

Phase coherence phenomena in superconducting films

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Superconducting films subject to an in-plane magnetic field exhibit a gapless superconducting phase. We explore the quasiparticle spectral properties of the gapless phase and comment on the transport properties. Of particular interest is the sensitivity of the quantum interference phenomena in this phase to the nature of the impurity scattering. We find that films subject to columnar defects exhibit a “Berry-Robnik” symmetry that changes the fundamental properties of the system. Furthermore, we explore the integrity of the gapped phase. As in the magnetic impurity system, we show that optimal fluctuations of the random impurity potential conspire with the in-plane magnetic field to induce a band of localized subgap states. Finally, we investigate the interplay of the proximity effect and gapless superconductivity in thin normal-metal–superconductor bilayers.

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I. INTRODUCTION

It is well established both theoretically and experimentally that bulk *s*-wave superconductivity is robust with respect to the addition of nonmagnetic impurities (Anderson theorem¹). By contrast, magnetic impurities affect a mechanism of pair breaking on the system which suppresses and eventually destroys the superconducting phase. Remarkably, in such systems, the quasiparticle energy gap vanishes more rapidly than the order parameter, admitting the existence of a gapless superconducting phase. The (mean-field) theory of the gap suppression was explored in a classic work by Abrikosov and Gor’kov.² Since this pioneering work, it is now realized that this scheme applies more widely, encompassing general time-reversal (*T*) symmetry-breaking perturbations (for a review see, e.g., Refs. 3 and 4). In the following, we will consider the general phenomenology of a disordered superconducting thin film subject to a homogeneous in-plane magnetic field.

Generally, a bulk superconductor subject to a weak magnetic field of magnitude $H < H_{c2}$ enters a Meissner phase:⁵ the field is expelled from the bulk, penetrating only a thin surface layer down to the London penetration depth. In the Meissner phase, the quasiparticle properties of the bulk states are not affected by the magnetic field and, thus, are largely insensitive to disorder. However, if the dimensions of the superconductor are diminished to a scale comparable to the penetration depth, field lines enter the sample and the superconductivity becomes strongly suppressed. Here the quasiparticle properties deviate significantly from those of the unperturbed system: in particular, the system can exhibit a gapless phase. At the level of the mean field, the properties of the thin-film superconductor mirror those of the magnetic impurity system and are described by the Abrikosov-Gor’kov (AG) phenomenology.³ However, beyond the level of the mean field, phase coherence effects due to normal disorder strongly influence the long-range properties of the system. In the following, we will investigate the properties of the gapless phase in the thin-film superconducting system focusing on two situations: in the first case, we will consider an arrangement of normal impurities drawn at random from a

δ -correlated distribution. In the second case, we will consider the superconducting system subject to a disordered array of columnar defects, i.e., an impurity potential that does not depend on the coordinate perpendicular to the plane. The two arrangements are depicted schematically in Fig. 1.

When applied to a superconductor, a magnetic field has two effects: first, it induces a Zeeman splitting and, second, it couples to the orbital motion of the electrons. Both have the effect of suppressing the superconductivity. Whereas, in a bulk system, the orbital effect usually dominates, in very thin films the opposite situation arises. The crossover can be estimated in the following way⁶: the critical magnetic field associated with the orbital effect is roughly determined by the condition that the flux threading an area spanned by the coherence length is of the order of one flux quantum, $H_{c2}\xi^2 \approx \phi_0$, where $\xi = \sqrt{D/(2\Delta)}$ is the coherence length in the dirty system with order parameter Δ and diffusion constant D . Now, if the width of the film d becomes smaller than the coherence length, $d \ll \xi$, this has to be replaced by the condition $H_{c2}^\parallel \xi d \approx \phi_0$, where H_{c2}^\parallel represents the in-plane field; i.e., the orbital critical field increases. The critical field H_Z associated with the Zeeman splitting is independent of the width of the system. H_Z is obtained from the condition that the energy splitting between up(\uparrow)- and down(\downarrow)-spins is roughly of the size of the order parameter, $g_L \mu_B H_Z \approx \Delta$, where g_L is the Landé *g* factor and $\mu_B = e/(2m)$ the Bohr magneton. Comparing the two, one concludes that the orbital effect is dominant in suppressing the superconductivity when

$$d > d_c \equiv g_L \frac{\lambda_F \xi}{l},$$

where λ_F is the Fermi wavelength and l the elastic mean free path.

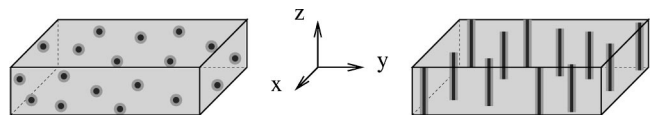


FIG. 1. Schematic picture of a thin film with δ -correlated disorder (left) and columnar defects (right).

In the following, we restrict our attention to systems where $d > d_c$ and the Zeeman splitting can be neglected. To be precise, we will consider a thin-film system where the relevant length scales are arranged in the following hierarchy:

$$\lambda_F \ll [d, l] \ll \xi. \quad (1)$$

The inequality $\lambda_F \ll l$ defines the quasiclassical regime, while $l \ll \xi$ is the condition for the dirty limit. Finally, $\lambda_F \ll d$ implies that the subband splitting due to size quantization is small, and many subbands are occupied. Thus, from the point of view of the normal electrons, the system is effectively three dimensional.

Before turning to the theoretical analysis, let us first briefly summarize the main results of this investigation. At the level of the mean field, the superconducting system with an in-plane magnetic field is described by the AG theory,² independent of the nature of the normal disorder potential (diffusive or columnar). The parameter governing the suppression of the quasiparticle energy gap is set by

$$\zeta \sim \frac{D(Hd)^2}{\Delta} \sim \Phi_{d\xi}^2,$$

where $\Phi_{d\xi}$ is the flux through an area perpendicular to the field spanned by the width of the film d and the superconducting coherence length ξ . As found for the magnetic impurity system,⁷ within the gapped phase, the sharp edge in the quasiparticle density of states (DOS) predicted by the mean field is softened by the nucleation of localized tail states that extend into the subgap region. In this sense, the mean-field gap edge becomes a mobility edge separating a region of bulk (weakly localized) quasiparticle states from strongly localized tail states. In the following, we will demonstrate that these subgap states are induced by optimal fluctuations of the random impurity potential and are reflected as inhomogeneous instanton field configurations of the mean-field equations. In the vicinity of the gap edge E_{gap} , the energy scaling of these tails is universal⁷ depending only on the distance from E_{gap} , the dimensionless parameter ζ and dimensionality. In 2D, one obtains the result

$$\nu(\epsilon < E_{\text{gap}}) \sim \exp\left[-a_2(\zeta) \nu_{\square} D \frac{E_{\text{gap}} - \epsilon}{\Delta}\right] \quad (2)$$

nonperturbative in the inverse dimensionless conductance $1/(\nu_{\square} D)$, where ν_{\square} is the DOS of the system when in the normal state. Here a_2 is a known dimensionless function of the control parameter ζ .

At the mean-field level, the choice of the disorder does not affect the results qualitatively. However, within the gapless phase, the low-energy physics depends sensitively on the character of the impurity potential and, in particular, on whether or not it respects inversion symmetry. Within the diffusive regime $1/\tau > \epsilon > D/L^2$ the quasiparticle states exhibit a localized or “spin-insulator” phase with a localization length ξ_{loc} , which depends on the fundamental symmetry of the system.^{8–10} However, it is within the *ergodic* regime $\epsilon < \max(D/\xi_{\text{loc}}^2, D/L^2)$, where the influence of the in-

version symmetry is most clearly exposed. Here the quasiparticle properties become universal, dependent only on the fundamental symmetry class.^{11,12} A complete classification scheme of ten symmetry classes,¹¹ corresponding to the ten large families of symmetric spaces identified by Cartan, exists; four of these symmetry classes are relevant in the context of superconductivity.¹² Possible symmetries are time-reversal, spin-rotation, particle-hole, and chiral symmetry. Now *s*-wave superconductors possess particle-hole symmetry but, generically, do not exhibit chiral symmetry. Furthermore, they may or may not possess time-reversal and/or spin-rotation symmetry. Here, as one might expect, one finds that the diffusive film in a magnetic field belongs to “symmetry class *C*” (according to Cartan’s notation), which corresponds to spin-rotation symmetry, but broken time-reversal symmetry. By contrast—despite the presence of the magnetic field—the film with columnar defects manifests a hidden symmetry placing the superconductor in the higher symmetry class CI, which is usually characteristic of systems possessing time-reversal invariance. The distinct symmetry classes manifest themselves in the energy dependence of the density of states for $\epsilon \rightarrow 0$. Thus, although superficially the suppression of superconductivity does not respond to geometrical symmetries, manifestations of the Berry-Robnik symmetry effect can be observed in the gapless phase. The distinct low-energy behavior predicted above is confirmed by numerics.

Finally, to complete our discussion, we turn to the consideration of the influence of an in-plane magnetic field on the spectral properties of a normal-metal–superconductor (NS) bilayer. Here the interplay of macroscopic quantum coherence phenomena in the superconductor and the mesoscopic metal leads to interesting effects.¹³ Generally, a superconductor brought into contact with a normal system tends to impact aspects of its superconducting character onto the latter. At the origin of this phenomenon is the Andreev reflection¹⁴ of electrons at the interface: an electron with energy smaller than Δ may be retroreflected off the boundary as a hole, and a Cooper pair added to the condensate in the superconductor. The coherent superposition of the incident electron with the reflected hole leads to a nonvanishing pair amplitude within the normal region. In particular, the density of states within the normal region may develop a gap—whose size depends on the coupling γ between the two systems. Here we consider the case of weak coupling, where the induced gap $E_{\text{gap}}^{(N)} = \gamma (\ll \Delta)$. A parallel magnetic field counteracts this phenomenon by suppressing the energy gap even within the superconductor. Below, we will show that the field necessary to suppress the induced gap in the normal region is much smaller than the field required to drive the superconductor into the gapless phase. However, the coupling to the normal region also induces a nonvanishing DOS at low energies $E_{\text{gap}}^{(N)} < \epsilon < \Delta$ in the superconductor. Here we consider two different geometries, namely, a planar NS bilayer as well as a NS cylinder with the magnetic field directed along the axis. In the second system the enclosed flux leads to a periodic modulation of the field effect as observed first in an experiment by Little and Parks.¹⁵ Only for small enough cylinders the system reaches the gapless phase in certain ranges

of the magnetic field. By further decreasing the radius, superconductivity is completely suppressed around half-integer flux quanta threading the cylinder, as predicted by de Gennes¹⁶ and very recently verified experimentally by Liu *et al.*¹⁷.

The paper is organized as follows: in Sec. II, we first study the suppression of superconductivity due to the parallel field at the mean-field level. In Sec. II B we show how the results for the subgap tail states, recently established for the magnetic impurity system, translate to thin films in parallel fields. Subsequently, in Secs. III and IV, we search for manifestations of the Berry-Robnik symmetry effect on the properties of the superconducting phase. In Sec. V, we discuss the properties of thin disordered NS bilayers. Finally, we conclude in Sec. VI.

II. DIFFUSIVE FILM

Before turning to the columnar defect system, let us begin with a discussion of the most generic case: a thin film with just a “normal” δ -correlated white-noise disorder potential. Here we are interested in the limit, where—in addition to the conditions (1) specified above— $l \ll d$, which implies diffusive motion in all three directions.

Building on the field-theory approach to the study of weakly disordered systems^{18–20} (for a review see, e.g., Ref. 21), the extension to the consideration of disordered superconducting systems follows straightforwardly.^{22–24} Therefore, here we will only briefly summarize the main elements in the construction of the field theory of the disordered superconductor in the framework of the nonlinear σ model (NL σ M). Using this formulation, we will hereafter investigate the response of the superconducting film to an in-plane magnetic field.

The superconducting system is described by the Gor’kov Hamiltonian

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_0 & \Delta \\ \Delta^* & -\mathcal{H}_0^T \end{pmatrix}_{PH}$$

with $\mathcal{H}_0 = \mathbf{p}^2/(2m) - \epsilon_F + V(\mathbf{r})$. Here, the matrix structure refers to the particle/hole (PH) space, ϵ_F denotes the Fermi energy, and $V(\mathbf{r})$ represents the quenched random impurity potential whose distribution is characterized by the mean scattering time τ . We consider the potential to be drawn from a Gaussian white-noise distribution, $\langle V(\mathbf{r})V(\mathbf{r}') \rangle_V = (2\pi\nu_0\tau)^{-1} \delta^{(3)}(\mathbf{r} - \mathbf{r}')$, where ν_0 is the average DOS of the normal system.

The starting point of the analysis is the functional field integral for the generating functional^{22–24,9}

$$\mathcal{Z}[j] = \int D[\Psi, \bar{\Psi}] \exp \left\{ -S[\bar{\Psi}, \Psi] + \int d\mathbf{r} (\bar{\Psi} j + \bar{j} \Psi) \right\},$$

where Ψ represent eight-component supervector fields, incorporating a boson/fermion space as well as the particle/hole and a charge conjugation space: to properly account for all channels of quantum interference, it is standard practice to affect a doubling of the field space to accommodate the particle-hole symmetry $\mathcal{H} = -\sigma_2^{PH} \mathcal{H}^T \sigma_2^{PH}$, where σ_2^{PH} is a

Pauli matrix in particle/hole space. This additional space is referred to as the charge conjugation sector. The introduction of commuting (bosonic) as well as anticommuting (fermionic) fields ensures the normalization of the generating functional in the absence of sources, $\mathcal{Z}[0] = 1$. Following Ref. 9, the superfields are not independent, but obey the condition $\bar{\Psi} = (i\sigma_2^{PH} \otimes \eta \Psi)^T$, where $\eta = \sigma_1^{CC} \otimes E_{BB} - i\sigma_2^{CC} \otimes E_{FF}$ [and $E_{BB} = \text{diag}(1, 0)_{BF}$, $E_{FF} = \text{diag}(0, 1)_{BF}$ are projectors onto the boson-boson and fermion-fermion block, respectively]. Taking our notation from Ref. 9, the action assumes the canonical form

$$S[\bar{\Psi}, \Psi] = i \int d\mathbf{r} \bar{\Psi} (\epsilon^- \sigma_3^{CC} - \mathcal{H}) \Psi,$$

where σ_3^{CC} is a Pauli matrix in charge conjugation space, $\epsilon^- = \epsilon - i0$, and \mathcal{H} represents the Gor’kov Hamiltonian introduced above.

To explore the low-energy properties of the superconducting system, after ensemble averaging over the random impurity distribution, the functional integral over the supervector fields Ψ can be traded for an integral involving supermatrix fields Q . Physically, the fields Q , which vary slowly on the scale of the mean free path l , describe the soft modes of density relaxation—the diffusion modes. In the quasiclassical limit ($\epsilon_F \gg 1/\tau$), the action for Q is dominated by the saddle-point field configuration. In the dirty limit ($1/\tau \gg \Delta$), the saddle-point equation admits the solution $Q_0 = \sigma_3^{PH} \otimes \sigma_3^{CC}$. However, in the limit $\epsilon \rightarrow 0$, the saddle point is not unique but spans an entire manifold parametrized by the unitary transformations $Q = T Q_0 T^{-1}$. Taking into account slow spatial and temporal fluctuations of the fields, the low-energy long-range properties of the weakly disordered superconducting system are described by an action of the nonlinear σ model type^{21–24,9}

$$S[Q] = -\frac{\pi\nu_0}{8} \int d\mathbf{r} \text{Str}[D(\partial Q)^2 - 4(i\epsilon^- \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta \sigma_2^{PH})Q], \quad (3)$$

where the supermatrix field obeys the nonlinear constraint $Q^2 = 1$. Furthermore, the matrices Q obey the symmetry relation $Q = \sigma_1^{PH} \otimes \eta Q^T (\sigma_1^{PH} \otimes \eta)^T$, reflecting the symmetry properties of the dyadic product $\Psi \otimes \bar{\Psi} \sigma_3^{PH}$. Finally, due to gauge invariance properties of the action, the incorporation of a magnetic field amounts to replacing the derivatives in Eq. (3) by their covariant counterpart $\tilde{\partial} = \partial - i\mathbf{A}[\sigma_3^{PH}, \cdot]$. This completes the formulation of the disordered superconducting system as a functional field integral involving the supersymmetric NL σ M action. Our interest here is in the thermodynamic DOS obtained as

$$\nu(\epsilon) = \pi^{-1} \int \frac{d\mathbf{r}}{V} \text{Im}[G(\mathbf{r}, \mathbf{r}; \epsilon^-)].$$

Making use of the generating functional, it is straightforward to show that the impurity averaged DOS can be obtained from the identity

$$\langle \nu(\epsilon) \rangle = \frac{\nu_0}{8} \int \frac{d\mathbf{r}}{V} \text{Re} \langle \text{Str}[\sigma_3^{BF} \otimes \sigma_3^{PH} \otimes \sigma_3^{CC} Q] \rangle_Q, \quad (4)$$

where $\langle \dots \rangle_Q = \int DQ \dots e^{-S[Q]}$.

Thus, the starting point of our analysis is the conventional NL σ M, Eq. (3), for a three-dimensional superconducting system subject to a magnetic field,^{21,24} where the vector potential reads $\mathbf{A} = -Hz\mathbf{e}_y$. The typical scale of variation of the Q fields is set by the coherence length ξ . Therefore, since $d \ll \xi$, the matrices Q are constant along the z direction, allowing integration along z to be performed explicitly. Making use of the identities

$$\frac{1}{d} \int_{-d/2}^{d/2} dz \mathbf{A} = 0, \quad \frac{1}{d} \int_{-d/2}^{d/2} dz \mathbf{A}^2 = \frac{1}{12} (Hd)^2,$$

one finds that

$$S = -\frac{\pi \nu_{\square}}{8} \int d^2r \text{Str} \left\{ D(\partial Q)^2 - \frac{\kappa}{2} [\sigma_3^{PH}, Q]^2 - 4(i\epsilon^{-} \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta \sigma_2^{PH}) Q \right\}, \quad (5)$$

where $\kappa = D(Hd)^2/6$ and ν_{\square} is the DOS of the two-dimensional system, i.e., here $\nu_{\square} = \nu_0 d$. In this planar geometry, the paramagnetic term, which is crucial for the Meissner effect, vanishes from the action.

Note that the choice of gauge is important here: the physical gauge to choose is the London gauge, $\nabla \cdot \mathbf{A} = 0$ and $A_z(\pm d/2) = 0$. Both conditions are fulfilled by $\mathbf{A} = -Hz\mathbf{e}_y$. The above requirements originate from the fact that, in a superconductor, the vector potential is associated with a supercurrent $\mathbf{j}_s = n_s \mathbf{A}/m$, where n_s is the density of Cooper pairs. The first condition tells us that no net current is generated while the second condition does not allow a supercurrent to flow through the superconductor-vacuum boundary. Thus, when integrating along z , we have fixed the gauge, i.e., the resulting action is not gauge invariant. Therefore, the magnetic field does not appear within a covariant derivative, but as an additional diamagnetic term $\sim \kappa [\sigma_3^{PH}, Q]^2$. This distinguishes the ‘‘thick’’ film, $d \gg \lambda_F$, from the single-channel case (i.e., the truly two-dimensional system), where the magnetic field can be gauged out and, thus, has no influence.

A. Mean-field analysis

As usual, at the mean-field level, the density of states is obtained by subjecting the action (5) to a saddle-point analysis. Varying the action with respect to Q , subject to the nonlinear constraint $Q^2 = 1$, one obtains the saddle-point equation

$$D\partial(Q\partial Q) - \frac{\kappa}{2} [\sigma_3^{PH} Q \sigma_3^{PH}, Q] - [i\epsilon^{-} \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta \sigma_2^{PH}, Q] = 0. \quad (6)$$

The latter can be identified as the Usadel equation²⁵ for the average quasiclassical Green function, supplemented by an

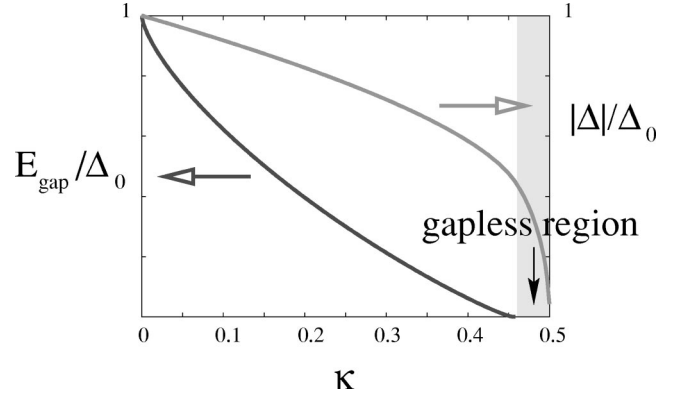


FIG. 2. Dependence of the energy gap E_{gap} and the order parameter Δ on κ . Gapless superconductivity occurs in the narrow range $\Delta_0 \exp(-\pi/4) \leq \kappa \leq \Delta_0/2$.

additional term due to the parallel magnetic field. With the ansatz $Q = \cosh \hat{\theta} \sigma_3^{PH} \otimes \sigma_3^{CC} + i \sinh \hat{\theta} \sigma_2^{PH}$, where $\hat{\theta} = \theta_B E_{BB} + i \theta_F E_{FF}$, one obtains

$$D\partial^2 \hat{\theta} - 2i(\epsilon \sinh \hat{\theta} - \Delta \cosh \hat{\theta}) - \kappa \sinh(2\hat{\theta}) = 0. \quad (7)$$

Taking θ to be homogeneous in space, and defining

$$\tilde{\epsilon} = \epsilon - \frac{i}{2} \kappa \cosh \hat{\theta}, \quad \tilde{\Delta} = \Delta + \frac{i}{2} \kappa \sinh \hat{\theta},$$

the equations for $\tilde{\epsilon}$ and $\tilde{\Delta}$ take the BCS form and, thus, admit the diagonal solution $\tilde{\epsilon}/\tilde{\Delta} = \coth \theta$. Then, in terms of the ‘‘bare’’ parameters ϵ and Δ , the saddle-point Eq. (7) can be brought to the conventional AG form²

$$\frac{\epsilon}{\Delta} = u \left(1 - \zeta \frac{1}{\sqrt{1-u^2}} \right), \quad (8)$$

where $u \equiv \coth \theta$. Following, for example, Ref. 3, by extremizing this equation and defining $\zeta = \kappa/\Delta$, one obtains the quasiparticle energy gap

$$E_{\text{gap}} = \Delta (1 - \zeta^{2/3})^{3/2}.$$

When combined with the self-consistency equation for the order parameter³

$$\Delta = \frac{\pi g \nu_{\square}}{\beta} \sum_n \frac{1}{\sqrt{1+u_n}},$$

where we have shifted to Matsubara frequencies, $i\epsilon^{-} \rightarrow \epsilon_n = (2n+1)\pi/\beta$, one finds that, at $T=0$, superconductivity is destroyed when $\kappa_{\Delta} = \Delta_0/2$, where Δ_0 is the order parameter in the absence of a magnetic field. The gapless phase sets in at the smaller value $\kappa_g = \Delta_0 \exp(-\pi/4)$. In the following, we will be able to determine the order parameter self-consistently, and Δ will be understood as the renormalized order parameter, even if not stated explicitly. The suppression of the energy gap E_{gap} and the order parameter Δ is shown in Fig. 2.

Finally, in the mean-field approximation, the DOS takes the form

$$\nu(\epsilon) = \frac{\nu_{\square}}{8} \int \frac{d^2r}{V} \text{Re Str}[\sigma_3^{BF} \otimes \sigma_3^{PH} \otimes \sigma_3^{CC} Q(\mathbf{r})] \quad (9)$$

$$= \nu_{\square} \text{Re}[\cosh \theta].$$

Inserting the solution determined by Eq. (8), this yields the characteristic AG DOS, exhibiting a square-root edge in the gapped phase.

B. Subgap states

According to the mean-field AG theory, the superconducting system is expected to exhibit a gapped phase over a wide range of parameters with the DOS varying as a square root of the energy difference $\epsilon - E_{\text{gap}}$ in the vicinity of the gap edge E_{gap} . Recently, studies in the related context of a superconducting system perturbed by magnetic impurities have shown that the hard edge predicted by the AG theory is untenable.^{26,7} In particular, in the diffusive system, it has been shown that optimal fluctuations of the random impurity potential lead to the nucleation of localized subgap quasiparticle states that soften the gap edge.⁷

Within the framework of the supersymmetric field theory, the precise mechanism of sub-gap state formation has been elucidated in Ref. 7. There, it was shown that, in addition to the homogeneous AG solution, the saddle-point Eq. (7) accommodates a degenerate manifold of spatially inhomogeneous instanton or bounce solutions. Referring to Ref. 7 for details, to exponential accuracy, the subgap DOS takes the form

$$\nu(\epsilon) \sim \exp\left[-c_2 \nu_{\square} D \zeta^{-2/3} (1 - \zeta^{2/3})^{-1/2} \frac{E_{\text{gap}} - \epsilon}{\Delta}\right], \quad (10)$$

where c_2 is a numerical constant. Within the same theory, one finds that the subgap states are confined to droplets of size

$$r_{\text{drop}}(\epsilon) = 6^{1/4} \xi \left(\frac{\Delta}{E_{\text{gap}}}\right)^{1/12} \left(\frac{E_{\text{gap}} - \epsilon}{\Delta}\right)^{-1/4}$$

diverging upon approaching the gap edge. The mean-field gap edge E_{gap} , therefore, assumes the significance of a mobility edge separating localized subgap states from extended bulk states.

Being confined to a region of size $r_{\text{drop}} \gtrsim \xi \gg l \gg \lambda_F$, it is evident that the physical mechanism of subgap state formation is quasiclassical in origin, relying on optimal fluctuations of the random impurity potential. Moreover, in contrast to Lifshitz semiconductor band-tail states,²⁷ each droplet leads to the nucleation of an entire band of localized states. Qualitatively, the physical mechanism of subgap state formation is connected to mesoscopic fluctuations in the phase sensitivity of the electron wave function. In regions where the sensitivity is high, the impact of a time-reversal symmetry breaking, such as that imposed by the magnetic impurities or parallel magnetic field, is stronger, and the effective scattering rate ζ is enhanced.

Note that, instead of applying a magnetic field, one might consider driving a supercurrent through the system. As em-

phasized in Ref. 3, this leads to the same AG mean-field results. However, in this case, due to the presence of a paramagnetic term in the action, the mean-field gap edge is robust, i.e., no subgap states are generated.

III. COLUMNAR DEFECTS

In the normal system, previous studies²⁸ have shown that the properties of a thin film in an in-plane magnetic field depend sensitively on the nature of the impurity potential. Do the quasiparticle properties of the disordered superconductor exhibit a similar sensitivity?

In the normal system, z -inversion symmetry effectively compensates for the time-reversal symmetry breaking of the magnetic field, driving the system into the orthogonal symmetry class.²⁹ As such, one might expect that the inversion symmetry restores the validity of the Anderson theorem, rendering the superconducting phase insensitive to the disorder potential. In fact, constraints imposed by self-consistency of the order parameter prevent the existence of a mechanism that could cancel the effect of the magnetic field. The reason is that here we are dealing with an interacting problem: with the formation of Cooper pairs, it is not possible to replace time-reversal symmetry by any other symmetry. Indeed, at the level of the mean-field AG theory, the influence of columnar defects is indistinguishable from that of the diffusive scatterers—only the parameter ζ is modified.

However, as we will see in Sec. IV, the Berry-Robnik phenomenon described in Ref. 29 is not completely ineffective in the superconducting phase. Taking into account fluctuations in the gapless phase, we will show that, while the diffusive film belongs to the fundamental symmetry class C (corresponding to a disordered superconductor in a magnetic field), with columnar defects, the system belongs to the higher symmetry class CI (characteristic of the time-reversal invariant superconductor). The result is a substantial modification of the low-energy behavior: in the diffusive film, quantum interference phenomena in the particle/hole channel induce a microgap structure with a DOS varying as $\nu(\epsilon) \sim \epsilon^2$, while in the film with columnar defects $\nu(\epsilon) \sim \epsilon$.

To be specific, let us consider a model of a thin-film superconductor subject to a random (impurity) potential that varies only along the in-plane directions. In the absence of a magnetic field or superconducting order parameter, the quasiparticle Hamiltonian can be subdivided into different subbands labeled by an index k . The spectral properties of each subband are described by a two-dimensional $NL\sigma M$ action of conventional type. The derivation of an effective low-energy action follows closely the normal case in Ref. 28. The Gor'kov Hamiltonian of the system now reads

$$\mathcal{H} = \left(-\frac{\tilde{\partial}^2}{2m} + W(z) - V(x, y) \right) \sigma_3^{PH} + \Delta(z) \sigma_2^{PH}, \quad (11)$$

where $\tilde{\partial} = \partial + iHze_y \sigma_3^{PH}$, W is the confining potential, and V represents an impurity potential drawn at random from the white-noise δ -correlated distribution with zero mean, and correlation $\langle V(\mathbf{r})V(\mathbf{r}') \rangle = (2\pi\nu_{\square}\tau)^{-1} \delta^{(2)}(\mathbf{r} - \mathbf{r}')$, where $\mathbf{r}^{(i)}$ are ‘‘in-plane’’ two-component vectors.

Diagonalizing the z -dependent part of the problem, and representing \mathcal{H} in the basis of the (real) eigenfunctions $\{\phi_k\}$, i.e., $\mathcal{H}_{kk'} = \int dz \phi_k \mathcal{H} \phi_{k'}$ where $[-\partial_z^2/(2m) + W(z) - \epsilon_k]\phi_k = 0$, the vector potential $\mathbf{A} = -Hz\mathbf{e}_y$ as well as the order parameter become matrices in k space,

$$\mathbf{A}_{kk'} = -H\mathbf{e}_y \int dz \phi_k(z) z \phi_{k'}(z),$$

$$\Delta_{kk'} = \int dz \phi_k(z) \Delta(z) \phi_{k'}(z).$$

Let us emphasize that, if the system possesses inversion symmetry $z \rightarrow -z$, the matrix element $A_{kk'}$ differs from zero only if $k+k'$ is odd; in particular, $A_{kk} = 0$. For simplicity, here we only consider the fully symmetric case.³⁰

Under the further assumption that the subband spacing $|\epsilon_k - \epsilon_{k'}|$ is larger than the scattering rate,³¹ one finds that only the diagonal components of the order parameter are nonvanishing. Starting from the conventional superconducting 2D NL σ M action for the k subbands and turning on an in-plane magnetic field, it is straightforward to show that the total effective action assumes the form

$$S = -\frac{\pi\nu_{\square}}{8} \int d^2r \sum_k \text{Str} \left[D_k (\partial Q_k)^2 - 4(i\epsilon^- \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta_{kk} \sigma_2^{PH}) Q_k - 2 \sum_{kk'} \mathcal{X}_{kk'} \sigma_3^{PH} Q_k \sigma_3^{PH} Q_{k'} \right], \quad (12)$$

where $\mathcal{X}_{kk'} = D_{kk'} A_{kk'} A_{k'k} / [1 + (E_{kk'} \tau)^2]$. Furthermore, $D_{kk'} = (D_k + D_{k'})/2$ (with D_k denoting the diffusion constant of subband k) and $E_{kk'} = \epsilon_k - \epsilon_{k'}$. Crucially, from this result we see that there exists no linear coupling of Q to the vector potential—a paramagnetic term does not appear.

To proceed, as before we subject the action (12) to a mean-field analysis. Varying the action with respect to fluctuations of Q_k , one obtains the modified (set of coupled) Usadel equations

$$D_k \partial(Q_k \partial Q_k) - [i\epsilon^- \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta_{kk} \sigma_2^{PH}, Q_k] - \sum_{k'} \mathcal{X}_{kk'} [\sigma_3^{PH} Q_k \sigma_3^{PH}, Q_{k'}] = 0.$$

Applying the ansatz $Q_k = \cosh \hat{\theta}_k \sigma_3^{PH} \otimes \sigma_3^{CC} + i \sinh \hat{\theta}_k \sigma_2^{PH}$ with θ_k homogeneous in the in-plane coordinates, the mean-field equation assumes the form

$$\epsilon^- \sinh \theta_k - \Delta_{kk} \cosh \theta_k - i \sum_{k'} \mathcal{X}_{kk'} \sinh(\theta_k + \theta_{k'}) = 0. \quad (13)$$

In principle, this equation has to be solved in parallel with the self-consistent equation for the order parameter

$$\Delta_{kk'} = \frac{\pi g \nu_{\square}}{\beta} \sum_n \sin \theta_{k,n} \delta_{kk'}, \quad (14)$$

where g is the effective BCS coupling constant and n is a Matsubara index.

Analyzing the saddle-point Eq. (13), it can be easily seen that the field dependent term $\sum_{k'} \mathcal{X}_{kk'} \sin(\theta_k + \theta_{k'})$ vanishes if we choose the solution $\theta_k = (-1)^k \theta$ (i.e., since $\mathcal{X}_{kk'} = 0$ for $k+k'$ even). Thus, there seems to be one mode that is not affected by the magnetic field. However, this would imply that the order parameter, too, must have an alternating sign, i.e., $\Delta_{kk} = (-1)^k \Delta$. Recalling the definition $\Delta_{kk} = \int dz \Delta(z) \phi_k^2$, this is not feasible. Thus, the above solution is ruled out³² and, therefore, on the mean-field level, the symmetry mechanism is ineffective.

A more natural choice seems to be a spatially homogeneous order parameter. Unfortunately, for a general model, the solution of the coupled Eqs. (13) and (14) does not seem to be readily accessible analytically. However, to gain some insight into the nature of the general solution, we will specialize further consideration to the particular case in which only the lowest two subbands are coupled.

With $\mathcal{X}_{12} = \mathcal{X}_{21} \equiv \mathcal{X}$ the equations for θ_1 and θ_2 coincide. Therefore, setting $\theta \equiv \theta_1 = \theta_2$, which implies that $\Delta_{11} = \Delta_{22} \equiv \Delta$, the mean-field equation takes the form reminiscent of the AG equation,

$$\epsilon \sinh \theta - \Delta \cosh \theta - i \mathcal{X} \sinh(2\theta) = 0.$$

As with the diffusive film, the application of a strong in-plane field suppresses the order parameter and allows for the existence of a gapless phase. According to the AG theory, the superconductor enters the gapless phase when $\zeta \equiv 2\mathcal{X}/\Delta = 1$.

If $E_{12}\tau \ll 1$, the parameter ζ is of the same form as that found in the diffusive case, i.e., $\zeta \sim D(Hd)^2/\Delta$. In the opposite limit, ζ is greatly reduced because the wide subband spacing restricts the motion in the z direction. Now, $\zeta \sim D(Hd)^2/[(E_{12}\tau)^2\Delta]$, and, thus, higher magnetic fields have to be applied in order to reach the gapless phase. As in the diffusive case, the hard edge in the gapped phase is compromised due to fluctuations—see the discussion above—and exponentially small tails in the subgap region arise.

More generally, for many subbands, one would expect the same qualitative picture to hold—although Δ_{kk} might slowly depend on k .

The effect of gap suppression is born out in a simple numerical simulation. Figure 3 shows the quasiparticle DOS for a two subband tight-binding model with $2 \times 20 \times 20$ sites when subject to an in-plane magnetic field. The energy is measured in units of the (unperturbed) order parameter. The three curves correspond to different values of the magnetic field. Details of the result at intermediate fields are magnified in Fig. 4. The mean-field square-root edge as well as the exponentially small tails are indicated. Furthermore, the inset shows the linear energy dependence of the subgap DOS exponent, cf. Eq. (10), on a linear-log scale: $\ln \nu(\epsilon < E_{\text{gap}}) \sim E_{\text{gap}} - \epsilon$.

IV. PHASE COHERENCE PROPERTIES OF THE GAPLESS PHASE: MASSLESS FLUCTUATIONS AND THE SOFT MODE ACTION

While, at the level of the mean field, all perturbations (i.e., magnetic impurities as well as parallel fields in films

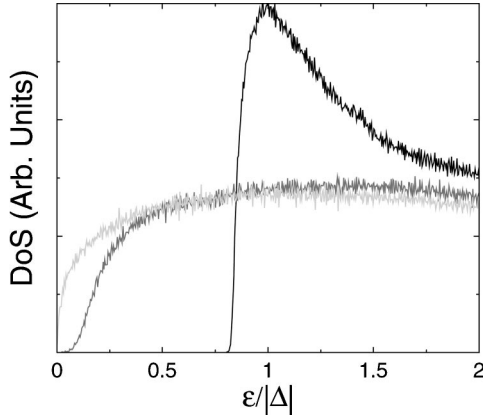


FIG. 3. Numerical results: DOS. Upon increasing the magnetic field, the energy gap closes and the BCS singularity disappears.

with different disorder potentials) follow the same AG phenomenology, it is interesting to note that differences show up in the spectrum of soft fluctuations. The latter are responsible for the long-range spectral and localization properties of the quasiparticles in the gapless phase. In contrast to the magnetic impurity model,^{2,7} which belongs to symmetry class D (due to broken time-reversal and *spin-rotation* symmetry), here the soft fluctuations around the mean-field should be described by an effective action belonging to symmetry class C (broken \mathcal{T} invariance).¹² Therefore, according to the considerations of Refs. 9 and 10, one expects localization of the quasiparticle states in the gapless phase. In fact, below we will see that the fluctuations are sensitive to the very nature of the impurity scattering. It turns out that the film with columnar defects is described by the higher symmetry class CI , implying a modified localization length.^{8,10}

To assess the low-energy properties of the system, we have to first identify the soft modes of the action. For frequencies $\epsilon \rightarrow 0$, the saddle point is not unique, but spans a degenerate manifold $Q = T Q_{SP} T^{-1}$ with $T = \exp(W)$ and $\{Q_{SP}, W\} = 0$. The symmetries of the system impose certain conditions on the generators W .

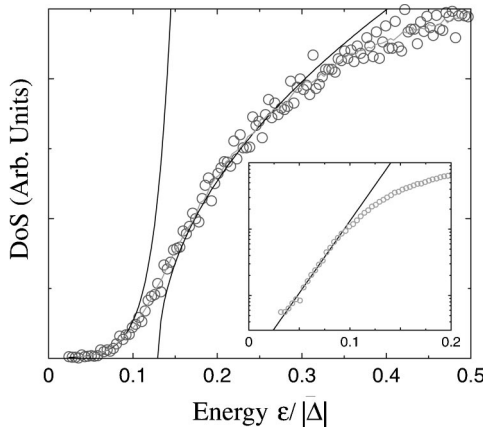


FIG. 4. Numerical results: subgap DOS. The solid lines show the square-root edge and the exponentially small tails. The same data are plotted on a linear-log scale in the inset.

A. Diffusive film

The choice of generators W is dictated by the presence of the order parameter and the magnetic field. This leads to the following conditions.

(1) First, W has to commute with the order parameter

$$[\sigma_2^{PH}, W] = 0.$$

(2) Second, since time-reversal symmetry is broken by the magnetic field, W has to fulfill the further restriction

$$[\sigma_3^{PH}, W] = 0.$$

Thus, $W = \mathbb{1}^{PH} \otimes W_s$. Taken together, these restrictions limit field fluctuations to those belonging to symmetry class C , which describes superconducting systems with spin-rotation symmetry, but broken \mathcal{T} invariance. The corresponding integration manifold of class C is $Osp(2|2)/GI(1|1)$. Expanded in the generators, the soft mode action reads

$$S_{Q_s} = -\frac{\pi\nu\Box}{4} \int d^2r \text{Str}[D \cosh^2\theta(\partial Q_s)^2 + 4i\epsilon \cosh\theta\sigma_3^{CC}Q_s], \quad (15)$$

where $Q_s = T_s \sigma_3^{CC} T_s^{-1}$ and $T_s = \exp(W_s)$.

On energy scales $\epsilon < E_c = D|\cosh\theta|/L^2$, the system enters the universal zero-dimensional regime. Here the properties of the action are dominated by the zero spatial mode and lead to¹²

$$\nu(\epsilon) = \nu(E_c) \left(1 - \frac{\sin(2\pi\epsilon/\delta)}{2\pi\epsilon/\delta} \right), \quad (16)$$

where $\delta = 1/[\nu(E_c)L^2]$; i.e. for $\epsilon \rightarrow 0$, the DOS vanishes quadratically,

$$\frac{\nu(\epsilon)}{\nu(E_c)} \simeq \frac{2}{3} \pi^2 \left(\frac{\epsilon}{\delta} \right)^2.$$

This is to be contrasted with the low-energy behavior of the DOS in the case of columnar defects, where the system possesses the \mathcal{P}_z symmetry.

B. Columnar defects

Here instead of a single generator W , one has to consider a set of generators W_k .

(1) Once again, W_k has to commute with the order parameter

$$[\sigma_2^{PH}, W_k] = 0.$$

(2) However, even though time-reversal symmetry is broken by the magnetic field, the generators do not have to obey $[\sigma_3^{PH}, W_k] = 0$. Due to \mathcal{P}_z symmetry, which causes all elements $\mathcal{X}_{kk'}$ with $k+k'$ even to vanish, it is sufficient to require

$$W_{k'} = \sigma_3^{PH} W_k \sigma_3^{PH} \quad \text{for } k+k' \text{ odd,}$$

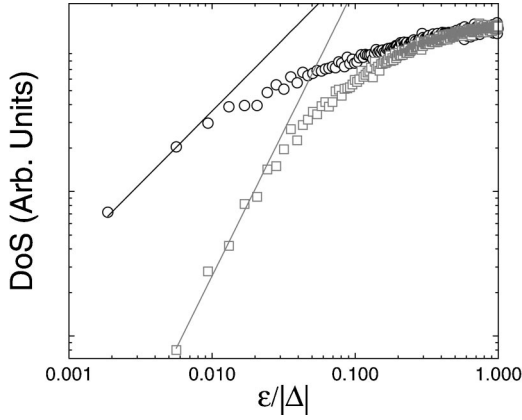


FIG. 5. Numerical results: log-log plot of the low-energy DOS in the gapless phase for a diffusive film (open squares) and a film with columnar defects (open circles).

i.e., one generator, take, e.g., W_0 , can be chosen “freely.” Then, the others are determined through the condition

$$W_k = (\sigma_3^{PH})^k W_0 (\sigma_3^{PH})^k,$$

or $W_k = W_0$ if $k \in 2\mathbf{N}$, and $W_k = \sigma_3^{PH} W_0 \sigma_3^{PH}$ if $k \in 2\mathbf{N} + 1$.

Thus, the second condition here only imposes certain relations between different W_k , but does not restrict the structure of W_k in particle-hole space. This corresponds to the higher symmetry class CI. Now the integration belongs to the group manifold $\text{Osp}(2|2)$. Again we find a manifestation of the Berry-Robnik symmetry phenomenon: the low-energy properties of the gapless phase are determined by the symmetry class associated with systems possessing time-reversal invariance.

Taking into account these fluctuations, the corresponding soft mode action reads

$$S_{Q_s} = -\frac{\pi\nu\Box}{8} \int d^2r \text{Str}[D_k \cosh^2 \theta_k (\partial Q_s)^2 + 4i\epsilon \cosh \theta_k \sigma_3^{PH} \otimes \sigma_3^{CC} Q_s], \quad (17)$$

where $Q_s = T_s \sigma_3^{PH} \otimes \sigma_3^{CC} T_s^{-1}$. Here $T_s = \exp(W_0)$, and W_0 fulfills the conditions specified above. Once again, properties of the class CI are available in the literature.¹² In particular, for small energies, one obtains

$$\frac{\nu(\epsilon)}{\nu(E_c)} = \frac{\pi}{2} \int_0^{\pi\epsilon/\delta} \frac{dz}{z} J_0(z) J_1(z) = \frac{\pi^2}{4} \epsilon/\delta + O(\epsilon^3), \quad (18)$$

showing the DOS to vanish linearly as $\epsilon \rightarrow 0$.

The predicted low-energy behavior can be verified numerically. In Fig. 5, the density of states at low energies is compared for the two cases. On the log-log scale one can read off the exponent α governing the energy dependence, $|\epsilon|^\alpha$. At low energies, the two lines with slope $\alpha_C = 2$ and $\alpha_{CI} = 1$ —characteristic of the symmetry classes C and CI—fit the data for the diffusive film and the film with columnar defects, respectively.

Having studied the influence of a parallel magnetic field on the properties of the superconducting film, we now turn

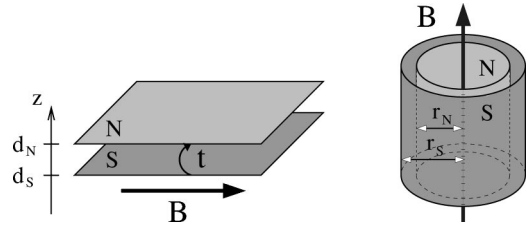


FIG. 6. Sketch of the hybrid NS systems considered in the text in the planar and cylindrical geometry.

our attention to the mean-field properties of normal-metal–superconductor hybrid systems.

V. NS HYBRID SYSTEMS

The properties of thin disordered NS bilayers have been studied in a recent work by Fominov and Feigel'man.¹³ By means of the coupled Usadel equations for the hybrid system, they investigated the density of states as well as the parallel and perpendicular critical fields of the bilayer as a function of the interface transparency. The asymptotics at high and low transparencies are accessible to analytical solutions while the results at intermediate transparencies were found numerically.

Here, we consider a different aspect of the properties of the hybrid system, namely, the interplay between gapless superconductivity and the proximity effect. In addition to the effect of the field on the individual system, in the NS bilayer it also affects the coupling. As we have seen in the preceding sections, an in-plane magnetic field gradually suppresses the gap in the single-particle DOS. On the other hand, in an NS structure, the proximity effect opens a gap in the DOS of the normal layer. Thus, we expect the magnetic field to weaken the proximity effect.

For simplicity, we consider here a hybrid system each consisting of a single N and S channel (see Fig. 6). That is, neglecting the finite width, the magnetic field does not influence the individual systems, and we can study the effect on the coupling alone. The coupling between the layers is described by a tunneling Hamiltonian $\mathcal{H}_T = \int d^2r (t \bar{\Psi}_N \Psi_S + \text{H.c.})$, where t is the tunneling matrix element (assumed to be spatially constant). Thus, the effective action for the NS system consists of a sum of the actions of the individual systems, S_N and S_S , and a coupling term that in the weak tunneling limit can be linearized. That is, the full action reads (see, e.g., Ref. 33)

$$S = -\frac{\pi\nu\Box}{8} \int d^2r \text{Str}[D_N (\tilde{\partial} Q_N)^2 - 4i\epsilon \sigma_3^{PH} \otimes \sigma_3^{CC} Q_N + D_S (\tilde{\partial} Q_S)^2 - 4(i\epsilon \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta \sigma_2^{PH}) Q_S - 4\gamma Q_N Q_S], \quad (19)$$

where $\gamma = |t|^2 \tau$ represents the transparency of the interface. Furthermore $\tilde{\partial} = \partial - i\mathbf{A}[\sigma_3^{PH}, \cdot]$, where the appropriate gauge for the vector potential $\mathbf{A}(\mathbf{r})$ will be specified shortly. In general, the order parameter may be complex, $\Delta = |\Delta| \exp(i\chi \sigma_3^{PH})$. Note that Δ here is the self-consistently

determined order parameter. The presence of the normal layer leads to a renormalization of the order parameter. At weak couplings, however, the proximity induced suppression of the order parameter is small. Therefore, we concentrate on the much more pronounced effect on the quasiparticle DOS.

Subjecting the action (19) to a saddle-point analysis, one obtains the following coupled Usadel-like equations:

$$D_N \tilde{\partial} (Q_N \tilde{\partial} Q_N) - [i \epsilon \sigma_3^{PH} \otimes \sigma_3^{CC}, Q_N] = \gamma [Q_S, Q_N], \quad (20)$$

$$D_S \tilde{\partial} (Q_S \tilde{\partial} Q_S) - [i \epsilon \sigma_3^{PH} \otimes \sigma_3^{CC} - \Delta \sigma_2^{PH}, Q_S] = \gamma [Q_N, Q_S].$$

As a simple guiding example, let us first consider the properties of the system in the absence of a magnetic field, where the order parameter can be chosen to be real. Now, in situations where the superconducting terminal is represented by a bulk system, the latter simply acts as a boundary condition for the normal region. However, in the present case, the single superconducting channel is itself heavily influenced by the contact with the normal region. As a result, Fominov and Feigel'man¹³ have shown that a gap develops in the normal region while in the superconductor quasiparticle states at energies down to the size of the proximity effect induced gap are generated.

To see this explicitly, let us employ the ansatz $Q_X = \cosh \hat{\theta}_X \sigma_3^{PH} \otimes \sigma_3^{CC} + i \sinh \hat{\theta}_X \sigma_2^{PH}$ with θ_X homogeneous ($X = N, S$). In this case, the saddle-point equations reduce to

$$-i \epsilon \sinh \hat{\theta}_N = \gamma \sinh(\hat{\theta}_N - \hat{\theta}_S), \quad (21)$$

$$-i(\epsilon \sinh \hat{\theta}_S - \Delta \cosh \hat{\theta}_S) = \gamma \sinh(\hat{\theta}_S - \hat{\theta}_N).$$

If the two systems are decoupled, $\gamma = 0$, the solution for the superconductor at energies well below the gap, $\epsilon \ll \Delta$, reads $\theta_S \approx i\pi/2$. At weak coupling, setting $\theta_S = i\pi/2 + \vartheta_S$ in the low-energy regime and expanding the equations above up to linear order in ϑ_S yields $\coth \theta_N = \epsilon/\gamma$ and $\vartheta_S = (\epsilon - i\gamma \cosh \theta_N)/\Delta$. Thus, at small energies, the density of states in the two layers is given as

$$\nu_N(\epsilon) = \nu_{\square} \operatorname{Re}[\cosh \theta_N] = \begin{cases} 0, & \epsilon < \gamma \\ \nu_{\square} \frac{\epsilon}{\sqrt{\epsilon^2 - \gamma^2}}, & \epsilon > \gamma, \end{cases} \quad (22)$$

$$\nu_S(\epsilon) = -\nu_{\square} \operatorname{Im}[\vartheta_S] = \frac{\gamma}{\Delta} \nu_N(\epsilon). \quad (23)$$

As expected, the superconductor induces an energy gap in the normal region of magnitude $E_{\text{gap}}^{(N)} = \gamma$. Furthermore, the contact with the normal region leads to the appearance of quasiparticle states in the superconductor at energies down to the proximity effect induced gap $E_{\text{gap}}^{(N)} \ll \Delta$. (Note that close to the singularity at $\epsilon = E_{\text{gap}}^{(N)}$, the approximations above are no longer valid.)

How do these characteristic features of the proximity effect change in the presence of a magnetic field? In the following, we will consider two different geometries as depicted in Fig. 6. In Sec. V A, a planar NS bilayer is investigated. Subsequently, in Sec. V B, we study a setup

where the bilayers are wrapped around a cylinder with the magnetic field directed along the cylinder axis. While in the first case the magnetic-field effect is due only to the flux enclosed between the N and S layers; here the system as a whole encloses magnetic flux, which leads to markedly different behavior.

A. Planar geometry

In the planar geometry, as before, the appropriate gauge is the London gauge $\mathbf{A} = -(Hz + c_0)\mathbf{e}_y$. Here the constant c_0 is determined through the condition that the supercurrent through a cross section of the bilayer vanishes,¹³

$$\int dz \mathbf{j}(z) = \mathbf{j}_N + \mathbf{j}_S = 0, \quad (24)$$

where $\mathbf{j}_X = n_X \mathbf{A}_X / m$. To a first approximation, $n_N = 0$. Thus, no supercurrent flows in the N region and, therefore, the supercurrent in the S region has to vanish as well, i.e., $\mathbf{A}_S = 0$ which implies $c_0 = -Hd_S$.

Using the same ansatz as for the field-free case, one obtains

$$-\frac{D_N}{4} (Hd)^2 \sinh(2\hat{\theta}_N) - i \epsilon \sinh \hat{\theta}_N = \gamma \sinh(\hat{\theta}_N - \hat{\theta}_S),$$

$$-i(\epsilon \sinh \hat{\theta}_S - \Delta \cosh \hat{\theta}_S) = \gamma \sinh(\hat{\theta}_S - \hat{\theta}_N),$$

where $d = d_N - d_S$ is the distance between the two layers.

As pointed out earlier, being a single-channel system, the superconductor alone does not feel the magnetic field. Again we are interested in the DOS at energies well below the gap, $\epsilon \ll \Delta$. As in the field-free case, an expansion in $\vartheta_S = \theta_S - i\pi/2$ leads to

$$\kappa_N \sinh(2\hat{\theta}_N) + 2i(\epsilon \sinh \hat{\theta}_N - \gamma \cosh \hat{\theta}_N) = 0, \quad (25)$$

where $\kappa_N = D_N (Hd)^2 / 2$. Furthermore, $\vartheta_S = (\epsilon - i\gamma \cosh \theta_N) / \Delta$ as before. Thus, in the two-channel case, the magnetic field leads to a suppression of the proximity effect. Equation (25) shows that the closing of the induced gap is described by the AG theory, where the relevant parameter is $\zeta_N = \kappa_N / \gamma$. Therefore, one finds that the characteristic field $\zeta_N(H_c) \equiv 1$ causing the proximity effect induced gap to vanish is much weaker than the field necessary to drive the superconductor into the gapless phase (i.e., taking into account the finite width of the individual layer).

B. Cylindrical geometry

While in the planar geometry a nonvanishing supercurrent is forbidden, in the cylindrical geometry a supercurrent can flow around the cylinder: in contrast to the previous case, the system now encloses magnetic flux.

Starting with a single superconducting layer, for the cylindrical geometry, the most convenient gauge to choose is the symmetric gauge $\mathbf{A} = \frac{1}{2} \mathbf{H} \times \mathbf{r} = -\frac{1}{2} H r \mathbf{e}_{\varphi}$, where we use cylindrical coordinates (r, φ, z) . Now the phase degree of

freedom has to be taken into account, and a more general ansatz for the matrices Q solving the saddle-point equations is needed,

$$Q_S = \cosh \hat{\theta}_S \sigma_3^{PH} \otimes \sigma_3^{CC} + i \sinh \hat{\theta}_S e^{in\varphi} \sigma_3^{PH} \sigma_2^{PH},$$

where $n \in \mathbf{Z}$, and the phase matches the phase of the order parameter, $\hat{\Delta} = |\Delta| \exp(in\varphi \sigma_3^{PH})$. Substituting this ansatz into the saddle-point equation, one obtains

$$\frac{D_S(n + Hr_S^2)^2}{2r_S^2} \sinh(2\hat{\theta}_S) + 2i(\epsilon \sinh \hat{\theta}_S - \Delta \cosh \hat{\theta}_S) = 0.$$

Thus, even in the absence of the NS coupling, the magnetic field affects the properties of the superconductor. The integer n has to be chosen such that it minimizes $|n + Hr_S^2|$, and, therefore, by increasing the magnetic field, the parameter governing the gap suppression

$$\zeta_S(H; n) = \frac{D_S(n + Hr_S^2)^2}{2r_S^2 \Delta}$$

varies periodically between 0 and $\zeta_S^{\max} = D_S / (8r_S^2 \Delta) = (\xi/d_S)^2$, where d_S is the diameter of the cylinder. The size of the energy gap is $E_{\text{gap}} = \Delta(1 - \zeta_S^{2/3})^{3/2}$, and the superconductor enters the gapless phase when $\zeta_S = 1$. This condition can only be fulfilled if $d_S < \xi$.

As pointed out earlier, the magnetic field not only suppresses the energy gap, but also renormalizes the order parameter Δ . That is the order parameter in the formulas above has to be determined self-consistently. At $T=0$, the self-consistency equation can be cast in the form³

$$\ln\left(\frac{\Delta}{\Delta_0}\right) = \begin{cases} -\frac{\pi}{4} \zeta_S, & \zeta_S \leq 1, \\ -\text{arcosh} \zeta_S - \frac{1}{2}(\zeta_S \arcsin \zeta_S^{-1} - \sqrt{1 - \zeta_S^{-2}}), & \zeta_S > 1. \end{cases} \quad (26)$$

Thus, the periodic modulation of ζ also leads to a periodic modulation of the order parameter. Similarly, the transition temperature T_c , which obeys³

$$\ln\left(\frac{T_c}{T_c^0}\right) = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\kappa}{2\pi T_c}\right), \quad (27)$$

is a periodic function of the applied field as observed first by Little and Parks.¹⁵ Furthermore, in small rings superconductivity is completely suppressed in a certain range of magnetic fields around half-integer flux quanta threading the cylinder.¹⁶ Only very recently it has been possible to manufacture small enough cylinders, where this prediction could be verified experimentally.¹⁷ Figure 7 shows the order parameter at $T=0$ and the transition temperature for a cylinder with $d_S < \xi_0$ (where ξ_0 is the coherence length at $H=0$),

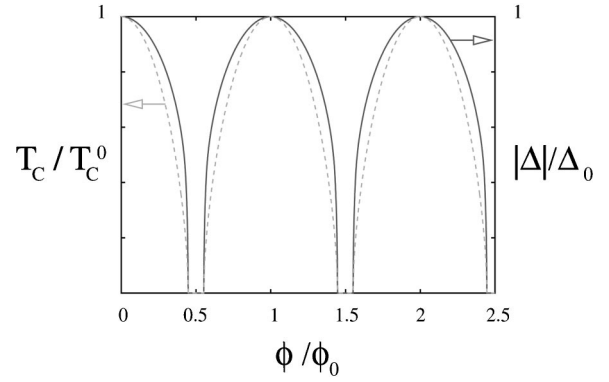


FIG. 7. The order parameter Δ and the transition temperature T_c as a function of the magnetic flux $\Phi = \pi Hr_S^2$ threading the cylinder.

plotted against magnetic flux Φ/ϕ_0 , where $\Phi = \pi Hr_S^2$. As expected, both vanish around half-integer flux quanta while they reach their unperturbed maximal values at integer flux quanta. The region around $\Phi/\phi_0 = 1/2$ is magnified in Fig. 8, where the energy gap and the order parameter are plotted. The system shows a crossover $S \rightarrow$ gapless $S \rightarrow N \rightarrow$ gapless $S \rightarrow S$ by increasing the magnetic field.

Now adding the normal layer, we again concentrate on energies much smaller than the gap. With the same approximation used earlier, the equation for the normal region reads

$$\frac{D_N(n + Hr_N^2)^2}{2r_N^2} \sinh(2\hat{\theta}_N) + 2i(\epsilon \sinh \hat{\theta}_N - \gamma \cosh \hat{\theta}_N) = 0.$$

Once again, the mean-field equation assumes the form of an AG equation with the parameter $\zeta_N(H; n) = D_N(n + Hr_N^2)^2 / (2r_N^2 \gamma)$. Comparing the two values ζ_S and ζ_N , we find that $\zeta_N^{\max} \gg \zeta_S^{\max}$, i.e., the proximity gap is suppressed before the superconductor itself would enter the gapless regime. At the same time the solution for the superconductor takes the form

$$\vartheta_S = \frac{\epsilon - i\gamma \cosh \theta_N}{\Delta(1 - \zeta_S)},$$

yielding $\nu_S = \nu_{\square} \gamma / \Delta(1 - \zeta_S)^{-1} \cosh \theta_N$. That is the combined influence of the presence of the normal region and the

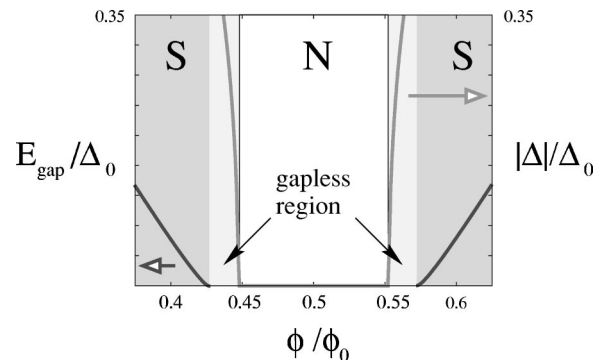


FIG. 8. Energy gap E_{gap} and order parameter Δ in the vicinity of $\Phi/\phi_0 = 1/2$.

magnetic field leads to an enhanced density of states at low energies in the superconductor.

This concludes our discussion of the mean-field properties of thin disordered NS hybrid systems. Taking into account the influence of fluctuations, it is straightforward to see that the low-energy properties of the quasiparticle states are dictated by the same theory obtained in the preceding section. Here, we assume the disorder in the N and S channel to be uncorrelated, which violates \mathcal{P}_z invariance. Thus, the gapless hybrid system is described by symmetry class C .

VI. CONCLUSIONS

To conclude, we have cast the properties of a disordered thin superconducting film subject to a parallel magnetic field in the framework of a statistical field theory. In the mean-field approximation, known results from the AG theory² are recovered. The same phenomenology applies to diffusive films as well as films with columnar defects. In the diffusive case, we have shown that—within the gapped phase—taking into account inhomogeneous instanton solutions of the saddle-point equation, the hard gap is destroyed. By analogy with the magnetic impurity problem,⁷ exponentially small tails within the gap region appear. The same is to be expected for the columnar defects. (For $M > 2$ the coupling between the different subbands complicates the analysis. However, the general behavior should not be affected qualitatively.)

Within the gapless phase, the Berry-Robnik symmetry phenomenon leads to different low-energy properties. As confirmed by numerics, for the diffusive film, the DOS vanishes quadratically for $\epsilon \rightarrow 0$ (class C) as one might expect

for superconducting systems where \mathcal{T} invariance is lifted. However, in the presence of columnar defects only, the DOS at small energies is linear in ϵ (class CI), a behavior characteristic of systems that possess time-reversal invariance. Although the \mathcal{P}_z symmetry cannot prevent the gradual destruction of superconductivity by the magnetic field, some compensation for the \mathcal{T} breaking is still effective.

In NS bilayers, we have shown that the coupling between the two systems leads to (i) an energy gap $E_{\text{gap}}^{(N)}$ in the DOS of the normal layer, and (ii) a finite density of states in the superconductor at energies $E_{\text{gap}}^{(N)} < \epsilon < \Delta$. In this geometry, a parallel magnetic field suppresses the induced proximity gap $E_{\text{gap}}^{(N)}$. The characteristic field $H_c(N)$ determining the occurrence of the gapless phase is greatly reduced as compared to the field $H_c(S)$ that drives the superconductor into the gapless phase, being roughly $H_c(N)/H_c(S) \approx (E_{\text{gap}}^{(N)}/\Delta)^{1/2}$. In a cylindrical geometry, the energy gap shows a periodic modulation with the magnetic field reminiscent of the Little-Parks effect: if the cylinder encloses multiples of the flux quantum ϕ_0 , this can be compensated by the phase of the order parameter. Thus, the variation of the energy gap is determined by the effective field $H_{\text{eff}} = \min_{n \in \mathbb{Z}} |H + n/r_s^2|$. The gapless phase can only be reached in sufficiently small systems, where the diameter of the cylinder fulfills the relation $d_S < \xi$.

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