

Critical universality and hyperscaling revisited for Ising models of general spin using extended high-temperature series

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(Received 16 October 2001; published 2 April 2002)

We have extended through β^{23} the high-temperature expansion of the second field derivative of the susceptibility for Ising models of general spin, with nearest-neighbor interactions, on the simple cubic and the body-centered cubic lattices. Moreover, the expansions for the nearest-neighbor correlation function, the susceptibility, and the second correlation moment have been extended up to β^{25} . Taking advantage of these new data, we can improve the accuracy of direct estimates of critical exponents and of hyperuniversal combinations of critical amplitudes such as the renormalized four-point coupling g_r or the quantity usually denoted by R_ξ^+ . In particular, we obtain $\gamma=1.2371(1)$, $\nu=0.6299(2)$, $\gamma_4=4.3647(20)$, $g_r=1.404(3)$, and $R_\xi^+=0.2668(5)$. We have used a variety of series extrapolation procedures and, in some of the analyses, we have assumed that the leading correction-to-scaling exponent θ is universal and roughly known. We have also verified, to high precision, the validity of the hyperscaling relation and of the universality property both with regard to the lattice structure and to the value of the spin.

DOI: 10.1103/PhysRevB.65.144431

PACS number(s): 75.10.Hk, 05.50.+q, 11.15.Ha, 64.60.Cn

I. INTRODUCTION

The numerical study of the critical properties of the spin- S Ising models with nearest-neighbor interactions had an important historical role in the chain of arguments leading to the modern formulation of the universality hypothesis^{1–6} for the critical phenomena and, in particular, to the concept of universality class. It was in a study of the susceptibility $\chi(\beta;S)$ for the general-spin Ising models on the face-centered-cubic (fcc) lattice by high-temperature (HT) expansions through β^6 that Domb and Sykes⁷ first pointed out that the exponent γ , which characterizes the divergence of χ , was roughly independent of S and guessed for it a universal “Daltonian” value $\gamma=5/4$. Later on, when longer series^{8–10} both for the fcc and for other lattices were derived, a weak dependence of the exponents γ and ν on S emerged from the HT analyses, but was soon correctly ascribed to the occurrence of nonanalytic “confluent corrections to scaling” (CCS) rather than to a failure of the universality. In those years the existence of CCS had been inferred by various authors both from the numerical analysis of HT series^{11–13} and from phenomenological fits^{14–16} to high-precision experimental data for some systems close to criticality. Eventually the status of the CCS was more firmly established¹⁷ in the context of the renormalization group (RG) theory.³ It was therefore recognized very early that accurate determination of the critical exponents in numerical or experimental studies and, as a consequence, the feasibility of stringent verifications both of the universality hypothesis and of the scaling and hyperscaling relations require a close control over the CCS. For many years, however, in two and in three dimensions, HT series were available only for few observables and, generally, they were barely sufficient to conjecture the presence of CCS, but definitely too short to make a numerically accurate discussion of these models possible.^{8,10,18,19} Still presently, the expansions of $\chi(\beta;S)$ and of the second mo-

ment of the correlation function $\mu_2(\beta;S)$ on the simplecubic (sc) lattice, for spin $S>1/2$, can be found explicitly in the literature⁹ only up to order β^{12} . The data files by Roskies and Sackett²⁰ made an extension of these series through β^{15} feasible for the sc and bcc lattices, but did not drastically change the situation. On the fcc lattice, the HT series initially derived through β^{12} in Ref. 9, were later extended in Ref. 21 to order β^{14} . Fortunately, in the case of the body-centered-cubic (bcc) lattice, decisive progress occurred already two decades ago, with the computation by Nickel²² of expansions for $\chi(\beta;S)$ and $\mu_2(\beta;S)$ through β^{21} . (To our knowledge, only the series for $S=1/2,1,2,\infty$ were published.²³)

By allowing to some extent for the leading CCS, the first modern analyses of the extended bcc series^{22–28} improved significantly the accuracy in the verification of universality with respect to the magnitude of the spin. Moreover, in the mentioned studies (as well as in later analyses²⁹ mainly devoted to the $S=1/2$ case), the central estimates of the susceptibility and the correlation-length exponents were reduced up to $\approx 1\%$ with respect to the values $\gamma=1.250(3)$ and $\nu=0.638(2)$, initially guessed in Ref. 7 and later confirmed^{31,32} by various studies. This development also contributed to settle⁵ a long-standing controversy raised by the results of Refs. 11 and 32–35, which stimulated the studies of Refs. 22–29 and 36–42, on the validity of hyperscaling and, more generally, on the consistency of the results from the HT analyses with the corresponding RG estimates,^{5,43–49} either in the ϵ -expansion approach⁴³ or in the fixed-dimension perturbative scheme.^{44,45}

One should also note that for the second field derivative of the susceptibility $\chi_4(\beta;S)$ and for the nearest-neighbor correlation function $G(\beta;S)$, the published data are even less abundant. On the sc lattice, series for $\chi_4(\beta;S)$ can be derived from the data files of Ref. 50 up to order β^{14} and up to β^{10} from the data of Ref. 34 for the bcc lattice. On the fcc lattice, series for $\chi_4(\beta;S)$ are available⁴¹ through β^{13} . For

TABLE I. Orders of high-temperature expansions, published (or obtainable from published data) before our work, for the nearest-neighbor correlation function $G(\beta;S)$, for the susceptibility $\chi(\beta;S)$, for the second moment of the correlation function $\mu_2(\beta;S)$, and for the second field derivative of the susceptibility $\chi_4(\beta;S)$, in the case of the Ising models with general spin S .

Observable	Lattice, order	Lattice, order	Lattice, order
$\chi(\beta;S)$	sc, 15 ^a	bcc, 21 ^b	fcc, 14 ^c
$\mu_2(\beta;S)$	sc, 15 ^a	bcc, 21 ^b	fcc, 14 ^c
$\chi_4(\beta;S)$	sc, 14 ^d	bcc, 10 ^e	fcc, 13 ^f
$G(\beta;S)$			fcc, 14 ^c

^aReference 20.

^dReference 50.

^bReference 23.

^eReference 34.

^cReference 21.

^fReference 41.

general spin, only expansions^{9,21} of $G(\beta;S)$ through β^{14} on the fcc lattice have been published. A summary of the HT expansions available until now for the Ising models of general spin appears in Table I.

We have been pursuing a long-term project to improve the algorithms and the codes for HT expansions in two-dimensional⁵¹ and in three-dimensional^{52–56} lattice spin models, keeping up with the steady increase of computer performances and periodically updating the numerical analyses whenever we could significantly extend the series. By using an appropriately renormalized linked-cluster method,^{34,50,57} we have now added from 4 up to 13 terms to the HT expansions for various observables of the general spin- S Ising models on sc and bcc lattices. In this paper we shall examine the expansions of $\chi(\beta;S)$ and $\mu_2(\beta;S)$ up to order β^{25} and of $\chi_4(\beta;S)$ up to β^{23} on both lattices. These data have been derived by slightly improving the thoroughly tested code which recently produced⁵⁶ our series through β^{23} for $\chi(\beta;1/2)$ and $\mu_2(\beta;1/2)$ on both lattices. The extension of the series is by far the hardest part of this work, but we will not enter here into the details of our procedure. To give an idea of the required computational effort, it will suffice to mention that our improved codes take minutes of CPU time on a COMPAQ Alpha XP1000 (500 MHz) single-processor workstation to reproduce the known series through β^{21} , whereas several days are necessary to add the following four orders. From the graph-theoretical point of view, it is the expansion of χ_4 through β^{23} which involves the most laborious part of the calculation: in the simplest vertex renormalized expansion scheme⁵⁰ it would require the generation and the evaluation of over 10^9 topologically inequivalent graphs. However, devising a careful strategy of in-depth renormalizations, the expected size of the calculation has been reduced by at least two orders of magnitude. On the other hand, from a purely computational standpoint, the calculation of the sc lattice constants for the second moment of the correlation function is the most demanding part of the job in terms of CPU time.

The correctness of our codes is ensured by numerous internal consistency checks, as well as by their ability to reproduce established results already available in simpler particular cases, such as the square-lattice two-dimensional spin-1/2

Ising model or the one-dimensional spin- S Ising models. Of course, our codes also reproduce the old computation of Ref. 22 for $S=1,2,\infty$ on square and bcc lattices and, as far as there is overlap, also the recent computation of Ref. 58 for $S=1/2$ on the bcc lattice.

Using this vast library of partially new high-order series data and in particular our significantly extended series for $\chi_4(\beta;S)$, we can resume from a vantage point the very accurate studies performed on series $O(\beta^{21})$ for χ and μ_2 in Refs. 22–26 and present an even more extensive and detailed survey of the critical behavior for the spin- S Ising models. In spite of the remarkable advances achieved by the calculations of Refs. 22–27 which removed away from the foreground the universality and the hyperscaling issues, further extensions of the HT data still remain of great interest. They are instrumental in the continuing efforts to gain a higher accuracy in the estimates of the critical parameters and, more generally, to perform more stringent tests of hyperscaling and of universality, with respect both to the value of the spin and to the lattice structure. These are certainly welcome results, since it is fair to say that the actual verification of such basic properties is still only moderately accurate, although no doubts persist anymore about their validity. Of course, one must be aware that the computational complexity of the calculation of higher-order series coefficients grows much faster than the precision in the evaluation of the critical parameters that can be obtained from them by the presently available numerical tools. Therefore the higher-order computations should be accompanied also by an effort to improve the techniques of analysis or, at least, by a careful comparison of the results obtained by a variety of methods.

The paper is organized as follows. In the next section, we set our notations and definitions. In Sec. III we state the assumptions underlying our analysis and its aims. The numerical procedures we have used—namely the modified-ratio methods introduced in Ref. 24 or the differential approximant methods^{59,60}—as well as the corresponding results of the series analysis are discussed in Secs. IV–VIII. In Sec. IX we compare our estimates with those of the most recent literature. The last few sections present our results for the critical amplitudes of the observables that have been expanded and for some (hyper)universal combinations⁶¹ of these amplitudes. In order to make our analysis completely reproducible and to provide a convenient source of data for further work, without overburdening this paper, we have collected into a separate report,⁶² available on request, the complete expansions of the nearest-neighbor correlation function, of the susceptibility, of the second moment of the correlation function, and of the second field derivative of the susceptibility for spin $S=1/2,1,3/2,2,5/2,3,7/2,4,5,\infty$, on square, sc, and bcc lattices.

II. THE SPIN- S ISING MODELS

The spin- S Ising models are defined by the Hamiltonian

$$H\{s\} = -\frac{J}{2} \sum_{\langle x, x' \rangle} s(\vec{x}) s(\vec{x}'), \quad (1)$$

where J is the exchange coupling, and $s(\vec{x}) = s^z(\vec{x})/S$ with $s^z(\vec{x})$ a classical spin variable at the lattice site \vec{x} , taking the $2S+1$ values $-S, -S+1, \dots, S-1, S$. The sum runs over all nearest-neighbor pairs of sites. We shall consider expansions in the usual HT variable $\beta = J/k_B T$ called ‘‘inverse temperature’’ for brevity.

In the high-temperature phase, the basic observables are the connected $2n$ -spin correlation functions. Here we shall limit our study to quantities related to the two-spin correlation functions $\langle s(\vec{x})s(\vec{y}) \rangle_c$ and to the four-spin correlation functions $\langle s(x)s(y)s(z)s(t) \rangle_c$.

In particular, we shall consider the nearest-neighbor correlation function

$$G(\beta; S) = \langle s(\vec{0})s(\vec{\delta}) \rangle_c = \sum_{r=0}^{\infty} h_r(S) \beta^r, \quad (2)$$

where $\vec{\delta}$ is a nearest-neighbor lattice vector.

The internal energy per spin is defined in terms of $G(\beta; S)$ by

$$U(\beta; S) = -\frac{qJ}{2} G(\beta; S), \quad (3)$$

where q is the lattice coordination number.

The specific heat is the temperature derivative of the internal energy at fixed zero external field:

$$C_H(\beta; S)/k_B = \frac{q\beta^2}{2} \frac{dG(\beta; S)}{d\beta}. \quad (4)$$

In terms of $\chi(\beta; S)$, the zero-field reduced susceptibility

$$\chi(\beta; S) = \sum_x \langle s(0)s(\vec{x}) \rangle_c = \sum_{r=0}^{\infty} c_r(S) \beta^r \quad (5)$$

and of $\mu_2(\beta; S)$, the second moment of the correlation function,

$$\mu_2(\beta; S) = \sum_x \vec{x}^2 \langle s(0)s(\vec{x}) \rangle_c = \sum_{r=1}^{\infty} d_r(S) \beta^r, \quad (6)$$

the ‘‘second-moment correlation length’’ $\xi(\beta; S)$ is defined by

$$\xi^2(\beta; S) = \frac{\mu_2(\beta; S)}{6\chi(\beta; S)} = \sum_{r=1}^{\infty} t_r(S) \beta^r. \quad (7)$$

The second field derivative of the susceptibility $\chi_4(\beta; S)$ is defined by

$$\chi_4(\beta; S) = \sum_{x,y,z} \langle s(0)s(x)s(y)s(z) \rangle_c = \sum_{r=0}^{\infty} e_r(S) \beta^r. \quad (8)$$

Notice that these definitions ensure the existence of a non-trivial limit as $S \rightarrow \infty$.

III. ASSUMPTIONS AND AIMS OF THE SERIES ANALYSIS

In the universality class of the spin- S Ising models, the asymptotic behavior of the susceptibility as $\beta \rightarrow \beta_c^\#(S)$ from below is expected to be

$$\chi^\#(\beta; S) \simeq C^\#(S) \tau^\#(S)^{-\gamma} [1 + a_\chi^\#(S) \tau^\#(S)^\theta + \dots + b_\chi^\#(S) \tau^\#(S) + \dots], \quad (9)$$

where $\tau^\#(S) = 1 - \beta/\beta_c^\#(S)$ is the reduced inverse temperature. We have introduced here the superscript $\#$ which stands for either sc or bcc, as appropriate, and will be used hereafter only when useful. Equation (9), often called the Wegner expansion,¹⁷ specifies how the dominant scaling behavior, characterized by the universal critical exponent γ and by the critical amplitude $C^\#(S)$, is modified by analytic and nonanalytic confluent corrections to scaling (CCS) in a close vicinity of the critical point. The leading nonanalytic CCS is characterized by a universal exponent θ and by an amplitude $a_\chi^\#(S)$. The critical amplitudes $C^\#(S)$, $a_\chi^\#(S)$, and $b_\chi^\#(S)$, as well as the inverse critical temperature $\beta_c^\#(S)$, are nonuniversal: namely, they depend on the spin S and on the lattice structure, as stressed by the notation. The analogous asymptotic behaviors of the correlation length,

$$\xi^\#(\beta; S) \simeq f^\#(S) \tau^\#(S)^{-\nu} [1 + a_\xi^\#(S) \tau^\#(S)^\theta + \dots + b_\xi^\#(S) \tau^\#(S) + \dots], \quad (10)$$

of the specific heat,

$$C_H^\#(\beta; S)/k_B \simeq A^\#(S) \tau^\#(S)^{-\alpha} [1 + a_C^\#(S) \tau^\#(S)^\theta + \dots + b_C^\#(S) \tau^\#(S) + \dots], \quad (11)$$

and of $\chi_4(\beta; S)$,

$$\chi_4^\#(\beta; S) \simeq -C_4^\#(S) \tau^\#(S)^{-\gamma_4} [1 + a_4^\#(S) \tau^\#(S)^\theta + \dots + b_4^\#(S) \tau^\#(S) + \dots], \quad (12)$$

as well as of the other singular observables, are characterized by different critical exponents and by different (nonuniversal) critical amplitudes $f^\#(S), a_\xi^\#(S)$, etc., but all contain the same leading confluent exponent θ . Notice that we have freely chosen in Eqs. (9)–(12) between the conventions of Ref. 63 and those of Ref. 61, since the notation for the amplitudes is not yet completely standardized.

Usually the exponent γ_4 is expressed in terms of γ and of Δ , the ‘‘gap’’ exponent associated with the critical behavior of the higher field derivatives of the free energy, as follows: $\gamma_4 = \gamma + 2\Delta$. Here $\Delta = \beta + \gamma$ and β denotes the magnetization exponent only in this formula and in the scaling relation to be quoted before Eq. (20). From RG calculations^{43–49,64,65} it is expected that $\theta \simeq 0.5$ for the Ising universality class.

For later use, we observe that, if the singularity closest to the origin in the complex β plane is the critical singularity, then Eq. (9) implies the following asymptotic behavior for the expansion coefficients $c_n^\#(S)$ of $\chi^\#(\beta; S)$:

$$c_n^\#(S) = C_\chi^\#(S) \frac{n^{\gamma-1}}{\Gamma(\gamma)} \beta_c^{-n}(S) \left[1 + \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} \frac{a_\chi^\#(S)}{n^\theta} + O(1/n) \right]. \quad (13)$$

In the Ising universality class, the expected $O(1/n^{2\theta})$ contributions in Eq. (13) should be practically degenerate with the analytic $1/n$ corrections. For bipartite lattices, the higher-order corrections in Eq. (13) include terms $O(1/n^{1+\gamma-\alpha})$ with alternating signs, which reflect the presence^{66–68} of a weak “antiferromagnetic” singularity at $\beta = -\beta_c^\#(S)$ with exponent $1 - \alpha$. Analogous formulas for the asymptotic behavior with respect to the order can be written for the expansion coefficients $t_n^\#(S)$ of $\xi^\#(\beta; S)^2$ and $e_n^\#(S)$ of $\chi_4^\#(\beta; S)$.

In terms of χ , ξ , and χ_4 , a “hyperuniversal” combination of critical amplitudes denoted by g_r and usually called the “dimensionless renormalized coupling constant” can be defined in d dimensions by the limiting value of the ratio

$$g^\#(\beta; S) \equiv - \frac{3v^\# \chi_4^\#(S; \beta)}{16\pi \xi^\#(S; \beta)^d \chi^\#(S; \beta)^2} \quad (14)$$

as $\tau^\#(S) \rightarrow 0+$. Here $v^\#$ denotes the volume per lattice site (in three dimensions $v^{sc} = 1$ and $v^{bcc} = 4/3\sqrt{3}$) and the normalization factor $3/16\pi$ is chosen in order to match the conventional field theoretic definition of g_r .⁴⁶ We shall call $g^\#(\beta; S)$ the “effective renormalized coupling constant” at the inverse temperature β .

By Eqs. (9)–(12), $g^\#(S; \beta)$ behaves as

$$g^\#(\beta; S) \approx g_r \tau^\#(S)^{\gamma+d\nu-2\Delta} [1 + a_g^\#(S) \tau^\#(S)^\theta + \dots] \quad (15)$$

when $\tau^\#(S) \rightarrow 0+$, with

$$g_r = - \frac{3v^\# C_4^\#(S)}{16\pi f^\#(S)^d C^\#(S)^2}. \quad (16)$$

The Guntton-Buckingham^{69–72} inequality

$$\gamma + d\nu - 2\Delta \geq 0, \quad (17)$$

together with the Lebowitz⁷³ inequality $\chi_4^\#(\beta; S) \leq 0$, ensures that $g^\#(\beta; S)$ remains bounded and non-negative as $\tau^\#(S) \rightarrow 0+$. The vanishing of $g^\#(\beta_c^\# - 0; S)$ is a sufficient condition for Gaussian behavior at criticality: namely, for the vanishing of the four-spin and of the higher-order connected correlation functions. In lattice field theory language, this corresponds to the “triviality,”^{3,74} numerically observed when $d = 4$ and proved when $d > 4$, for the continuum field theory defined by the lattice model (with ferromagnetic couplings^{75,76}) in the critical limit. If the inequality (17) holds as an equality

$$\gamma + d\nu - 2\Delta = 0 \quad (18)$$

(called the “hyperscaling relation”), if there are no logarithmic corrections to the scaling behavior,^{77,78} and if $\chi_4^\#(\beta; S)$ is nonvanishing, we have

$$g^\#(\beta; S) \approx g_r [1 + a_g^\#(S) \tau^\#(S)^\theta + \dots]; \quad (19)$$

namely, the “effective coupling” $g^\#(\beta; S)$ tends to a universal nonzero limiting value g_r as $\tau^\#(S) \rightarrow 0+$.

Using the Essam-Fisher^{79–81} scaling relation $\alpha + 2\beta + \gamma = 2$, Eq. (18) can be rewritten as a relation^{82,83} between α and ν :

$$\alpha = 2 - d\nu. \quad (20)$$

As a consequence, also the combination of critical amplitudes $A^\#(S)$ and $f^\#(S)$,

$$R_\xi^+ = \left(\frac{\alpha A^\#(S)}{v^\#} \right)^{1/d} f^\#(S), \quad (21)$$

is hyperuniversal, as pointed out⁸⁴ by Stauffer, Ferer, and Wortis.

The ratios a_ξ/a_χ , a_4/a_χ , a_c/a_χ , etc., of the amplitudes of the leading CCS are less studied,⁶¹ but not less interesting, universal^{85–87} critical observables.

In the rest of this paper we shall employ our HT series to estimate the critical parameters defined by Eqs. (9)–(12), (19), and (21), to check the validity of Eqs. (18) and (20) and of the universality property with respect to the value of the spin and to the lattice structure.

In the actual numerical analysis of finite-order HT expansions, the presence of the CCS will generally become manifest^{8,22–24} by small apparent violations both of the hyperscaling relations and of the universality properties: namely, by a weak apparent dependence of the universal quantities on the lattice structure and on the value S of the spin. We shall point out this fact by explicitly indicating, in our numerical estimates of the universal quantities, the value of the spin and the lattice structure for the series used in the analysis. For instance, $\gamma^{bcc}(S)$ will denote the numerical estimate of the universal exponent γ obtained from the series $\chi^{bcc}(\beta; S)$. This notation will help to emphasize how small the mentioned effects of apparent nonuniversality are reduced if the HT expansions can be pushed to a sufficiently high order, provided that the numerical tools of the analysis can, at least approximately, allow for the leading CCS.

Part of our analysis will rely upon the main assumption that the exponent θ of the leading CCS is universal and roughly known. A recent accurate RG recalculation of universal critical data^{64,65} predicts the value $\theta = 0.504(8)$ in the fixed-dimension perturbative approach, while within the ϵ -expansion scheme, the updated estimate is $\theta = 0.512(13)$. In the rest of this paper, we shall adopt as a reference value the fixed-dimension RG estimate $\theta^{ref} = 0.504(8)$ when computing the central values of critical parameters by procedures biased with θ . Even if one has no compelling reason to suppose that the uncertainty of the RG prediction of θ is largely underestimated (but this possibility is advocated in Ref. 88), the reliability of the θ -biased analyses presented here will be greater whenever their results are not too sensitive to the precise value of θ . In the following sections, it will be clear that, in most cases, we can tolerate an uncertainty of this exponent even several times larger than above indicated. We

will show that most of our estimates biased with $\theta^{ref} = 0.504(8)$ will be compatible also with higher values, such as $\theta = 0.52(3)$, proposed in Refs. 23,27, and $\theta = 0.53(2)$ in Ref. 89, or even $\theta = 0.54(3)$ from Refs. 25,26 and 90–92. We can add that both our direct HT evaluation of θ and the part of our series analysis which is not biased with the value of the exponent θ will be completely consistent with the above assumption.

If appropriate, we shall provide detailed information on the θ -biased numerical results reported in our tables for a given quantity P by indicating together with the central estimate also the derivative $\partial P/\partial\theta$ evaluated at the reference value chosen for θ . Similarly, in the cases where the parameter estimates are biased with the value of a critical inverse temperature $\beta_c^{\#ref}(S)$ and/or of a critical exponent—for instance, γ —we shall report the corresponding derivatives $\partial P/\partial\beta_c$ and/or $\partial P/\partial\gamma$ computed at the specified reference values. As an example, for the critical amplitude of the susceptibility $C^\#(S)$, our final estimate can be read as

$$C^\#(S)(\text{error}) + [\partial C^\#(S)/\partial\beta_c](\beta_c - \beta_c^{ref}) + [\partial C^\#(S)/\partial\gamma](\gamma - \gamma^{ref}). \quad (22)$$

Here both the estimate and its derivatives are evaluated for sharp values of $\beta_c = \beta_c^{ref}$ and of $\gamma = \gamma^{ref}$ and the error attached to the first term does not allow for the uncertainty of the bias parameters. Since the above expression describes how the central estimate of $C_\chi^\#(S)$ changes under small variations of the bias parameters, comparisons with previous results in the literature, often based on slightly different assumptions, are made straightforward.

As a final general remark, it is worth mentioning that, due to the higher coordination number of the lattice, the bcc series approach their asymptotic structure, Eq. (13), generally faster than the sc series. For this reason, the bcc series are usually observed to yield more accurate estimates of the critical parameters than the corresponding sc series with the same number of coefficients and are often said to have a greater “effective length.” This fact will be confirmed here and will be one of the reasons to draw our final best estimates from the analysis of the bcc series. Nevertheless, the sc lattice series remain very interesting, in particular because the nonuniversal information obtained from them is directly comparable to the data from simulation studies, traditionally performed on the sc lattice. It is also interesting to notice that, for both lattices, the asymptotic behaviors set in more slowly in the most widely studied $S = 1/2$ case. This is not surprising since the number of degrees of freedom per site is proportional to the magnitude of the spin. A slower convergence is observed also for the higher moments of the correlation function, since in their construction larger weights are given to the correlations between farther sites for which the expansions are effectively shorter. As a consequence, on both sc and bcc lattices, the expansions of ξ show a slower convergence than those of χ .

Based on the assumptions indicated above, our analysis will aim to exhibit, within the family of the spin- S Ising models, some consequences of the universality property, of

the scaling, and of the hyperscaling laws for the correlation functions, for the exponents, and for various universal combinations of critical amplitudes. In particular our accurate verification of the universality property will strengthen the justification of a technique advocated long ago by Zinn-Justin²⁴ and independently by Chen, Fisher, and Nickel²⁵ to improve the precision in the computation of the universal critical parameters of the Ising model. These authors argued that numerical studies should address appropriate families of spin models parametrized by a continuous auxiliary variable and belonging to the same universality class as the Ising model. For specific values of this variable it is possible to select representative models for which the amplitudes of the leading confluent corrections to scaling are negligible and, as a consequence, the determination of the universal critical parameters can be more accurate. By relying on a similar prescription, we will also obtain very accurate estimates of some universal critical parameters.

IV. ESTIMATES OF THE CRITICAL POINTS

In this section we shall examine the HT expansion of the susceptibility in zero field, for several values of the spin S on sc and bcc lattices. The series coefficients of the susceptibility generally show a very smooth dependence on the order of expansion and a relatively fast approach to their asymptotic forms. Therefore they are best suited to an accurate determination of the critical temperatures. The estimates so obtained will also be adopted to bias the calculation of the critical exponents and amplitudes.

As we have already argued in our previous study⁵⁶ of the HT series for the $S = 1/2$ case, the modified-ratio method introduced by Zinn-Justin²⁴ (see also Ref. 59) can lead to estimates of the critical inverse temperatures with an accuracy comparable or sometimes higher than the traditional differential approximant (DA) methods. Perhaps, the potential of this tool has not been properly appreciated, because so far it could not be used with series long enough.

The method consists in evaluating $\beta_c^\#(S)$ from the approximant sequence

$$(\beta_c^\#(S))_n = \left(\frac{c_{n-2}c_{n-3}}{c_n c_{n-1}} \right)^{1/4} \exp \left[\frac{s_n + s_{n-2}}{2s_n(s_n - s_{n-2})} \right], \quad (23)$$

with

$$s_n = \left[\ln \left(\frac{c_{n-2}^2}{c_n c_{n-4}} \right)^{-1} + \ln \left(\frac{c_{n-3}^2}{c_{n-1} c_{n-5}} \right)^{-1} \right] / 2. \quad (24)$$

Since the expected value of θ is very nearly 1/2, by using Eq. (13), the asymptotic behavior of the approximant sequence can be expressed as follows:

$$(\beta_c^\#(S))_n = \beta_c^\#(S) \left[1 - \frac{\Gamma(\gamma)}{2\Gamma(\gamma - \theta)} \frac{\theta^2(1 - \theta)a_\chi^\#(S)}{n^{1+\theta}} + O\left(\frac{1}{n^2}\right) \right]. \quad (25)$$

TABLE II. Estimates of the critical inverse temperatures for the spin- S Ising models on sc and bcc lattices, obtained in this work by the modified-ratio method biased with the leading correction-to-scaling exponent θ . The sensitivity of the estimates to the bias value of θ is characterized by the derivatives of β_c . For comparison, we have also reported some results obtained by simulation methods or from the analysis of shorter series in the recent literature.

	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$\beta_c^{sc}(S)$	0.221655(2)	0.312867(2)	0.368657(2)	0.406352(3)	0.433532(3)	0.454060(3)	0.601271(3)
$\partial\beta_c^{sc}(S)/\partial\theta$	7×10^{-6}	4×10^{-6}	4×10^{-6}	4×10^{-6}	6×10^{-6}	6×10^{-6}	6×10^{-6}
$\beta_c^{bcc}(S)$	0.1573725(10)	0.224656(1)	0.265641(1)	0.293255(2)	0.313130(2)	0.328119(2)	0.435085(3)
$\partial\beta_c^{bcc}(S)/\partial\theta$	3×10^{-6}	10^{-8}	-3×10^{-6}	-6×10^{-6}	-6×10^{-6}	-6×10^{-6}	-6×10^{-6}
$\beta_c^{sc}(S)^a$	0.2216546(10)						
$\beta_c^{sc}(S)^b$	0.2216595(15)						
$\beta_c^{sc}(S)^c$	0.221655(1)						
$\beta_c^{bcc}(S)^d$	0.157374(7)	0.224657(4)		0.293258(6)			0.435089(11)
$\beta_c^{bcc}(S)^e$	0.157373	0.224654		0.293255			

^aReference 93.

^bReference 97.

^cReference 98.

^dReference 24.

^eReference 28.

It is interesting to observe also that, if $\theta=1/2$, the coefficient of the $O(1/n^2)$ correction in Eq. (25) is equal to $[a_\chi^\#(S)]^2$ times a very small positive factor. Moreover, the coefficient of the $1/n$ term in Eq. (13) enters into Eq. (25) only at next higher orders. Since $a_\chi^\#(S)$ is expected to be small (though generally not negligible), these remarks help to understand how this method works and why it is much more efficient than conventional ratio prescriptions. We can say that Eq. (25) provides an estimate of the leading “finite-order effects”: namely, of the corrections due to using series of finite length n . These are strictly analogous to the well known “finite-size effects” which have to be carefully considered to improve the data from simulations of finite systems. At the orders of expansion presently available Eq. (25) has already a reasonably accurate quantitative meaning.

Although devised specifically to deal with the expected structure of the singularities, the procedure we have sketched is *unbiased*: namely, no additional accurate information on other critical parameters must be used together with the series in order to get the estimate sequence. However, at the present orders of expansion, the n dependence of $(\beta_c(S))_n$ is not saturated and, for sufficiently large n , the successive estimates show an evident residual trend, very nearly linear on a $1/n^{1+\theta}$ plot, as expected from Eq. (25). Small odd-even oscillations are superimposed onto the main monotonic trend as a consequence of the above-mentioned antiferromagnetic singularity [see the comments to Eq. (13)]. These observations suggest that one can do something better than taking the highest-order available term of the sequence, Eq. (23), as the final estimate of $\beta_c^\#(S)$. The most obvious improvement consists in using the assumed known value of θ to fit the asymptotic behavior of the sequence and in taking the extrapolated value of the sequence as a better estimate of $\beta_c^\#(S)$. As usual, one should separately extrapolate to large n the odd and the even subsequences of $(\beta_c(S))_n$, in order to deal properly also with the oscillations due to the antiferromagnetic singularity. Our extrapolation will be based on the successive pairs of terms in the approximant subsequences.

Eventually a further minor adjustment of the results might be performed by a second (purely visual) extrapolation in order to allow also for a very small residual curvature of the plots due to the higher corrections in Eq. (25). For instance, in the $S=1/2$ case on the sc lattice, the highest-order estimate from the extrapolation of the last pair in the odd-approximant subsequence is 0.221 656 46. In order to allow for the small residual curvature of the extrapolation sequence, this figure should probably be slightly reduced, to yield our final (and very conservative) estimate $\beta_c^{sc}(1/2)=0.221\ 655(2)$.

The set of our estimates for $\beta_c^\#(S)$ is reported in Table II. The errors we have indicated are small multiples (2–4) of the differences between the extrapolations of the two highest-order pairs of terms in the odd subsequences. In the same table, we have also reported $\partial\beta_c^\#(S)/\partial\theta$ evaluated at $\theta=\theta^{ref}$. As shown by our data, the above-mentioned uncertainty in the value of θ^{ref} turns out to be unimportant in the whole procedure, because it contributes only a small fraction of the final uncertainty of the estimates.

In order to give an idea of the qualitative features of the method, for each value S of the spin examined in this study, we have plotted in Fig. 1 the corresponding “normalized” approximant sequence $(\beta_c^{bcc}(S))_n/N(S)$ vs $1/n^{1+\theta}$. We have taken the average of the extrapolated values of the even and odd subsequences as the normalization factor $N(S)$, introduced only to make the various plots easily comparable and conveniently fit all of them into a single figure. We have drawn as solid lines the extrapolants of the last odd pair of terms in the sequences, whereas the dashed lines indicate the extrapolants of the last even pair. The difference between the extrapolated values of the odd and even normalized subsequences, which is generally very small (for instance, it is $\approx 10^{-6}$ in the bcc lattice case and at most 4 times as large in the sc case), provides a first rough indication that the oscillating corrections due to the antiferromagnetic singularity give only a small contribution to the uncertainty of the results. The final relative errors reported in Table II are generally much larger.

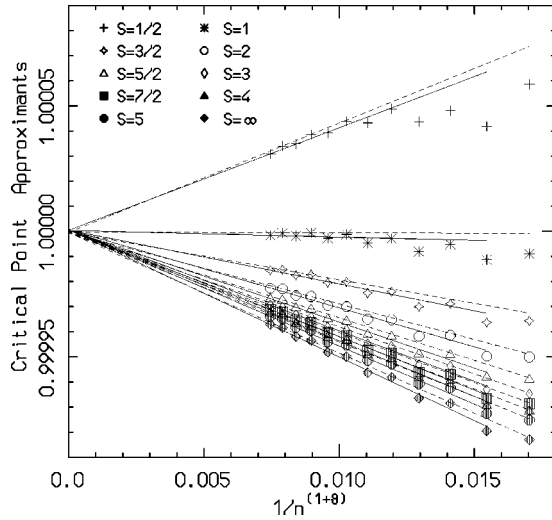


FIG. 1. The “normalized” modified-ratio approximant sequences $(\beta_c^{bcc}(S))_n/N^{bcc}(S)$ for the critical inverse temperature of the spin- S Ising models on the bcc lattice, plotted vs $1/n^{1+\theta}$, with $\theta = \theta^{ef} = 0.504$. They are obtained from Eq. (23) using the coefficients $c_n^{bcc}(S)$ of the susceptibility series for the bcc lattice. In order to fit into a single figure the sequences for different values of the spin, each sequence has been normalized by the average $N^{bcc}(S)$ of the critical inverse temperatures obtained extrapolating separately the even and odd subsequences. We have indicated by solid lines the extrapolants of the odd subsequences, based on the last odd pair of approximants, while the dashed lines indicate the extrapolants of the even subsequences, based on the last even pair of approximants.

If we refer to Eq. (25), the plots in Fig. 1 strongly suggest that $a_\chi^{bcc}(1/2) < 0$, whereas $a_\chi^{bcc}(S) > 0$ for $S \geq 2$. Since $|a_\chi^{bcc}(1)|$ and $|a_\chi^{bcc}(3/2)|$ are very small, we cannot yet be completely sure about their sign. The smallness of these confluent corrections is confirmed observing that $|\partial\beta_c^{bcc}(1)/\partial\theta|$ and $|\partial\beta_c^{bcc}(3/2)/\partial\theta|$ are much smaller than for the other values of S .

Simple model series with a structure specified by Eq. (13) (including the antiferromagnetic oscillating corrections) can mimic rather accurately the behavior of the spin- S Ising series for sufficiently high orders. Therefore numerical experimentation with these model series can give us some intuition on the virtues and the limitations of the modified-ratio method and help to assess its accuracy. These tests add further confidence in our estimates of the relative error of β_c . On the other hand, it may take series significantly longer than those presently available to determine $a_\chi(S)$ with a precision better than a few percent, since the slopes of the approximant sequences provide only “effective” values of these amplitudes due to the residual influence of the higher-order corrections. Actually, the relative uncertainty of $a_\chi(S)$ can be larger, particularly so if its absolute value is very small.

The sc lattice series have been studied in the same fashion and the results are illustrated by Fig. 2. The main difference with respect to the bcc case is that all approximant sequences are decreasing, so that $a_\chi^{sc}(S) < 0$ for all S . It is also clear that, for this lattice, the rate of convergence of the approxi-

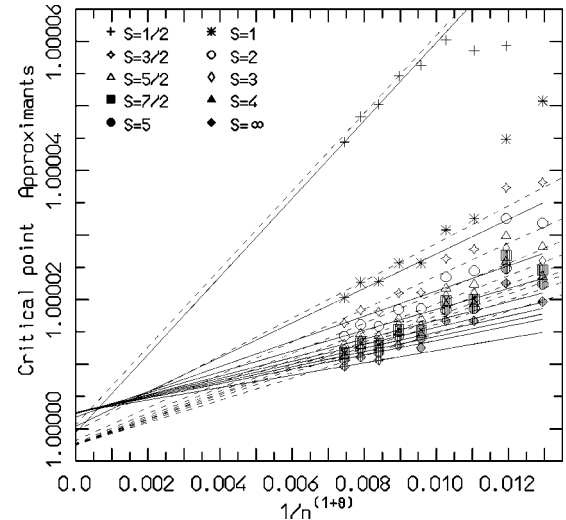


FIG. 2. The same as in Fig. 1, but for the “normalized” modified-ratio approximant sequences $(\beta_c^{sc}(S))_n/N^{sc}(S)$ formed from the coefficients $c_n^{sc}(S)$ of the susceptibility series for the sc lattice.

mant sequences to their asymptotic behavior is distinctly slower than in the bcc case.

In order to gain further confidence in the estimates by the modified-ratio method, we must confirm at least their main features also by numerical tests of a different nature or involving different assumptions, thus reducing the probability of being misled by only apparent convergence. We have therefore performed also a more traditional *unbiased* analysis by first- and second-order inhomogeneous DA’s yielding values of the critical inverse temperatures in essential agreement, to within their uncertainties, with those obtained from the modified-ratio method. In the case of the bcc lattice, for spin $S = 1/2$, the highest-order available DA estimates are slightly larger than the estimates from the modified-ratio method. Nevertheless, the estimates from DA’s using r series coefficients show a slowly decreasing trend as r increases. For $S > 2$ the highest-order DA estimates are slightly smaller than the corresponding results from the modified-ratio method, but the estimate sequences show an increasing trend. If we make the reasonable assumption⁶³ that also for DA’s the dominant finite-order corrections are proportional to the amplitudes of the leading nonanalytic corrections to scaling, these features of the results can be simply explained by the pattern of signs and sizes of these amplitudes previously observed in the analysis. Taking account of these trends and performing some purely visual extrapolation of the DA estimate sequences, we can reconcile the DA and the analyses by the modified-ratio method. We shall not report in Table II the DA results, but simply quote here the average of the highest-order DA data for a few values of S . The presence of residual trends in the sequence of DA estimates will be indicated by asymmetric uncertainties roughly corresponding to the range of possible extrapolations. For instance, from second-order unbiased DA’s, we

obtain $\beta^{bcc}(1/2)=0.157\,376^{(+1)}_{(-4)}$, $\beta^{bcc}(1)=0.224\,655(2)$, $\beta^{bcc}(3/2)=0.265\,640(2)$, $\beta^{bcc}(2)=0.293\,255(2)$, and $\beta^{bcc}(\infty)=0.435\,082^{(+4)}_{(-1)}$.

For the same reasons as above indicated, in the sc lattice case, the values of $\beta_c^{sc}(S)$ obtained from the DA's are generally slightly larger than those suggested by the modified-ratio analysis. This upward shift is more pronounced in the case $S=1/2$, in which we obtain $\beta_c^{sc}(1/2)=0.221\,665^{(+2)}_{(-10)}$, whereas for higher values of S the differences from the results by modified-ratio method are much smaller: for example, $\beta_c^{sc}(1)=0.312\,870(3)$, $\beta_c^{sc}(3/2)=0.368\,660(3)$, and $\beta_c^{sc}(\infty)=0.601\,271(3)$.

In conclusion, our modified-ratio method (biased with θ) and the unbiased DA estimates of the critical inverse temperatures on the sc and bcc lattices are consistent and compare fairly well with, but sometimes are more accurate than, those already available in the literature for a few values of the spin and also reported in the table. A wider discussion of other estimates by different methods in the $S=1/2$ case^{29,30,55,92-98} can be found in our previous paper⁵⁶ and in recent reviews⁹⁹⁻¹⁰¹ of Monte Carlo simulations and other studies of spin models. It is interesting to mention at this point that the two most extensive simulations on the sc lattice, by a static⁹³ and by a kinetic⁹⁷ method, yield the estimates $\beta_c^{sc}(1/2)=0.221\,654\,6(10)$ and $\beta_c^{sc}(1/2)=0.221\,659\,5(15)$, respectively, which agree only within two standard deviations.

V. MODIFIED-RATIO ESTIMATES OF THE CRITICAL EXPONENTS

Modified-ratio methods can lead also to fairly good estimates⁵⁶ of the exponents γ and ν . Let us first focus on the calculation of the exponent γ to recall the prescription of Refs. 24 and 59. An analogous procedure can be used for other exponents. For each value of S , we form the approximant sequence

$$(\gamma(S))_n = 1 + \frac{2(s_n + s_{n-2})}{(s_n - s_{n-2})^2}, \quad (26)$$

where s_n is still defined by Eq. (24) in terms of the expansion coefficients $c_n(S)$ of $\chi(\beta;S)$.

Using Eq. (13), we can compute the asymptotic behavior of the sequence $(\gamma(S))_n$ as follows:

$$(\gamma(S))_n = \gamma(S) - \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} \frac{\theta(1-\theta^2)a_\chi^\#(S)}{n^\theta} + O\left(\frac{1}{n}\right). \quad (27)$$

If $\theta=1/2$, the $1/n$ term in Eq. (27) has a coefficient equal to $[a_\chi^\#(S)]^2$ times a small positive factor. The higher-order corrections contain powers of γ . As a consequence, for the Ising model, the first important correction is $O(1/n^{1+\theta})$ and, in general, the convergence of the sequence, Eq. (26), will be slower whenever the exponent under study is ≥ 1 . This the case of γ_4 and, actually, we have observed that, at the presently available orders, this procedure is not convenient for

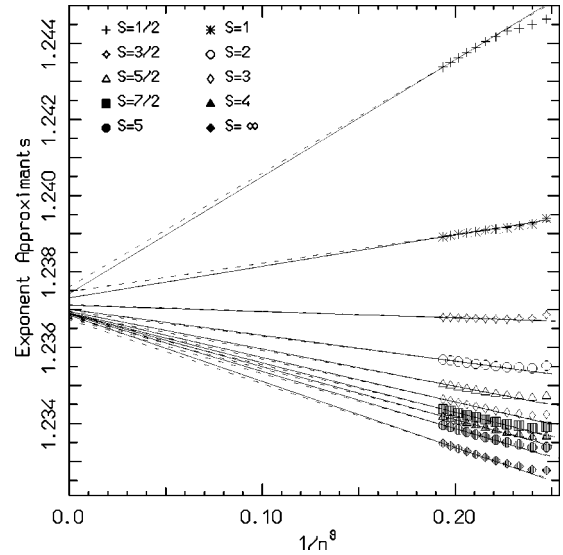


FIG. 3. The modified-ratio approximant sequences $(\gamma^{bcc}(S))_n$ of the susceptibility critical exponents for various values of the spin S , plotted vs $1/n^\theta$, with $\theta = \theta^{ef} = 0.504$. They are obtained using Eq. (26) from the susceptibility series coefficients $c_n^{bcc}(S)$. For each value of the spin, we have indicated by a solid line the extrapolation to large n of the sequence, linearly in $1/n^\theta$, based on the last odd pair of terms $\{(\gamma^{bcc}(S))_{23}, (\gamma^{bcc}(S))_{25}\}$. A dashed line indicates the extrapolation based on the previous odd pair $\{(\gamma^{bcc}(S))_{21}, (\gamma^{bcc}(S))_{23}\}$.

estimating γ_4 , whereas a directly biased variant, to be described in the next section, is more successful. On the other hand, for the calculation of the specific-heat exponent α , there are difficulties of a different nature: the critical singularity is very weak and the number of nonzero coefficients of the HT expansion of $C_H^\#(\beta;S)$ is still too small. Because of that, we have not been able to improve by modified-ratio methods the accuracy of the current *direct* HT estimates^{52,102,103} of α .

For sufficiently large n , the sequence of approximants defined by Eq. (26) is very nearly linear on a $1/n^\theta$ plot. Therefore, arguing like in the previous section, we are led to improve our estimates by extrapolating the odd (or even) subsequences linearly in $1/n^\theta$. The higher-order corrections for the exponents are expected to be more important than in the calculation of β_c and this is reflected in a larger uncertainty of the extrapolation procedure. Just like in the formula for β_c , the limiting value of the approximant sequence is asymptotically approached from above, if the amplitude $a_\chi^\#(S)$ of the leading nonanalytic confluent correction to scaling is negative, or from below, if it is positive.

In Fig. 3 we have plotted the approximant sequences $(\gamma^{bcc}(S))_n$ for several values of S between $1/2$ and ∞ . The structure of the plots is generally consistent with the pattern of signs of the CCS amplitudes already emerged from the study of $\beta_c^{bcc}(S)$. For each sequence $(\gamma^{bcc}(S))_n$ plotted in the figure, we have drawn as a solid line the extrapolant based on the last odd pair of approximants, whereas a dashed line represents the extrapolant based on the previous odd pair. The small residual curvature of the approximant subsequences, which is due to the higher-order corrections in Eq.

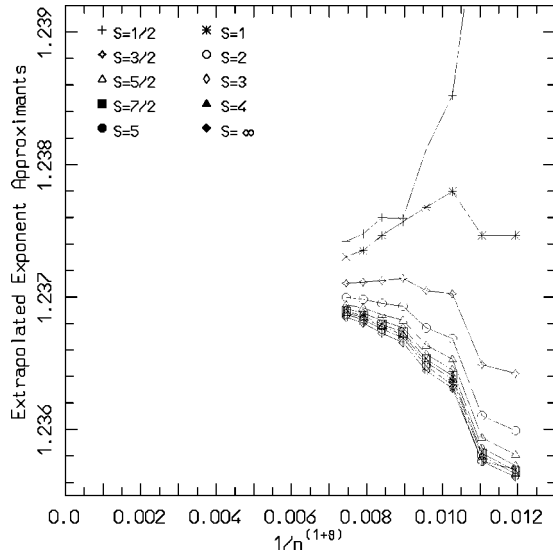


FIG. 4. The sequences of the extrapolations of the successive odd pairs of modified-ratio approximants $(\gamma^{bcc}(S))_n$ vs $1/n^{1+\theta}$. The solid lines are only guides to the eye.

(27), is made manifest in Fig. 3 by the splitting of the solid and the dashed-line extrapolants. It can also be exhibited more directly by plotting (see Fig. 4) the sequence of extrapolations of the successive odd (or even) pairs of approximants.

In Table III, for both lattices and for several values of the spin, we have reported the numerical values of the extrapolated exponents with an error corresponding to a small multiple of the difference between the solid and dashed extrapolations. We have also reported the derivatives of these estimates with respect to θ , computed at the reference value $\theta^{ref}=0.504$. For comparison, the same table also shows the exponent estimates obtained from DA's, while the results obtained in other recent numerical studies using Monte Carlo methods, by shorter HT series, or in the RG approach, will be further discussed in Secs. VIII and IX and are collected in Table IV.

From Figs. 3 and 4, it is clear that not only $a_\chi^{bcc}(S)$, but also the amplitudes of the main subleading CCS change sign as S varies between 1 and 2. This very favorable circumstance, which can also be confirmed numerically—for example, by fitting the approximant subsequences to the simple asymptotic form $\gamma+c_1(S)/n^\theta+c_2(S)/n^{1+\theta}$ —makes us very confident about the accuracy of the exponent estimates presented below. For each value of S , a simple monotonic behavior appears to have set in, since as shown in Fig. 4, the subleading asymptotic correction in Eq. (27) generally works in the expected “right” direction. Namely, it tends to lower the extrapolated exponent values obtained from the decreasing approximant subsequences for spin $S=1/2$ and $S=1$, while it tends to raise the extrapolated values obtained from the increasing subsequences for $S\geq 2$. Only in the $S=3/2$ case, in which both the amplitudes of the leading and of the subleading correction have the smallest absolute value, the approximant sequence is very slowly increasing and the sequence of extrapolated exponents is very slowly decreasing. Thus we can expect that, as the number of available coefficients grows large, the range of variation with respect to S of the extrapolated estimates of $\gamma^{bcc}(S)$ will continue to shrink, further improving the verification of the universality of the exponent with regard to the spin. More precisely, assuming that the general features of the behavior we have described persist as the order of the series increases, the successive extrapolations of the sequences $(\gamma^{bcc}(1/2))_n$ and $(\gamma^{bcc}(1))_n$ should provide decreasing sequences of upper bounds, while those of the sequences $(\gamma^{bcc}(2))_n$, $(\gamma^{bcc}(5/2))_n$, etc., should give increasing sequences of lower bounds for γ .

At the present order of expansion, the exponent estimates obtained by our extrapolation prescription range orderly from 1.237 42 for $S=1/2$ to 1.236 84 for $S=\infty$. Therefore, if we now *assume* that universality is valid, in particular that γ is independent of S , the previous remarks suggest to take simply the average $\gamma=1.2371(4)$ of these extrema as a first rough approximation of the exponent with an uncertainty corresponding to the half-width of the range of variation. We

TABLE III. Estimates of the critical exponents γ and ν for the spin- S Ising model series on sc and bcc lattices obtained from extrapolation of the modified-ratio (MR) approximant sequences [defined by Eq. (26)]. We have also reported the estimates obtained in this paper by second-order inhomogeneous differential approximants (DA's) biased with β_c .

Exponent	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$\gamma^{sc}(S)(MR)$	1.2375(6)	1.2378(7)	1.2371(8)	1.2367(10)	1.2364(10)	1.2363(10)	1.2359(15)
$\partial\gamma^{sc}(S)/\partial\theta$	0.016	0.009	0.007	0.006	0.006	0.006	0.006
$\gamma^{bcc}(S)(MR)$	1.23742(20)	1.23730(16)	1.23710(3)	1.23699(10)	1.23694(10)	1.23691(10)	1.23685(15)
$\partial\gamma^{bcc}(S)/\partial\theta$	0.012	0.003	-0.0005	-0.002	-0.003	-0.004	-0.006
$\gamma^{sc}(S)(DA)$	1.238(2)	1.239(2)	1.240(3)	1.239(2)	1.239(2)	1.239(2)	1.238(2)
$\gamma^{bcc}(S)(DA)$	1.2378(8)	1.2385(15)	1.2370(4)	1.2365(4)	1.2366(4)	1.2366(4)	1.2367(4)
$\nu^{sc}(S)(MR)$	0.6277(30)	0.6279(30)	0.6283(20)	0.6285(20)	0.6286(20)	0.6286(20)	0.6288(20)
$\partial\nu^{sc}(S)/\partial\theta$	0.021	0.014	0.010	0.0095	0.009	0.008	0.006
$\nu^{bcc}(S)(MR)$	0.6283(20)	0.6294(8)	0.6297(6)	0.6298(6)	0.6299(6)	0.6299(6)	0.6301(5)
$\partial\nu^{bcc}(S)/\partial\theta$	0.006	0.001	-0.001	-0.0015	-0.002	-0.0022	-0.003
$\nu^{sc}(S)(DA)$	0.632(2)	0.631(1)	0.631(1)	0.631(1)	0.631(1)	0.631(1)	0.631(1)
$\nu^{bcc}(S)(DA)$	0.6308(10)	0.631(1)	0.6299(4)	0.6295(4)	0.6296(4)	0.6295(4)	0.6294(6)

TABLE IV. Estimates of the exponents γ and ν obtained in the recent literature by various kinds of analyses of shorter high-temperature series, by Monte Carlo methods, or by renormalization-group methods. The estimates marked with an asterisk are obtained by procedures implying or assuming universality.

Exponent	HT	Monte Carlo	RG ϵ exp.	RG fixed- D exp.
γ	1.237(2)* ^a			
	1.2385(25)* ^b			
	1.2385(15)* ^c	1.2367(11) ^d		1.2378(6)* ^e
	1.2395(4)* ^f	1.237(2)* ^g	1.2380(50)* ^h	1.2396(13)* ^h
	1.2378(6) ⁱ	1.2372(17)* ^j		
	1.2373(1) ^k 1.2371(4) ^l			
ν	0.6300(15)* ^a			
	0.6305(15)* ^b	0.6296(7) ^d		0.6301(5)* ^e
	0.632(1)* ^f	0.6301(8)* ^g	0.6305(25)* ^h	0.6304(13)* ^h
	0.6311(3)* ⁱ	0.6303(6)* ^j		
	0.6300(2) ^k 0.63002(23)* ^l	0.6294(10) ^m		

^aReference 23.
^bReference 24.
^cReference 25.
^dReference 89.
^eReference 45.
^fReference 26.
^gReference 92.

^hReference 64.
ⁱReference 27.
^jReference 93.
^kReference 28.
^lReference 113.
^mReference 114.

can further refine this estimate observing that, for values of the spin between 1 and 2, both the leading and the main subleading CCS are very small, as it appears observing that the exponent approximant sequences have very small slopes, clearly positive for $S=1$ and negative for $S=3/2$ and $S=2$. Moreover the extrapolated exponent estimates are fairly insensitive to the bias value of θ [for instance, we have $\partial\gamma^{bcc}(1)/\partial\theta \approx 0.003$, $\partial\gamma^{bcc}(3/2)/\partial\theta \approx -0.0005$, and $\partial\gamma^{bcc}(2)/\partial\theta \approx -0.002$ at $\theta = \theta^{eff}$]. Since $(\gamma^{bcc}(1))_n$ and $(\gamma^{bcc}(2))_n$ are very close, a better estimate for γ should lie in between. The extrapolation of the last odd pair of terms in the sequence $(\gamma^{bcc}(1))_n$ yields 1.237 30, whereas for the sequence $(\gamma^{bcc}(2))_n$ it leads to 1.236 99, and therefore the rough estimate given above can be improved to $\gamma = 1.237 15(15)$. Consideration also of the sequence $(\gamma^{bcc}(3/2))_n$ suggests that we take $\gamma = 1.2371(1)$ as our final best estimate.

A closely related procedure was proposed long ago in Refs. 26. These authors analyzed, but never published, extensive two-variable series in power of β and of a continuous Ising spin variable (made available by B.G. Nickel), using partial-differential approximant methods which indicated an “effective fixed point” around $S = 3/2$. Within the precision of the present calculations, the simple prescription of taking the average of the extrapolations of $(\gamma^{bcc}(1))_n$ and of $(\gamma^{bcc}(2))_n$, or the extrapolation of $(\gamma^{bcc}(3/2))_n$ as the best approximation of γ , should be equally (or perhaps more) effective since it also takes advantage of the vanishing of the main subleading correction.

Since it is not difficult to show that the leading CCS for any observable must also vanish for the same value of S , the

same prescription can be used for extracting the best value of ν from the approximant sequences $(\nu^{bcc}(S))_n$ formed by the series coefficients of $\xi^2(\beta; S)$ and shown in Fig. 5. Figure 6 shows the sequence of the extrapolations of the successive odd pairs of approximants. The slower convergence of the approximants to the correlation-length exponent should not be surprising, simply because $\mu_2(\beta; S)$ enters into the definition of ξ^2 . At the present orders of expansion, the behavior

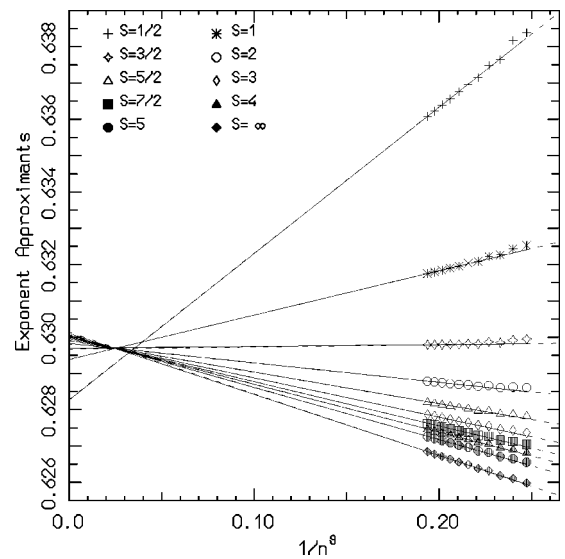


FIG. 5. Same as in Fig. 3, but for the modified-ratio approximant sequences $(\nu^{bcc}(S))_n$ of the correlation-length critical exponent, as obtained from the expansion of $\xi^{bcc}(\beta; S)$.

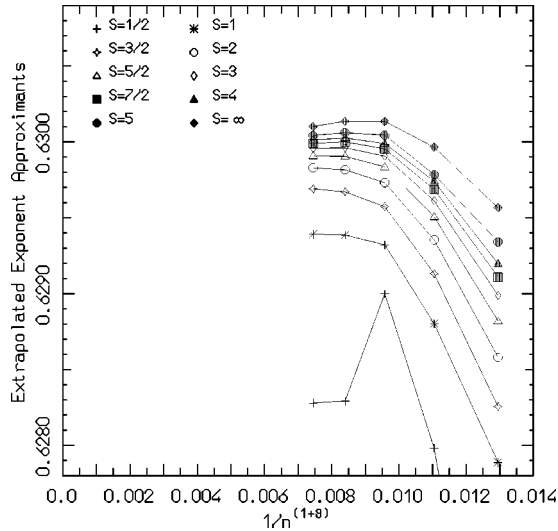


FIG. 6. Same as in Fig. 4, but for the sequences of the extrapolations of the successive odd pairs of modified-ratio approximants $(\nu^{bcc}(S))_n$ vs $1/n^{1+\theta}$.

of the sequence of the extrapolated exponent values is clearly not yet asymptotic for $S=1/2$ and $S=1$, while it is much smoother and shows a slowly increasing trend for $3/2 \leq S \leq 3$ and a slowly decreasing trend for $S > 3$. Thus arguing as before, we can conclude that $\nu = 0.6299(2)$.

If we bias the extrapolation procedure with a larger value of θ^{ef} , the range of variation of $\gamma^{bcc}(S)$ with S will be expanded, to an extent that can be easily figured out from the data reported in Table III, but the estimated central value of γ^{bcc} will be practically unchanged. For instance, if we adopt the significantly larger value $\theta^{ef} = 0.54$, we find $\gamma^{bcc}(1/2) = 1.23782$ and $\gamma^{bcc}(\infty) = 1.23661$, whereas $\gamma^{bcc}(1) = 1.23741$, $\gamma^{bcc}(3/2) = 1.23708$, and $\gamma^{bcc}(2) = 1.23691$ are changed to a smaller extent. Averaging $(\gamma^{bcc}(1))_n$ and $(\gamma^{bcc}(2))_n$ yields $\gamma = 1.23716(25)$ and consideration also of $(\gamma^{bcc}(3/2))_n$ leads to essentially the same final estimate as the one obtained for $\theta^{ef} = 0.504$. In the case of the exponent ν , the estimated central value is slightly lowered to 0.6298 , well within the error bars of our previous estimate.

Using the Fisher scaling relation,¹⁰⁴ the exponent η describing the large distance falloff of the two-spin correlation function at the critical temperature can be estimated as $\eta^{bcc} = 2 - \gamma^{bcc}/\nu^{bcc} = 0.0360(8)$.

In Figs. 7 and 8 we show the results of the analogous procedure of extrapolation for $(\gamma^{sc}(S))_n$ and $(\nu^{sc}(S))_n$. The main features are similar to the bcc case, except, unfortunately, for the sign pattern of the amplitudes of the leading CCS, all of which now appear to be negative, consistently with the study of $\beta_c^{sc}(S)$ by the modified-ratio method. In complete analogy with the bcc case, for both exponents γ and ν , the residual curvature of the approximant sequences tends to correct the extrapolations in the direction expected in order that universality be realized. The accuracy of the exponent estimates is notably smaller, due to the anticipated slower convergence of the sc lattice series coefficients to their asymptotic form and perhaps to the presence of larger subleading corrections in Eq. (27). The values of the expo-

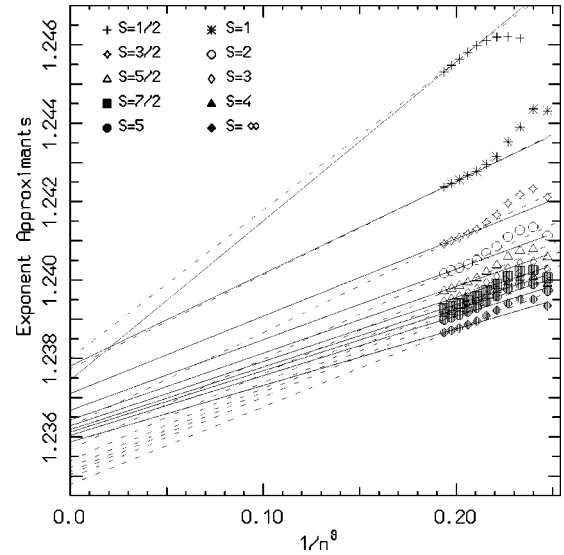


FIG. 7. Same as in Fig. 3, but for the modified-ratio approximant sequences $(\gamma^{sc}(S))_n$ of the susceptibility critical exponent as obtained from $\chi^{sc}(\beta; S)$ using Eq. (26).

ponents γ and ν obtained from the sc series data are consistent with those from the bcc data, but they are affected by significantly larger uncertainties: we can roughly estimate $\gamma = 1.2368(10)$ and $\nu = 0.6285(20)$.

The numerical progress achieved in this study is best appreciated by comparing our Figs. 3 and 5 with the analogous Figs. 1 and 2 of Ref. 24. We should first observe that in Ref. 24 a straightforward extrapolation linear in $1/n$ was implied for the sequences $(\gamma^{bcc}(S))_n$ and $(\nu^{bcc}(S))_n$. Due to this choice of the plotting variable and to the smaller extension of the bcc series available two decades ago, the “relative maximal spreads” with respect to the spin S of the extrapolated exponent values are

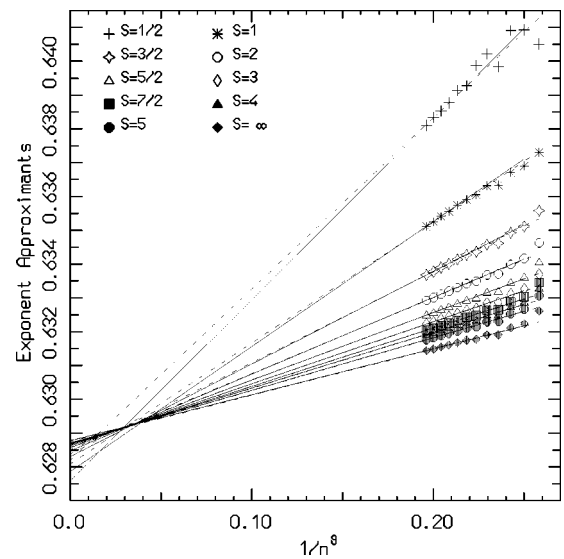


FIG. 8. Same as in Fig. 3, but for the modified-ratio approximant sequences $(\nu^{sc}(S))_n$ of the correlation-length critical exponent, as obtained from $\xi^{sc}(\beta; S)$.

$$\frac{\gamma(1/2) - \gamma(\infty)}{\gamma(1/2) + \gamma(\infty)} \approx 2.5 \times 10^{-3}$$

and

$$\frac{\nu(\infty) - \nu(1/2)}{\nu(1/2) + \nu(\infty)} \approx 7.6 \times 10^{-3},$$

respectively. In our study of the same lattice, the corresponding figures are smaller by nearly one order of magnitude; namely, the relative spread is now $\approx 2.3 \times 10^{-4}$ in the case of γ and $\approx 1.4 \times 10^{-3}$ for ν . The values of these spreads can be taken as rough accuracy limits for the verification of the universality with respect to S , which is thereby convincingly corroborated by the new analysis.

We close this discussion with a few remarks. Our extension of the series to order β^{25} has been crucial in showing that, in the bcc lattice case, the asymptotic structure of the HT expansion coefficients is already well stabilized, since the last six or seven modified-ratio method approximants of the critical inverse temperature or of the critical exponents show remarkably regular trends. Also in the sc lattice case, there are indications from the last three or four approximants, obtained by the same method, that a similar trend is setting in, but clearly the convergence is not as fast as for the bcc lattice.

Some numerical experimentation with model series suggests, also for the exponent analysis, that our error estimates are reasonable and quite conservative.

For both lattices, the CCS amplitudes can be estimated from the slopes of the exponent approximant sequences, as will be further discussed in Sec. IX.

In conclusion, this simple modified-ratio approach confirms accurately the universality of γ and ν with respect to the magnitude of S and to the lattice structure and, conversely, *assuming* universality and using the bcc series data, it yields very accurate estimates for these exponents.

VI. BIASED MODIFIED-RATIO METHOD FOR THE EXPONENTS

In Ref. 24, Zinn-Justin proposed also a more direct modified-ratio procedure for biasing the exponent estimates with the value of θ , in order to eliminate or strongly reduce the influence of the leading confluent corrections to scaling. The prescription involves the quantities

$$\bar{s}_n = (s_n + s_{n-1})/2 \tag{28}$$

and

$$b_n = \left(\frac{1}{\theta} (\bar{s}_n^{\theta/2} - \bar{s}_{n-2}^{\theta/2}) \right)^{2/(\theta-1)}. \tag{29}$$

In terms of b_n the following approximant sequence can be formed:

$$(\hat{\gamma}(S))_n = 1 + \frac{(b_n - b_{n-2})^2}{2(b_n + b_{n-2})}. \tag{30}$$

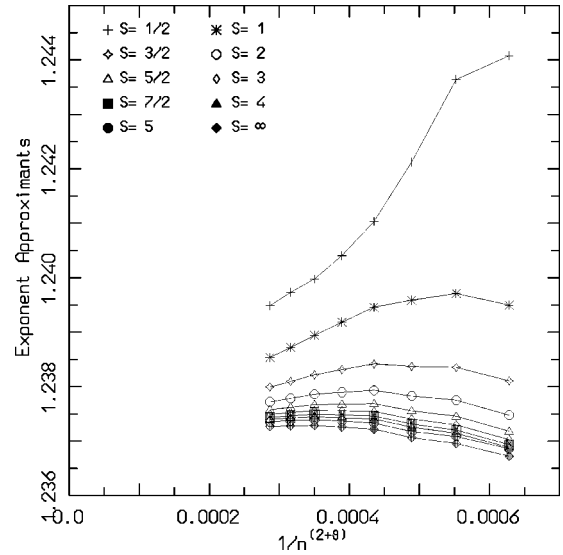


FIG. 9. The *directly* biased modified-ratio approximant sequences $(\hat{\gamma}^{bcc}(S))_n$ plotted vs $1/n^{2+\theta}$. They are obtained from Eq. (30), using as a bias the value of θ in order to reduce the influence of the confluent corrections to scaling. The solid lines are only guides to the eye.

If we make the simplifying assumption that θ is exactly $1/2$, also the correction terms $O(1/n^{2\theta})$ will be eliminated by this prescription, along with the regular correction $O(1/n)$, and therefore

$$(\hat{\gamma}(S))_n = \gamma(S) + O\left(\frac{1}{n^{3/2}}\right). \tag{31}$$

By the remarks made at the beginning of the preceding section, these are not decisive improvements in the calculation of γ and ν , and indeed, for both sc and bcc lattices, the results obtained by this procedure are consistent with but not more accurate than those of our previous analysis by modified-ratio methods. See, for example, Figs. 9 and 10, where, for convenience, we have plotted $(\hat{\gamma}(S))_n$ vs $1/n^{2+\theta}$ rather than vs $1/n^{1+\theta}$, because the plots of the approximants appear to be more nearly linear (although with somewhat large corrections) with respect to former than to the latter variable.

On the other hand, this biased variant of the modified-ratio method is more successful in the analysis of the expansions that we have computed for χ_4 . The sequences of biased approximants $(\hat{\gamma}_4^\#(S))_n$ for the exponent of $\chi_4^\#(\beta, S)$ are shown in Fig. 11 for the bcc lattice and in Fig. 12 for the sc lattice. In order to avoid confusing the plots, we have indicated only the extrapolations, linear in $1/n^{2+\theta}$, based on the last odd pair of terms $\{(\hat{\gamma}_4^\#(S))_{21}, (\hat{\gamma}_4^\#(S))_{23}\}$ in the approximant sequences. When the spin S varies between $1/2$ and ∞ , the extrapolated values of $\gamma_4^{sc}(S)$ range from 4.366 for $S = 1/2$ to 4.372 for $S = \infty$. Similarly, in the case of the bcc lattice the values of $\gamma_4^{bcc}(S)$ vary between 4.369 and 4.375. We have reported in Table V the results obtained by this method for several values of S . From the sc lattice data we

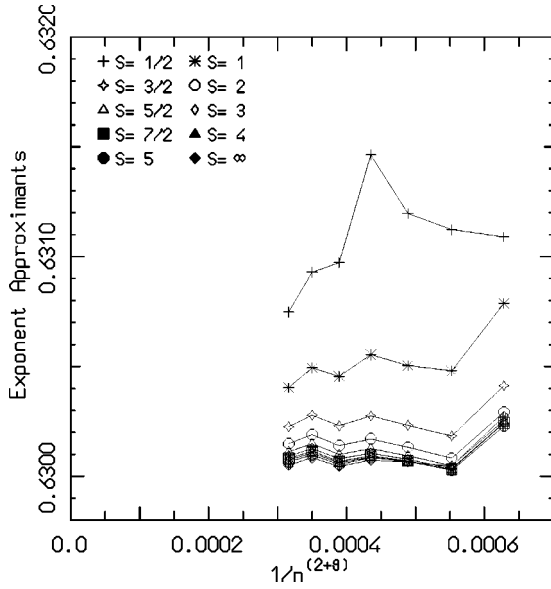


FIG. 10. Same as Fig. 8, but for the *directly* biased modified-ratio approximant sequences $(\hat{\nu}^{bcc}(S))_n$ plotted vs $1/n^{2+\theta}$.

can conclude that $\gamma_4^{sc} = \gamma + 2\Delta = 4.369(8)$ and from the bcc lattice data $\gamma_4^{bcc} = \gamma + 2\Delta = 4.372(8)$.

The accuracy in the verification of the validity of hyperscaling is often characterized quantitatively by quoting the value of the right-hand side of Eq. (18): from our estimates we have $\gamma + 3\nu - 2\Delta = 2\gamma + 3\nu - \gamma_4 = -0.0099(160)$ in the sc lattice case and, analogously, $\gamma + 3\nu - 2\Delta = -0.0081(88)$ for the bcc lattice.

These results give strong support to the validity of the hyperscaling relation and of the universality of γ_4 with respect both to the lattice structure and to the value of S .

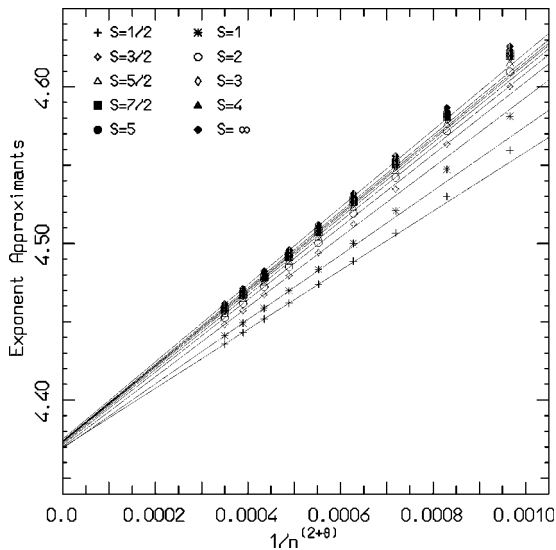


FIG. 11. The *directly* biased modified-ratio approximant sequences $(\hat{\gamma}_4^{bcc}(S))_n$ for the critical exponent of $\chi_4^{bcc}(\beta; S)$ plotted vs $1/n^{2+\theta}$. In order to keep the figure readable we have indicated only the extrapolations of the odd approximant subsequences.

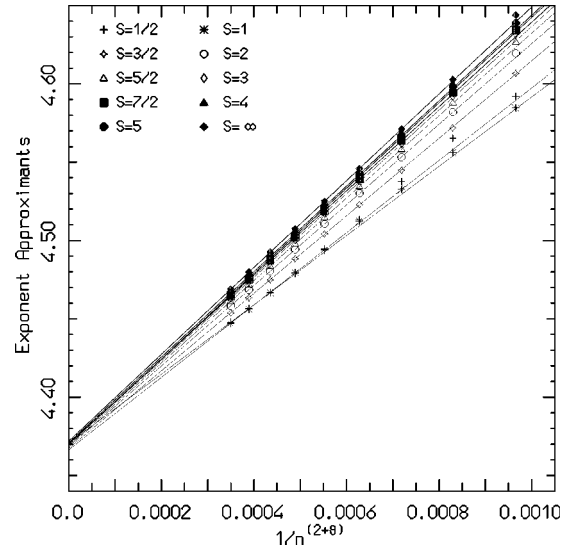


FIG. 12. Same as in Fig. 11, but for the *directly* biased modified-ratio approximant sequences $(\hat{\gamma}_4^{sc}(S))_n$ for the critical exponent of $\chi_4^{sc}(\beta; S)$ plotted vs $1/n^{2+\theta}$.

VII. RATIO ESTIMATES FOR THE EXPONENT OF THE LEADING CONFLUENT SINGULARITY

Assuming that θ is universal, the simplest prescription for estimating this exponent is based on the series with coefficients

$$q_n(S_1, S_2) = \frac{c_n(S_1)d_n(S_2)}{c_n(S_2)d_n(S_1)} \quad (32)$$

for $n > 0$ and, of course, $S_1 \neq S_2$. Here $c_n(S)$ are the coefficients of the susceptibility and $d_n(S)$ the coefficients of the second correlation moment for spin S . From Eq. (13) we can observe that, for large n ,

$$q_n(S_1, S_2) = A(S_1, S_2) \left[1 + \frac{B(S_1, S_2)}{n^\theta} + O(1/n) \right] \quad (33)$$

and therefore

$$r_n(S_1, S_2) = \frac{q_n(S_1, S_2)}{q_{n+2}(S_1, S_2)} = 1 + \frac{A'}{n^{\theta+1}} + O(1/n^2) \quad (34)$$

so that

$$(\theta(S_1, S_2))_n = \frac{1}{2} \left[n \left(\frac{r_n(S_1, S_2) - 1}{r_{n+2}(S_1, S_2) - 1} - 1 \right) - 2 \right] \quad (35)$$

is an approximant sequence for θ . In the bcc lattice case, if

TABLE V. Estimates of the critical exponent γ_4 from the spin- S Ising model high-temperature series on sc and bcc lattices obtained in this work by the modified-ratio (MR) method *directly* biased with the leading correction-to-scaling exponent θ following Eq. (30) or by second-order differential approximants biased with β_c . For comparison, we have also reported some results obtained by simulation methods or from shorter series in the recent literature.

Exponent	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$\gamma_4^{sc}(S)$ (DA)	4.372(8)	4.368(8)	4.369(8)	4.369(8)	4.368(8)	4.367(4)	4.366(4)
$\hat{\gamma}_4^{sc}(S)$ (MR)	4.3703(12)	4.3662(16)	4.3671(9)	4.3683(9)	4.3691(9)	4.3697(9)	4.3719(2)
$\partial \hat{\gamma}_4^{sc}(S)/\partial \theta$	0.027	0.023	0.015	0.009	0.004	0.002	-0.007
$\gamma_4^{bcc}(S)$ (DA)	4.376(20)	4.3666(10)	4.3638(10)	4.3629(10)	4.3632(10)	4.3631(10)	4.3631(10)
$\hat{\gamma}_4^{bcc}(S)$ (MR)	4.3696(3)	4.3690(11)	4.3702(12)	4.3714(12)	4.3722	4.3729(12)	4.3749(10)
$\partial \hat{\gamma}_4^{bcc}(S)/\partial \theta$	0.02	-0.008	-0.014	-0.023	-0.028	-0.032	-0.043
$\gamma_4^{sc}(S)$ ^a	4.361(8)						
$\gamma_4^{bcc}(S)$ ^a	4.366(6)						
$\gamma_4^{bcc}(S)$ ^b	4.370(14)						

^aReference 53.

^bReference 42.

we choose $S_1 = 1/2$, $S_2 > 2$, and extrapolate only the even (or, equivalently, the odd) subsequences linearly in $1/n^{1+\theta}$, we obtain Fig. 13. The results indicate very suggestively that $\theta = 0.50_{-1}^{+3}$, independently of the value of S_2 .

VIII. ANALYSIS OF THE EXPONENTS BY DIFFERENTIAL APPROXIMANTS

The modified-ratio methods employed in the last sections have proved successful and suggestive both for the determination of the critical temperatures and for the calculation of the exponents γ , ν , and γ_4 . Let us now turn to the more

traditional differential-approximant-based procedures of series analysis after recalling that their main difficulty is the necessity of some further extrapolation with respect to the order of the series used, which is not straightforward, due mainly to the lack of simple estimates for the finite-order corrections and to the spread of the various DA estimates at a given order of approximation. This fact also hampers the assessment of the errors, which can be realistic if not only they reflect the spread of the values of the highest-order approximants, but also allow for the possible residual trends. In this respect, the modified-ratio methods might be easier to use, as we have suggested in the previous sections. We have already discussed in Sec. IV the DA estimates for the critical points. For measuring the exponents, we have preferred series analyses using the inhomogeneous first- and second-order DA's *biased* with β_c (or in some cases with $\pm \beta_c$), or sometimes the *simplified* inhomogeneous first-order differential approximants defined in Ref. 52, in which we have fixed also the correction-to-scaling exponent θ . The extrapolations of the results from the biased DA's and from the simplified DA's may be performed with a smaller uncertainty, because the spread of the estimates tends to be narrower than for unbiased approximants. Moreover, in order to understand, at least qualitatively, how the estimates on a given lattice depend on the spin and to improve them, it will be sufficient to assume that the leading finite-order corrections are proportional to the amplitudes of the leading nonanalytic corrections to scaling.

A simpler approach^{105,106} similar to the simplified DA's consists in forming the conventional Padé approximants after subjecting the series to the biased variable change $w^\#(S) = 1 - \tau^\#(S)^\theta$ in order to regularize the leading CCS. The results obtained either by simplified DA's or by Padé approximants (PA's) in the variable $w^\#(S)$ are sometimes numerically comparable, but the latter are generally affected by a larger uncertainty.

We have computed also the *effective exponents*, introduced long ago in Ref. 107 and more recently reconsidered

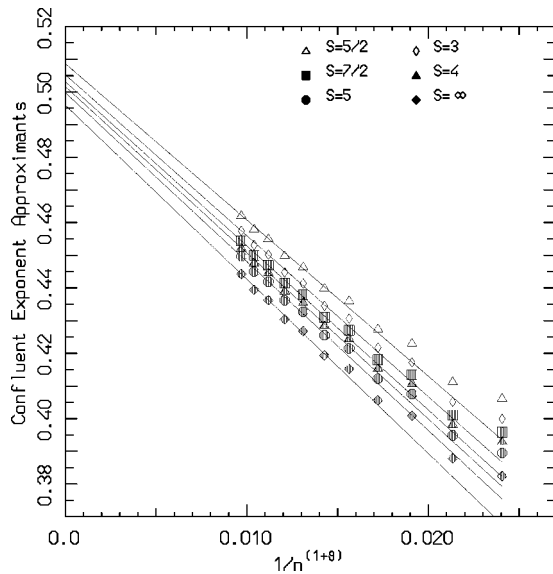


FIG. 13. Approximant sequences $(\theta^{bcc}(S_1, S_2))_n$ for the correction-to-scaling exponent θ as obtained using Eq. (35) for fixed $S_1 = 1/2$ and $S_2 = 5/2, 3, 7/2, 4, 5, \infty$. The symbols refer to the values of S_2 . The sequences are plotted vs $1/n^{1+\theta}$, with $\theta = 0.504$. In order to keep the figure readable we have indicated only the extrapolations of the odd approximant subsequences.

and systematically studied in Refs. 14,63,108, and 109, for the susceptibility

$$\begin{aligned}\gamma_{eff}^{\#}(\beta;S) &\equiv -\frac{d \ln \chi^{\#}(\beta;S)}{d \ln \tau^{\#}(S)} \\ &= \gamma(S) - \theta a_{\chi}^{\#}(S) \tau^{\#}(S)^{\theta} + O(\tau^{\#}(S)),\end{aligned}\quad (36)$$

for the correlation length

$$\begin{aligned}\nu_{eff}^{\#}(\beta;S) &\equiv -[1 - \tau^{\#}(S)] \frac{d \ln \xi^{\#}(\beta;S)}{d \ln \tau^{\#}(S)} \\ &= \nu(S) - \theta a_{\xi}^{\#}(S) \tau^{\#}(S)^{\theta} + O(\tau^{\#}(S)),\end{aligned}\quad (37)$$

and for the second field derivative of the susceptibility

$$\begin{aligned}\gamma_{4eff}^{\#}(\beta;S) &\equiv -\frac{d \ln \chi_4^{\#}(\beta;S)}{d \ln \tau^{\#}(S)} \\ &= \gamma_4(S) - \theta a_4^{\#}(S) \tau^{\#}(S)^{\theta} + O(\tau^{\#}(S)).\end{aligned}\quad (38)$$

The critical exponents γ , ν , and γ_4 are estimated by extrapolating the effective exponents to the critical singularity. Of course, the factor $[1 - \tau(S)]$ in Eq. (37) is introduced only to compensate for the singularity of $d \ln \xi(\beta;S)/d \ln \tau(S)$ at $\beta=0$ and is unimportant at the critical point.

It is interesting to plot the effective exponents over a wide vicinity of $\beta_c^{\#}(S)$, not only to gain information on the leading correction amplitudes $a_{\chi}^{\#}(S)$, $a_{\xi}^{\#}(S)$, and $a_4^{\#}(S)$ through Eqs. (36)–(38), by examining whether and how fast they approach the critical limit from above or from below, but also simply in order to display the variety¹¹⁰ of preasymptotic critical behaviors which can occur within the same universality class. The parametrizations of the approach to the critical behavior, proposed within various field-theoretical approaches^{111,112} to the RG, must confront also with this phenomenology.

In Figs. 14 and 15 we have shown the highest-order simplified DA's of the effective exponents $\gamma_{eff}^{bcc}(S)$ and, respectively, $\nu_{eff}^{bcc}(S)$ for spin $S=1/2, 1, \dots, \infty$, over wide ranges of inverse temperatures. In order to make the curves conveniently comparable for all values of the spin, we have plotted the effective exponents against the variable $[\tau^{bcc}(S)]^{\theta}$. The sign of the leading CCS is revealed by the slope of the plots near $\tau^{bcc}(S)=0$.

While the simplified DA's are quite sufficient to give a general view of the behavior of the effective exponents, more accurate results for the exponent γ are obtained extrapolating the effective exponent expansions by second-order inhomogeneous DA's biased with β_c . The estimates thus obtained for $\gamma(S)$ range from $\gamma^{bcc}(1/2)=1.2385(6)$ to $\gamma^{bcc}(\infty)=1.2367(5)$. They are reported in Table III. Our best DA estimate $\gamma=1.2373(4)$ is obtained simply by averaging $\gamma^{bcc}(1)$ and $\gamma^{bcc}(2)$ and taking into account also the value of $\gamma^{bcc}(3/2)$. It agrees well with the estimate by modified-ratio methods. The corresponding results for the correlation-length exponent range from $\nu^{bcc}(1/2)=0.6314(20)$ to $\nu^{bcc}(\infty)=0.6294(5)$ and our best estimate is $\nu=0.6301(4)$.

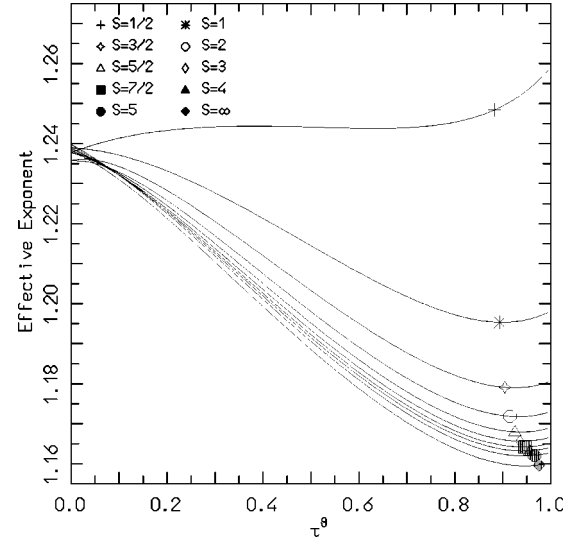


FIG. 14. Highest-order simplified-differential approximants of the effective exponent $\gamma_{eff}^{bcc}(\beta;S)$ of the susceptibility $\chi^{bcc}(\beta;S)$ as defined by Eq. (36). For each value of the spin S the effective exponent is plotted vs $\tau^{bcc}(S)^{\theta}=[1 - \beta/\beta_c^{bcc}(S)]^{\theta}$. As indicated by the symbols attached to them, the curves refer, from the highest downwards, to the spin values $S=1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 5, \infty$.

In the sc lattice case, the analogous (but less well converged) plots for $\gamma_{eff}^{sc}(\beta;S)$ and for $\nu_{eff}^{sc}(\beta;S)$, obtained by simplified DA's, are shown in Figs. 16 and 17. This analysis also confirms that, on the sc lattice, the amplitudes of the leading CCS do not have a dependence on S similar to the bcc case, but remain negative for all values of the spin. The estimates of $\gamma^{sc}(S)$ and of $\nu^{sc}(S)$ obtained by second-order biased DA's are also reported in Table III.

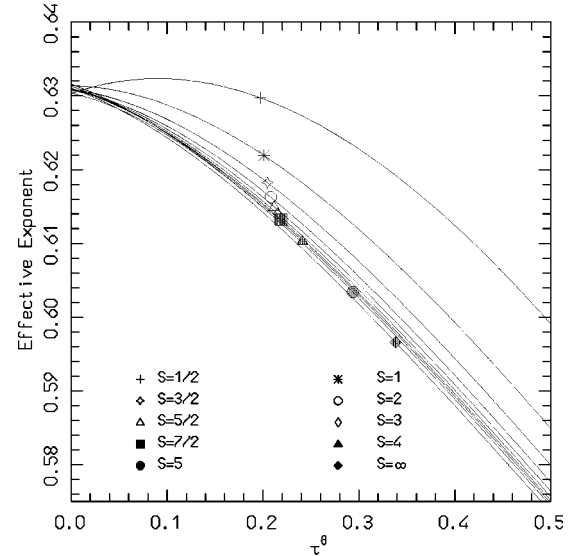


FIG. 15. Highest-order simplified-differential approximants of the effective exponent $\nu_{eff}^{bcc}(\beta;S)$ of the correlation length $\xi^{bcc}(\beta;S)$ as defined by Eq. (36). For each value of the spin S the effective exponent is plotted vs $\tau^{bcc}(S)^{\theta}=[1 - \beta/\beta_c^{bcc}(S)]^{\theta}$. As indicated by the symbols attached to them, the curves refer, from the highest downwards, to the spin values $S=1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 5, \infty$.

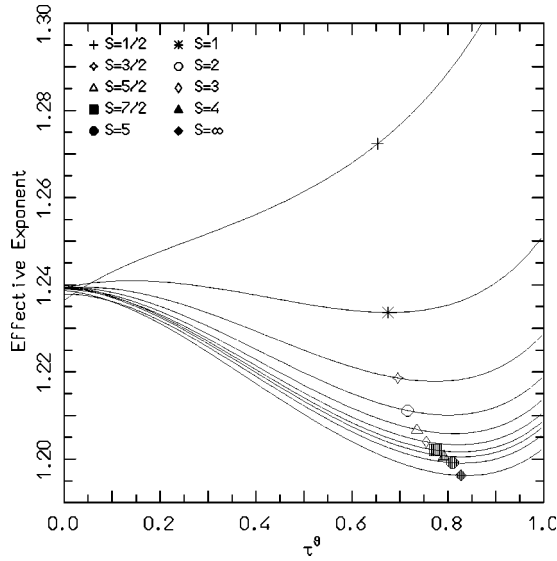


FIG. 16. Highest-order simplified-differential approximants of the effective exponent $\gamma_{eff}^{sc}(\beta; S)$ of the susceptibility $\chi^{sc}(\beta; S)$ as defined by Eq. (36). For each value of the spin S the effective exponent is plotted vs $\tau^{sc}(S)^\theta = [1 - \beta/\beta_c^{sc}(S)]^\theta$. As indicated by the symbols attached to them, the curves refer, from the highest downwards, to the spin values $S = 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 5, \infty$.

The simplified-DA analysis of the effective exponent $\gamma_{4eff}^{bcc}(S)$ of $\chi_4^{bcc}(\beta; S)$ yields estimates of $\gamma_4^{bcc}(S)$ ranging between 4.3647 and 4.3653. It also indicates that $a_4^{bcc}(1/2)$ and $a_4^{bcc}(1)$ are negative, whereas $a_4^{bcc}(S)$ is positive for $S > 3/2$. On the sc lattice, the corresponding estimates of $\gamma_4^{sc}(S)$ vary between 4.363 and 4.373 and $a_4^{sc}(S) > 0$ for all S . We can conclude that $\gamma_4 = 4.366(2)$, independently of the spin and the lattice, and in good agreement with the results

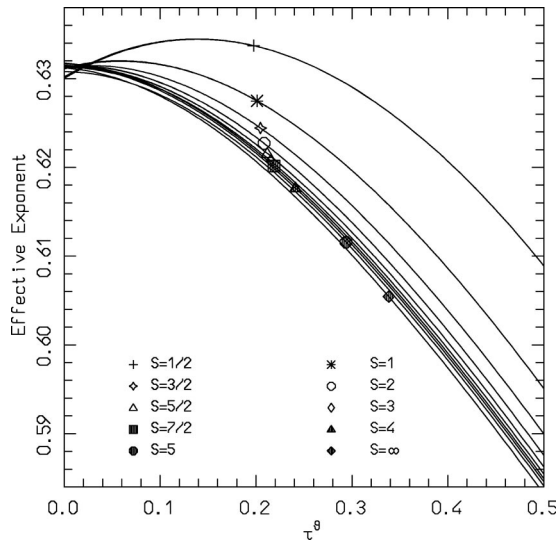


FIG. 17. Highest-order simplified-differential approximants of the effective exponent $\nu_{eff}^{sc}(\beta; S)$ of the correlation length $\xi^{sc}(\beta; S)$ as defined by Eq. (36). For each value of the spin S the effective exponent is plotted vs $\tau^{sc}(S)^\theta = [1 - \beta/\beta_c^{sc}(S)]^\theta$. As indicated by the symbols attached to them, the curves refer, from the highest downwards, to the spin values $S = 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 5, \infty$.

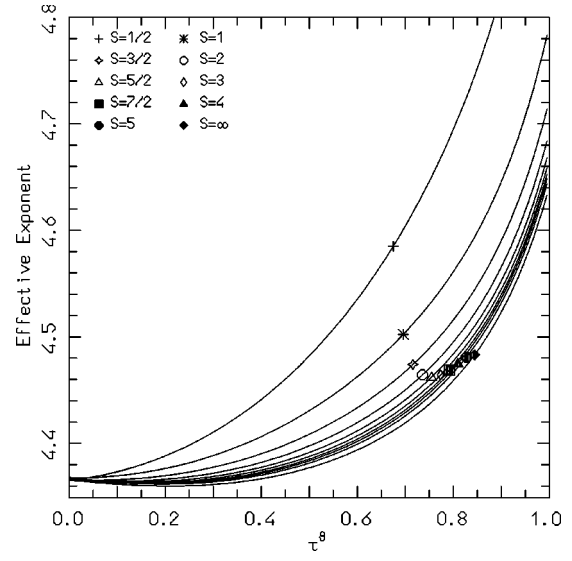


FIG. 18. Highest-order simplified-differential approximants of the effective exponent $\gamma_{4eff}^{bcc}(\beta; S)$ computed from $\chi_4^{bcc}(\beta; S)$. For each value of the spin S the effective exponent is plotted vs the corresponding reduced inverse temperature $\tau^{bcc}(S)^\theta = [1 - \beta/\beta_c^{bcc}(S)]^\theta$.

of the analysis by modified-ratio methods. The accuracy in the verification of hyperscaling is now slightly improved with respect to the biased modified-ratio methods of sec. VI, since we have $\gamma + 3\nu - 2\Delta = -0.0021(28)$.

As shown in Figs. 18 and 19, the pattern of signs for the confluent amplitudes of $\chi_4^\#(\beta; S)$ is consistent with the corresponding results for $\chi^\#(\beta; S)$, as it must, since the ratios $a_4^\#(S)/a_\chi^\#(S)$ are expected to be universal.

The exponent $\gamma_4^\#(S)$ can also be evaluated extrapolating the effective exponents by inhomogeneous second-order DA's biased with $\beta_c^\#(S)$. On the bcc lattice, our results, which appear in Table V, range between 4.376(8) for S

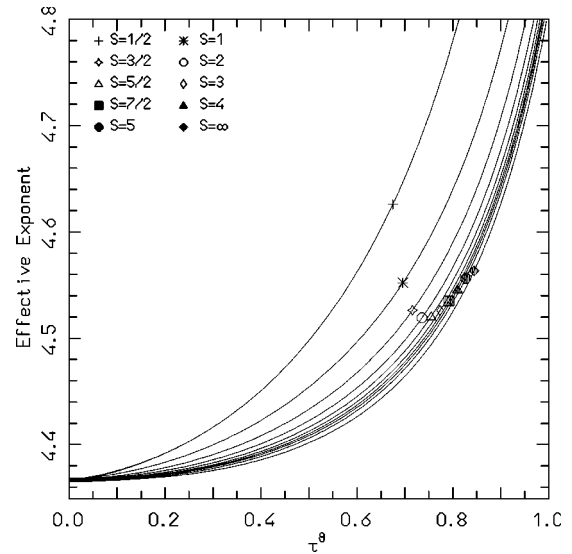


FIG. 19. Same as Fig. 18, but for the effective exponent $\gamma_{4eff}^{sc}(\beta; S)$ computed from $\chi_4^{sc}(\beta; S)$.

$=1/2$ and $4.3631(4)$ for $S=\infty$. In particular we find $\gamma_4^{bcc}(1)=4.3666(10)$, $\gamma_4^{bcc}(3/2)=4.3638(10)$ and $\gamma_4^{bcc}(2)=4.3629(10)$. By the same arguments used in the modified-ratio method analysis of the bcc lattice series, the best value for the exponent γ_4 should lie between the estimates for $S=1$ and $S=2$. This leads to our final estimate $\gamma_4=4.3647(20)$, which is even more accurately consistent with hyperscaling, since for the bcc lattice data we have $2\gamma+3\nu-\gamma_4=-0.0008(28)$.

Finally, it is worth mentioning briefly that the ratio ν/γ can be determined, to a good precision, also studying the log-derivative ratios $D \ln(\chi_4)/D \ln(\chi)$ and $D \ln(\mu_2/\beta)/D \ln(\chi)$, either by DA's biased in β_c or by simplified DA's biased in β_c and θ . The values thus obtained from the bcc lattice expansions (except for $S=1/2$) fall within the error bars of our best result $\nu/\gamma=0.5092(2)$ from modified-ratio methods. The accuracy of the estimates can be further improved by focusing on the bcc lattice case and arguing as usual that the best value of ν/γ is simply an average of the estimates for $S=1$ and $S=2$. We thus arrive to the value $\nu/\gamma=0.5091(1)$. The estimates of this ratio obtained from the sc series lie within twice the expected error bars for $S>2$, but are slightly worse for smaller values of S . In the bcc lattice case, also the DA estimate of γ obtained from the analysis of the ratio of the log derivatives of χ' and χ , whose value at the critical point is $1+1/\gamma$, agrees very closely with our best results by modified-ratio methods.

We have also examined the term-by-term divided series

$$Q(x;S)=\sum_{r\geq 0} \frac{e_r(S)}{p_r(S)} x^r, \quad (39)$$

where $e_r(S)$ are the expansion coefficients of $\chi_4(\beta;S)$ and $p_r(S)$ those of $\chi^2(\beta;S)$. Using Eq. (13), it is easily shown that the auxiliary function $Q(x;S)$ has a critical point at $x=1$ with exponent $3\nu+1$, if hyperscaling holds. A second-order biased DA analysis of the effective exponent yields the estimate $\nu=0.6300(4)$ independently of S and in complete agreement with hyperscaling.

In conclusion, we have observed that if the sequences of modified-ratio method approximants are carefully extrapolated using as bias the value of θ derived by RG methods, the estimates of the exponents obtained by the modified-ratio method, for all S and on both lattices under consideration, show good agreement with the results from DA's biased only with β_c or simplified DA's biased with both β_c and θ . The close consistency between the critical parameter estimates obtained by a number of different procedures adds further confidence that the HT series have now reached a fairly safe extension and that we are not being misled by accidental apparent convergence, so that the uncertainties of the HT estimates can be significantly reduced. The small residual dependence of the exponent estimates on the spin S and on the lattice structure can be confidently used to characterize how accurately universality is respected.

IX. COMPARISON WITH OTHER EXPONENT ESTIMATES

The agreement of our HT estimates with the values $\gamma=1.2396(13)$ and $\nu=0.6304(13)$, obtained in the context of the RG by Borel summation of coupling-constant seventh-order fixed-dimension perturbative expansions^{45,64,65} or with the values $\gamma=1.2380(50)$ and $\nu=0.6305(25)$ obtained by Borel summation of the fifth-order ϵ expansion,^{64,65} is still reasonable. The values $\gamma=1.2378(6)+0.18(g_r-1.40)$ and $\nu=0.6301(5)+0.12(g_r-1.40)$ proposed in Ref. 45 on the basis of a slightly different resummation of the fixed-dimension perturbative RG expansion are perhaps even closer. At the presently available orders of HT expansion, our series estimates prefer central values for γ and ν which are only slightly lower. It is appropriate to mention that very similar central estimates, though with larger uncertainties, were already obtained quite some time ago^{23,25-28} from bcc lattice series of order β^{21} by the method^{24,25} of Chen, Fisher, Nickel, and Zinn-Justin. For instance, the analysis of Ref. 23 yielded $\gamma=1.237(2)$ and $\nu=0.6300(15)$. A more recent study¹¹³ of HT series through $O(\beta^{20})$ for the sc lattice, along the same lines as in Refs. 23, 25 and 26, indicates $\gamma=1.2371(4)$ and $\nu=0.63002(23)$. All these results are also quoted for comparison in our Table IV.

The technique²⁴⁻²⁶ of focusing the analysis on some particular model in the Ising universality class with negligible amplitudes of the leading confluent corrections to scaling was advantageously adapted also to Monte Carlo simulations in Refs. 92 and 93, which report $\gamma=1.2372(17)$ and $\nu=0.6303(6)$. This procedure was further improved in Ref. 89, in which it led to the estimates $\nu=0.6296(7)$ and $\eta=0.0358(9)$, implying $\gamma=1.2367(20)$. Even lower central estimates of the exponents—namely, $\gamma=1.2353(25)$ and $\nu=0.6294(10)$ —have been measured in a more conventional Monte Carlo simulation of the spin-1/2 Ising model supplemented by a finite-size scaling analysis¹¹⁴ which allows also for the corrections to scaling. It is tempting to conjecture that, for $S=1/2$ on the sc lattice, our results from the extrapolations of the modified-ratio approximants and the best finite-size scaling analyses of the Monte Carlo simulations on the largest accessible lattices are subject to errors of the same nature. This would explain the rather small central values of the quoted Monte Carlo estimates of γ and indicate the need of simulations of a significantly larger scale in order to obtain from spin-1/2 systems on the sc lattice exponent values in closer agreement with our bcc series estimates.

Let us now comment briefly on the existing results for γ_4 . We recall that the validity of the hyperscaling relation, Eq. (18), for the spin-1/2 Ising model was questioned³³⁻³⁵ on the basis of an analysis of 10–12 term series on sc, bcc, and fcc lattices, yielding the estimate $\gamma+3\nu-2\Delta=0.038(12)$. This result was at the time interpreted as an indication of a small, but clear, failure of hyperscaling. As already mentioned in the Introduction, the problem was convincingly settled only when the HT series for χ and ξ^2 on the bcc lattice, extended²² up to order 21, were analyzed with careful allowance²²⁻³⁰ for the CCS and indicated the insufficient accuracy of the “classical” HT estimates $\gamma=1.250(3)$, ν

$=0.638(2)$, and $\alpha=1/8$ generally accepted^{31,32,36} until then.

For general spin, a single study⁴¹ of χ_4 , performed with series $O(\beta^{13})$ on the fcc lattice, can be found in the literature. The log derivative of the series $Q(x;S)$, defined by Eq. (39), was examined by PA's. The analysis produced estimates of the exponent $2\Delta - \gamma$ ranging from 1.887(1) for $S=1/2$ to 1.893(1) for $S=9/2$. Of course, if hyperscaling is valid, $2\Delta - \gamma = 3\nu$. Thus the final estimate $2\Delta - \gamma = 1.890(3)$ indicated the absence of hyperscaling violations of the size predicted from Refs. 33–35, provided that the central value $\nu = 0.630$ suggested by the RG, rather than the ‘‘classical’’ HT estimate $\nu = 0.638(2)$, was adopted.

The expansion of χ_4 on the sc lattice, for $S=1/2$, was at the time already available¹¹⁵ up to order 17, but it was analyzed only later in Ref. 42. It yielded the estimate $\gamma_4 = 4.370(5)$, still confirming the validity of hyperscaling, provided that the revised values obtained from the RG in those years were assumed for γ and ν . The series on the bcc lattice, for $S=1/2$, was extended to the same order only much later⁵³ and its analysis also confirmed the above conclusion. Further support of these results came also from various more recent Monte Carlo tests.^{116–119}

It should be stressed that the finite-order effects are stronger in the calculation of χ_4 (and of related quantities like g_r) than in the calculation of quantities defined in terms of two-spin correlations, as we have already remarked in Ref. 53, and therefore that the accuracy of the results is correspondingly smaller. Our comparison with previous studies shows, however, that we have achieved some improvement not only in the precision of the estimates of γ and ν , but mainly of γ_4 , by taking advantage of our significantly extended expansions of χ_4 .

X. ESTIMATES OF CRITICAL AMPLITUDES

For proper reference and for comparison with the earlier studies, we have reported in Table VI a set of updated estimates for the critical amplitudes of $\chi^\#(\beta;S)$, of $\xi^\#(\beta;S)$, of $C_H^\#(\beta;S)$, and of $\chi_4^\#(\beta;S)$ as defined by Eqs. (9)–(12).

We have evaluated the critical amplitude $C^\#(S)$ of $\chi^\#(\beta;S)$ as follows. We have adopted as a bias the value $\gamma = 1.2371$ and our estimates of $\beta_c^\#(S)$ to compute the HT series of the ‘‘effective amplitude’’:

$$C^\#(\beta;S) = \tau^\gamma \chi^\#(\beta;S) \simeq C^\#(S) [1 + a_\chi^\#(S) \tau^\#(S)^\theta + \dots + b_\chi^\#(S) \tau^\#(S) + \dots]. \quad (40)$$

The amplitude $C^\#(S)$ is then estimated by extrapolating the effective amplitude $C^\#(\beta;S)$ to the critical point. The extrapolation has been performed either by first-order inhomogeneous simplified DA's biased with $\beta_c^\#(S)$ and with θ in order to allow for the confluent corrections to scaling or, more traditionally, by using second-order inhomogeneous DA's biased with $\pm \beta_c^\#(S)$. Since these two procedures yield fully consistent estimates, we have reported in Table VI only the results obtained by the usual differential approximants, which do not need to be biased also with θ . By the same procedure, we have also studied the effective ampli-

tudes for the correlation length and for χ_4 in order to evaluate the corresponding critical amplitudes $[f^\#(S)]^2 = \lim_{\tau \rightarrow 0^+} \tau^{2\nu} \xi^\#(\beta;S)^2$ and $C_4^\#(S) = -\lim_{\tau \rightarrow 0^+} \tau^{\gamma+2\Delta} \times \chi_4^\#(\beta;S)$. The results are reported in the same table.

The above-mentioned difficulties in the analysis of the critical behavior of the specific heat also result in larger errors of the critical amplitude $A^\#(S)$. Therefore it seems more convenient to compute this quantity from the second derivative of $G^\#(\beta;S)$, which presents a sharper singularity.

Other estimates for some of the mentioned critical amplitudes, obtained from shorter HT series and under slightly different biasing assumptions or by other numerical methods, can also be found in earlier studies.^{28,42,120–122} For instance, from Ref. 120 we have cited in Table VI the estimates of $A^{sc}(1/2)$, $C^{sc}(1/2)$, and $f^{sc}(1/2)$, obtained from series $O(\beta^{17})$, $O(\beta^{19})$, and $O(\beta^{12})$, respectively, under the assumptions $\alpha=0.104$, $\gamma=1.237$, $\nu=0.6325$, and $\beta_c^{sc}(1/2) = 0.221620$. From the same reference, we have also reported the estimates of $A^{bcc}(1/2)$ obtained from a series $O(\beta^{17})$ and of $C^{bcc}(1/2)$ and $f^{bcc}(1/2)$ obtained from series $O(\beta^{21})$ by assuming $\beta_c^{bcc}(1/2) = 0.157362$ and the same values as above for α , γ , and ν .

Under various assumptions on the value of α , estimates of $A^{sc}(1/2)$ were derived in Ref. 123 from a simulation in which the energy and the specific heat were measured. By straightforward interpolation, we can conclude that, for $\alpha = 0.11$, these data would imply the estimate $A^{sc}(1/2) = 1.368(7)$, in reasonable agreement with ours.

From Ref. 121, we have quoted estimates of $C_4^{sc}(1/2)$ and of $C_4^{bcc}(1/2)$ obtained from the series $O(\beta^{17})$ and $O(\beta^{13})$, respectively, assuming $\gamma_4 = 4.375$ and $\beta_c^{sc}(1/2) = 0.221630(16)$, $\beta_c^{bcc}(1/2) = 0.157368(7)$.

In the same table, we have also reported estimates²⁸ from series $O(\beta^{21})$ for $C^{bcc}(1/2)$, $C^{bcc}(1)$, and $C^{bcc}(2)$ assuming $\gamma = 1.237$ and the estimates of $f^{bcc}(1/2)$, $f^{bcc}(1)$, and $f^{bcc}(2)$, obtained assuming $\nu = 0.6297$, $\nu = 0.6298$, and $\nu = 0.6300$, respectively, together with the values of $\beta_c^{bcc}(S)$ quoted in the same reference and reported in Table II.

In Ref. 122 the values of $C^{sc}(1/2)$ and of $f^{sc}(1/2)$ have been computed by a Monte Carlo method assuming $\gamma = 1.237$, $\nu = 0.628$, and $\beta_c^{sc}(1/2) = 0.22165$.

This brief review of some existing results shows how sensitively the estimates of the critical amplitudes depend on the bias values chosen for the critical exponents and temperatures and, of course, on the length of the series. If we also allow properly for these sources of uncertainty, which generally are not included in the error bars quoted in the literature, many of the cited estimates can be considered essentially compatible among themselves and with ours.

XI. ESTIMATES OF THE RENORMALIZED COUPLING

The value of the hyperuniversal renormalized coupling constant g_r can be obtained from our estimates of the critical amplitudes. Alternatively, without biasing the computation also with the critical exponents, g_r can be computed, with a smaller uncertainty by extrapolating to critical point the expansion of the auxiliary function

TABLE VI. Estimates of the critical amplitudes $A^\#(S)$ of the specific heat $C_H^\#(\beta, S)$, $C^\#(S)$ of the susceptibility $\chi^\#(\beta; S)$, $f^\#(S)$ of the correlation length $\xi^\#(\beta; S)$ and $C_4^\#(S)$ of $\chi_4^\#(\beta; S)$ for the spin- S Ising models on sc and bcc lattices. They are obtained by differential approximants biased with the critical inverse temperatures reported in Table II and with the critical exponents estimated in this work. For comparison, we have also reported a few estimates obtained by simulation methods or from shorter series in the recent literature. For some of them, no indication of error is available.

	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$A^{sc}(S)$	1.34(1)	1.80(3)	2.00(2)	2.09(1)	2.15(2)	2.18(2)	2.28(1)
$\partial A^{sc}(S)/\partial\beta_c^{sc}(S)$	640	980	170	550	300	750	440
$\partial A^{sc}(S)/\partial\alpha$	-20	-26	-28	-32	-32	-32	-33
$A^{bcc}(S)$	1.302(6)	1.732(6)	1.911(6)	2.000(6)	2.051(6)	2.082(6)	2.171(6)
$\partial A^{bcc}(S)/\partial\beta_c^{bcc}(S)$	800	1000	950	900	1200	870	830
$\partial A^{bcc}(S)/\partial\alpha$	-19	-26	-29	-30	-31	-32	-33
$A^{sc}(S)^a$	1.464(7)						
$A^{sc}(S)^b$	1.368(7)						
$A^{bcc}(S)^a$	1.431(9)						
$C^{sc}(S)$	1.127(3)	0.682(1)	0.545(1)	0.482(1)	0.443(1)	0.4184(6)	0.3073(2)
$\partial C^{sc}(S)/\partial\beta_c^{sc}(S)$	1900	1800	1900	1000	600	220	150
$\partial C^{sc}(S)/\partial\gamma$	-9	-8	-7	-7	-5.5	-5	-3
$C^{bcc}(S)$	1.042(1)	0.622(1)	0.4967(3)	0.4379(3)	0.4045(4)	0.3826(4)	0.2817(4)
$\partial C^{bcc}(S)/\partial\beta_c^{bcc}(S)$	3900	-1300	170	750	500	480	310
$\partial C^{bcc}(S)/\partial\gamma$	-9	-5	-4.5	-3.6	-3.1	-3	-2.2
$C^{sc}(S)^a$	1.1025(10)						
$C^{sc}(S)^c$	1.093(13)						
$C^{bcc}(S)^d$	1.026	0.620		0.4346			
$C^{bcc}(S)^a$	1.0312(10)						
$f^{sc}(S)$	0.506(1)	0.458(1)	0.443(1)	0.436(1)	0.432(1)	0.430(1)	0.423(1)
$\partial f^{sc}(S)/\partial\beta_c^{sc}(S)$	290	280	200	180	400	120	120
$\partial f^{sc}(S)/\partial\nu$	-4	-4	-3.5	-4	-4	-3	-3.5
$f^{bcc}(S)$	0.4686(4)	0.4262(8)	0.4112(4)	0.4047(4)	0.4013(4)	0.3992(4)	0.3937(4)
$\partial f^{bcc}(S)/\partial\beta_c^{bcc}(S)$	500	450	500	230	200	170	1700
$\partial f^{bcc}(S)/\partial\nu$	-4	-5	-3	-3	-3	-3	-3
$f^{sc}(S)^a$	0.496(4)						
$f^{sc}(S)^c$	0.501(2)						
$f^{sc}(S)^d$	0.5192						
$f^{bcc}(S)^a$	0.4590(1)						
$f^{bcc}(S)^d$	0.46821	0.42605		0.4038			
$C_4^{sc}(S)$	3.87(1)	1.05(1)	0.606(1)	0.456(2)	0.375(1)	0.327(1)	0.169(1)
$\partial C_4^{sc}(S)/\partial\beta_c^{sc}(S)$	5000	3200	2900	2700	800	-1300	350
$\partial C_4^{sc}(S)/\partial\gamma_4$	-20	-9	-7	-6	-5	-1.5	-1.5
$C_4^{bcc}(S)$	3.410(8)	0.912(3)	0.523(1)	0.3884(6)	0.3230(5)	0.2847(5)	0.1478(5)
$\partial C_4^{bcc}(S)/\partial\beta_c^{bcc}(S)$	15000	8000	460	630	470	450	190
$\partial C_4^{bcc}(S)/\partial\gamma_4$	-24	-9.5	-3	-2.5	-2	-2	-1
$C_4^{sc}(S)^e$	3.70(3)						
$C_4^{sc}(S)^f$	3.630 ₍₋₁₇₎ ⁽⁺³⁾						
$C_4^{bcc}(S)^f$	3.236(2)						

^aReference 120.

^bReference 123.

^cReference 122.

^dReference 28.

^eReference 42.

^fReference 121.

$$y^\#(\beta; S) = [g_r^\#(\beta; S)]^{-2/3} \quad (41)$$

or of the function

$$z^\#(\beta; S) = [\beta/\beta_c^\#(S)]^{3/2} g_r^\#(\beta; S). \quad (42)$$

Unlike the effective coupling $g_r^\#(\beta; S)$, both $y^\#(\beta; S)$ and $z^\#(\beta; S)$ are regular analytic at $\beta=0$, so that they can be expanded in powers of β and extrapolated to the critical point by Padé approximants, DA's, or simplified DA's. Due

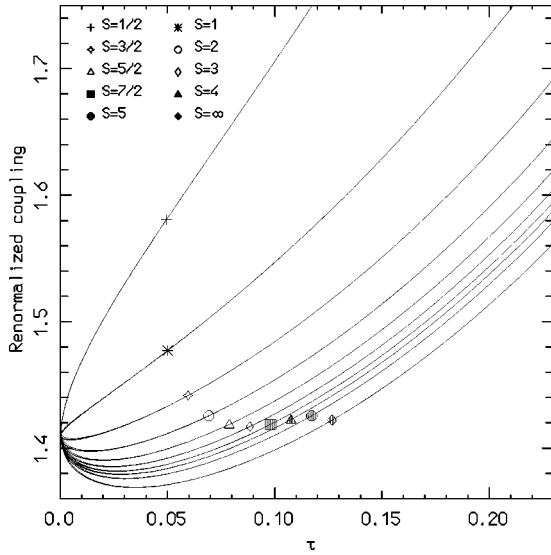


FIG. 20. Highest-order simplified differential approximants of the effective dimensionless renormalized coupling constant $g_r(\beta;S)$, as obtained from the bcc lattice series. The effective coupling is computed from the auxiliary function $z^{bcc}(\beta;S)$ defined in Eq. (42) and is plotted vs $\tau^{bcc}(S) = 1 - \beta/\beta_c^{bcc}(S)$.

to the finite extension of the series, the numerical estimates of g_r derived from Eq. (41) or from Eq. (42) are of course (very) slightly different. In order to allow for the expected leading confluent corrections to scaling, in our calculations we have used first- and second-order DA's biased in $\beta_c^\#(S)$ or simplified DA's biased with $\beta_c^\#(S)$ and with the confluent exponent θ . In Fig. 20 we have plotted vs $\tau^{bcc}(S)$ the effective coupling $g_r^{bcc}(\beta;S)$ as obtained from the function $z^{bcc}(\beta;S)$ for various values of the spin S . The curves, computed in the simplest way by simplified DA's, show the strong influence of the CCS nearby the critical point and indicate that $g_r^{bcc}(S)$ is independent of S within a very good approximation. Comparison with Fig. 21, which shows the effective coupling $g_r^{sc}(\beta;S)$ plotted vs $\tau^{sc}(S)$, similarly indicates that the renormalized coupling $g_r^{sc} \approx g_r^{bcc} \approx 1.41$ is independent not only of S , but also of the lattice structure. Using Eq. (19), we can infer from Fig. 20 that the amplitudes of the CCS are generally large and, more precisely, that $a_g^{bcc}(S) > 0$ for $S = 1/2$ and 1, whereas $a_g^{bcc}(S) < 0$ for $S > 2$. Analogously, from Fig. 21 we can conclude that $a_g^{sc}(S) > 0$ for all S . These qualitative conclusions are consistent with RG estimates^{61,124} in the fixed-dimension approach indicating that a_g/a_χ lies in a range from ≈ -3 to ≈ -2 .

In order to reach higher precision in the calculation of g_r , we have preferred to use first- and second-order DA's biased with β_c . In the bcc lattice case, we notice that, for $S = 1/2$ and 1, approximants which use an increasing number of coefficients show a residual slowly decreasing trend, while, for $S \geq 2$, they show an increasing trend. We shall indicate by asymmetric uncertainties these features of the approximant sequences. Again arguing as for the critical exponents γ, ν , and γ_4 , we can expect that the most reliable estimate of g_r obtained from the bcc lattice series will be nearly equal to

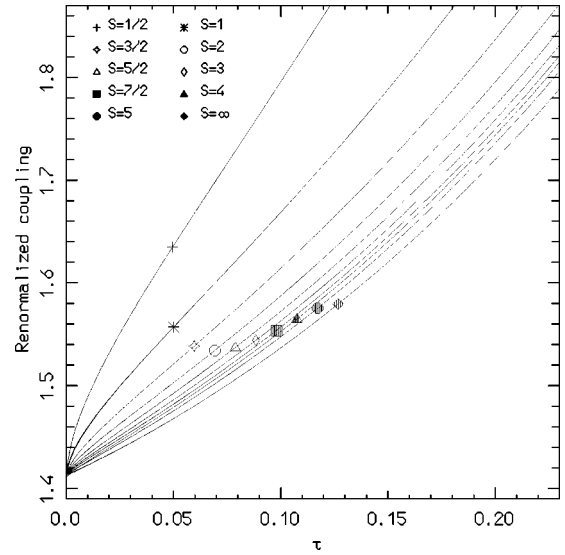


FIG. 21. Highest-order simplified differential approximants of the effective dimensionless renormalized coupling constant $g_r(\beta;S)$ as obtained from the sc lattice series. The effective coupling is computed from the auxiliary function $z^{sc}(\beta;S)$ defined by Eq. (42) and is plotted vs $\tau^{sc}(S) = 1 - \beta/\beta_c^{sc}(S)$.

the average of $g_r^{bcc}(1)$ and $g_r^{bcc}(2)$ or to the value $g_r^{bcc}(3/2)$. Thus we conclude that $g_r = 1.404(3)$. The central value of our updated estimate is slightly lower than our previous result⁵³ $g_r = 1.407(6)$, based on a series $O(\beta^{17})$ for the bcc lattice in the $S = 1/2$ case, in which the convergence is slowest. However, our revised result is slightly closer to the value $g_r = 1.400$ advocated by Murray and Nickel⁴⁵ in the context of the RG fixed-dimension approach and is compatible with the more recent HT result¹¹³ $g_r = 1.402(2)$ obtained by the method of Refs. 24 and 25 from sc lattice series extending to order β^{18} . Our numerical estimates of $g_r^\#(S)$, for several values of S and on both lattices, are reported in Table VII. Notice that in the bcc case the DA estimates are larger or smaller (in the sc case generally larger) than the expected best value, consistently with the signs of the CCS amplitudes. In the same table we have quoted for comparison also some recent Monte Carlo estimates^{116,118} of g_r , as well as other results from the HT and RG methods.

XII. ESTIMATES OF R_ξ^+

The combination of critical amplitudes R_ξ^+ , defined by Eq. (21), can be computed either from the estimates of the critical amplitudes $A^\#, f^\#$ and of the exponent $\alpha = 2 - 3\nu$ or, more directly, by extrapolating to the critical point the expansion of the auxiliary function

$$F(\beta;S) = \frac{-q\nu^3\beta_c(S)}{2\nu\beta} \frac{d^2G(\beta;S)}{d\beta^2} \left(\frac{\beta^{3/2}}{\beta_c^{3/2}(S)} \frac{d(1/\xi(\beta;S))}{d\beta} \right)^{-3} \quad (43)$$

TABLE VII. Estimates of the universal renormalized coupling g_r using the auxiliary function $z^\#(\beta;S)$ of Eq. (42) for the spin- S Ising model series on sc and bcc lattices. They are obtained by differential approximants biased with the modified-ratio estimates of the critical inverse temperatures reported in Table II. For comparison, we have also reported other estimates obtained by simulation methods or from shorter series in the recent literature. (For some of them, no indication of error is available.) The values marked with an asterisk have been obtained either by renormalization-group methods or by high-temperature methods which assume universality and therefore they refer to the Ising universality class although, for simplicity, they are reported in the column of the $S=1/2$ results.

	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$g_r^{sc}(S)$ (DA)	1.40(1)	1.410(6)	1.404(6)	1.414(10)	1.415(10)	1.414(10)	1.412(10)
$g_r^{bcc}(S)$ (DA)	$1.408_{(-4)}^{(+1)}$	1.409(4)	1.404(3)	$1.401_{(-1)}^{(+4)}$	$1.400_{(-1)}^{(+5)}$	$1.400_{(-1)}^{(+5)}$	$1.398_{(-1)}^{(+6)}$
$g_r^{bcc\ a}$	1.401(8)						
$g_r^{sc\ a}$	1.403(7)						
$g_r^{bcc\ b}$	1.459(9)						
g_r^c	1.402(2)*						
g_r^d	1.40*						
g_r^e	1.411(4)*						

^aReference 116.

^bReference 121.

^cReference 113.

^dReference 45.

^eReference 64.

since the validity of the hyperscaling relation, Eq. (20) implies $F(\beta_c(S)-0;S)=(R_\xi^+)^3+O(1/n^\theta)$. We have assumed $\nu=0.6299$ and evaluated $F(\beta;S)$ by first-order differential approximants biased with $\pm\beta_c(S)$. The estimates of R_ξ^+ obtained by this prescription are shown in Table VIII. Within a fair approximation, they are independent of S and of the lattice structure and compatible with the estimates obtained combining the amplitudes reported in Table VI.

Our final estimate is $R_\xi^+=0.2668(5)$. This result is slightly smaller than that the value $R_\xi^+=0.272(4)$ reported in our previous study⁵² of the single $S=1/2$ case employing shorter series.

In Table VIII we have also shown values of R_ξ^+ obtained via the RG, either by the fixed-dimension perturbative expansion to fifth order¹²⁵ or by ϵ -expansion to second order.¹²⁶ We have also quoted the estimate $R_\xi^+=0.270(4)$ that would be obtained from the Monte Carlo measures of Ref. 123 assuming $\alpha=0.11$ and the recent HT result $R_\xi^+=0.2659(4)$ taken from Ref. 113 as a representative of various nearly equal central estimates from studies^{52,113,120} performed at different times, with different techniques, under different assumptions on the values of ν and α and using series of different extensions. The discrepancy from our estimate should probably be taken as an indication of the re-

TABLE VIII. Estimates of the universal quantity R_ξ^+ using the auxiliary function F of Eq. (43) for the spin- S Ising model series on sc and bcc lattices. They are obtained by differential approximants biased with the critical inverse temperatures reported in Table II and with the value of ν obtained in this study. We have also reported some estimates obtained by other methods or from shorter series in the recent literature. The estimates marked with an asterisk are obtained by renormalization-group methods or by high-temperature methods that assume universality and therefore they refer to the Ising universality class although, for simplicity, they are reported in the column of the $S=1/2$ results.

	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$R_\xi^{+sc}(S)$	0.2664(10)	0.2669(12)	0.2671(11)	0.2673(12)	0.2679(15)	0.2674(10)	0.2673(10)
$R_\xi^{+bcc}(S)$	0.2664(4)	0.2667(3)	0.2668(3)	0.2669(4)	0.2669(4)	0.2669(4)	0.2670(4)
R_ξ^{+a}	0.2659(4)						
R_ξ^{+b}	0.270(4)						
R_ξ^{+c}	0.270(1)*						
R_ξ^{+d}	0.27*						
R_ξ^{+e}	0.284(18)						
R_ξ^{+f}	0.265(6)						

^aReferences 84, 120, and 113.

^bReference 123.

^cReference 125.

^dReference 126.

^eReference 127.

^fReference 128.

TABLE IX. Estimates of the universal ratio a_ξ/a_χ for the spin- S Ising model series on the bcc lattice. For comparison, we have also reported some estimates obtained by renormalization-group methods. (For some of them, no indication of error is available.) Although they refer to the Ising universality class, for simplicity, they are reported in the column of the $S=1/2$ results.

	$S=1/2$	$S=1$	$S=3/2$	$S=2$	$S=5/2$	$S=3$	$S=\infty$
$a_\chi^{bcc}(S)$	-0.129(3)	-0.0363(10)	0.0079(8)	0.0307(10)	0.0436(10)	0.0515(10)	0.0742(20)
$a_\xi^{bcc}(S)$	-0.100(4)	-0.0279(15)	0.0061(6)	0.0233(10)	0.0331(20)	0.0390(20)	0.0560(20)
$a_\chi^{bcc}(S)^a$	-0.119	-0.034		0.023			
$a_\xi^{bcc}(S)^a$	-0.1085	-0.033		0.0225			
$a_\chi^{bcc}(S)^b$	-0.13						
$a_\xi^{bcc}(S)^b$	-0.11						
$a_\xi^{bcc}(S)/a_\chi^{bcc}(S)$	0.78(5)	0.77(6)	0.77(16)	0.76(6)	0.76(6)	0.76(5)	0.76(5)
a_ξ/a_χ^b	0.85						
a_ξ/a_χ^c	0.65(5)*						
a_ξ/a_χ^d	0.65*						

^aReference 28.

^bReference 23.

^cReference 129.

^dReference 124.

maining difficulty of accurately evaluating the specific-heat amplitude.

Finally, it is worth while to quote two recent very accurate measurements on binary mixtures: $R_\xi^+ = 0.284(18)$, performed in a microgravity experiment,¹²⁷ and $R_\xi^+ = 0.265(6)$, obtained in a conventional environment.¹²⁸

XIII. ESTIMATING THE RATIOS OF CONFLUENT-SINGULARITY AMPLITUDES

From the extended series presented here, we have also tried to evaluate the universal ratio $a_\xi(S)/a_\chi(S)$. We recall that, for the bcc lattice, our analysis of χ and ξ by modified-ratio methods had shown that, as the spin S varies between 1 and 2, the leading correction amplitudes $a_\chi(S)$ and $a_\xi(S)$ vary from small negative values to small positive values, whereas in the sc lattice case no change of the sign of the confluent amplitudes is observed. As we have emphasized, some knowledge of these amplitudes is necessary to understand how the various numerical estimates obtained for each value of S approach the true values of the universal critical parameters. A simple prescription to compute accurately the universal quantities consists in using series on the bcc lattice with spins between 1 and 2, for which the amplitudes of the leading confluent corrections to scaling are very small. Conversely, the numerical methods to evaluate the amplitudes and the exponent of the leading CCS can be expected to work with fair accuracy only when the confluent corrections are *not* too small. For $S=1/2$ the size of the leading CCS is largest, but unfortunately also the higher-order corrections seem to be still important, as shown by the steep behavior of the extrapolated sequences in Fig. 4. Therefore, the most reliable results are likely to come from the bcc series for $S>2$.

We have obtained reasonably accurate estimates of the CCS amplitudes for χ simply by fitting the asymptotic form $\gamma + c_1(S)/n^\theta + c_2(S)/n^{1+\theta}$, suggested by Eq. (27), to the exponent approximant sequences, Eq. (26), under the assump-

tions that $\gamma=1.2371$ and $\theta=0.504$. A similar procedure can be repeated in the case of ξ^2 , assuming $\nu=0.6299$. The estimates of the CCS amplitudes and of the universal ratios $a_\xi^{bcc}(S)/a_\chi^{bcc}(S)$ thus obtained are shown in Table IX. The values of the amplitudes $a_\chi^{bcc}(S)$ obtained by this procedure are generally consistent within a few percent, and those of $a_\xi^{bcc}(S)$ within $\approx 10\%$ – 20% in the worst cases, with those evaluated by simplified DA's of the log derivatives of χ and ξ biased with β_c and θ . Moreover, for $S>2$, the estimates of $a_\chi^{bcc}(S)$ obtained from the modified-ratio approximants for γ are consistent within a few percent with those obtained from the corresponding approximants for β_c . As expected, for smaller values of S , we have consistency only within 20% – 40% because the rate of convergence of the series is slower and/or the subleading confluent corrections are more important.

The ratios $a_\xi^{bcc}(S)/a_\chi^{bcc}(S)$ appear to be approximately independent of the spin S , as they should, and suggest the final estimate $a_\xi/a_\chi=0.76(6)$. The error includes also the uncertainties of the bias parameters γ , ν , and θ . In the sc lattice case, the same analysis leads to amplitude ratios which show larger uncertainties, but agree within the errors with the bcc results.

For $S=1/2$, the series $O(\beta^{21})$ of Ref. 23 yielded the estimates (without indication of error) $a_\xi^{bcc}(1/2)=-0.11$, which is 10% larger than ours, and $a_\chi^{bcc}=-0.13$, which agrees closely with ours.

Using the same series, Ref. 28 obtained estimates of a_χ^{bcc} and a_ξ^{bcc} for $S=1/2,1,2$, also quoted in Table IX and in good agreement with ours.

Our central estimate of the ratio a_ξ/a_χ is somewhat smaller than our previous estimate⁵⁶ $a_\xi/a_\chi=0.87(6)$, based on shorter $S=1/2$ series, than the old HT estimates $a_\xi/a_\chi=0.83(5)$ of Ref. 27 and $a_\xi/a_\chi=0.85$ of Refs. 23 and 28 (reported without indication of error), but it is larger than the HT estimate $0.71(7)$ of Ref. 24 and the earlier estimate⁸⁵ $a_\xi/a_\chi=0.70(2)$ based on the fcc series $O(\beta^{12})$ for general

spin tabulated in Refs. 8 and 9. We should also mention the estimate $a_\xi/a_\chi=0.65(5)$ obtained by the RG in the perturbative fixed-dimension approach at sixth order.¹²⁹ The ϵ -expansion scheme (extending to second order) yielded¹²⁴ the estimate $a_\xi/a_\chi=0.65$.

Finally, let us note that our results confirm the observations of Refs. 23, 25–27 and 52 and the arguments presented in Ref. 63 that the amplitudes of the leading CCS have a negative sign, both for the susceptibility and for the correlation length, in the case of the spin-1/2 Ising model, on sc and bcc lattices.

XIV. CONCLUSIONS

For the Ising models of general spin S , on sc and bcc lattices, we have produced extended HT expansions of the nearest-neighbor correlation function, of the susceptibility, of the second correlation moment, and of the second field derivative of the susceptibility.

Our procedure of series analysis differs somewhat from the most traditional ones, but leads to completely consistent conclusions. At least for the models studied here, we are also confident that it yields very accurate direct estimates of the various critical parameters. Our updated results $\gamma = 1.2371(1)$, $\nu = 0.6299(2)$, $\gamma_4 = 4.3647(20)$, $g_r = 1.404(3)$, and $R_\xi^+ = 0.2668(5)$ are in good agreement with the latest calculations by other approximate methods, including perturbative field-theoretic RG approaches. At the same time, our new series data have proved to be sufficiently rich that we can obtain fairly tight checks of the conventional expectations about hyperscaling and universality, with regard both to the spin S and to the lattice structure.

ACKNOWLEDGMENTS

This work has been partially supported by the Ministry of Education, University and Research.

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