms of so-called momentum currents (as opposed to real currents). As before, we find in our formalism the object X_0 that spoils the unitary nature of these operators. This object though troublesome, is very important since without it our program would be inconsistent. To understand the need for this object more clearly let us write (as we rightly anticipate) $P_\mathbf{k}=X_0+\widetilde{P}_\mathbf{k}$, where $\widetilde{P}_\mathbf{k}$ is manifestly self-adjoint whereas X_0 is not self-adjoint and obeys the relation $[X_0,\hat{N}]=i$. If X_0 is interpreted as being self adjoint, then the resulting formula for $A_\mathbf{k}^\dagger(\mathbf{q})A_\mathbf{k}(\mathbf{q})$ obtained from Eq. (43) would be inconsistent with Eq. (19) unless $n_\mathbf{k}=n_F(\mathbf{k})\hat{\mathbf{1}}$, a state of affairs realized only in the noninteracting system at zero temperature. Therefore, it is imperative that we do not interpret X_0 as being self-adjoint. However, just as in the Bose case we may still choose the Hermitian part of X_0 provided we interpret the polar decomposition in Eq. (42) in the distribution theoretic sense to be made clear below. Furthermore, it is not clear what the role of idempotence is in this formalism. In Appendix B, we write down *closed* commutation rules obeyed by the sea-displacement operators. These are the *exact* rules. The derivation of these do not require the use of idempotence, but we have to make use of the fact $(c_\mathbf{k})^2=0$. This and other similar observations such as idempotence require point splitting techniques. This procedure is well known to the traditional bosonizing community, mainly because it is indispensable in dealing with relativistic systems in $1+1$ dimensions which have a different mathematical structure than the nonrelativistic systems that we consider here. Perhaps these techniques can even be made mathematically rigorous in those systems. We had to consider the field operators in real space to be operator-

valued distributions,[1] which required the introduction of a separable one-particle Hilbert space, which amounts to placing the system in a box with periodic boundary conditions. Similarly, we are now forced, it seems, to do the same with momentum space, that is, we postulate in *addition* a short distance cutoff that amounts to introducing a lattice constant. Space is now discretized and finite as well. There is a large macroscopic length scale L (dimension of the box) and a small microscopic length scale a (the lattice spacing). In the end we set $L\rightarrow\infty$ and $a\rightarrow0$ and hope that our edifice remains intact. Let us try and make more sense out of Eq. (43). First, we would like a formula for X_0 in terms of the Fermi fields *without* involving $\Theta([n];\mathbf{k})$. To this end, let us postulate the existence of a ''super boson.'' It is defined as

$$\hat{C}=e^{-iX_0}\sqrt{\hat{N}}, \qquad (44)$$

where $\hat{N}=\Sigma_\mathbf{k}c_\mathbf{k}^\dagger c_\mathbf{k}$ and $[X_0,\hat{N}]=i$. If X_0 is strictly self-adjoint, then $\hat{C}$ is indeed an exact boson. Otherwise it is a quasiboson. We would not like to have X_0 as self-adjoint for reasons already alluded to. The claim is that the filled Fermi sea is obtained by creating N^0 number of these bosons from the vacuum of the fermions:

$$(C^\dagger)^{N^0}|0\rangle=|\text{FS}\rangle=\left(\prod_{|\mathbf{k}|<k_F}c_\mathbf{k}^\dagger\right)|0\rangle. \qquad (45)$$

We may rewrite the filled Fermi sea as follows:

$$|\text{FS}\rangle=\left(\prod_{|\mathbf{k}|<k_F}c_\mathbf{k}^\dagger\right)|0\rangle=\exp\left(\sum_\mathbf{k}n_F(\mathbf{k})\ln(c_\mathbf{k}^\dagger)\right)|0\rangle. \qquad (46)$$

From this we may deduce

$$C^\dagger=\exp\left(\frac{1}{N^0}\sum_\mathbf{k}n_F(\mathbf{k})\ln(c_\mathbf{k}^\dagger)\right)\approx\sqrt{N^0}e^{iX_0^\dagger} \qquad (47)$$

up to some phases that we will soon show are unimportant. From this we may read off a formula for X_0:

$$X_0=\frac{i}{N^0}\sum_\mathbf{k}n_F(\mathbf{k})\ln(c_\mathbf{k}). \qquad (48)$$

From this it is clear that X_0 is not self-adjoint and further we have to show that it is a conjugate to $\hat{N}$. For this let us compute

$$e^{i\lambda\hat{N}}X_0e^{-i\lambda\hat{N}}=X_0+\lambda. \qquad (49)$$

Using the definition in Eq. (48) we have

$$e^{i\lambda\hat{N}}X_0e^{-i\lambda\hat{N}}=\frac{i}{N^0}\sum_\mathbf{k}n_F(\mathbf{k})\ln(e^{i\lambda\hat{N}}c_\mathbf{k}e^{-i\lambda\hat{N}})=X_0+\lambda. \qquad (50)$$

Therefore the conjugate obeys all the expected properties, independent of the meaning of the logarithm. We would now like to ascertain how this object commutes with other objects such as $n_\mathbf{P}=c_\mathbf{P}^\dagger c_\mathbf{P}$:

$$e^{i\lambda n_P}X_0 e^{-i\lambda n_P} = \frac{i}{N^0}\sum_{\mathbf{k}} n_F(\mathbf{k})\ln(e^{i\lambda n_P}c_{\mathbf{k}}e^{-i\lambda n_P})$$

$$= X_0 + \lambda\frac{n_F(\mathbf{p})}{N^0}. \tag{51}$$

Now we would like a formula for the self-adjoint operator $\tilde{P}_{\mathbf{k}}$. First we would like to point out some obvious pitfalls. If one takes the point of view advocated in our earlier work,[1] then we have to face up to the fact that the number operator obeys idempotence since only the real space quantities are distributions whereas the momentum space quantities are bonafide operators. Idempotence, unfortunately is inconsistent with the existence of a canonical conjugate such as $P_{\mathbf{k}}$ as has been argued earlier. This then means that somehow we have to eschew idempotence in favor of retaining the conjugate. It is still unclear to the authors why idempotence is such a big hurdle and how to overcome it. We take the point of view that any $\tilde{P}_{\mathbf{k}}$ should result in a $A_{\mathbf{k}}(\mathbf{q})$ that obeys Eq. (19) and furthermore we also demand as indicated earlier, the identity in Eq. (26). This matter has been studied by us quite thoroughly and we are unable to find a completely satisfactory answer. However, we write down one possibility. This definition involves the problematic line integral encountered in our earlier work as well.[1] The resolution to this may be obtained by transforming to the real space analogous to the approach used for making sense of the DPVA.[1] The quantity $\tilde{P}_{\mathbf{k}}$ may be written down as follows:

$$\tilde{P}_{\mathbf{k}} = \int^{\mathbf{k}} d\mathbf{l}(-1/n_{\mathbf{p}})\mathbf{I}(\mathbf{p}) + \Theta([n];\mathbf{k}) - \int^{\mathbf{k}} d\mathbf{l}[-i\Theta, \nabla\tilde{P}](\mathbf{p}). \tag{52}$$

The line integral is to be interpreted as being carried out after transforming the quantities to real space analogous to the DPVA.[1] The quantity $\mathbf{I}(\mathbf{p}) = (1/2i)[c_{\mathbf{p}}^\dagger \nabla c_{\mathbf{p}} - (\nabla c_{\mathbf{p}}^\dagger)c_{\mathbf{p}}]$. Finally, we have to write down a prescription for Θ. Since we have discretised the momenta by placing the system in a box, we may use the following natural ansatz. There is no need to make contact with the free theory very likely since in the number-phase respresentation the free theory is recovered automatically. Let $\mathbf{k_n} = (2\pi/L)(n_1, n_2, n_3)$. Define the mapping $t(\mathbf{k_n}) = 2^{n_1}\times 3^{n_2}\times 5^{n_3}$. It may be seen that t is a bijection. From this we may write,[9]

$$\Theta([n];\mathbf{k_n}) = \pi\sum_{\mathbf{p_m}}\theta[t(\mathbf{k_n}) - t(\mathbf{p_m})]n_{\mathbf{p_m}}. \tag{53}$$

Here $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ is $x \leq 0$ is the usual Heaviside step function. This obeys the required recursion relation below, necessary for ensuring that Fermi statistics are obeyed:

$$\Theta([n_{\mathbf{p}_1} - \delta_{\mathbf{p}_1,\mathbf{k}'}\}];\mathbf{k}) + \Theta([n];\mathbf{k}') - \Theta([n];\mathbf{k})$$

$$- \Theta([\{n_{\mathbf{p}_1} - \delta_{\mathbf{p},\mathbf{k}}\}];\mathbf{k}')$$

$$= \pm\pi. \tag{54}$$

B. Short-range interactions

We have found a curious feature in our investigations of the nature of the RPA that is worth mentioning. It seems that depending upon how one groups the Fermi field operators, one encounters an enigmatic duality between repulsion and attraction (see Ref. 10, Chap. 4). It is as follows. Consider the interaction term[11] in jellium:[10]

$$H_I = \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'} c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}/2}^\dagger c_{\mathbf{k}'+\mathbf{q}/2}, \tag{55}$$

$$= -\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'} c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}'+\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}$$

$$= -\sum_{\mathbf{P}\neq\mathbf{P}'}\sum_{\mathbf{Q}\neq 0}\frac{v_{\mathbf{P}-\mathbf{P}'}}{2V} c_{\mathbf{P}+\mathbf{Q}/2}^\dagger c_{\mathbf{P}-\mathbf{Q}/2}c_{\mathbf{P}'-\mathbf{Q}/2}^\dagger c_{\mathbf{P}'+\mathbf{Q}/2}. \tag{56}$$

The last result is obtained simply by relabeling the indices. If we set (see the end of Appendix B) $c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} =_{\text{RPA}} [A_{\mathbf{k}}(-\mathbf{q}) + A_{\mathbf{k}}^\dagger(\mathbf{q})]$ with equality to the level of RPA, then we may write

$$H_I = \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'} [A_{\mathbf{k}}(-\mathbf{q}) + A_{\mathbf{k}}^\dagger(\mathbf{q})][A_{\mathbf{k}'}(\mathbf{q}) + A_{\mathbf{k}'}^\dagger(-\mathbf{q})]. \tag{57}$$

Alternatively, we may also write

$$H_I = -\sum_{\mathbf{q}\neq 0}\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{2V}[A_{\mathbf{k}}(-\mathbf{q}) + A_{\mathbf{k}}^\dagger(\mathbf{q})][A_{\mathbf{k}'}(\mathbf{q})$$

$$+ A_{\mathbf{k}'}^\dagger(-\mathbf{q})]. \tag{58}$$

The first alternative yields results identical to the ones presented in our earlier work.[1] The momentum distributions derived from this interaction term possess sharp discontinuities. The second alternative is more interesting. It has a minus sign signifying an apparent change from repulsion to attraction (only apparently so). It has been argued in the literature[10] that if we assume that $v_{\mathbf{q}} = w_0$ is independent of $\mathbf{q}$ (in other words, a δ-function interaction in real space), then we may conclude by examining Eqs. (55) and (56), that $H_I = -H_I$. Therefore, $H_I = 0$. The exchange correlation energy of a system of fermions interacting via a δ-function interaction is identically zero. This is not surprising since we know that two fermions cannot be at the same point and the interaction is zero unless they are at the same point, therefore one obtains an answer zero. This means that for such systems the Hamiltonian is a simple functional of the number operator:

$$H = \sum_{\mathbf{k}}\epsilon_{\mathbf{k}}n_{\mathbf{k}} - \frac{w_0}{2V}\sum_{\mathbf{k},\mathbf{q}\neq 0}n_{\mathbf{k}+\mathbf{q}/2}n_{\mathbf{k}-\mathbf{q}/2}. \tag{59}$$

Unfortunately, this Hamiltonian does not yield results too different from the noninteracting case. However, we have found that if one relaxes the assumption that $v_{\mathbf{q}}$ is strictly independent of $\mathbf{q}$ and instead say, puts in a weak dependence on $\mathbf{q}$, it is still a good approximation to ignore the exchange

correlation energy (at least in comparison with the kinetic energy and the exchange self-energy), and we obtain some rather interesting results for the one-particle Green function. For instance, we could assume that the potential in real space instead of being a strict δ function is somewhat smeared. For example in 1D, we might take $V(x)=e^2(a/\pi)(x^2+a^2)^{-1}$. We merely quote the final results for the one particle properties as this is just an aside. In order to derive these results one has to make use of the Schwinger's functional approach to dealing with quantum field theories.[12] This exercise highlights the importance of fluctuations in the momentum distribution in determining the salient features of the one-particle Green function in systems that have very short range interactions and are not too strong. Following the notation of Kadanoff and Baym,[12] we write down the spectral function and the spectral width as

$$A(\mathbf{p},\omega)=\frac{\sqrt{-\kappa(\mathbf{p},\omega)}}{F(\mathbf{p})}, \qquad (60)$$

$$\Gamma(\mathbf{p},\omega)=\sqrt{-\kappa(\mathbf{p},\omega)}, \qquad (61)$$

where $\kappa(\mathbf{p},\omega)=(\omega-\tilde{\epsilon}_\mathbf{P}+\mu)^2-4F(\mathbf{p})<0$. In the event $\kappa(\mathbf{p},\omega)>0$, $\sqrt{-\kappa(\mathbf{p},\omega)}=0$, by definition. Here $F(\mathbf{k})$ is related to the number-number correlation function $N(\mathbf{k},\mathbf{k}')=\langle n_\mathbf{k}n_{\mathbf{k}'}\rangle-\langle n_\mathbf{k}\rangle\langle n_{\mathbf{k}'}\rangle$ where $\tilde{\epsilon}_\mathbf{k}$ is the modified dispersion relation: $\tilde{\epsilon}_\mathbf{k}=\epsilon_\mathbf{k}-V^{-1}\Sigma\mathbf{q}\neq0 v_\mathbf{q}\langle n_{\mathbf{k}-\mathbf{q}}\rangle$. Furthermore, $F(\mathbf{k})=V^{-2}\Sigma\mathbf{q},\mathbf{q}'\neq0 v_\mathbf{q}v_{\mathbf{q}'}N(\mathbf{k}-\mathbf{q},\mathbf{k}-\mathbf{q}')$. The momentum distribution has to be determined self-consistently:

$$\langle n_\mathbf{p}\rangle=\left(\frac{2}{\pi}\right)\int_{-\pi/2}^{\pi/2}d\theta\,\frac{\cos^2\theta}{e^{\beta(\tilde{\epsilon}_\mathbf{p}-\mu)}e^{2\beta\sqrt{F(\mathbf{p})}\sin\theta}+1}. \qquad (62)$$

These formulas are incomplete unless one has a prescription for the object $N(\mathbf{k},\mathbf{k}')$. We note some properties of this object. First, $N(\mathbf{k},\mathbf{k})=\langle n_\mathbf{k}\rangle(1-\langle n_\mathbf{k}\rangle)$. It may occur to the reader that since $[n_\mathbf{k},H]=0$, the ground state of the system should be an eigenstate of $n_\mathbf{k}$, and therefore $N(\mathbf{k},\mathbf{k}')=0$ for $\mathbf{k}\neq\mathbf{k}'$. This quantity then signifies the importance of terms beyond exchange energy. We have tried a number of ansatzs

for this object. In particular, if we ignore the dependence on the angle $\mathbf{k}\cdot\mathbf{k}'$, we may write a reasonable-looking formula for this object ($\mathbf{k}\neq\mathbf{k}'$):

$$N(\mathbf{k},\mathbf{k}')=[\langle n_\mathbf{k}\rangle-n_\beta(\mathbf{k})][\langle n_{\mathbf{k}'}\rangle-n_\beta(\mathbf{k}')]. \qquad (63)$$

This has the attractive feature of conserving the number of particles and also the net momentum of the electrons $[\Sigma_\mathbf{k}N(\mathbf{k},\mathbf{k}')\approx0,\Sigma_\mathbf{k}\mathbf{k}N(\mathbf{k},\mathbf{k}')\approx0]$ and further since $0<n_\mathbf{k}<1$ we expect $-1<N(\mathbf{k},\mathbf{k}')<1$ which is also obeyed. It has been pointed out that this object $N(\mathbf{k},\mathbf{k}')$ is singular in the BCS case.[9] In particular, it is identically zero unless $\mathbf{k}=\pm\mathbf{k}'$. This means that for the BCS case which is a special case we may write

$$N(\mathbf{k},\mathbf{k}')=\Delta(\mathbf{k})(\delta_{\mathbf{k},\mathbf{k}'}-\delta_{\mathbf{k},-\mathbf{k}'}). \qquad (64)$$

Here $\Delta(\mathbf{k})$ is related to the gap function. Ideally, we would like a formula for this object that in the normal Fermi liquid regime reduces to an unremarkable function of $|\mathbf{k}|,|\mathbf{k}'|$, and $\mathbf{k}\cdot\mathbf{k}'$ and exhibits a phase transition to the singular appearance as one crosses over into the superconducting regime. In fact it may be seen that the BCS correlation functions supports a net momentum in the sense of a quantum fluctuation $\langle\vec{P}^2\rangle_{BCS}=2\Sigma_\mathbf{k}k^2\Delta(\mathbf{k})\neq0$. Bare short-range interactions are not of much interest in continuum systems so we leave it at that [however Eqs. (63) and (64) are meant to be universally valid].

C. Long-range interactions

For long-range interactions, if one considers an approach similar to the one used for deriving the traditional RPA dielectric function, but this time including possible fluctuations in the momentum distribution, we find a result that is different from the usual RPA-dielectric function but one that reduces to it in the weak coupling limit. This derivation is found in Appendix D. We merely quote the final answer here:

$$\epsilon_{\text{eff}}(\mathbf{q},\omega)=\epsilon_{g-\text{RPA}}(\mathbf{q},\omega)-\left(\frac{v_\mathbf{q}}{V}\right)^2\frac{P_2(\mathbf{q},\omega)}{\epsilon_{g-\text{RPA}}(\mathbf{q},\omega)}. \qquad (65)$$

Here,

$$P_2(\mathbf{q},\omega) \qquad (66)$$

$$=\sum_{\mathbf{k},\mathbf{k}'}\frac{N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}/2)-N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}/2)-N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}/2)+N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}/2)}{(\omega-\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}+\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})(\omega-\tilde{\epsilon}_{\mathbf{k}'-\mathbf{q}/2}+\tilde{\epsilon}_{\mathbf{k}'+\mathbf{q}/2})}, \qquad (67)$$

$$\epsilon_{g-\text{RPA}}(\mathbf{q},\omega)=1+(v_\mathbf{q}/V)\sum_\mathbf{k}\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega-\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}+\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}} \qquad (68)$$

and

$$\tilde{\epsilon}_\mathbf{k}=\epsilon_\mathbf{k}-\sum_{\mathbf{q}\neq0}\frac{v_\mathbf{q}}{V}\langle n_{\mathbf{k}-\mathbf{q}}\rangle. \qquad (69)$$

The momentum distribution in such a system with long-range interactions may be written down as follows. This formula is derived by assuming that it is a good approximation to totally ignore the form of the exchange correlation Hamiltonian in Eq. (58) in favor of retaining Eq. (57) exclusively. In fact, this assumption is valid only in systems in 3D or 2D where the interaction is of the form $4\pi e^2/q^2$ or $2\pi e^2/|q|$. The term that breaks Fermi liquid behavior is the one that couples the left movers to the right movers in the usual bosonization scheme.[10] This is present only in Eq. (58). This term may be ignored since $4\pi e^2/(k-k')^2$ or $2\pi e^2/|k-k'|$ are vanishingly small when $k \approx -k' \approx k_F$ in the large k_F limit. However, in 1D the inverse square interaction of the Calogero Sutherland model,[13] for example, has $v_q = -\pi\beta(\beta-1)|q|/2m$. This makes Eq. (58) the more important form. Moving on, we may write in 2D or 3D, the following formula for the momentum distribution:

$$\bar{n}_\mathbf{k} = n_F(\mathbf{k})F_1(\mathbf{k}) + [1 - n_F(\mathbf{k})]F_2(\mathbf{k}), \quad (70)$$

$$F_1(\mathbf{k}) = \frac{1}{1 + \dfrac{S_B(\mathbf{k})}{1 + S_A(\mathbf{k})}}, \quad (71)$$

$$F_2(\mathbf{k}) = \frac{1}{1 + \dfrac{1 + S_B(\mathbf{k})}{S_A(\mathbf{k})}}, \quad (72)$$

$$S_A(\mathbf{k}) = \sum_{\mathbf{q},i} \frac{\bar{n}_{\mathbf{k}-\mathbf{q}}}{[\omega_i(-\mathbf{q}) + \mathbf{k}\cdot\mathbf{q}/m - \epsilon_\mathbf{q}]^2} g_i^2(-\mathbf{q}), \quad (73)$$

$$S_B(\mathbf{k}) = \sum_{\mathbf{q},i} \frac{1 - \bar{n}_{\mathbf{k}+\mathbf{q}}}{[\omega_i(-\mathbf{q}) + \mathbf{k}\cdot\mathbf{q}/m + \epsilon_\mathbf{q}]^2} g_i^2 - \mathbf{q}], \quad (74)$$

$$g_i(\mathbf{q}) = \left[\sum_\mathbf{k} \frac{\bar{n}_{\mathbf{k}-\mathbf{q}} - \bar{n}_{\mathbf{k}+\mathbf{q}}}{\left[\omega_i(\mathbf{q}) - \dfrac{\mathbf{k}\cdot\mathbf{q}}{m}\right]^2} \right]^{-1/2}. \quad (75)$$

This quantity $g_i(\mathbf{q})$ is a residue of complex integration[1] and is given as the frequency derivative of the polarization

$$g_i(\mathbf{q}) = \frac{V}{v_\mathbf{q}} \left(\frac{\partial}{\partial\omega} \right)_{\omega=\omega_i(\mathbf{q})} \epsilon_{\text{eff}}^P(\mathbf{q},\omega). \quad (76)$$

Here ϵ_{eff}^P is the principal part of ϵ_{eff}. Moreover ω_i are the zeros of this dielectric function

$$\epsilon_{\text{eff}}(\mathbf{q},\omega_i) = 0. \quad (77)$$

As it stands, the above equation is ill defined since the dielectric function is complex. Interpreting ω_i to be the zero of the principal part results in capturing only the collective mode. The particle-hole mode which is completely lost in this approach is very important. In order not to lose this mode we have to interpret the zeros in a special manner. We shall take the point of view that all positive energies are allowed as zeros but each comes with a weight correspond-

ing to the strength of the dynamical structure factor at that energy. Define the weight to be

$$W(\mathbf{q},\omega) = \text{Im}\left(\frac{1}{\epsilon_{\text{eff}}(\mathbf{q},\omega - i0^+)} \right). \quad (78)$$

Therefore, sum over modes is now interpreted as

$$\sum_{\mathbf{q},i} f(\mathbf{q},\omega_i) = \frac{\displaystyle\int_0^\infty d\omega\, W(\mathbf{q},\omega) f(\mathbf{q},\omega)}{\displaystyle\int_0^\infty d\omega\, W(\mathbf{q},\omega)}. \quad (79)$$

It is worth emphasizing that Eq. (65) is valid for long range interactions where it is wrong to ignore the exchange correlation term. The one-particle properties that were derived in Eq. (60) are valid for short range interactions only. The interaction strength had to be weak so that one could ignore the exchange-correlation term in the short-range case. This was also needed in the long-range case to ensure that one could ignore the quadratic terms in the sea displacements. However, in arriving at Eq. (65) use was made of the form in Eq. (55) rather than Eq. (56). There really is no *a priori* reason to prefer one over the other. It is possible that one has to use a combination of the two (in fact, this has been shown in a Ref. 14). Using the second form exclusively [Eq. (56)] leads to a dielectric function that is totally different from the RPA-dielectric function (we have not been able to write down an explicit formula for this), thereby casting doubt on the sanctity of the traditional methods of deriving dielectric functions. This issue has probably not been addressed fully by the many-body community even though a well known textbook mentions some of these facts (Mahan[10]: end of Chap. 4). One may argue that the latter form, namely, in Eq. (56) at least for Coulomb repulsion ($1/r$) in three spatial dimensions is negligible at high density. This is because in such a case we have two independent length scales k_F^{-1} and a_B and if $k_F a_B \gg 1$, we may express all states as linear combinations of low lying excited states of the noninteracting system and then $|\mathbf{k}| \approx |\mathbf{k}'| \approx k_F$ and $0 < |\mathbf{q}| \approx a_B^{-1} \ll k_F$, which means that $|\mathbf{k} - \mathbf{k}'|^2 \gg (a_B^{-1})^2 \approx q^2$. This means that Eq. (56) is negligible compared to Eq. (55). This ambiguity in operator ordering that leads to very different looking Hamiltonians as in Eqs. (55) and (56) suggests that this problem will persist in all cases where electron-electron interactions are present, in other words, in nearly all of many-body physics. This suggests to the authors that one must look elsewhere to find a more natural home for these techniques. We have some reason to believe that quantum electrodynamics is such a place. There, electron-electron interactions come about indirectly via coupling to gauge fields where the operator ordering ambiguity is absent. Furthermore, a phenomenon important to condensed matter physics, namely, magnetism, being primarily due to the spin of the electron, is taken into account naturally in the relativistic theory, since spin is a consequence of the Dirac equation.[15] In the next section, we study charge-conserving electron-hole systems that is a precursor to this more ambitious program of reworking gauge theories

in the sea-displacement language. To conclude this section, we point out another definition of the sea displacement that gives us the illusion of understanding the meaning of the square root. This is similar to Eq. (10) for bosons:

$$A_{\mathbf{k}}(\mathbf{q}) = n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)]$$

$$\times \left(\hat{\mathbf{1}} - \sum_{\mathbf{q}_1 \neq 0} A_{\mathbf{k} - \mathbf{q}/2 + \mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) A_{\mathbf{k} - \mathbf{q}/2 + \mathbf{q}_1/2}(\mathbf{q}_1) \right)^{-1/2}$$

$$\times c_{\mathbf{k} - \mathbf{q}/2}^{\dagger} c_{\mathbf{k} + \mathbf{q}/2}. \tag{80}$$

The reason why this formula is not too illuminating is because a power series expansion around the unit operator and an iterative solution, yields just as in the Bose case, a result that amounts to ignoring the presence of the square root. That is,

$$A_{\mathbf{k}}(\mathbf{q}) = {}_? n_F(\mathbf{k} - \mathbf{q}/2)[1 - n_F(\mathbf{k} + \mathbf{q}/2)] c_{\mathbf{k} - \mathbf{q}/2}^{\dagger} c_{\mathbf{k} + \mathbf{q}/2}. \tag{81}$$

This is not the definition that is intended.[4] Instead, one must also consider regimes in which the square root is close to zero rather than close to unity. The exact definition of $A_{\mathbf{k}}(\mathbf{q})$ is the "self-consistent" solution of the nonlinear operator equation Eq. (80). Readers with superior mathematical skills may finally be able to solve this operator equation and map out its domain and range in Fock space and explain how this object acts on elements of its domain.

D. Comparison with other truly exactly solved models

One of the other criticisms that has been expressed against the sea-boson method is that it is unable to reproduce Luttinger liquid behavior. The authors have convinced themselves that the Calogero-Sutherland model (CSM) is in fact a Luttinger liquid. A Luttinger liquid is characterized by a lack of discontinuity in the momentum distribution, and the divergence (negative infinity) and continuity of the slope of momentum distribution across the Fermi surface. The correlation functions of this model has been derived by Ha *et al.* and Forrester.[13] However, the correlation functions of this model depend on the use of a singular gauge transformation that forces a particular statistics (in our case, fermionic) on the correlation functions. The authors who have computed the correlation functions have used the natural statistics for the CSM which is in general, fractional. This makes comparison with our approach difficult. The CSM for statistical parameter $\beta = 3$ (in the notation of Lesage *et al.*[13]) is the simplest model that describes interacting fermions. However, this choice makes the potential energy of the same order as the kinetic energy making the system strongly interacting. In our approach, the sea-bosons were written down in the plane wave basis with the tacit assumption that this is a good choice. It is easy to see that this choice is likely to be good only for weakly nonideal systems. The fermionic correlation functions for the weakly coupled CSM has yet to be written down. According to our expectations, this should result in a sharp Fermi surface for sufficiently weak repulsion. The Hubbard model in 1D (Ref. 16) is another system that we have to compare our results with. Its correlation functions are

reducible to quadratures only for half filling. The correlation functions are hard to deduce in a closed form away from half filling, but it is believed that this is a Luttinger liquid for arbitrarily weak repulsion. This latter result is the one we have the greatest difficulty in proving. In any event, in order for the community to accept the sea-boson technique, we have to somehow reproduce Luttinger liquid features. Work is in progress to do exactly this. It will be reported in a future publication.

IV. ELECTRON-HOLE SYSTEMS: EXCITON-EXCITON INTERACTIONS

In this section, we study the exciton Green function using the sea-displacement technique. In fact, the biexciton Green function would be more interesting since it determines nonlinear optical response. This technique is ideally suited to study this quantity since exciton-exciton interactions are crucial in determining the true nature of the biexciton. We find that exciton-exciton interactions are not of the two-body type as it is often assumed in the literature but rather more complicated. These interactions are mediated by other bosons that are present in the system, namely, the intraband particle hole excitations and the zero-momentum interband particle-hole excitation. All this will be made clear in the present section. Let us now move on to the details. As we have said before, a generalization is possible of these techniques to two-component charge-conserving electron-hole systems. One may define sea-displacement operators $A_{\mathbf{k}\sigma}(\mathbf{q}\sigma')$ that include discrete internal degrees of freedom that may be metaphorically called spin. These objects are defined below. Since the definition in Eq. (15) was valid for both $\mathbf{q} = 0$ and $\mathbf{q} \neq 0$, we have

$$A_{\mathbf{k}\sigma}(\mathbf{q}\sigma') = n_F(\mathbf{k} - \mathbf{q}/2\sigma)[1 - n_F(\mathbf{k}$$

$$+ \mathbf{q}/2\sigma')] \frac{1}{\sqrt{n_{\mathbf{k} - \mathbf{q}/2\sigma}}} c_{\mathbf{k} - \mathbf{q}/2\sigma}^{\dagger} c_{\mathbf{k} + \mathbf{q}/2\sigma'}. \tag{82}$$

We now make the following identifications. Let $c_{\mathbf{k}}$ be the operator that annihilates an electron from the conduction band and $d_{-\mathbf{k}}^{\dagger}$ be the operator that creates a hole in valence band. Then we write

$$c_{\mathbf{k}\uparrow} = c_{\mathbf{k}}, \tag{83}$$

$$c_{\mathbf{k}\downarrow} = d_{-\mathbf{k}}^{\dagger}. \tag{84}$$

Then we have the following formulas for the electron-hole sea-displacement operators (again for both $\mathbf{q} = 0$ and $\mathbf{q} \neq 0$)

$$A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow) = [1 - n_F^h(-\mathbf{k} + \mathbf{q}/2)][1 - n_F^e(\mathbf{k} + \mathbf{q}/2)]$$

$$\times \frac{1}{\sqrt{\hat{\mathbf{1}} - d_{-\mathbf{k} + \mathbf{q}/2}^{\dagger} d_{-\mathbf{k} + \mathbf{q}/2}}} d_{-\mathbf{k} + \mathbf{q}/2} c_{\mathbf{k} + \mathbf{q}/2}, \tag{85}$$

$$A_{\mathbf{k}\uparrow}(\mathbf{q}\downarrow) = n_F^h(\mathbf{k} + \mathbf{q}/2) n_F^e(\mathbf{k} - \mathbf{q}/2)$$

$$\times \frac{1}{\sqrt{c_{\mathbf{k} - \mathbf{q}/2}^{\dagger} c_{\mathbf{k} - \mathbf{q}/2}}} c_{\mathbf{k} - \mathbf{q}/2}^{\dagger} d_{-\mathbf{k} - \mathbf{q}/2}^{\dagger}, \tag{86}$$

$$A_{\mathbf{k}\uparrow}(\mathbf{q}\uparrow) = n_F^e(\mathbf{k}-\mathbf{q}/2)[1-n_F^e(\mathbf{k}+\mathbf{q}/2)]$$

$$\times \frac{1}{\sqrt{c_{\mathbf{k}-\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}}} c_{\mathbf{k}-\mathbf{q}/2}^\dagger c_{\mathbf{k}+\mathbf{q}/2}, \qquad (87)$$

$$A_{\mathbf{k}\downarrow}(\mathbf{q}\downarrow) = -n_F^h(-\mathbf{k}-\mathbf{q}/2)[1-n_F^h(-\mathbf{k}+\mathbf{q}/2)]$$

$$\times \frac{1}{\sqrt{\hat{\mathbf{1}}-d_{-\mathbf{k}+\mathbf{q}/2}^\dagger d_{-\mathbf{k}+\mathbf{q}/2}}} d_{-\mathbf{k}-\mathbf{q}/2}^\dagger d_{-\mathbf{k}+\mathbf{q}/2}.$$

$$(88)$$

If the system is undoped, we are tempted to set $n_F^e(\mathbf{k}) = n_F^h(\mathbf{k}) = 0$. But this would be unwise. For such a choice would make all the sea-displacement operators but one, identically zero and this would lead to the conclusion that the exciton-exciton interactions are of the two-body type exclusively. We will soon argue that this underestimates the strength and importance of exciton-exciton interactions. The reason for this fallacy it seems is that we have to careful with the order in which we take limits. At any nonzero temperature, we expect that the $\mathbf{k}=0$ state of the noninteracting system is always occupied even in a fully undoped system. Thermal fluctuations lead to a nonempty band. This in turn facilitates exciton-exciton interaction, since we may now contemplate intraband particle-hole excitations competing for prominence with excitons and scattering off them and so on. In order to make all this more concrete, let us proceed as follows. We know that for a noninteracting Fermi system,

$$n_F^e(\mathbf{k}) = \theta(k_F - |\mathbf{k}|), \quad n_F^h(\mathbf{k}) = \theta(k_F - |\mathbf{k}|). \qquad (89)$$

If we go to the limit of an undoped system, $k_F \to 0$ we find the following result:

$$n_F^e(\mathbf{k}) = n_F^h(\mathbf{k}) = \delta_{\mathbf{k},0}. \qquad (90)$$

That is, only the zero momentum state is occupied, the rest are unoccupied. This choice enables us to have a scheme by which even in an undoped system, exciton-exciton interactions may be present and may contribute to the line shape of the exciton. With this simplification we may write (for both $\mathbf{q}=0$ and $\mathbf{q}\neq 0$)

$$A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow) = (1-\delta_{\mathbf{k},\mathbf{q}/2})(1-\delta_{\mathbf{k},-\mathbf{q}/2})$$

$$\times \frac{1}{\sqrt{\hat{\mathbf{1}}-d_{-\mathbf{k}+\mathbf{q}/2}^\dagger d_{-\mathbf{k}+\mathbf{q}/2}}} d_{-\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2}, \quad (91)$$

$$A_{\mathbf{k}\uparrow}(\mathbf{q}\downarrow) = \delta_{\mathbf{q},0}\delta_{\mathbf{k},0}\frac{1}{\sqrt{c_0^\dagger c_0}}c_0^\dagger d_0^\dagger, \qquad (92)$$

$$A_{\mathbf{k}\uparrow}(\mathbf{q}\uparrow) = \delta_{\mathbf{k},\mathbf{q}/2}(1-\delta_{\mathbf{k},-\mathbf{q}/2})\frac{1}{\sqrt{c_0^\dagger c_0}}c_0^\dagger c_{\mathbf{q}}, \qquad (93)$$

$$A_{\mathbf{k}\downarrow}(\mathbf{q}\downarrow) = -\delta_{\mathbf{k},-\mathbf{q}/2}(1-\delta_{\mathbf{k},\mathbf{q}/2})\frac{1}{\sqrt{\hat{\mathbf{1}}-d_{\mathbf{q}}^\dagger d_{\mathbf{q}}}}d_0^\dagger d_{\mathbf{q}}. \quad (94)$$

Let us now ascertain the commutation rules obeyed by these objects. Following the prescription outlined in Appendix B, we may write down the RPA-like commutation rules by replacing the right side of the approximate ones by the leading order results. If $\mathbf{k}\neq \pm\mathbf{q}/2$ and $\mathbf{k}'\neq \pm\mathbf{q}'/2$ and $\mathbf{q}\neq 0$ then,

$$[A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow), A_{\mathbf{k}'\downarrow}(\mathbf{q}'\uparrow)] = 0,$$

$$[A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow), A_{\mathbf{k}'\downarrow}^\dagger(\mathbf{q}'\uparrow)] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}. \qquad (95)$$

Also,

$$[A_{\mathbf{k}\downarrow}(0\uparrow), A_{\mathbf{k}'\downarrow}(0\uparrow)] = 0, \quad [A_{\mathbf{k}\downarrow}(0\uparrow), A_{\mathbf{k}'\downarrow}^\dagger(0\uparrow)] = \delta_{\mathbf{k},\mathbf{k}'},$$

$$(96)$$

$$[A_{0\uparrow}(0\downarrow), A_{0\uparrow}^\dagger(0\downarrow)] = 1, \qquad (97)$$

$$[A_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow), A_{\mathbf{q}'/2\uparrow}(\mathbf{q}'\uparrow)] = 0,$$

$$[A_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow), A_{\mathbf{q}'/2\uparrow}^\dagger(\mathbf{q}'\uparrow)] = \delta_{\mathbf{q},\mathbf{q}'}, \qquad (98)$$

$$[A_{\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow), A_{\mathbf{q}'/2\downarrow}(\mathbf{q}'\downarrow)] = 0,$$

$$[A_{\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow), A_{\mathbf{q}'/2\downarrow}^\dagger(\mathbf{q}'\downarrow)] = \delta_{\mathbf{q},\mathbf{q}'}. \qquad (99)$$

Finally, $A_{\mathbf{k}\uparrow}(0\uparrow) = A_{\mathbf{k}\downarrow}(0\downarrow) = 0$. All other commutators involving any two of these sea-displacement operators are zero. In deriving these commutation rules, use has been made of the following approximate formulas. On the right-hand side of commutation rules we are obliged to set the off diagonal terms to be identically zero and $c_0^\dagger c_0 = d_0^\dagger d_0 \approx 1$. $c_{\mathbf{k}}^\dagger c_{\mathbf{k}} = c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \approx 0$. It is worthwhile to verify some of these commutation rules explicitly. For example,

$$[A_{0\uparrow}(0\downarrow), A_{0\uparrow}^\dagger(0\downarrow)] \approx [c_0^\dagger d_0^\dagger, d_0 c_0] = c_0^\dagger c_0 - d_0 d_0^\dagger = c_0^\dagger c_0$$

$$+ d_0^\dagger d_0 - 1 \approx 2 - 1 = 1. \qquad (100)$$

The rest are reasonably straightforward. Using these facts, we may write the following correspondence for the number conserving Fermi bilinears (the total number of electrons and holes commutes with these objects; see Appendix B for some hints as to how to derive these formulas). Here $\mathbf{q}\neq 0$, and we have singled out $\mathbf{q}=0$ as a special case. If $\mathbf{k}\neq \pm\mathbf{q}/2$

$$c^\dagger_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=\lambda A^\dagger_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2\uparrow)A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2\uparrow),+\lambda\sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2}A^\dagger_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}-\mathbf{q}_1/2\downarrow}(-\mathbf{q}+\mathbf{q}_1\uparrow),$$
$$(101)$$

$$d^\dagger_{-\mathbf{k}+\mathbf{q}/2}d_{-\mathbf{k}-\mathbf{q}/2}=\lambda\, A^\dagger_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}(-\mathbf{k}+\mathbf{q}/2\downarrow)A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2\downarrow)+\lambda\sum_{\mathbf{q}_1\neq-\mathbf{k}+\mathbf{q}/2}A^\dagger_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}_1/2\downarrow}(-\mathbf{q}+\mathbf{q}_1\uparrow),$$
$$(102)$$

$$c^\dagger_0 c_{-\mathbf{q}}=\sqrt{c^\dagger_0 c_0}A_{-\mathbf{q}/2\uparrow}(-\mathbf{q}\uparrow)\approx A_{-\mathbf{q}/2\uparrow}(-\mathbf{q}\uparrow),$$
$$(103)$$

$$c^\dagger_{\mathbf{q}}c_0=A^\dagger_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow)\sqrt{c^\dagger_0 c_0}\approx A^\dagger_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow),$$
$$(104)$$

$$d_{-\mathbf{q}}d^\dagger_0=(\hat{\mathbf{1}}-d^\dagger_{-\mathbf{q}}d_{-\mathbf{q}})^{1/2}A_{\mathbf{q}/2\downarrow}(-\mathbf{q}\downarrow)\approx A_{\mathbf{q}/2\downarrow}(-\mathbf{q}\downarrow),$$
$$(105)$$

$$d_0 d^\dagger_{\mathbf{q}}=A^\dagger_{-\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow)(\hat{\mathbf{1}}-d^\dagger_{\mathbf{q}}d_{\mathbf{q}})^{1/2}\approx A^\dagger_{-\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow).$$
$$(106)$$

Also we have for the number operators ($\mathbf{k}\neq 0$):

$$c^\dagger_0 c_0=\hat{\mathbf{1}}-\lambda\sum_{\mathbf{q}_1\neq 0}A^\dagger_{\mathbf{q}_1/2\uparrow}(\mathbf{q}_1\uparrow)A_{\mathbf{q}_1/2\uparrow}(\mathbf{q}_1\uparrow)-\lambda A^\dagger_{0\uparrow}(0\downarrow)A_{0\uparrow}(0\downarrow),$$
$$(107)$$

$$c^\dagger_{\mathbf{k}}c_{\mathbf{k}}=\lambda\, A^\dagger_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow)A_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow)+\lambda\sum_{\mathbf{q}_1\neq\mathbf{k}}A^\dagger_{\mathbf{k}-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow),$$
$$(108)$$

$$d^\dagger_0 d_0=\hat{\mathbf{1}}-\lambda\sum_{\mathbf{q}_1\neq 0}A^\dagger_{-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\downarrow)A_{-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\downarrow)-\lambda A^\dagger_{0\uparrow}(0\downarrow)A_{0\uparrow}(0\downarrow),$$
$$(109)$$

$$d^\dagger_{-\mathbf{k}}d_{-\mathbf{k}}=\lambda A^\dagger_{\mathbf{k}/2\downarrow}(-\mathbf{k}\downarrow)A_{\mathbf{k}/2\downarrow}(-\mathbf{k}\downarrow)+\lambda\sum_{\mathbf{q}_1\neq-\mathbf{k}}A^\dagger_{\mathbf{k}+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow).$$
$$(110)$$

Now for the charge-conserving Fermi bilinear (the total charge operator commutes with these objects). If $\mathbf{k}\neq\pm\mathbf{q}/2$ then (with $\mathbf{q}\neq 0$)

$$d_{-\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=(\hat{\mathbf{1}}-d^\dagger_{-\mathbf{k}-\mathbf{q}/2}d_{-\mathbf{k}-\mathbf{q}/2})^{1/2}A_{\mathbf{k}\downarrow}(-\mathbf{q}\uparrow)$$
$$\approx A_{\mathbf{k}\downarrow}(-\mathbf{q}\uparrow)\qquad(111)$$

if, in addition, we have $\mathbf{k}\neq 0$

$$d_{-\mathbf{k}}c_{\mathbf{k}}=(\hat{\mathbf{1}}-d^\dagger_{-\mathbf{k}}d_{-\mathbf{k}})^{1/2}A_{\mathbf{k}\downarrow}(0\uparrow)\approx A_{\mathbf{k}\downarrow}(0\uparrow)\qquad(112)$$

if $\mathbf{k}=0$

$$d_0 c_0=A^\dagger_{0\uparrow}(0\downarrow)\sqrt{c^\dagger_0 c_0}\approx A^\dagger_{0\uparrow}(0\downarrow),\qquad(113)$$

$$d_0 c_{-\mathbf{q}}=\lambda A^\dagger_{0\uparrow}(0\downarrow)A_{-\mathbf{q}/2\uparrow}(-\mathbf{q}\uparrow)$$
$$+\lambda\sum_{\mathbf{q}_1\neq 0}A^\dagger_{-\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\downarrow)A_{-\mathbf{q}/2-\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow),$$
$$(114)$$

$$d_{-\mathbf{q}}c_0=-\lambda A^\dagger_{0\uparrow}(0\downarrow)A_{\mathbf{q}/2\downarrow}(-\mathbf{q}\downarrow)$$
$$-\lambda\sum_{\mathbf{q}_1\neq 0}A^\dagger_{\mathbf{q}_1/2,\uparrow}(\mathbf{q}_1\uparrow)A_{\mathbf{q}/2+\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow).$$
$$(115)$$

Let us now verify some of these correspondences. For example, we know that in the Fermi language,

$$[c^\dagger_0 c_0,c^\dagger_0 c_{-\mathbf{q}}]=c^\dagger_0 c_{-\mathbf{q}}.\qquad(116)$$

In the sea-displacement language we have

$$[c^\dagger_0 c_0,c^\dagger_0 c_{-\mathbf{q}}]=\sum_{\mathbf{q}_1\neq 0}[A_{-\mathbf{q}/2\uparrow}(-\mathbf{q}\uparrow),A^\dagger_{\mathbf{q}_1/2\uparrow}(\mathbf{q}_1\uparrow),$$
$$A_{\mathbf{q}_1/2\uparrow}(\mathbf{q}_1\uparrow)]=A_{-\mathbf{q}/2\uparrow}(-\mathbf{q}\uparrow)\approx c^\dagger_0 c_{-\mathbf{q}}\qquad(117)$$

as required. Similarly, we have

$$[c^\dagger_0 c_0,d_0 c_0]=-d_0 c_0.\qquad(118)$$

In the sea-displacement language we have

$$[c_0^\dagger c_0, d_0 c_0] = -[A_{\mathbf{0}\uparrow}^\dagger(\mathbf{0}\downarrow)A_{\mathbf{0}\uparrow}(\mathbf{0}\downarrow), A_{\mathbf{0}\uparrow}^\dagger(\mathbf{0}\downarrow)]$$

$$= -A_{\mathbf{0}\uparrow}^\dagger(\mathbf{0}\downarrow) \approx -d_0 c_0. \quad (119)$$

Next, we write down the Hamiltonian of the electron-hole system interacting via repulsion and attractive interactions.

We show how excitons emerge naturally from the formalism. We are also able to pin down the precise nature of exciton-exciton interactions. The total Hamiltonian may be split into several parts. First is the kinetic energy plus the part of the potential energy that leads to the exciton:

$$H_0 = \sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2m_e}+E_g\right)c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2m_h}\right)d_{-\mathbf{k}}^\dagger d_{-\mathbf{k}} + E_g c_0^\dagger c_0 - \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}\sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq 0}c_{\mathbf{k}+\mathbf{q}/2}^\dagger d_{-\mathbf{k}'-\mathbf{q}/2}^\dagger d_{-\mathbf{k}'+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}, \quad (120)$$

$$H_I = \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}';\mathbf{k}\pm\mathbf{q}/2\neq 0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq 0}c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}'-\mathbf{q}/2}^\dagger c_{\mathbf{k}'+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}$$

$$+\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}';\mathbf{k}\pm\mathbf{q}/2\neq 0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq 0}d_{-\mathbf{k}-\mathbf{q}/2}^\dagger d_{-\mathbf{k}'+\mathbf{q}/2}^\dagger d_{-\mathbf{k}'-\mathbf{q}/2}d_{-\mathbf{k}+\mathbf{q}/2} - \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}n_{\mathbf{k}+\mathbf{q}/2}^{(e)}n_{\mathbf{k}-\mathbf{q}/2}^{(e)}$$

$$-\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}n_{-\mathbf{k}-\mathbf{q}/2}^{(h)}n_{-\mathbf{k}+\mathbf{q}/2}^{(h)}, \quad (121)$$

$$H_{I,0} = \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}(c_0^\dagger c_{\mathbf{q}}+c_{-\mathbf{q}}^\dagger c_0)c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} + \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}(d_{-\mathbf{q}}^\dagger d_0 + d_0^\dagger d_{\mathbf{q}})d_{-\mathbf{k}+\mathbf{q}/2}^\dagger d_{-\mathbf{k}-\mathbf{q}/2}$$

$$-\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0}(c_{\mathbf{k}+\mathbf{q}/2}^\dagger d_0^\dagger d_{\mathbf{q}}c_{\mathbf{k}-\mathbf{q}/2}+c_{\mathbf{k}+\mathbf{q}/2}^\dagger d_{-\mathbf{q}}^\dagger d_0 c_{\mathbf{k}-\mathbf{q}/2}) - \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{V}\sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq 0}(c_0^\dagger d_{-\mathbf{k}'+\mathbf{q}/2}^\dagger d_{-\mathbf{k}'-\mathbf{q}/2}c_{\mathbf{q}}$$

$$+c_{-\mathbf{q}}^\dagger d_{-\mathbf{k}'+\mathbf{q}/2}^\dagger d_{-\mathbf{k}'-\mathbf{q}/2}c_0). \quad (122)$$

The part H_I when written out in terms of sea displacements have four bosons in them. Whereas the term $H_{I,0}$ has three bosons. The kinetic energy operator has only two bosons. Thus we may systematically regard the kinetic energy as being more important than $H_{I,0}$ which in turn is more important than H_I. In the sea-displacement language, we may write up to additive constants

$$H_0 = \sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2m_e}\right)A_{\mathbf{k}/2\uparrow}^\dagger(\mathbf{k}\uparrow)A_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow) + \sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2m_h}\right)$$

$$\times A_{\mathbf{k}/2\downarrow}^\dagger(-\mathbf{k}\downarrow)A_{\mathbf{k}/2\downarrow}(-\mathbf{k}\downarrow) - E_g A_{\mathbf{0}\uparrow}^\dagger(\mathbf{0}\downarrow)A_{\mathbf{0}\uparrow}(\mathbf{0}\downarrow)$$

$$+\sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2\mu}+E_g\right)A_{\mathbf{k}\downarrow}^\dagger(\mathbf{0}\uparrow)A_{\mathbf{k}\downarrow}(\mathbf{0}\uparrow) - \sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}$$

$$\times A_{\mathbf{k}\downarrow}^\dagger(\mathbf{0}\uparrow)A_{\mathbf{k}'\downarrow}(\mathbf{0}\uparrow) + \sum_{\mathbf{k},\mathbf{q}\neq 0}\omega_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}\downarrow}^\dagger(\mathbf{q}\uparrow)A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow)$$

$$-\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}\sum_{\mathbf{q}\neq 0}A_{\mathbf{k}\downarrow}^\dagger(\mathbf{q}\uparrow)A_{\mathbf{k}'\downarrow}(\mathbf{q}\uparrow), \quad (123)$$

where

$$\omega_{\mathbf{k}}(\mathbf{q}) = \left[\frac{k^2}{2\mu}+E_g+\frac{\mathbf{k}\cdot\mathbf{q}}{2}\left(\frac{1}{m_e}-\frac{1}{m_h}\right)+\frac{1}{4}\frac{q^2}{2\mu}\right]$$

$$\times(1-\delta_{\mathbf{k},\mathbf{q}/2})(1-\delta_{\mathbf{k},-\mathbf{q}/2}). \quad (124)$$

This Hamiltonian has a very appealing form. The term containing $A_{\mathbf{k}\downarrow}(\mathbf{0}\uparrow)$ has been singled out since it highlights the exciton. This is nothing but the Hamiltonian of the free exciton with center-of mass momentum equal to zero. The other Hamiltonian involving $A_{\mathbf{k}\downarrow}(\mathbf{q}\uparrow)$ corresponds to an exciton with nonzero center-of-mass motion. If for example, we write,

$$A_{\mathbf{k}\downarrow}(\mathbf{0}\uparrow) = \sum_I \tilde{\varphi}_I(\mathbf{k})b_I(0), \quad (125)$$

where $\tilde{\varphi}_I(\mathbf{k})$ is the Fourier transform of excitonic (hydrogenic) wave functions, then the Hamiltonian may be recast in the diagonal form

$$\sum_{\mathbf{k}\neq 0}\left(\frac{k^2}{2\mu}+E_g\right)A_{\mathbf{k}\downarrow}^\dagger(\mathbf{0}\uparrow)A_{\mathbf{k}\downarrow}(\mathbf{0}\uparrow) - \sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}$$

$$\times A_{\mathbf{k}\downarrow}^\dagger(\mathbf{0}\uparrow)A_{\mathbf{k}'\downarrow}(\mathbf{0}\uparrow)$$

$$= \sum_I E_I(0)b_I^\dagger(0)b_I(0), \quad (126)$$

where $E_I(0)$ are the energy levels of the exciton. We shall assume that a similar transformation has been performed for the full Hamiltonian including center-of-mass motion

$$H_0 = \sum_{I,\mathbf{q}} E_I(\mathbf{q}) b_I^\dagger(\mathbf{q}) b_I(\mathbf{q}) - E_g A_{0\uparrow}^\dagger(0\downarrow) A_{0\uparrow}(0\downarrow)$$

$$+ \sum_{\mathbf{k}\neq 0} \left(\frac{k^2}{2m_e}\right) A_{\mathbf{k}/2\uparrow}^\dagger(\mathbf{k}\uparrow) A_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow)$$

$$+ \sum_{\mathbf{k}\neq 0} \left(\frac{k^2}{2m_h}\right) A_{\mathbf{k}/2\downarrow}^\dagger(-\mathbf{k}\downarrow) A_{\mathbf{k}/2\downarrow}(-\mathbf{k}\downarrow). \qquad (127)$$

Of particular interest in the term $-E_g A_{0\uparrow}^\dagger(0\downarrow) A_{0\uparrow}(0\downarrow)$. It suggests that the ground state of the system has one of these bosons present. This is consistent with the earlier observation since we have $\mathbf{k}=0$ occupied by one electron and one hole. Since $[A_{0\uparrow}(0\downarrow)]^2=0$ it is clear that we can have only one of these bosons present. Let $|0\rangle$ be the vacuum of the fermions. Thus the $A_{0\uparrow}(0\downarrow)$ creates a solitron. The vacuum of the fermions may be written as

$$c_\mathbf{k}|0\rangle = 0, \quad d_{-\mathbf{k}}|0\rangle = 0. \qquad (128)$$

It may be seen that this vacuum is the vacuum of all the bosons as well *except* $A_{0\uparrow}(0\downarrow)$. The action of $A_{0\uparrow}(0\downarrow)$ on $|0\rangle$

produces an electron-hole pair at $\mathbf{k}=0$. Since we have argued earlier that this corresponds to the ground state of the free theory, we have the following fact. The ground state of H_0 is given by

$$|G\rangle = A_{0\uparrow}(0\downarrow)|0\rangle. \qquad (129)$$

It may be seen quite easily that

$$c_0^\dagger c_0|G\rangle = d_0^\dagger d_0|G\rangle = 1|G\rangle. \qquad (130)$$

The object $A_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow)$ annihilates a conductron. Finally the operator $A_{-\mathbf{k}/2\downarrow}(\mathbf{k}\downarrow)$ annihilates a valeron. Now we would like to see how this evolves under the presence of the exciton-exciton interaction terms. A few words regarding these are in order. We will see here perhaps for the first time, the true nature of exciton-exciton interactions. They do not interact via simple two-body interactions rather the interaction is mediated by other bosons such as $A_{\mathbf{k}/2\uparrow}(\mathbf{k}\uparrow)$ which are intraband particle-hole excitations. Further we have also coupling to the object $A_{0\uparrow}^\dagger(0\downarrow)$ which is an interband zero momentum electron-hole excitation. Thus exciton-exciton interactions are mediated by these other bosons and the interactions are rather more complex than the simple two-body variety. Let us write down precisely what they are.

$$H_{I,0} = \lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} (A_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow) + A_{-\mathbf{q}/2\uparrow}^\dagger(-\mathbf{q}\uparrow)) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}^\dagger(\mathbf{k}+\mathbf{q}/2,\uparrow) A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2,\uparrow)$$

$$+\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} (A_{\mathbf{q}/2\uparrow}(\mathbf{q}\uparrow) + A_{-\mathbf{q}/2\uparrow}^\dagger(-\mathbf{q}\uparrow)) \sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2} A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}^\dagger(\mathbf{q}_1,\uparrow) A_{\mathbf{k}-\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow)$$

$$+\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} (-A_{\mathbf{q}/2\downarrow}^\dagger(-\mathbf{q}\downarrow) - A_{-\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow)) A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}^\dagger(-\mathbf{k}+\mathbf{q}/2,\downarrow) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2,\downarrow)$$

$$+\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} (-A_{\mathbf{q}/2\downarrow}^\dagger(-\mathbf{q}\downarrow) - A_{-\mathbf{q}/2\downarrow}(\mathbf{q}\downarrow)) \sum_{\mathbf{q}_1\neq -\mathbf{k}+\mathbf{q}/2} A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2\downarrow}^\dagger(\mathbf{q}_1,\uparrow) A_{\mathbf{k}+\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow)$$

$$-\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}^\dagger(\mathbf{k}+\mathbf{q}/2,\uparrow) A_{0,\uparrow}(0,\downarrow) A_{(1/2)(\mathbf{k}-\mathbf{q}/2-\mathbf{q}),\downarrow}(\mathbf{k}+\mathbf{q}/2,\uparrow)$$

$$-\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} A_{(1/2)(\mathbf{k}-\mathbf{q}/2-\mathbf{q}),\downarrow}^\dagger(\mathbf{k}+\mathbf{q}/2,\uparrow) A_{0,\uparrow}^\dagger(0,\downarrow) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2,\uparrow)$$

$$+\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq 0} A_{(1/2)(\mathbf{k}'-\mathbf{q}/2),\downarrow}^\dagger(-\mathbf{k}'+\mathbf{q}/2,\downarrow) A_{0,\uparrow}(0,\downarrow) A_{(1/2)(\mathbf{k}'+\mathbf{q}/2+\mathbf{q}),\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\uparrow)$$

$$+\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq 0} A_{(1/2)(\mathbf{k}'+\mathbf{q}/2+\mathbf{q}),\downarrow}^\dagger(-\mathbf{k}'+\mathbf{q}/2,\uparrow) A_{0,\uparrow}^\dagger(0,\downarrow) A_{(1/2)(\mathbf{k}'-\mathbf{q}/2)\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\downarrow)$$

$$-\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} \sum_{\mathbf{q}_1\neq 0} A_{(1/2)(\mathbf{k}+\mathbf{q}/2)-\mathbf{q}_1/2,\downarrow}^\dagger(\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1,\uparrow) A_{-\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\downarrow) A_{(1/2)(\mathbf{k}-\mathbf{q}/2-\mathbf{q}),\downarrow}(\mathbf{k}+\mathbf{q}/2,\uparrow)$$

$$-\lambda \sum_{\mathbf{q}\neq 0} \frac{v_\mathbf{q}}{V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} \sum_{\mathbf{q}_1\neq 0} A_{(1/2)(\mathbf{k}-\mathbf{q}/2-\mathbf{q}),\downarrow}^\dagger(\mathbf{k}+\mathbf{q}/2,\uparrow) A_{-\mathbf{q}_1/2,\downarrow}^\dagger(\mathbf{q}_1\downarrow) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)-\mathbf{q}_1/2,\downarrow}(\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1,\uparrow)$$

$$+\lambda\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{V}\sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq0}A^{\dagger}_{(1/2)(\mathbf{k}'-\mathbf{q}/2)+\mathbf{q}_1/2,\downarrow}(-\mathbf{k}'+\mathbf{q}/2+\mathbf{q}_1,\uparrow)A_{\mathbf{q}_1/2,\uparrow}(\mathbf{q}_1\uparrow)A_{(1/2)(\mathbf{k}'+\mathbf{q}/2+\mathbf{q}),\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\uparrow)$$

$$+\lambda\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{V}\sum_{-\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq0}A^{\dagger}_{(1/2)(\mathbf{k}'+\mathbf{q}/2+\mathbf{q}),\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\uparrow)A^{\dagger}_{\mathbf{q}_1/2,\uparrow}(\mathbf{q}_1\uparrow)A_{(1/2)(\mathbf{k}'-\mathbf{q}/2)+\mathbf{q}_1/2,\downarrow}(-\mathbf{k}'+\mathbf{q}/2+\mathbf{q}_1,\uparrow),\quad(131)$$

$$H_I=\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2,\uparrow)A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2,\uparrow)A^{\dagger}_{(1/2)(\mathbf{k}'-\mathbf{q}/2)\uparrow}$$

$$\times(\mathbf{k}'-\mathbf{q}/2,\uparrow)A_{(1/2)(\mathbf{k}'+\mathbf{q}/2)\uparrow}(\mathbf{k}'+\mathbf{q}/2,\uparrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}-\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow)A^{\dagger}_{(1/2)(\mathbf{k}'-\mathbf{q}/2)\uparrow}$$

$$\times(\mathbf{k}'-\mathbf{q}/2,\uparrow)A_{(1/2)(\mathbf{k}'+\mathbf{q}/2)\uparrow}(\mathbf{k}'+\mathbf{q}/2,\uparrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2,\uparrow)A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}$$

$$\times(\mathbf{k}-\mathbf{q}/2,\uparrow)A^{\dagger}_{\mathbf{k}'-\mathbf{q}/2-\mathbf{q}_2/2,\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}'-\mathbf{q}_2/2,\downarrow}(\mathbf{q}+\mathbf{q}_2,\uparrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2}\sum_{\mathbf{q}_2\neq\mathbf{k}'-\mathbf{q}/2}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}-\mathbf{q}_1/2,\downarrow}$$

$$\times(-\mathbf{q}+\mathbf{q}_1,\uparrow)A^{\dagger}_{\mathbf{k}'-\mathbf{q}/2-\mathbf{q}_2/2,\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}'-\mathbf{q}_2/2,\downarrow}(\mathbf{q}+\mathbf{q}_2,\uparrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}(-\mathbf{k}+\mathbf{q}/2,\downarrow)A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2,\downarrow)A^{\dagger}_{(1/2)(\mathbf{k}'+\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}'-\mathbf{q}/2,\downarrow)A_{(1/2)(\mathbf{k}'-\mathbf{q}/2)\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\downarrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq-\mathbf{k}+\mathbf{q}/2}A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}_1/2,\downarrow}(-\mathbf{q}+\mathbf{q}_1,\uparrow)A^{\dagger}_{(1/2)(\mathbf{k}'+\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}'-\mathbf{q}/2,\downarrow)A_{(1/2)(\mathbf{k}'-\mathbf{q}/2)\downarrow}(-\mathbf{k}'+\mathbf{q}/2,\downarrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_2\neq-\mathbf{k}'-\mathbf{q}/2}A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}(-\mathbf{k}+\mathbf{q}/2,\downarrow)A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}-\mathbf{q}/2,\downarrow)A^{\dagger}_{\mathbf{k}'+\mathbf{q}/2+\mathbf{q}_2/2,\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}'+\mathbf{q}_2/2,\downarrow}(\mathbf{q}+\mathbf{q}_2,\uparrow)$$

$$+\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{k}'\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq-\mathbf{k}+\mathbf{q}/2}\sum_{\mathbf{q}_2\neq-\mathbf{k}'-\mathbf{q}/2}A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2,\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}_1/2,\downarrow}$$

$$\times(-\mathbf{q}+\mathbf{q}_1,\uparrow)A^{\dagger}_{\mathbf{k}'+\mathbf{q}/2+\mathbf{q}_2/2,\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}'+\mathbf{q}_2/2,\downarrow}(\mathbf{q}+\mathbf{q}_2,\uparrow)$$

$$-\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2\uparrow)A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2\uparrow)A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2\uparrow)A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2\uparrow)$$

$$-\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_2\neq\mathbf{k}-\mathbf{q}/2}A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2\uparrow)A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\uparrow}(\mathbf{k}+\mathbf{q}/2\uparrow)A^{\dagger}_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow)$$

$$-\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\uparrow}(\mathbf{k}-\mathbf{q}/2,\uparrow)A_{(1/2)(\mathbf{k}-\mathbf{q}/2),\uparrow}(\mathbf{k}-\mathbf{q}/2\uparrow)$$

$$-\lambda^2\sum_{\mathbf{q}\neq0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}\pm\mathbf{q}/2\neq0}\sum_{\mathbf{q}_1\neq\mathbf{k}+\mathbf{q}/2}\sum_{\mathbf{q}_2\neq\mathbf{k}-\mathbf{q}/2}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow)A^{\dagger}_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow)A_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow)$$

$$-\lambda^2 \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{-\mathbf{k}\pm\mathbf{q}/2\neq 0} A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2\downarrow) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2\downarrow) A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}+\mathbf{q}/2\downarrow) A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}(-\mathbf{k}+\mathbf{q}/2\downarrow)$$

$$-\lambda^2 \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{-\mathbf{k}\pm\mathbf{q}/2\neq 0} \sum_{\mathbf{q}_2\neq -\mathbf{k}+\mathbf{q}/2} A^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}(-\mathbf{k}-\mathbf{q}/2\downarrow) A_{(1/2)(\mathbf{k}+\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}-\mathbf{q}/2\downarrow) A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow) A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow)$$

$$-\lambda^2 \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{-\mathbf{k}\pm\mathbf{q}/2\neq 0} \sum_{\mathbf{q}_1\neq -\mathbf{k}-\mathbf{q}/2} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow) A_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow) A^{\dagger}_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}$$

$$\times(-\mathbf{k}+\mathbf{q}/2,\downarrow) A_{(1/2)(\mathbf{k}-\mathbf{q}/2)\downarrow}(-\mathbf{k}+\mathbf{q}/2,\downarrow)$$

$$-\lambda^2 \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}\pm\mathbf{q}/2\neq 0} \sum_{\mathbf{q}_1\neq -\mathbf{k}-\mathbf{q}/2} \sum_{\mathbf{q}_2\neq -\mathbf{k}+\mathbf{q}/2} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow) A_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2\downarrow}(\mathbf{q}_1\uparrow) A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow) A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_2/2\downarrow}(\mathbf{q}_2\uparrow).$$

$$(132)$$

It is clear that this Hamiltonian is rather more complex than the simple two-body variety. In particular, the leading contribution which is first order in λ has three operators signifying a nonconservation of exciton number. In one such process, an exciton recombines with a solitron to produce a conductron. Only excitons couple with external fields and we are able to infer the existence of the other excitations indirectly, by studying the binding energy of biexcitons that are due to these exotic many-body processes. From the above form of the interaction terms we see that the ground state of H_0 does not evolve with time under the action of the perturbation. However, if we first create an exciton (or two excitons) and then evolve the state under the action of the full Hamiltonian, then the state does indeed evolve and produces all the effects that we expect, such as a possible exciton-line width, biexciton (bound state of two excitons), and so on. Let us now try to verify these expectations rigorously.

A. The Exciton Green Function

In this section, we compute the exciton Green function as resulting from exciton-exciton interactions and compare the lifetime of the exciton arising from exciton-exciton interactions with the lifetime from usual radiative recombination processes. Consider the initial state which corresponds to an exciton in internal state I and whose center of mass is moving with momentum $\mathbf{Q}$. Then we may write

$$|I,\mathbf{Q}\rangle = b^{\dagger}_I(\mathbf{Q})|G\rangle. \qquad (133)$$

Let us now examine how this state evolves with time.

$$e^{-i\,t(H+\lambda H_{I,0}+\lambda^2 H_I)}|I,\mathbf{Q}\rangle \approx e^{-i\,t H_0}\left(\hat{\mathbf{1}} - i\lambda \int_0^t dt_1 \hat{H}_{I,0}(t_1) + (-i)^2 \lambda^2 \int_0^t dt_1 \hat{H}_{I,0}(t_1) \right.$$

$$\left. \times \int_0^{t_1} dt_2 \hat{H}_{I,0}(t_2) - i\lambda^2 \int_0^t dt_1 \hat{H}_I(t_1) \right) b^{\dagger}_I(\mathbf{Q})|G\rangle. \qquad (134)$$

There are many final states possible. We would like to compute the Green function

$$G_I(\mathbf{Q};\omega) = i \int_0^{\infty} dt\, e^{i\,\omega t} \langle I,\mathbf{Q}|e^{-it\,H}|I,\mathbf{Q}\rangle \qquad (135)$$

The exciton line shape is given by plotting

$$L(\omega;\mathbf{Q}) = -\sum_I \mathrm{Im}[G_I(\mathbf{Q};\omega - i\,0^+)]. \qquad (136)$$

We may evaluate the matrix element as

$$\langle I,\mathbf{Q}|e^{-i\,t\,(H_0+\lambda H_{I,0}+\lambda^2 H_I)}|I,\mathbf{Q}\rangle=e^{-i\epsilon_I(\mathbf{Q})t}+(-i)^2\lambda^2 e^{-i\epsilon_I(\mathbf{Q})t}\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)^2\varphi_I^*(\mathbf{k}_1+\mathbf{q}_1/2-\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}(\mathbf{k}_1-\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})$$

$$\times\varphi_{I_2}^*(\mathbf{k}_2-\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_I(\mathbf{k}_2+\mathbf{q}_1/2-\mathbf{Q}/2;\mathbf{Q})\int_0^t dt_1 e^{-i\epsilon_e(\mathbf{q}_1)t_1}e^{i\epsilon_I(\mathbf{Q})t_1}e^{-i\epsilon_{I_2}(-\mathbf{q}_1+\mathbf{Q})t_1}$$

$$\times\int_0^{t_1}dt_2 e^{i\epsilon_e(\mathbf{q}_1)t_2}e^{i\epsilon_{I_2}(\mathbf{Q}-\mathbf{q}_1)t_2}e^{-i\epsilon_I(\mathbf{Q})t_2}+(-i)^2\lambda^2 e^{-i\epsilon_I(\mathbf{Q})t}\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)^2$$

$$\times\varphi_I^*(\mathbf{k}_1-\mathbf{q}_1/2+\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}(\mathbf{k}_1+\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_{I_2}^*(\mathbf{k}_2+\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_I(\mathbf{k}_2-\mathbf{q}_1/2$$

$$+\mathbf{Q}/2;\mathbf{Q})\int_0^t dt_1 e^{-i\epsilon_h(\mathbf{Q}_1)t_1}e^{i\epsilon_I(\mathbf{Q})t_1}e^{-i\epsilon_{I_2}(-\mathbf{Q}_1+\mathbf{Q})t_1}\int_0^{t_1}dt_2 e^{i\,\epsilon_h(\mathbf{Q}_1)t_2}e^{i\epsilon_{I_2}(\mathbf{Q}-\mathbf{q}_1)t_2}e^{-i\epsilon_I(\mathbf{Q})t_2}$$

$$+(-i)^2\lambda^2 e^{-i\epsilon_I(\mathbf{Q})t}\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)\left(\frac{v_{\mathbf{q}_2}}{V}\right)\varphi_I^*[(1/2)\mathbf{Q}-\mathbf{Q}_1;\mathbf{Q}]\varphi_{I_2}[(1/2)(\mathbf{Q}-\mathbf{Q}_1)$$

$$-\mathbf{q}_1;\mathbf{Q}-\mathbf{Q}_1]\varphi_{I_2}^*[(1/2)(\mathbf{Q}-\mathbf{Q}_1)-\mathbf{q}_2;\mathbf{Q}-\mathbf{Q}_1]\varphi_I[(1/2)\mathbf{Q}-\mathbf{Q}_1;\mathbf{Q}]$$

$$\times\int_0^t dt_1 e^{-i\epsilon_h(\mathbf{Q}_1)t_1}e^{i\epsilon_I(\mathbf{Q})t_1}e^{-i\epsilon_{I_2}(-\mathbf{Q}_1+\mathbf{Q})t_1}\int_0^{t_1}dt_2 e^{i\epsilon_h(\mathbf{Q}_1)t_2}e^{i\epsilon_{I_2}(\mathbf{Q}-\mathbf{Q}_1)t_2}e^{-i\epsilon_I(\mathbf{Q})t_2}$$

$$+(-i)^2\lambda^2 e^{-i\epsilon_I(\mathbf{Q})t}\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)\left(\frac{v_{\mathbf{q}_2}}{V}\right)\varphi_I^*(\mathbf{Q}_1-\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}[(1/2)(\mathbf{Q}_1-\mathbf{Q})$$

$$+\mathbf{q}_1;\mathbf{Q}-\mathbf{Q}_1]\varphi_{I_2}^*[(1/2)(\mathbf{Q}_1-\mathbf{Q})+\mathbf{q}_2;\mathbf{Q}-\mathbf{Q}_1]\varphi_I(\mathbf{Q}_1-\mathbf{Q}/2;\mathbf{Q})$$

$$\times\int_0^t dt_1 e^{-i\,\epsilon_e(\mathbf{Q}_1)t_1}e^{i\epsilon_I(\mathbf{Q})t_1}e^{-i\epsilon_{I_2}(-\mathbf{Q}_1+\mathbf{Q})t_1}\int_0^{t_1}dt_2 e^{i\epsilon_e(\mathbf{Q}_1)t_2}e^{i\epsilon_{I_2}(\mathbf{Q}-\mathbf{Q}_1)t_2}e^{-i\epsilon_I(\mathbf{Q})t_2}. \quad (137)$$

We may rewrite the above equation retaining only the most singular parts as

$$\langle I\vec{Q}|e^{-it(H_0+\lambda H_{I0}+\lambda^2 H_I)}|I\vec{Q}\rangle=e^{-i\,t\,\epsilon_I(\mathbf{Q})}-i\,F_I(\mathbf{Q})\,t\,e^{-i\,t\,\epsilon_I(\mathbf{Q})}\approx e^{-i\,t[\epsilon_I(\mathbf{Q})+F_I(\mathbf{Q})]}, \quad (138)$$

where

$$F_I(\mathbf{Q})=-\lambda^2\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)^2\frac{\varphi_I^*(\mathbf{k}_1+\mathbf{q}_1/2-\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}(\mathbf{k}_1-\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_{I_2}^*(\mathbf{k}_2-\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_I(\mathbf{k}_2+\mathbf{q}_1/2-\mathbf{Q}/2;\mathbf{Q})}{\epsilon_e(\mathbf{q}_1)+\epsilon_{I_2}(\mathbf{Q}-\mathbf{q}_1)-\epsilon_I(\mathbf{Q})}$$

$$-\lambda^2\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)^2\frac{\varphi_I^*(\mathbf{k}_1-\mathbf{q}_1/2+\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}(\mathbf{k}_1+\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_{I_2}^*(\mathbf{k}_2+\mathbf{Q}/2;-\mathbf{q}_1+\mathbf{Q})\varphi_I(\mathbf{k}_2-\mathbf{q}_1/2+\mathbf{Q}/2;\mathbf{Q})}{\epsilon_h(\mathbf{q}_1)+\epsilon_{I_2}(\mathbf{Q}-\mathbf{q}_1)-\epsilon_I(\mathbf{Q})}$$

$$-\lambda^2\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)\left(\frac{v_{\mathbf{q}_2}}{V}\right)$$

$$\times\frac{\varphi_I^*(\mathbf{Q}_1-\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}[(1/2)(\mathbf{Q}_1-\mathbf{Q})+\mathbf{q}_1;\mathbf{Q}-\mathbf{Q}_1]\varphi_{I_2}^*[(1/2)(\mathbf{Q}_1-\mathbf{Q})+\mathbf{q}_2;\mathbf{Q}-\mathbf{Q}_1]\varphi_I(\mathbf{Q}_1-\mathbf{Q}/2;\mathbf{Q})}{\epsilon_e(\mathbf{Q}_1)+\epsilon_{I_2}(\mathbf{Q}-\mathbf{Q}_1)-\epsilon_I(\mathbf{Q})}$$

$$-\lambda^2\sum\left(\frac{v_{\mathbf{q}_1}}{V}\right)\left(\frac{v_{\mathbf{q}_2}}{V}\right)$$

$$\times\frac{\varphi_I^*(-\mathbf{Q}_1+\mathbf{Q}/2;\mathbf{Q})\varphi_{I_2}[(1/2)(\mathbf{Q}-\mathbf{Q}_1)-\mathbf{q}_1;\mathbf{Q}-\mathbf{Q}_1]\varphi_{I_2}^*[(1/2)(\mathbf{Q}-\mathbf{Q}_1)-\mathbf{q}_2;\mathbf{Q}-\mathbf{Q}_1]\varphi_I(-\mathbf{Q}_1+\mathbf{Q}/2;\mathbf{Q})}{\epsilon_h(\mathbf{Q}_1)+\epsilon_{I_2}(\mathbf{Q}-\mathbf{Q}_1)-\epsilon_I(\mathbf{Q})}. \quad (139)$$

Here $\epsilon_e(\mathbf{k})=k^2/2m_e$ and $\epsilon_h(\mathbf{k})=k^2/2m_h$. The line shape of the exciton may be written as

$$L(\mathbf{Q},\omega)=\sum_I \frac{\text{Im}[F_I(\mathbf{Q})]}{\{\omega-\epsilon_I(\mathbf{Q})-\text{Re}[F_I(\mathbf{Q})]\}^2+\{\text{Im}[F_I(\mathbf{Q})]\}^2}.$$

(140)

In this article, we merely point out the feasibility of these computations. In our next article we intend to explore the practical consequences more thoroughly, specifically, the biexciton Green functions and nonlinear optical susceptibilities.

V. CONCLUSIONS

In this article, we have laid down the ground work for a very promising new approach for understanding excitons in semiconductors and charge-conserving systems such as electrons and positrons. We have shown how to compute the exciton line shape that (if broadened) is also due to nonradiative many-body processes. We have identified the elementary excitations in the system and precisely pointed out the role each of them play in the theory. The elementary entities in the two-component Fermi system may be thought of in two equivalent ways. One may consider them to be electrons and holes interacting via two-body attractive and repulsive interactions or we may consider the system to be made of excitons, solitrons, valerons, and conductrons all interacting with each other by somewhat complicated, but purely local Hamiltonians. The formidable technical challenges have been overcome and now a clear path has been mapped out for a systematic exploration of nonperturbative phenomena in many-body physics.

ACKNOWLEDGMENTS

APPENDIX A:

In this section, we prove the claims made in the first section, namely, that X_0 is canonically conjugate to $\hat{N}_0$. Let $\hat{O}=(b_0-\sqrt{\hat{N}_0})(1/\sqrt{\hat{N}_0})$.

$$X_0=\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\hat{O}^n-\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\hat{O}^{\dagger n},\quad (\text{A1})$$

$$[X_0,\hat{N}_0]=\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}[\hat{O}^n,\hat{N}_0]$$
$$-\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}[\hat{O}^{\dagger n},\hat{N}_0],\quad (\text{A2})$$

$$[\hat{O}^n,\hat{N}_0]=\sum_{m=1}^{n}\hat{O}^{n-m}[\hat{O},\hat{N}_0]\hat{O}^{m-1}.\quad (\text{A3})$$

But,

$$[\hat{O},\hat{N}_0]=\hat{\mathbf{1}}+\hat{O}.\quad (\text{A4})$$

Therefore,

$$[\hat{O}^n,\hat{N}_0]=n\,\hat{O}^{n-1}(\hat{\mathbf{1}}+\hat{O})\quad (\text{A5})$$

It is a relatively simple matter to sum this series and we may write

$$[X_0,\hat{N}_0]=\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}n\,\hat{O}^{n-1}(\hat{\mathbf{1}}+\hat{O})$$
$$+\frac{i}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}n\hat{O}^{\dagger n-1}(\hat{\mathbf{1}}+\hat{O}^{\dagger})$$
$$=\frac{i}{2}(\hat{\mathbf{1}}+\hat{O})\frac{\hat{\mathbf{1}}}{\hat{\mathbf{1}}+\hat{O}}+\frac{i}{2}(\hat{\mathbf{1}}+\hat{O}^{\dagger})\frac{\hat{\mathbf{1}}}{\hat{\mathbf{1}}+\hat{O}^{\dagger}}=i\hat{\mathbf{1}}.$$

(A6)

Therefore, much to our relief, we find

$$[X_0,\hat{N}_0]=i\hat{\mathbf{1}}.\quad (\text{A7})$$

Now we move on to fermions.

APPENDIX B:

Here we would like to prove some facts about fermions that were claimed in the main text. Writing down the commutation rules obeyed by $A_{\mathbf{k}}(\mathbf{q})$ involves knowing the precise meaning of the square root in the denominator. Since we have not been successful in pinning down its true meaning, we shall have to take the point of view that the meaning is uniquely fixed by *demanding* that the following commutation rules be obeyed. This is not very satisfying but the authors have exhausted their meager capabilities. Let us consider the various possibilities.

(a) $n_F(\mathbf{k}+\mathbf{q}/2)=0$, $n_F(\mathbf{k}-\mathbf{q}/2)=1$:

$$\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})$$
$$=\left(1-\sum_{\mathbf{q}_1\neq 0}A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\right)^{1/2}$$
$$\times A_{\mathbf{k}}(\mathbf{q})=c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}.\quad (\text{B1})$$

We have to convince ourselves that the left hand side obeys the same commutation rules as the right-hand side.

(b) $n_F(\mathbf{k}+\mathbf{q}/2)=1,n_F(\mathbf{k}-\mathbf{q}/2)=0$:

$$A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}$$
$$=A^{\dagger}_{\mathbf{k}}(-\mathbf{q})$$
$$\times\left(1-\sum_{\mathbf{q}_1\neq 0}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\right)^{1/2}$$
$$=c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}.\quad (\text{B2})$$

(c) $n_F(\mathbf{k}+\mathbf{q}/2)=0,n_F(\mathbf{k}-\mathbf{q}/2)=0$:

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=\sum_{\mathbf{q}_1\neq\mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1).$$
$$(B3)$$

(d) $n_F(\mathbf{k}+\mathbf{q}/2)=1, n_F(\mathbf{k}-\mathbf{q}/2)=1$:

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=-\sum_{\mathbf{q}_1\neq\mathbf{q},0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1).$$
$$(B4)$$

Similarly, we have
(a') $n_F(\mathbf{k}'+\mathbf{q}'/2)=0, n_F(\mathbf{k}'-\mathbf{q}'/2)=1$:

$$\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')=c^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2}c_{\mathbf{k}'+\mathbf{q}'/2}, \qquad (B5)$$

and so on for the other cases. We would now like to write down some statements that would be analogous to commutation rules. Let us first define the following statements(it goes without saying that $\mathbf{q}\neq 0$ and $\mathbf{q}'\neq 0$).

$$S1:\mathbf{k}+\mathbf{q}/2=\mathbf{k}'+\mathbf{q}'/2,$$

$$S2:\mathbf{k}+\mathbf{q}/2=\mathbf{k}'-\mathbf{q}'/2,$$

$$S3:\mathbf{k}-\mathbf{q}/2=\mathbf{k}'+\mathbf{q}'/2,$$

$$S4:\mathbf{k}-\mathbf{q}/2=\mathbf{k}'-\mathbf{q}'/2,$$

$$SS1:\mathbf{k}=\mathbf{k}' \quad \text{and} \quad \mathbf{q}=\mathbf{q}',$$

$$SS2:\mathbf{k}=\mathbf{k}' \quad \text{and} \quad \mathbf{q}=-\mathbf{q}'.$$

Consider the object

$$\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}'). \qquad (B6)$$

We would now like to ascertain the meaning of this object when say, $S1$ is true but $S2, S3$, and $S4$ are false. This state of affairs in symbolic logic is written as $S1\wedge\neg S2\wedge\neg S3\wedge\neg S4$. Let us define

$$AS1=S1\wedge\neg S2\wedge\neg S3\wedge\neg S4,$$

$$AS2=S2\wedge\neg S1\wedge\neg S3\wedge\neg S4,$$

$$AS3=S3\wedge\neg S1\wedge\neg S2\wedge\neg S4,$$

$$AS4=S4\wedge\neg S1\wedge\neg S2\wedge\neg S3,$$

$$AA0=\neg S1\wedge\neg S2\wedge\neg S3\wedge\neg S4.$$

(1) If $AS1$ is true then we have

$$\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')=\sqrt{n_{\mathbf{k}'-\mathbf{q}'/2}}A_{\mathbf{k}'}(\mathbf{q}')\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})=0, \qquad (B7)$$

$$\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})A^{\dagger}_{\mathbf{k}'}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}=A^{\dagger}_{\mathbf{k}'}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})=0, \qquad (B8)$$

$$\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})\left(\sum_{\mathbf{q}_1\neq\mathbf{q}',0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}'-\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)$$
$$=c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}'/2}=\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}-\mathbf{q}'/2}(\mathbf{q}-\mathbf{q}')\left(\hat{\mathbf{1}}-\sum_{\mathbf{q}_1\neq 0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\right), \qquad (B9)$$

$$\left(\sum_{\mathbf{q}_1\neq\mathbf{q}',0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}'-\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})$$
$$=c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}'/2}c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}=-\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}-\mathbf{q}'/2}(\mathbf{q}-\mathbf{q}')\left(\sum_{\mathbf{q}_1\neq 0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\right), \qquad (B10)$$

$$-\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})\left(\sum_{\mathbf{q}_1\neq\mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}'+\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)=0, \qquad (B11)$$

$$-\left(\sum_{\mathbf{q}_1\neq\mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}'+\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}}(\mathbf{q})=0, \qquad (B12)$$

$$A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A^{\dagger}_{\mathbf{k}'}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}=A^{\dagger}_{\mathbf{k}'}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}A^{\dagger}_{\mathbf{k}'}(-\mathbf{q}')\sqrt{n_{\mathbf{k}'+\mathbf{q}'/2}}=0, \qquad (B13)$$

$$A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}\left(\sum_{\mathbf{q}_1\neq\mathbf{q}',0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}'-\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)=0, \qquad (B14)$$

$$\left(\sum_{\mathbf{q}_1 \neq \mathbf{q}',0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}'-\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right) A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}=0, \tag{B15}$$

$$-A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}\left(\sum_{\mathbf{q}_1 \neq \mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}'+\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right)$$

$$=\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\right)\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}-\mathbf{q}'/2}(\mathbf{q}-\mathbf{q}') \tag{B16}$$

$$-\left(\sum_{\mathbf{q}_1 \neq \mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}'+\mathbf{q}_1/2}(-\mathbf{q}'+\mathbf{q}_1)\right) A^{\dagger}_{\mathbf{k}}(-\mathbf{q})\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}$$

$$=-\left(\hat{\mathbf{1}}-\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\right)\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}A_{\mathbf{k}-\mathbf{q}'/2}(\mathbf{q}-\mathbf{q}'), \tag{B17}$$

$$\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\right)\left(\sum_{\mathbf{q}_2 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_2/2}(\mathbf{q}_2) A_{\mathbf{k}'-\mathbf{q}_2/2}(-\mathbf{q}'+\mathbf{q}_2)\right)=0, \tag{B18}$$

$$\left(\sum_{\mathbf{q}_2 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_2/2}(\mathbf{q}_2) A_{\mathbf{k}'-\mathbf{q}_2/2}(-\mathbf{q}'+\mathbf{q}_2)\right)\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\right)=0, \tag{B19}$$

$$\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\right)\left(\sum_{\mathbf{q}_2 \neq \mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_2/2}(\mathbf{q}_2) A_{\mathbf{k}'+\mathbf{q}_2/2}(-\mathbf{q}'+\mathbf{q}_2)\right)=0, \tag{B20}$$

$$\left(\sum_{\mathbf{q}_2 \neq \mathbf{q}',0} A^{\dagger}_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_2/2}(\mathbf{q}_2) A_{\mathbf{k}'+\mathbf{q}_2/2}(-\mathbf{q}'+\mathbf{q}_2)\right)\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\right)=0. \tag{B21}$$

Similarly, the reader may write down the corresponding relations when (2) AS2 is true, (3) AS3 is true, and (4) when AS4 is true. If AA0 is true then we have

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}(\mathbf{q}')]=[A_{\mathbf{k}}(\mathbf{q}),A^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]=0. \tag{B22}$$

If SS1 is true then we have

$$\left[\left(1-\sum_{\mathbf{q}_1 \neq 0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)\right)^{1/2} A_{\mathbf{k}}(\mathbf{q})\right]^2=0 \tag{B23}$$

and, for example,

$$\left(\sum_{\mathbf{q}_1 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\right)$$

$$\times\left(\sum_{\mathbf{q}_2 \neq \mathbf{q},0} A^{\dagger}_{\mathbf{k}-\mathbf{q}_2/2}(-\mathbf{q}+\mathbf{q}_2) A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)\right)$$

$$=\left(\sum_{\mathbf{q}_1 \neq 0} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1) A_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)\right)$$

$$\times\left(1-\sum_{\mathbf{q}_2 \neq 0} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2) A_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}_2/2}(\mathbf{q}_2)\right). \tag{B24}$$

Similarly, the reader can fill in the rest of the rules once the main techniques for deducing these rules have been laid down as we have done here. We see here that the objects $A_{\mathbf{k}}(\mathbf{q})$ and $A^{\dagger}_{\mathbf{k}}(\mathbf{q})$ obey *exact, closed* commutation rules. This enables us to treat any theory involving fermions obeying simple fermion commutation rules in terms of a theory involving sea displacements obeying these rather complicated-looking commutation rules. However, in all cases of practical interest we shall adopt an approximation that in the one-component Fermi system is equivalent to the RPA or its generalizations. In the multicomponent case, we may write down the following formula for the Fermi bilinears. This has been used to derive the corresponding formulas for the charge conserving electron-hole systems in the main text. If $\mathbf{q} \neq 0$ then,

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2\sigma} c_{\mathbf{k}-\mathbf{q}/2\sigma'}$$

$$=\sqrt{n_{\mathbf{k}+\mathbf{q}/2\sigma}}A_{\mathbf{k}\sigma}(-\mathbf{q}\sigma')+A^{\dagger}_{\mathbf{k}\sigma'}(\mathbf{q}\sigma)\sqrt{n_{\mathbf{k}-\mathbf{q}/2\sigma'}}$$

$$+\sum_{\mathbf{q}_1 \neq \mathbf{q},0\sigma_1} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2\sigma_1}(\mathbf{q}_1\sigma) A_{\mathbf{k}-\mathbf{q}_1/2\sigma_1}(-\mathbf{q}+\mathbf{q}_1\sigma')$$

$$-\sum_{\mathbf{q}_1 \neq \mathbf{q},0\sigma_1} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2\sigma'}(\mathbf{q}_1\sigma_1) A_{\mathbf{k}+\mathbf{q}_1/2\sigma}$$

$$\times(-\mathbf{q}+\mathbf{q}_1\sigma_1), \tag{B25}$$

$$c_{\mathbf{k}\sigma}^{\dagger}c_{\mathbf{k}\sigma}=n_F(\mathbf{k}\sigma)\hat{\mathbf{1}}+\sum_{\mathbf{q}_1\neq 0,\sigma_1}A_{\mathbf{k}-\mathbf{q}_1/2\sigma_1}^{\dagger}(\mathbf{q}_1\sigma)A_{\mathbf{k}-\mathbf{q}_1/2\sigma_1}(\mathbf{q}_1\sigma)$$

$$-\sum_{\mathbf{q}_1\neq 0,\sigma_1}A_{\mathbf{k}+\mathbf{q}_1/2\sigma}^{\dagger}(\mathbf{q}_1\sigma_1)A_{\mathbf{k}+\mathbf{q}_1/2\sigma}(\mathbf{q}_1\sigma_1). \quad (B26)$$

Approximate commutation rules : The random phase approximation. The commutation rules presented above together with prescriptions such as Eqs. (26) and (24) form a complete and closed system of rules. These rules, however, are far from simple to use. It is desirable to make some approximations in a systematic manner that enables us to simplify these rules. One such natural approximation is the RPA of Bohm and Pines.[7] The way to do this is to make the following assertion: The RPA is obtained by retaining only the lowest order sea-displacement terms in Eq. (17) and setting $n_{\mathbf{k}}=n_F(\mathbf{k})\hat{\mathbf{1}}$ in Eq. (19). This means that in the exact definition in Eq. (15) we have to replace the number operator by the unit operator. When this is done, we have the RPA result for $A_{\mathbf{k}}(\mathbf{q})$

$$A_{\mathbf{k}}(\mathbf{q})=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]c_{\mathbf{k}-\mathbf{q}/2}^{\dagger}c_{\mathbf{k}+\mathbf{q}/2}. \quad (B27)$$

Ignoring higher order terms amounts to the following [from Eqs. (B3) and (B4)]:

$$c_{\mathbf{k}+\mathbf{q}/2}^{\dagger}c_{\mathbf{k}-\mathbf{q}/2}=0 \quad (B28)$$

when $n_F(\mathbf{k}+\mathbf{q}/2)=0$ and $n_F(\mathbf{k}-\mathbf{q}/2)=0$ or when $n_F(\mathbf{k}+\mathbf{q}/2)=1$ and $n_F(\mathbf{k}-\mathbf{q}/2)=1$. Let us now use these simplifications to obtain a set of closed commutation rules valid in the RPA sense for the object $A_{\mathbf{k}}(\mathbf{q})$. From Eq. (B27) we have

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}(\mathbf{q}')]$$

$$=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]n_F(\mathbf{k}'-\mathbf{q}'/2)$$

$$\times[1-n_F(\mathbf{k}'+\mathbf{q}'/2)][c_{\mathbf{k}-\mathbf{q}/2}^{\dagger}c_{\mathbf{k}+\mathbf{q}/2},c_{\mathbf{k}'-\mathbf{q}'/2}^{\dagger}c_{\mathbf{k}'+\mathbf{q}'/2}]$$

$$=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]n_F(\mathbf{k}'-\mathbf{q}'/2)$$

$$\times[1-n_F(\mathbf{k}'+\mathbf{q}'/2)](c_{\mathbf{k}-\mathbf{q}/2}^{\dagger}c_{\mathbf{k}'+\mathbf{q}'/2}\delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}$$

$$-c_{\mathbf{k}'-\mathbf{q}'/2}^{\dagger}c_{\mathbf{k}+\mathbf{q}/2}\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2})=0. \quad (B29)$$

Now the right-hand side is identically zero as we see from this argument. When $\mathbf{k}+\mathbf{q}/2=\mathbf{k}'-\mathbf{q}'/2$ we have the factor $[1-n_F(\mathbf{k}+\mathbf{q}/2)]n_F(\mathbf{k}'-\mathbf{q}'/2)=0$. When $\mathbf{k}-\mathbf{q}/2=\mathbf{k}'+\mathbf{q}'/2$ we have $n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}'+\mathbf{q}'/2)]=0$. Therefore,

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}(\mathbf{q}')]_{\text{RPA}}=0. \quad (B30)$$

Now let us compute

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]$$

$$=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]n_F(\mathbf{k}'-\mathbf{q}'/2)$$

$$\times[1-n_F(\mathbf{k}'+\mathbf{q}'/2)][c_{\mathbf{k}-\mathbf{q}/2}^{\dagger}c_{\mathbf{k}+\mathbf{q}/2},c_{\mathbf{k}'+\mathbf{q}'/2}^{\dagger}c_{\mathbf{k}'-\mathbf{q}'/2}]$$

$$=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]n_F(\mathbf{k}'-\mathbf{q}'/2)$$

$$\times[1-n_F(\mathbf{k}'+\mathbf{q}'/2)](c_{\mathbf{k}-\mathbf{q}/2}^{\dagger}c_{\mathbf{k}'-\mathbf{q}'/2}\delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2}$$

$$-c_{\mathbf{k}'+\mathbf{q}'/2}^{\dagger}c_{\mathbf{k}+\mathbf{q}/2}\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}) \quad (B31)$$

Because of Eq. (B28), the right-hand side of the above equation [Eq. (B31)] is identically zero unless $\mathbf{k}=\mathbf{k}'$ and $\mathbf{q}=\mathbf{q}'$. Therefore we may write

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}(n_{\mathbf{k}-\mathbf{q}/2}-n_{\mathbf{k}+\mathbf{q}/2}). \quad (B32)$$

If we retain the number operator as it is in the right-hand side of Eq. (B32) we are dealing with the generalized RPA. The generalized RPA pays attention to possible fluctuations in the momentum distribution around a nonideal mean.[1] However, in the simple case we are obliged to set $n_{\mathbf{k}}=n_F(\mathbf{k})\hat{\mathbf{1}}$. When this is done we find the following simple answer:

$$[A_{\mathbf{k}}(\mathbf{q}),A_{\mathbf{k}'}^{\dagger}(\mathbf{q}')]_{\text{RPA}}=n_F(\mathbf{k}-\mathbf{q}/2)[1-n_F(\mathbf{k}+\mathbf{q}/2)]$$

$$\times\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\hat{\mathbf{1}}. \quad (B33)$$

Finally, we would like to address a rather important question, namely is the RPA as described above a controlled approximation? Indeed, what is a "controlled approximation" really? And finally, does it matter whether some approximation is controlled or not? In order to answer these questions it is important to first define what is meant by a controlled approximation. A definition that seems reasonable is as follows: An expansion in powers of a dimensionless parameter small compared to unity that is obtained by combining the various dimensionful parameters of the theory, is a controlled approximation. By this definition it is clear that the RPA is not a controlled approximation. Rather than expanding in powers of a dimensionless *parameter* it seems that we are expanding in powers of a dimensionless *operator*, namely, $A_{\mathbf{k}}(\mathbf{q})$. For this to be justifiable, we have to show that this object is in some sense small compared to unity. This means that the matrix elements of this object have to be small compared to unity.[17] This is possible only if we restrict our Hilbert space to be one that contains low lying excited stated of the noninteracting system. However, we would like to know *beforehand* whether or not such an assumption is justified given the type of interaction and its strength. In fact, in our earlier article[1] we made some rather unfortunate remarks that may be forgiven since it was the first article in the series and we were going to fix the technical aspects later anyway. The formula for the "sea-boson" given there in terms of the Fermi fields and justly criticized Cune and Apostol[2] is totally wrong despite its appealing form. The other remarks in the Appendix[1] justifying the controlled nature of the RPA ap-

proximation provided one restricts the functional form of the potential to obey the constraints outlined earlier[1] so far seem alright. In any event, the moral of this discussion is that the RPA is a strange kind of approximation. It is not possible to make useful statements about when such an approximation breaks down. To say that RPA is valid if all states of the interacting system are expressible as a linear combination of low lying states is as illuminating as saying that the best strategy to win a game of chess is to play it so that the opponent's king is placed under check and cannot move and the opponent cannot block or eliminate the check.

APPENDIX C:

Consider the Hamiltonian

$$H = \sum_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}} n_{\mathbf{k}} + \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k} \neq \mathbf{k}'} c^\dagger_{\mathbf{k}+\mathbf{q}/2} c^\dagger_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2},$$
$$(C1)$$

where the exchange energy (minus the term quartic in the sea displacements) has been absorbed into the kinetic energy

$$\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{V} \langle n_{\mathbf{k}-\mathbf{q}} \rangle. \qquad (C2)$$

Let $n_{\mathbf{q}}(\mathbf{k}) = c^\dagger_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}$ and $n_0(\mathbf{k}) = c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$. Let us now write down the equation of motion for $n_{\mathbf{q}}(\mathbf{k})$. Introduce a source

$$H_{\text{ext}}(t) = \sum_{\mathbf{q}} [U_{\text{ext}}(\mathbf{q}t) + U^*_{\text{ext}}(-\mathbf{q}t)] \sum_{\mathbf{k}} n_{\mathbf{q}}(\mathbf{k}), \quad (C3)$$

where $U_{\text{ext}}(\mathbf{q}t) = e^{-i\omega t} U_0(\mathbf{q})$.

$$i\frac{\partial}{\partial t} n^t_{\mathbf{q}}(\mathbf{k}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}) n^t_{\mathbf{q}}(\mathbf{k})$$
$$+ \sum_{\mathbf{q}' \neq 0} \frac{v_{\mathbf{q}'}}{2V} [n_{\mathbf{q}}(\mathbf{k}), \rho_{\mathbf{q}'}] \rho^t_{-\mathbf{q}'}$$
$$+ \sum_{\mathbf{q}' \neq 0} \frac{v_{\mathbf{q}'}}{2V} \rho^t_{\mathbf{q}'} [n_{\mathbf{q}}(\mathbf{k}), \rho_{-\mathbf{q}'}] + \sum_{\mathbf{q}' \neq 0} [U_{\text{ext}}(\mathbf{q}'t)$$
$$+ U^*_{\text{ext}}(-\mathbf{q}'t)] [n_{\mathbf{q}}(\mathbf{k}), \rho_{\mathbf{q}'}], \qquad (C4)$$

$$[n_{\mathbf{q}}(\mathbf{k}), \rho_{\mathbf{q}'}] = \sum_{\mathbf{k}'} [c^\dagger_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}, c^\dagger_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}'-\mathbf{q}'/2}]$$
$$= \sum_{\mathbf{k}'} c^\dagger_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}'-\mathbf{q}'/2} \delta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}'/2}$$
$$- \sum_{\mathbf{k}'} c^\dagger_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}-\mathbf{q}/2} \delta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}'-\mathbf{q}'/2}$$
$$\approx \delta_{\mathbf{q},-\mathbf{q}'} [n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)], \quad (C5)$$

$$i\frac{\partial}{\partial t} n^t_{\mathbf{q}}(\mathbf{k}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}) n^t_{\mathbf{q}}(\mathbf{k}) + \frac{v_{\mathbf{q}}}{V} [n_0(\mathbf{k}+\mathbf{q}/2)$$
$$- n_0(\mathbf{k}-\mathbf{q}/2)] \rho^t_{\mathbf{q}} + [U_{\text{ext}}(-\mathbf{q}t) + U^*_{\text{ext}}(\mathbf{q}t)]$$
$$\times [n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)]. \qquad (C6)$$

Let us make a first pass at the computation of the dielectric function. Here, we make use of mean-field theory, that is, replace $\langle n_0(\mathbf{k}') \rho_{\mathbf{q}} \rangle = \langle n_0(\mathbf{k}') \rangle \langle \rho_{\mathbf{q}} \rangle$:

$$\omega \langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}) \langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle + \frac{v_{\mathbf{q}}}{V} (\langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle$$
$$- \langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle) \langle \rho^t_{\mathbf{q}} \rangle + [U_{\text{ext}}(-\mathbf{q}t) + U^*_{\text{ext}}(\mathbf{q}t)]$$
$$\times [\langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle], \qquad (C7)$$

$$\langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle = \frac{v_{\mathbf{q}}}{V} \frac{\langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}} \langle \rho^t_{\mathbf{q}} \rangle$$
$$+ [U_{\text{ext}}(-\mathbf{q}t) + U^*_{\text{ext}}(\mathbf{q}t)]$$
$$\times \frac{\langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}}. \qquad (C8)$$

This means that

$$\langle \rho_{-\mathbf{q}} \rangle = U_{\text{ext}}(\mathbf{q}t) \frac{P_0(\mathbf{q}, \omega)}{\epsilon(\mathbf{q}, \omega)}, \qquad (C9)$$

$$P_0(\mathbf{q}, \omega) = \sum_{\mathbf{k}} \frac{\langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}}, \quad (C10)$$

$$\epsilon(\mathbf{q}, \omega) = 1 - \frac{v_{\mathbf{q}}}{V} P_0(\mathbf{q}, \omega). \qquad (C11)$$

From this and the fact that

$$\epsilon_{g-\text{RPA}}(\mathbf{q}, \omega) = \frac{U_{\text{ext}}(\mathbf{q}t)}{U_{\text{eff}}(\mathbf{q}t)} = \epsilon(\mathbf{q}, \omega). \qquad (C12)$$

Next we would like to include fluctuations. Let us do this differently this time via the use of the BBGKY hierarchy:

$$i\frac{\partial}{\partial t} \langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}) \langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle + \frac{v_{\mathbf{q}}}{V} [\langle n_0(\mathbf{k}+\mathbf{q}/2) \rangle$$
$$- \langle n_0(\mathbf{k}-\mathbf{q}/2) \rangle] \langle \rho^t_{\mathbf{q}} \rangle + \frac{v_{\mathbf{q}}}{V} [F_{2A}(\mathbf{k}+\mathbf{q}/2, \mathbf{q})$$
$$- F_{2A}(\mathbf{k}-\mathbf{q}/2, \mathbf{q})] + [U_{\text{ext}}(-\mathbf{q}t) + U^*_{\text{ext}}(\mathbf{q}t)]$$
$$\times [n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)]. \qquad (C13)$$

Here,

$$F_{2A}(\mathbf{k}', \mathbf{q}; t) = \langle n_0(\mathbf{k}') \rho^t_{\mathbf{q}} \rangle - \langle n_0(\mathbf{k}') \rangle \langle \rho^t_{\mathbf{q}} \rangle, \qquad (C14)$$

$$F_2(\mathbf{k}'; \mathbf{k}, \mathbf{q}; t) = \langle n_0(\mathbf{k}') n^t_{\mathbf{q}}(\mathbf{k}) \rangle - \langle n_0(\mathbf{k}') \rangle \langle n^t_{\mathbf{q}}(\mathbf{k}) \rangle, \qquad (C15)$$

$$F_{2A}(\mathbf{k}',\mathbf{q};t) = \sum_{\mathbf{k}} F_2(\mathbf{k}';\mathbf{k},\mathbf{q};t), \qquad \text{(C16)}$$

$$i\frac{\partial}{\partial t}F_2(\mathbf{k}';\mathbf{k},\mathbf{q};t)$$

$$= (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})F_2(\mathbf{k}';\mathbf{k},\mathbf{q};t)$$

$$+ \frac{v_{\mathbf{q}}}{V}[N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2)$$

$$- N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)]\langle \rho_{\mathbf{q}}^t\rangle + \frac{v_{\mathbf{q}}}{V}[\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle$$

$$- \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle]F_{2A}(\mathbf{k}',\mathbf{q};t) + [U_{\text{ext}}(-\mathbf{q}t)$$

$$+ U_{\text{ext}}^*(\mathbf{q}t)][N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)) \qquad \text{(C17)}$$

Let us write,

$$F_2(\mathbf{k}';\mathbf{k},\mathbf{q};t) = U_{\text{ext}}(-\mathbf{q}t)F_{2,a}(\mathbf{k}';\mathbf{k},\mathbf{q})$$

$$+ U_{\text{ext}}^*(\mathbf{q}t)F_{2,b}(\mathbf{k}';\mathbf{k},\mathbf{q}), \qquad \text{(C18)}$$

$$\langle \rho_{\mathbf{q}}^t\rangle = U_{\text{ext}}(-\mathbf{q}t)\langle \rho_{\mathbf{q}}'\rangle + U_{\text{ext}}^*(\mathbf{q}t)\langle \rho_{\mathbf{q}}''\rangle. \qquad \text{(C19)}$$

Also, define

$$N(\mathbf{k},\mathbf{k}') = \langle n_0(\mathbf{k})n_0(\mathbf{k}')\rangle - \langle n_0(\mathbf{k})\rangle\langle n_0(\mathbf{k}')\rangle, \qquad \text{(C20)}$$

$$\omega F_{2,a}(\mathbf{k}';\mathbf{k},\mathbf{q}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})F_{2,a}(\mathbf{k}';\mathbf{k},\mathbf{q}) + \frac{v_{\mathbf{q}}}{V}[N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)]\langle \rho_{\mathbf{q}}'\rangle$$

$$+ \frac{v_{\mathbf{q}}}{V}[\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle]F_{2A}^a(\mathbf{k}',\mathbf{q}) + [N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)] \qquad \text{(C21)}$$

$$-\omega F_{2,b}(\mathbf{k}';\mathbf{k},\mathbf{q}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})F_{2,b}(\mathbf{k}';\mathbf{k},\mathbf{q}) + \frac{v_{\mathbf{q}}}{V}[N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)]\langle \rho_{\mathbf{q}}''\rangle$$

$$+ \frac{v_{\mathbf{q}}}{V}[\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle]F_{2A}^b(\mathbf{k}',\mathbf{q}) + [N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)], \qquad \text{(C22)}$$

$$\epsilon(\mathbf{q},\omega)F_{2A}^a(\mathbf{k}',\mathbf{q}) = \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \frac{N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}}\langle \rho_{\mathbf{q}}'\rangle + \sum_{\mathbf{k}} \frac{N(\mathbf{k}',\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}',\mathbf{k}-\mathbf{q}/2)}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}}, \qquad \text{(C23)}$$

$$\omega C_{\mathbf{q}}(\mathbf{k}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})C_{\mathbf{q}}(\mathbf{k}) + \frac{v_{\mathbf{q}}}{V}[\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle]\langle \rho_{\mathbf{q}}'\rangle + \frac{v_{\mathbf{q}}}{V}[F_{2A}^a(\mathbf{k}+\mathbf{q}/2) - F_{2A}^a(\mathbf{k}-\mathbf{q}/2)]$$

$$+ [\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle], \qquad \text{(C24)}$$

$$C_{\mathbf{q}}(\mathbf{k}) = \frac{v_{\mathbf{q}}}{V}\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}}\langle \rho_{\mathbf{q}}'\rangle + \frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2}} + \frac{v_{\mathbf{q}}}{V}\frac{F_{2A}^a(\mathbf{k}+\mathbf{q}/2,\mathbf{q}) - F_{2A}^a(\mathbf{k}-\mathbf{q}/2,\mathbf{q})}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}}. \qquad \text{(C25)}$$

After all this, it may be shown that the overall dielectric function including possible fluctuations in the momentum distribution is given by

$$\epsilon_{\text{eff}}(\mathbf{q},\omega) = \epsilon_{g-\text{RPA}}(\mathbf{q},\omega) - \left(\frac{v_{\mathbf{q}}}{V}\right)^2 \frac{P_2(\mathbf{q},\omega)}{\epsilon_{g-\text{RPA}}(\mathbf{q},\omega)}. \qquad \text{(C26)}$$

Here,

$$P_2(\mathbf{q},\omega) = \sum_{\mathbf{k},\mathbf{k}'} \frac{N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}'+\mathbf{q}/2) - N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}/2) - N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}/2) + N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}'-\mathbf{q}/2)}{(\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})(\omega - \tilde{\epsilon}_{\mathbf{k}'-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}'+\mathbf{q}/2})}, \qquad \text{(C27)}$$

$$\epsilon_{g-\text{RPA}}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}}. \qquad \text{(C28)}$$

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[4] The reason why replacing the square root of the number operator by the unit operator does not work is because, for example, Eq. (20) is inconsistent with Eq. (19) except when $n_{\mathbf{k}} = n_F(\mathbf{k})\hat{\mathbf{1}}$. Therefore, Eq. (20) is correct in the sense of the RPA.

[5] The authors are not clear as to the role played by idempotence in this framework. It is likely that idempotence is violated by the sea-boson approach. It is likely to be obeyed only in some restricted sense. In any event the reader is advised not to invoke idempotence anywhere else in this article, or else all assertions are inconsistent.

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[17] This issue is also not so simple. If one considers the restricted Hilbert space to be so small as to contain only the ground state of the noninteracting system, then trivially $A_{\mathbf{k}}(\mathbf{q})$ is small since $\langle \mathrm{FS} | A_{\mathbf{k}}(\mathbf{q}) | \mathrm{FS} \rangle = 0$. If one starts adding states that are low lying excitations to this restricted Hilbert space, then $\langle I | A_{\mathbf{k}}(\mathbf{q}) | J \rangle = 0$ unless $\mathbf{k} - \mathbf{q}/2$ is just below the Fermi surface and $\mathbf{k} + \mathbf{q}/2$ is just above the Fermi surface. This means that $|\mathbf{k}| \approx k_F$ and $|\mathbf{q}| \ll k_F$. Since the matrix elements of $A_{\mathbf{k}}(\mathbf{q})$ are zero "most of the time," the matrix representation of $A_{\mathbf{k}}(\mathbf{q})$ is sparse and in this sense the operator $A_{\mathbf{k}}(\mathbf{q})$ may be regarded as being small even though some elements of this matrix may be of order unity.