PHYSICAL REVIEW B, VOLUME 65, 134416

Thermodynamics of the anisotropic two-channel Kondo problem

Gergely Zaránd,[1,2] Theo Costi,[3,4] Andres Jerez,[3,5] and Natan Andrei[6]
[1]Lyman Physics Laboratory, Harvard University, Cambridge, Massachusetts 02138
[2]Research Group of the Hungarian Academy of Sciences, Institute of Physics, TU Budapest, H-1521 Hungary
[3]Institut Laue–Langevin, 6 rue Jules Horowitz, B.P. 156, 38042 Grenoble Cedex 9, France
[4]Institut für Theorie der Kondensierten Materie, Universität Karlsruhe (TH), D-76128 Karlsruhe, Germany
[5]European Synchrotron Radiation Facility, B.P. 220, 38042 Grenoble Cedex, France
[6]Center of Materials Theory, Rutgers University, Piscataway, New Jersey 08854-0849

(Received 10 October 2001; published 19 March 2002)

We construct and solve numerically the thermodynamic Bethe-ansatz equations for the spin-anisotropic two-channel Kondo model in arbitrary external field h. At high temperatures the specific heat and the susceptibility show power-law dependence. For $h\to 0$ and at temperatures below the Kondo temperature T_K a two-channel Kondo effect develops characterized by a Wilson ratio 8/3, and a logarithmic divergence of the susceptibility and the linear specific-heat coefficient. A finite magnetic field, $h>0$, drives the system to a Fermi-liquid fixed point with an unusual Wilson ratio which depends sensitively on h.

DOI: 10.1103/PhysRevB.65.134416 PACS number(s): 75.20.Hr, 71.10.Hf, 71.27.+a, 72.15.Qm

I. INTRODUCTION

The two-channel Kondo model (2CKM) attracted a lot of interest during the past few years.[1] This intense interest is mostly triggered by the rather unusual properties of this model: The existence of finite residual entropy in zero external field, the vanishing of single-particle scattering amplitude at $T=0$ temperature, and the logarithmic singularity of various thermodynamic quantities. All these unusual properties appear due to the presence of a hidden and conserved *flavor* quantum number of the electrons.

Several physical systems have been proposed to realize the 2CKM. An unambiguous realization of the 2CKM is provided by dilute uranium and cerium-based heavy fermion compounds.[1] In these alloys the combination of strong spin-orbit interaction and crystal-field symmetry effects leads to an effective 2CKM.

It has been also suggested that noncommutative tunneling centers may form two-level systems (TLS's) and realize the 2CKM.[2] In this case the localized spin is replaced by the position of the tunneling center, while the angular momenta of the conduction electrons play the role of the electron spin in the 2CKM. The electron spins in the TLS problem play the role of silent flavor indices. Though there have been several concerns raised concerning the microscopic structure of the TLS and the possibility of observing the two-channel Kondo behavior,[3,4] many experiments may be consistently interpreted in terms of such dynamical two-channel Kondo impurities.[5–7]

Another realization of the 2CKM is provided by quantum dots near to their charge degeneracy point.[8] In this case the charging states of the dot replace the impurity spin states and they couple to the position variable of the conduction electrons. Again, the spin of the electrons acts as a flavor variable. Though it appears to be extremely difficult to observe experimentally the non-Fermi-liquid behavior associated with the two-channel Kondo fixed point,[9] some fingerprints of the two-channel Kondo effect have been observed in the capacitance of such semiconducting quantum dots.[10]

The above physical realizations of the 2CKM have the common feature that the 2CKM that describes them is in all cases generically strongly spin anisotropic, and breaks the full SU(2) spin symmetry. While this SU(2) symmetry breaking is known to be irrelevant at the two-channel Kondo fixed point,[1] it affects both qualitatively and quantitatively the properties of the model at energies around and above the Kondo temperature T_K, and has to be taken into account to make comparison with experiments.

In the present paper we generalize the Bethe-ansatz results of Ref. 11 to compute both analytically and numerically the thermodynamics of the anisotropic 2CKM at arbitrary temperatures and magnetic fields for various values of the anisotropy. Below the Kondo temperature T_K we find that the thermodynamic properties of the model are very similar to those of the isotropic 2CKM. This observation is in full agreement with the irrelevance of the anisotropy at the 2CKM. However, above T_K the behavior of the model depends crucially on spin anisotropy. We find that above T_K all thermodynamic quantities display a *power-law* behavior with an exponent determined by the value of the anisotropy. Furthermore, we find that at finite magnetic fields the system flows to an unusual Fermi-liquid fixed point with an anisotropy-dependent Wilson ratio.

The paper is organized as follows. In Sec. II we introduce the 2CKM. In Sec. III we construct the thermodynamic Bethe-ansatz equations for the anisotropic 2CKM that we analyze in detail in Sec. IV. Some details of the computation are given in the Appendix.

II. MODEL

The anisotropic 2CKM consists of a spin $S_0=1/2$ impurity that couples dynamically to the spin of the conduction electrons through a strongly anisotropic exchange interaction, and is described by the following first quantized Hamiltonian ($\hbar=k_B=\mu_B=1$):[2]

$$H=\sum_{j=1}^{N_e}\left\{-i\partial/\partial x_j+\sum_{\alpha=x,y,z}\delta(x_j)J_\alpha S_j^\alpha S_0^\alpha\right\}+h\left(gS_0^z+g'\sum_j S_j^z\right). \tag{1}$$

In this equation x_j is the coordinate of the jth conduction electron, $N_e=N\times f$ is the total number of electrons, and the S_j's ($j=0,\ldots,N_e$) denote the spin-1/2 operators of the electrons. Electrons also have a conserved flavor quantum number, $m=\{1,\ldots,f\}$, implicit in Eq. (1). In the following we discuss the case $f\geqslant 2$ as well though for the physical realizations mentioned in the Introduction $f=2$. (The $f=1$ case has been analyzed in Refs. 12 and 11.)

The meaning of the various terms in this Hamiltonian depends on the particular physical realization. The external field h couples to the local moment and the conduction electron spins with different g factors. In the TLS problem and the quantum dot case, the external field h represents the asymmetry of the TLS and the splitting between the two charging states of the quantum dot, respectively, and $g'\equiv 0$. For heavy fermion systems, depending on the particular realization, h can describe an external magnetic or strain field, and may also couple to $\Sigma_j S_j^z$. In the present paper we mostly study the *local* susceptibility, corresponding to $g'=0$.

In the original formulation of the Bethe ansatz, similar to the conformal field theory solution of the Kondo problem,[13] the impurity spin is "fused" with the spin degrees of freedom of the conduction electrons, and the external field couples to the total spin, corresponding to $g'=1$. Thus the Bethe-ansatz calculates the impurity contribution to the *global* susceptibility with $g'=g$. Lowenstein made a remarkable attempt to treat the $g'\neq 1$ case,[14] however, his results were not fully conclusive.[14] In the present paper we use a different strategy to cope with the $g'=0$ situation, and show that, similar to the case of the single-channel anisotropic Kondo problem,[15] the $g'=0$ *local* susceptibility (i.e., the susceptibility arising when the field couples only to the impurity spin) is simply proportional to the global susceptibility. Therefore, after determining the appropriate normalization factor, we are able to extract the exact *local* susceptibility from the Bethe-ansatz calculation with $g=g'$.

In our analysis we shall assume that $J_z\neq J_x=J_y=J_\perp$ and thus the Hamiltonian has a U(1) spin symmetry. This is true for the quantum dot and heavy fermion realizations, however, for the TLS problem $J_z\gg J_x>J_y=0$.[2] Fortunately, it is not necessary to treat the general Bethe-ansatz equations in order to determine the universal features of the TLS Hamiltonian in the TLS case either: Under scaling the terms J_x and J_y, describing electron-assisted tunneling processes, become *rapidly* equal[2] and therefore it suffices to consider an equivalent effective model with $J_x^{\rm eff}=J_y^{\rm eff}=J_\perp=J_x/2$ to determine the thermodynamics. This effective model has U(1) orbital symmetry and can be much more easily analyzed by Bethe-ansatz techniques than the fully anisotropic one.

In constructing the wave function we implicitly have to introduce a further electron-electron interaction,

$$H^{el-el}=\sum_{1\leqslant i<j\leqslant N_e}\sum_{\alpha=x,y,z}\delta(x_i-x_j)J_\alpha S_i^\alpha S_j^\alpha. \tag{2}$$

Since electrons move with the same velocity and in the same direction, this interaction does not modify the thermodynamics in a crucial way for $J_z>J_\perp$. In particular, we have shown that the specific heat of the electrons is unaffected by this interaction. However, it does *rescale* the g factor in Eq. (1): $g\rightarrow g^{\rm eff}$. Fortunately, we shall be able to *compensate* this effect by the renormalization of g factors in the magnetic-field term.

In the present paper we restrict our discussion to the case $J_z>J_\perp$. Our calculations are therefore not directly applicable to the quantum dot case.

In Eq. (1), only a term $\sim hS_0^z$ has been included, which acts as a local field in the z direction. In reality, however, the field can also point into the "perpendicular" direction corresponding to a term $\sim h_\perp S_0^x$. In the TLS case this term describes spontaneous tunneling between the TLS positions. The general effect of the latter term is very similar to that of hS_0^z, but quantitatively it behaves somewhat differently. For a TLS, e.g., in a typical situation, h is believed to be much larger than $h_\perp$,[4] and furthermore, its *effective* value is orders of magnitude reduced by polaronic effects.[2,27] Therefore in most situations it is enough to include only the term $\sim hS_0^z$ in the Hamiltonian.

III. BETHE-ANSATZ SOLUTION

The most important ingredients of the algebraic Bethe ansatz (BA) are the various scattering matrices. The impurity-conduction electron S matrix can be constructed directly from Eq. (1):

$$R_{0j}=U_{0j}^{(spin)}(\lambda_0-\lambda_j)_{\sigma_0\sigma_0';\sigma_j\sigma_j'}\otimes Id^{(flavor)}, \tag{3}$$

where $\lambda_0=-1$ and $\lambda_j=\rightarrow 0$ denote the "rapidities" of the impurity and conduction electron j, and U is the U(1) scattering matrix,

$$U_{0j}(\lambda)=a(\lambda)P_{\uparrow\uparrow}+b(\lambda)P_{\uparrow\downarrow}+\frac{1}{2}c(\lambda)(S_0^+S_j^-+S_0^-S_j^+),$$

$$\frac{a(\lambda)}{c(\lambda)}=\frac{\sinh(i\mu+\lambda\vartheta)}{\sinh(i\mu)},\quad \frac{a(\lambda)}{b(\lambda)}=\frac{\sinh(i\mu+\lambda\vartheta)}{\sinh(\lambda\vartheta)}, \tag{4}$$

with $P_{\uparrow\uparrow}$ and $P_{\uparrow\downarrow}$ being the projection operators for parallel and opposite spins. Excepting the small coupling regime, the connection of the parameters μ and ϑ with the bare couplings J_z and $J_\perp$ is ambiguous[12,16] and depends on the regularization procedure of the Dirac delta and the cutoff scheme used. Therefore, instead of J_z and $J_\perp$, it is rather μ and θ that should be viewed as the basic parameters of the BA solution: μ turns out to be connected to the renormalized phase shift while the ratio μ/θ determines the Kondo temperature, below which non-Fermi-liquid correlations appear:

$$T_K=\left(1-f\frac{\mu}{\pi}\right)2D\exp\{-\pi\theta/\mu\}, \tag{5}$$

with $D=N/L$.

Since electrons move with the same velocity we have the liberty to define their scattering matrix in a way to maintain integrability:

$$R_{ij}=U_{ij}(\lambda_i-\lambda_j)\otimes F_{ij}(\lambda_i-\lambda_j). \tag{6}$$

Here $U(\lambda)$ is given by Eq. (4) and F describes scattering in the flavor sector:

$$F_{ij}(\lambda_i-\lambda_j)=\frac{\lambda_i-\lambda_j+icX_{ij}}{\lambda_i-\lambda_j+ic}, \tag{7}$$

with X_{ij} the flavor exchange operator of particle i and j and c an arbitrary constant to be defined later.

Starting from these scattering matrices we used the algebraic BA to determine the nested BA equations and then applied the dynamical fusion procedure of Ref. 17 to eliminate the flavor degrees of freedom. The fused equations considerably simplify with the choice $c\equiv\mu/\vartheta$. Then the rapidities $\{\lambda_\alpha;\alpha=1,\dots,M\}$ describing the spin sector of the wave function satisfy

$$\frac{\sinh\left(\mu\left(\lambda_\alpha+\frac{i}{2}\right)+\vartheta\right)}{\sinh\left(\mu\left(\lambda_\alpha-\frac{i}{2}\right)+\vartheta\right)}\left[\frac{\sinh\mu\left(\lambda_\alpha+i\frac{f}{2}\right)}{\sinh\mu\left(\lambda_\alpha-i\frac{f}{2}\right)}\right]^N$$

$$=-\prod_{\beta=1}^{M}\frac{\sinh\mu(\lambda_\alpha-\lambda_\beta+i)}{\sinh\mu(\lambda_\alpha-\lambda_\beta-i)}. \tag{8}$$

The momenta of the electrons and thus the total energy is determined by the periodic boundary conditions

$$e^{ik_ALf}=\prod_{\alpha=1}^{M}\frac{\sinh\mu\left(\lambda_\alpha+i\frac{f}{2}\right)}{\sinh\mu\left(\lambda_\alpha-i\frac{f}{2}\right)};\qquad E=\sum_{A=1}^{N}fk_A, \tag{9}$$

where fk_A denotes the total momentum of the fused f-electron composites, and L is the system size.

In the thermodynamic limit, $L,N\rightarrow\infty$, $N/L=D$, the "spin rapidities" λ_α in Eq. (8) are organized into strings[12,11] of length r and parity $v=\pm$: $\lambda\rightarrow\{\lambda_q^{(r,v)};\ q=1,\dots,r\}$ with

$$\lambda^{(r,v)}\leftrightarrow\lambda_q^{(r,v)}=\lambda^{(r,v)}+\left[\frac{r+1}{2}-q\right]+i\frac{\pi}{4\mu}(1-v). \tag{10}$$

We have verified that to obtain a stable solution for $\mu<\pi/f$, v and r must satisfy the *same stability condition* as for $f=1$,[18,12]

$$v\,\sin(\mu q)\sin[\mu(r-q)]>0,\qquad q=1,\dots,[r/2]. \tag{11}$$

As shown by Takahashi and Susuki,[18] the allowed (r,v) strings can be classified on the basis of an infinite (or finite) fraction expansion of μ/π. To be specific, here we only discuss the simplest case $\mu=\pi/\nu$ and $f<\nu$, where only ν different stable string configurations exist: $n=(r,v)=(1,+),(2,+),\dots,(\nu-1,+)$ and $(1,-)$. The case $\mu=\pi/f$ represents a singular limit:[19] For $f=1$ it corresponds to the decoupling point[20] while for $f=2$ it can be identified with the Emery-Kivelson point (see below).

IV. THERMODYNAMICS

A. Thermodynamic Bethe-ansatz equations

To derive the thermodynamic BA equations in the continuum limit $L,N\rightarrow\infty$ and $D\equiv N/L=cst$ we proceeded in the usual way. We first defined the density of rapidities (rapidity holes), $\varrho_n(\lambda)$ $[\tilde{\varrho}_n(\lambda)]$. These are related to the "excitation energies" $\epsilon_n(\lambda)$ through $\eta_n\equiv\tilde{\varrho}_n/\varrho_n\equiv e^{\varepsilon_n/T}$. The functions $\epsilon_n(\lambda)$ are determined by the following integral equations for $\nu>f$:

$$\varepsilon_\nu/T=gh\nu/2T-s*\ln(1+e^{\varepsilon_{\nu-2}/T})+\delta_{\nu,f+1}\Theta(\lambda),$$

$$\varepsilon_{\nu-1}/T=gh\nu/2T+s*\ln(1+e^{\varepsilon_{\nu-2}/T})-\delta_{\nu,f+1}\Theta(\lambda),$$

$$\varepsilon_j/T=s*\ln[(1+e^{\varepsilon_{j+1}/T})(1+e^{\varepsilon_{j-1}/T})]$$
$$+\delta_{j,\nu-2}s*(1+e^{-\varepsilon_\nu/T})-\delta_{j,f}\Theta(\lambda)\qquad(j<\nu-1),$$

where $s*$ denotes convolution with the Kernel $s(\lambda)=1/\cosh(\pi\lambda)$, the driving term is given by $\Theta(\lambda)=2D/T\arctan(e^{\pi\lambda})$ and $\varepsilon_0\rightarrow-\infty$. The impurity contribution to the free energy is given by

$$F^{\mathrm{imp}}=-T\int_{-\infty}^{\infty}s\left(\lambda+\frac{\mu}{\vartheta}\right)\ln\{1+\exp[\varepsilon_1(\lambda)/T]\}d\lambda,$$

and, in principle, all thermodynamic quantities can be calculated by taking the derivatives of F^{imp}.

B. Analytical results

Many of the thermodynamic properties can be determined from the asymptotic form of the η_n's. Using the ansatz $\eta_n(\lambda\rightarrow\pm\infty)\approx\eta_n^{\pm}+b_n^{\pm}e^{\mp\pi\tau_\pm\lambda}$ one obtains a set of algebraic equations for the $\eta_n^{\pm}$'s, $b_n^{\pm}$'s, and the exponents $\tau_\pm$. The latter exponents govern the scaling of the free energy in the vicinity of the low- and high-energy fixed points and are given by $\tau_+=4/(2+f)$ and $\tau_-=2\mu/\pi$. The crossover between the two regimes occurs at the Kondo scale Eq. (5), which emerges naturally if one rewrites the thermodynamic BA equations above in a "universal" form by removing the cutoff D.[21,19,22] The asymptotic form of the impurity free energy for $h\ll T\ll T_K$ is given by

$$-F^{\mathrm{imp}}/T\sim\begin{cases}S^{\mathrm{imp}}+\left[a+b\left(\frac{gh}{T}\right)^2\right]\left(\frac{T}{T_K}\right)^{4/2+f} & f>2,\\ S^{\mathrm{imp}}+\left[a+b\left(\frac{gh}{T}\right)^2\right]\left[\frac{T}{T_K}\ln\left(\frac{T}{T_K}\right)\right] & f=2,\end{cases}$$

implying the divergence of the linear specific heat coefficient c/T at $h=0$ and the susceptibility as $T\rightarrow0$. The constants a and b above depend on the specific value of f and μ, and the residual entropy S^{imp} is the same as in the isotropic case,[17]

$$S^{\rm imp}=\ln\left(\frac{\sin\left(\frac{f\pi}{f+2}\right)}{\sin\left(\frac{\pi}{f+2}\right)}\right). \tag{12}$$

To determine the renormalization of the g factor and the Wilson ratio, we calculated the linear specific-heat coefficient and the bulk magnetization $M_z^{\rm tot}\equiv-\partial F^{\rm tot}/\partial h$ in the absence of the impurity spin (but with $g'=g$). Similar to the case $f=1$,[12] the linear specific-heat coefficient agrees with that of the spin sector of noninteracting free electrons. However, the magnetization does not. Following similar lines as in Ref. 12, the total spin can be related to $\eta_\nu^\pm$ and $\eta_{\nu-1}^\pm$ and thus the magnetization is simply given by

$$M^z=-\frac{\partial}{\partial h}F^{\rm tot}=g\left\langle\sum_{i=1}^{N_e}S_i^z\right\rangle=g^2\frac{fL}{4\pi}\frac{h}{1-f\mu/\pi}, \tag{13}$$

with the term $g^2fL/4\pi$, the Pauli susceptibility of free electrons.

From this equation it immediately follows by integration that at zero temperature

$$F^{\rm tot}=-\frac{1}{2}g^2\frac{fL}{4\pi}\frac{h^2}{1-f\mu/\pi}. \tag{14}$$

Thus the g factor is *renormalized* due to the electron-electron interaction as $g\rightarrow g/(1-\mu f/\pi)^{1/2}$, and to compensate the effect of Eq. (2), we have to choose

$$g\equiv(1-\mu f/\pi)^{1/2}.$$

Having thus compensated the effect of the artificial electron-electron interaction of Eq. (2) by rescaling g, we can proceed to calculate the impurity contribution to the *global* susceptibility (defined with $g'=1$, $g=1$ but *no* electron-electron interaction). We find indeed that with the choice $g\equiv(1-\mu f/\pi)^{1/2}$ the low-temperature (global) Wilson ratio, defined in terms of this global impurity susceptibility, takes on a universal value

$$R_{\rm glob}^{\rm imp}=\lim_{T\rightarrow0}\lim_{h\rightarrow0}\frac{c^{\rm bulk}}{\chi^{\rm bulk}}\frac{\chi_{\rm glob}^{\rm imp}}{c_{\rm glob}^{\rm imp}}=\frac{8}{3}, \tag{15}$$

as in the isotropic case,[23] proving again that exchange anisotropy is irrelevant at the two-channel Kondo fixed point.[24–26] In the following we shall always denote quantities that were calculated without (i.e., compensating) the artificial electron-electron interaction by the superscript "imp."

To capture the meaning of the parameter μ we also determined the impurity contribution to the *global* susceptibility in the high-temperature regime:

$$\chi_{\rm glob}^{\rm imp}=\frac{g^2}{4T}=\frac{1-f\mu/\pi}{4T}. \tag{16}$$

Using Abelian Bosonization techniques we were also able to prove analytically that for $J_\perp\ll J_z$ at high temperatures,

$$\chi_{\rm glob}^{\rm imp}(T\rightarrow\infty)=\left(1-2f\frac{\delta}{\pi}\right)^2\frac{1}{4T}, \tag{17}$$

with δ the phase shift generated by J_z (see the Appendix for the derivation and the precise definition of the phase shift).[27,25]

This immediately implies the important relation

$$\frac{\mu}{\pi}=4\frac{\delta}{\pi}-4f\frac{\delta^2}{\pi^2}. \tag{18}$$

Comparing this expression with the results of Ref. 27 we notice immediately that μ/π is nothing but the *scaling dimension* of $J_\perp$, which satisfies the following scaling equation at energy scales ω well above T_K:[27]

$$\frac{d\ln J_\perp}{d\ln(\omega_0/\omega)}=\frac{\mu}{\pi},\quad(\omega\gg T_K). \tag{19}$$

Here ω_0 is a high-energy cutoff. For a TLS it is of the order of the Debye temperature, while for heavy fermions it is usually of the order of the Fermi energy. For a quantum dot the cutoff is the charging energy, $\omega_0\sim E_c$. The effective perpendicular coupling $J_\perp$ at energy scale ω can be obtained by simply integrating this equation.

Equation (18) is further confirmed by noticing that for $f=2$ at the Emery-Kivelson line, $\delta=\pi/4$, the global susceptibility Eq. (16) identically vanishes, in complete agreement with the results of Refs. 28. The point $\delta=\pi/2f$ corresponding to $\mu=\pi/f$ is highly singular, and needs special care:[19] At this particular point the amplitude of the leading irrelevant operator, responsible for the divergence of the susceptibility and the linear specific-heat coefficient, becomes zero.[28]

The global susceptibility $\chi_{\rm glob}^{\rm imp}$ and the associated global Wilson ratio $R_{\rm glob}^{\rm imp}$, defined in Eq. (15), are useful for magnetic Kondo systems. However, for quantum dots and TLS's it is rather the *local* impurity susceptibility that is of interest, i.e., the response of the system to an external field coupled only to the impurity spin. Hence we studied the impurity contribution to the susceptibility when $g=1$ and $g'=0$ (and in the absence of the artificial electron-electron interaction). In order to determine this we generalized the path-integral derivation of Ref. 15 to show that

$$F^{\rm imp}(h,T,g,g'=g)=F^{\rm imp}(\tilde{h},T,g,g'=0), \tag{20}$$

where $\tilde{h}=h(1-2f\delta/\pi)=h(1-\mu f/\pi)^{1/2}$. To establish the second equality we used Eq. (18). Thus we can calculate the local impurity properties (those for $g=1$, $g'=0$ and no electron-electron interaction) from the BA equations with $g'=g=(1-f\mu/\pi)^{1/2}$ by simply rescaling the field h according to Eq. (20).[29]

Henceforth, unless otherwise stated, we shall denote the local impurity susceptibility obtained from the BA solution in combination with Eq. (20) by $\chi^{\rm imp}$ and the associated local Wilson ratio by $R^{\rm imp}$ (see below).

Following the above procedure we find at high temperature

$$\chi^{\rm imp} = \frac{1}{4T}\left[1 - B\left(\frac{T_K}{T}\right)^{2\mu/\pi}\right], \tag{21}$$

$$C^{\rm imp} \sim \left(\frac{T_K}{T}\right)^{2\mu/\pi}. \tag{22}$$

Note that at high temperatures the specific heat exhibits a *power-law* behavior which crosses over to a logarithmic behavior in the isotropic case $\mu \to 0$. Similarly, the corrections to the susceptibility about the free behavior are power-law like and these power laws give way to logarithmic corrections in the isotropic limit $\mu \to 0$. We note that the above exponent $2\mu/\pi$ is formally the same as found for the $f=1$ case.[11] However, the relation between μ and the bare couplings is quite different in the two cases and involves the channel number f [see Eq. (18)].

The power-law corrections can be very easily understood from the scaling picture. Expanding the free energy in terms of $J_\perp$ one finds that the leading correction is second order in $J_\perp$. Making use of the scaling equation (19) it immediately follows that the leading corrections to the free energy behave as $T^{1-2\mu/\pi}$, implying Eq. (22). Similar arguments lead to the conclusion that the impurity-induced resistivity correction behaves at high temperatures as

$$\rho^{\rm imp} = A + B\left(\frac{T_K}{T}\right)^{2\mu/\pi} \tag{23}$$

for $T \gg T_K$.

At low temperatures, $T \ll T_K$, some care is required in discussing thermodynamic properties. In contrast to the case $f=1$, where F is analytical around $T=h=0$,[12] here the $h=0$, $T=0$ point is an essential singularity and the two limits $T\to 0$ and $h \to 0$ are not interchangeable. Taking the limit $h \to 0$ first, we find the following non-Fermi-liquid behavior,

$$\chi^{\rm imp} \sim \frac{1}{T_K}\ln\left(\frac{T}{T_K}\right), \tag{24}$$

$$C^{\rm imp} \sim \frac{T}{T_K}\ln\left(\frac{T}{T_K}\right), \tag{25}$$

$$\tilde{R}^{\rm imp} = \lim_{T\to 0}\lim_{h\to 0}\frac{c^{\rm bulk}}{\chi^{\rm bulk}}\frac{\chi^{\rm imp}}{C^{\rm imp}} = \frac{1}{1-f\mu/\pi}\frac{8}{3}. \tag{26}$$

Here, we have defined a local Wilson ratio $\tilde{R}^{\rm imp}$. It is expressed in terms of the local impurity susceptibility $\chi^{\rm imp}$, and the impurity contribution to the specific heat $c^{\rm imp}$. It differs from the Wilson ratio defined in Eq. (15) (which is the usual definition for this quantity in the magnetic Kondo problem) by having $g'=0$. We see that this local Wilson ratio depends on anisotropy μ and thus the phase shift δ, and is therefore *not* universal.

We now consider the limit of $T\to 0$, with h remaining either finite, or taken to zero subsequently. In this case, and for $h \ll T_K$, we find from the numerical results of the next section that there is a low-energy scale $T_{\rm FL} = h^2/T_K$, below which the thermodynamics is Fermi-liquid like. In particular for $T \ll T_{\rm FL}$ and small magnetic fields $h \ll T_K$ we find

$$\chi^{\rm imp}(h, T \ll T_{\rm FL}) \sim -\ln(h/T_K), \tag{27}$$

$$C^{\rm imp}(h, T\ll T_{\rm FL}) \sim \frac{T}{T_{\rm FL}}, \tag{28}$$

with $T_{\rm FL} \sim h^2/T_K$ (for $h \ll T_K$). A local Wilson ratio for arbitrary local magnetic field h can be defined as

$$R^{\rm imp}(h) = \lim_{T\to 0}\frac{c^{\rm bulk}(h)}{\chi^{\rm bulk}(h)}\frac{\chi^{\rm imp}(h)}{C^{\rm imp}(h)}. \tag{29}$$

In contrast to the $f=1$ case, for which this quantity is independent of h, but dependent on anisotropy [being given by $R^{\mathrm{imp},\, f=1}(h) = 2/(1-\mu/\pi)$[11,31]], for the present $f=2$ case it depends explicitly on *both* h and anisotropy (Fig. 6). This important result, which is consistent with the result for the isotropic case,[32] will be discussed in the following section on the numerical solution. We note here, however, that the local Wilson ratio for the $f=2$ case agrees with the $f=1$ local Wilson ratio in the case of asymptotically large magnetic fields, $h \gg T_K$, i.e., in this case we have

$$R^{\rm imp}(h \gg T_K) = \frac{2}{1-\dfrac{\mu}{\pi}}, \tag{30}$$

although the meaning of μ is different in the two cases [see Eq. (18)]. The detailed dependence of this local Wilson ratio on h will be discussed in the next section.

C. Numerical solution

In order to obtain the thermodynamics at all temperatures, it was necessary to solve the thermodynamic BA equations of Sec. IV A numerically. A procedure for doing this, which is valid for arbitrary values of the magnetic field h and temperature has been developed in Ref. 11 for $f=1$. With small modifications, the same procedure applies also to the present case. We considered anisotropies given by $\mu/\pi = 1/\nu$ with $\nu = 3, 4, 5, 6$. In Figs. 1(a)–(c) we show the thermodynamics of the anisotropic 2CKM for a large anisotropy ($\nu=3$) as one might have in a realistic system. The characteristic non-Fermi-liquid behavior, in particular the ln(2)/2 entropy[17] and the logarithmically divergent $\chi^{\rm imp}(T)$ and $C^{\rm imp}(T)/T$, are found at zero field. A finite field, $h>0$, restores Fermi-liquid behavior at temperatures below a low-energy scale $T_{\rm FL} = h^2/T_K$, as found for the isotropic, $\nu=\infty$, case.[32] The non-Fermi-liquid behavior for $0<h<T_K$ is therefore restricted to an intermediate range of temperatures, $T_{\rm FL}<T<T_K$, and we see that a clear signature of such behavior [such as the two peaks in the specific heat with each peak having only ln(2)/2 entropy, or a ln(T) behavior of $\chi(T)$ for a temperature range below T_K] is possible, even at moderate magnetic fields, $h \sim T_K/16$.

In Figs. 2(a)–(c) and Figs. 3(a)–(c) we show the effect of different anisotropies on the thermodynamics for both a small external field ($h \ll T_K$, Fig. 2) and a large magnetic

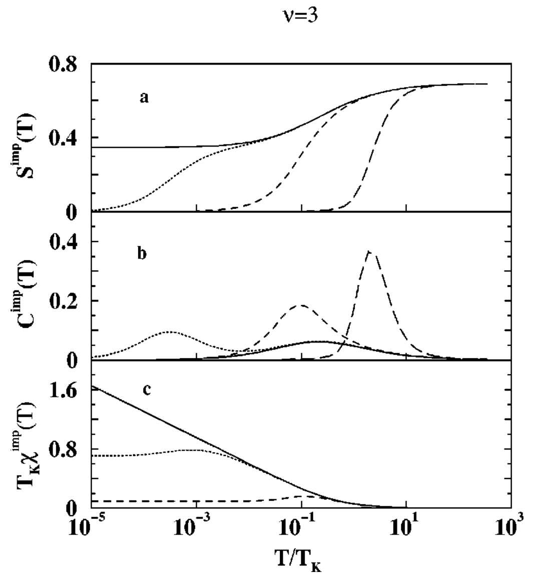

FIG. 1. The impurity contribution to the entropy $S^{\rm imp}$, specific heat $C^{\rm imp}$, and (local) susceptibility $\chi^{\rm imp}$, as functions of T/T_K for magnetic fields $h/T_K=2^{-6}$ (solid), $h/T_K=2^{-4}$ (dotted), $h/T_K=1$ (dashed), and $h/T_K=2^{4}$ (long dashed) for the case $\nu=3$ corresponding to the largest anisotropy studied.

field ($h\gg T_K$, Fig. 3). At $h\ll T_K$, the main effect of anisotropy is to modify the thermodynamics at intermediate, $T_{\rm FL}\lesssim T\lesssim T_K$, and high temperatures, $T>T_K$. For large magnetic field, $h\gg T_K$, the anisotropy modifies the thermodynamic properties at temperatures, $T\gtrsim T_K$ (Fig. 3).

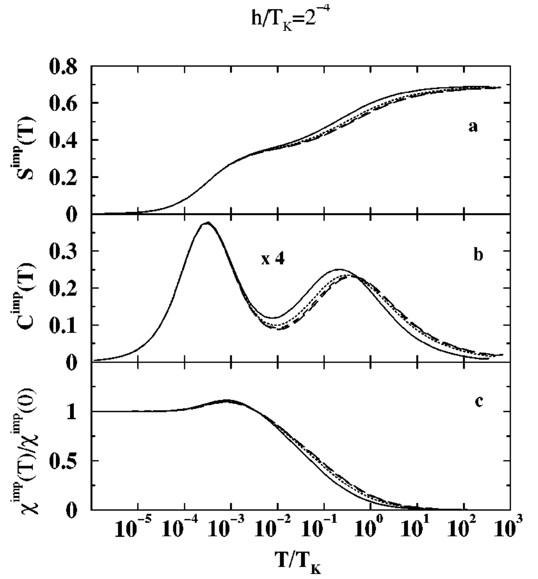

FIG. 2. The anisotropy dependence of the entropy, specific heat, and susceptibility at small magnetic fields ($h/T_K=2^{-4}$) [note that (b) is scaled by a factor 4 for comparison with the corresponding case of large magnetic fields shown in Fig. 3]. The anisotropies shown are for $\mu/\pi=1/\nu$ with $\nu=3$ (solid), $\nu=4$ (dotted), $\nu=5$ (dashed), and $\nu=6$ (long dashed).

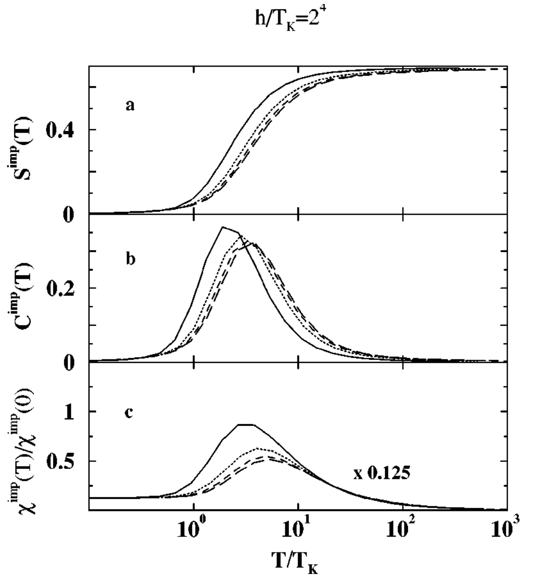

FIG. 3. The anisotropy dependence of the entropy, specific heat, and susceptibility at large magnetic fields ($h/T_K=2^{4}$) [note that (c) is scaled by a factor 0.125 for comparison with the corresponding case for small magnetic fields shown in Fig. 2]. The anisotropies shown are for $\mu/\pi=1/\nu$ with $\nu=3$ (solid), $\nu=4$ (dotted), $\nu=5$ (dashed), and $\nu=6$ (long dashed).

A characteristic feature of the present model ($f=2$), is that, similar to the case dissipative two state systems,[11] it has *power-law* corrections in its thermodynamics at high temperatures ($T\gg T_K$). The exponents are uniquely related to the anisotropy parameter μ [cf. Eq. (22)]. This is in contrast to the corresponding isotropic models which have logarithmic corrections at high temperatures. Figures 4 and 5 show this for the susceptibility and specific heat, respectively.

Figure 6 shows the remarkable magnetic-field dependence of the local Wilson ratio $R^{\rm imp}(h)$. This magnetic-field dependence is consistent with the result for the isotropic case $\mu\to 0$.[32] It is quite unexpected for the following reasons.

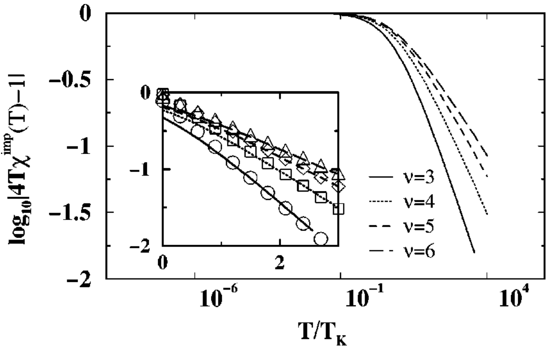

FIG. 4. The susceptibility at high temperatures, $T\gg T_K$, for anisotropies $\mu/\pi=1/\nu$ and ν=3, 4, 5, 6. The impurity susceptibility at $T\gg T_K$ has the form $\chi^{\rm imp}(T)=(1/T)[1/4-B(T_K/T)^{2\mu/\pi}]$, i.e., the corrections to the free behavior are power laws with exponents $2\mu/\pi=2/3$, 1/2, 2/5, 1/3 for $\nu=3$, 4, 5, 6. This is illustrated in the inset, which shows $\log_{10}4[T\chi^{\rm imp}(T)-1]$ versus $\log_{10}(T/T_K)$. Straight lines with slopes $-2\mu/\pi=-1/3$, $-2/5$, $-1/2$, $-2/3$ for ν=3, 4, 5, 6 are indicated by symbols and are well reproduced by the numerical results.

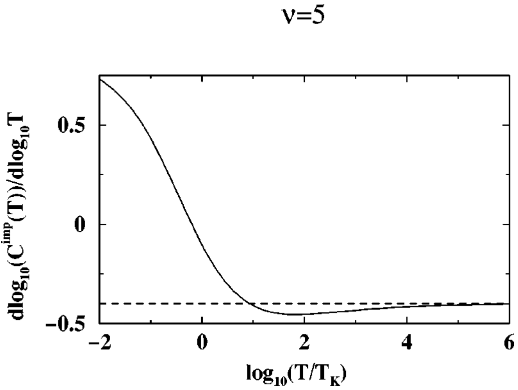

FIG. 5. The specific heat at high temperatures, $T \gg T_K$, shows power-law behavior $C^{\mathrm{imp}}(T) \sim (T_K/T)^{2\mu/\pi}$ with $\mu/\pi = 1/\nu$. The logarithmic derivative of this, $d \log_{10} C^{\mathrm{imp}}(T)/d \log_{10} T$, (solid line) approaches $-2\mu/\pi$ at $T \gg T_K$ and is shown here for $\nu=5$ ($\mu/\pi=1/5$). It is seen to approach $-2\mu/\pi = -2/5$ (dashed line) at high temperatures.

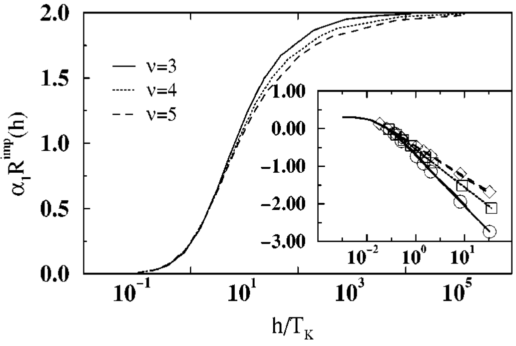

FIG. 6. The magnetic-field (h) dependence of the (local) Wilson ratio $R^{\mathrm{imp}}(h)$ for different anisotropies $\mu/\pi = 1/\nu$ [$\nu=3$ (solid), $\nu=4$ (dotted), $\nu=5$ (dashed)]. At $h \gg T_K$, $\alpha_1 R^{\mathrm{imp}}(h) = 2 - a(T_K/h)^{2\mu/\pi}$. This is illustrated in the inset, which shows $\log_{10}[2 - R^{\mathrm{imp}}(h)]$ versus $\log_{10}(h/T_K)$. Straight lines with slopes $-2\mu/\pi = -2/3$, $-1/2$, $-2/5$, for $\nu=3$, 4, 5, are indicated by symbols and are well reproduced by the numerical results.

First, the corresponding local Wilson ratio for the $f=1$ case (discussed in Refs. 12,11 and 33) is independent of the magnetic field and is given by

$$R^{\mathrm{imp},\, f=1} = \frac{2}{\alpha_1} = \frac{2}{(1-\mu/\pi)}.$$

It depends only on the anisotropy μ (that corresponds to the dissipation strength α_1 in the equivalent dissipative two-state system[11]). In contrast, for the present case ($f=2$), the local Wilson ratio depends both on the anisotropy μ and on the magnetic field h. Thus, even though the regime $T \ll T_{\mathrm{FL}}$ describes a Fermi liquid, in the sense of Eq. (29), the Fermi-liquid state appears different to that for the $f=1$ case. We note that the local Wilson ratio for $f=2$ deviates increasingly from the usual $f=1$ Fermi-liquid Wilson ratio with decreasing h, i.e., as the range over which non-Fermi-liquid behavior dominates increases. The numerical results for $R^{\mathrm{imp}}(h)$ in Fig. 6 indicate that $R^{\mathrm{imp}}(h)$ vanishes as $h \to 0$. This result is consistent with the numerical analysis of the $h \to 0$ estimates of the susceptibility and specific heat in Eq. (28). In contrast, from Eq. (26) $\widetilde{R}^{\mathrm{imp}}(T \to 0, h=0)$ is finite, so we see again that the two limits $T \to 0$, $h=0$ and $T=0$, $h \to 0$ cannot be exchanged. The inset to Fig. 6 shows that $R^{\mathrm{imp}}(h)$ exhibits power-law behavior at large magnetic fields. For $h \gg T_K$ we find

$$\alpha_1 R^{\mathrm{imp}} = 2 - a(T_K/h)^{2\mu/\pi}.$$

Finally, for completeness, we show in Fig. 7 the impurity magnetization (or polarizability) in the presence of a local field, $M_z(h) = \langle S_0^z \rangle(h)$. For each anisotropy μ, $M_z(h)$ is a universal function of h/T_K [the same holds for $R^{\mathrm{imp}}(h)$]. The approach of $M_z(h)$ to the free value for $h \gg T_K$ depends on anisotropy and is found numerically to behave like

$$M_z(h) = 1/2 - b(T_K/h)^{2\mu/\pi}.$$

We expect that in the isotropic limit $\mu \to 0$ the above power laws will give way to logarithmic corrections in the same way that the power-law corrections to the high-temperature thermodynamics gave rise to logarithmic corrections in this limit.

V. CONCLUSIONS

In summary, we presented a detailed analysis of the thermodynamics of spin-anisotropic two-channel Kondo model by using the Bethe-ansatz technique combined with Bosonization and renormalization-group arguments, and discussed quantitatively the role of the anisotropy and the magnetic field.

We showed that at high temperatures the thermodynamics is very different from that of the isotropic model: The local impurity susceptibility is essentially free-impurity like, however, the coefficient of the global susceptibility is nonuniversal, and is related to the phase shift δ generated by the coupling J_z. In particular, the global susceptibility vanishes at the Emery-Kivelson line, $\delta = \pi/2f$. More interestingly, the impurity specific heat (and the corrections to the susceptibil-

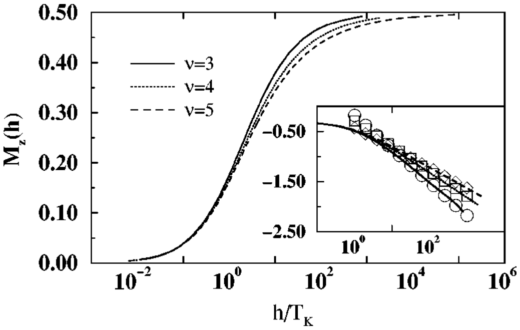

FIG. 7. The magnetic-field (h) dependence of the impurity magnetization $M_z(h, T=0)$ for different anisotropies $\mu/\pi = 1/\nu$ [$\nu=3$ (solid), $\nu=4$ (dotted), $\nu=5$ (dashed)]. At $h \gg T_K$, $M_z(h) = 1/2 - b(T_K/h)^{2\mu/\pi}$. This is illustrated in the inset, which shows $\log_{10}[1/2 - M_z(h)]$ versus $\log_{10}(h/T_K)$. Straight lines with slopes $-2\mu/\pi = -2/3$, $-1/2$, $-2/5$, for $\nu=3$, 4, 5, are indicated by symbols and are well reproduced by the numerical results.

ity about the free behavior) exhibits a power-law behavior at high temperature,

$$C^{\mathrm{imp}}(T \gg T_K, h) \sim \left(\frac{T_K}{T}\right)^{2\mu/\pi}. \tag{31}$$

The anomalous exponent μ is the anisotropy parameter in the Bethe ansatz. We have shown that μ/π is just the anomalous scaling exponent of the spin-flip term $J_\perp$ at high temperature and that it is related to the phase shifts generated by J_z through[27]

$$\frac{\mu}{\pi} = 4\frac{\delta}{\pi} - 4f\frac{\delta^2}{\pi^2}. \tag{32}$$

On general scaling arguments,[2,27] a similar power-law dependence is expected to appear in the impurity resistivity. For $f=2$ we find

$$\rho^{\mathrm{imp}}(T \gg T_K) \sim \left(\frac{T_K}{T}\right)^{2\mu/\pi}. \tag{33}$$

rather than a simple logarithmic scaling.

For $h=0$, and for temperatures below the Kondo temperature, the thermodynamics is governed by the isotropic 2CK fixed point and most of the thermodynamic properties resemble very much those of the fully isotropic model, even for strong anisotropies.

For finite h we showed that the non-Fermi-liquid behavior found for $h=0$ persists for an intermediate region of temperatures $T_{\mathrm{FL}} < T < T_K$, provided $h < T_K$ so that the new scale $T_{\mathrm{FL}} \sim h^2/T_K$ is well below T_K, just as in the isotropic case.[32] We also showed that the Fermi-liquid behavior below T_{FL} is unusual in that the Wilson ratio, $R^{\mathrm{imp}}(h)$, depended very sensitively on the magnetic field h (in contrast to the $f=1$ case, which is completely independent of h) and we calculated the detailed dependence of this quantity for several anisotropies.

While here we focused our attention to specific values of the anisotropy, our calculations can be easily generalized to other anisotropy values and serve as a basis for any interpolation necessary for cases where there is wide distribution of anisotropies present.

ACKNOWLEDGMENTS

The authors are grateful to A. M. Tsvelik and A. Zawadowski for useful discussions. This research has been supported by the Hungarian Grant Nos. OTKA T026327, OTKA F030041, and OTKA T029813, and NSF Grant No. DMR 99-81283. The hospitality of the Institut-Laue Langevin during a visit of G.Z. is gratefully acknowledged.

APPENDIX: DERIVATION OF EQ. (17)

To prove Eq. (17) let us consider the limit $J_\perp \to 0$ of Eq. (1). In this case the interaction part of the Hamiltonian becomes

$$H_{\mathrm{int}} = \frac{J_z}{2} \sum_{j=1}^{f} [\psi^\dagger_{\uparrow,j}(0)\psi_{\uparrow,j}(0) - \psi^\dagger_{\downarrow,j}(0)\psi_{\downarrow,j}(0)] S_0^z, \tag{A1}$$

where the electronic field operator $\psi^\dagger_{\alpha,j}$ creates chiral (right-moving) electron with spin $\alpha = \{+,-\} = \{\uparrow,\downarrow\}$ and channel index $j = \{1, \dots, f\}$. Thus the interaction simply produces a spin-dependent potential scattering, and gives rise to a phase shift δ,

$$\psi_{\alpha,j}(x=0^+) = \psi_{\alpha,j}(x=0^-) e^{-4i\alpha\delta S_0^z}. \tag{A2}$$

In general, the connection between J_z and δ depends on the particular cutoff scheme used except for the small coupling limit, $J_z \ll 1$. To be specific, here we shall use Abelian Bosonization on a system of finite size L and the cutoff scheme associated with it.[30] In the Bosonization procedure we rewrite the Hamiltonian as $(g = g' = 1)$:

$$\begin{aligned} H = {} & \sum_{\alpha,j} \int \frac{dx}{4\pi} (\partial_x \Phi_{\alpha,j})^2 + \frac{2\pi}{L}\frac{1}{2}\sum_{j,\alpha} N^2_{\alpha,j} \\ & + \frac{J_z}{2} S_0^z \sum_{\alpha,j} \left(\frac{\alpha}{2\pi}\partial_x \Phi_{\alpha,j} + \frac{1}{L}\alpha N_{\alpha,j} \right) \\ & + h\left(\sum_{\alpha,j} \frac{\alpha}{2} N_{\alpha,j} + S_0^z \right), \end{aligned} \tag{A3}$$

where the external field h couples to the *total* pseudospin of the system, $N_{\alpha,j}$ denotes the total number of electrons with respect to the ground state with spin α in channel j, and the free Bosonic fields satisfy

$$[\partial_x \Phi_{\alpha,j}(x), \Phi_{\alpha',j'}(x')] = -i2\pi \delta_{jj'} \delta_{\alpha\alpha'} \delta(x-x'). \tag{A4}$$

The original fermion fields can be represented as

$$\psi_{\alpha,j}(x) = \frac{1}{\sqrt{a}} F_{\alpha,j} e^{-i\Phi_{\alpha,j}(x)}, \tag{A5}$$

where $F_{\alpha,j}$ denotes the Klein factor, and a is a small distance cutoff of the order of the lattice spacing.

The phase shift can be most easily calculated by introducing charge and spin fields and quantum numbers:

$$\begin{pmatrix} \Phi_{c,j} \\ \Phi_{s,j} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Phi_{\uparrow,j} \\ \Phi_{\downarrow,j} \end{pmatrix}, \tag{A6}$$

$$N_{c,j} \equiv N_{\uparrow,j} + N_{\downarrow,j}, \tag{A7}$$

$$N_{s,j} \equiv \frac{1}{2}(N_{\uparrow,j} - N_{\downarrow,j}), \tag{A8}$$

and performing a unitary transformation on the Hamiltonian by $U = e^{i\Sigma_j J_z \Phi_{s,j}(0) S_0^z/(2\pi\sqrt{2})}$, resulting in

$$\partial_x \Phi_{s,j}(x) \to \partial_x \Phi_{s,j}(x) - \frac{J_z}{\sqrt{2}} S_0^z \delta(x), \tag{A9}$$

and the ‘‘noninteracting’’ Hamiltonian:

$$H=H_0+H_s, \tag{A10}$$

$$H_0=\sum_{j=1}^{f}\sum_{\mu=c,s}\int\frac{dx}{4\pi}(\partial_x\Phi_{\mu,j})^2+\frac{2\pi}{L}\frac{1}{4}\sum_j N_{c,j}^2, \tag{A11}$$

$$H_s=hS_0^z+\sum_j\left(hN_{s,j}+\frac{2\pi}{L}N_{s,j}^2+J_z\frac{N_{s,j}}{L}S_0^z\right). \tag{A12}$$

From Eqs. (A9), (A6), (A5), and (A2) immediately follows that in the Bosonization cutoff scheme simply

$$\delta=\frac{J_z}{8}. \tag{A13}$$

To prove Eq. (17) we observe that the external field only appears in Eq. (A12). Therefore the partition function factorizes as

$$Z(\beta)=Z_0(\beta)\times Z_s(\beta,h),$$

$$Z_s(\beta,h)=\sum_{N_s}\sum_{S_0^z=\pm 1/2}e^{-\beta H_s}.$$

It is easy to evaluate the sum above in the $L\to\infty$ limit giving

$$Z_s\sim e^{\beta f(L/2\pi)(1/4)h^2}\times\cosh\left[\frac{\beta h}{2}\left(1-f\frac{J_z}{4\pi}\right)\right]. \tag{A14}$$

The first term just generates the Pauli susceptibility of a free-electron gas, while the second corresponds to a free spin coupled to a renormalized magnetic field and gives a Curie susceptibility:

$$\chi_{\rm glob}^{\rm TLS}=\frac{(1-fJ_z/4\pi)^2}{4T}. \tag{A15}$$

Together with Eq. (A13), this yields Eq. (17).

As a further test, one can compare this result with the exact relation, Eq. (16) in the small coupling limit. Within the Bethe-ansatz cutoff scheme[12] $\cos(\mu)=\cos(J_z/2)/\cos(J_\perp/2)$, which in the appropriate small coupling limit gives $\mu=J_z/2+\mathcal{O}(J_z^2)=4\delta+\mathcal{O}(\delta^2)$. Substituting this expression into Eq. (16) we indeed recover the exact relation Eq. (17) in linear order in δ.

---

[1] For a review, see D. L. Cox and A. Zawadowski, Adv. Phys. **47**, 599 (1998).

[2] K. Vladár and A. Zawadowski, Phys. Rev. B **28**, 1564 (1983); **28**, 1582 (1983); **28**, 1596 (1983).

[3] I. L. Aleiner, B. L. Altshuler, Y. M. Galperin, and T. A. Shutenko, Phys. Rev. Lett. **86**, 2629 (2001).

[4] I. Smolyarenko and N. S. Wingreen, Phys. Rev. B **60**, 9675 (1999).

[5] D. C. Ralph, A. W. W. Ludwig, Jan von Delft, and R. A. Buhrman, Phys. Rev. Lett. **72**, 1064 (1994).

[6] S. K. Upadhyay, R. N. Louie, and R. A. Buhrman, Phys. Rev. B **56**, 12 033 (1997).

[7] G. Zaránd, Jan von Delft, and A. Zawadowski, Phys. Rev. Lett. **80**, 1353 (1998).

[8] L. I. Glazman and K. A. Matveev, Zh. Eksp. Teor. Fiz. **98**, 1834 (1970) [Sov. Phys. JETP **71**, 1031 (1990)].

[9] G. Zaránd, G. T. Zimányi, and F. Wilhelm, Phys. Rev. B **62**, 8137 (2000).

[10] D. Berman, N. B. Zhetinev, and R. C. Ashoori, Phys. Rev. Lett. **82**, 161 (1999).

[11] T. A. Costi and G. Zaránd, Phys. Rev. B **59**, 12 398 (1999).

[12] A. M. Tsvelik and P. B. Wiegmann, Adv. Phys. **32**, 453 (1983).

[13] I. Affleck and A. W. W. Ludwig, Phys. Rev. B **48**, 7297 (1993).

[14] J. H. Lowenstein, Phys. Rev. **29**, 4120 (1984).

[15] P. B. Vigmann and A. M. Finkel’steĭn, Zh. Eksp. Teor. Fiz. **75**, 204 (1978) [Sov. Phys. JETP **48**, 102 (1978)].

[16] N. Andrei and A. Jerez, Phys. Rev. Lett. **74**, 4507 (1995).

[17] N. Andrei and C. Destri, Phys. Rev. Lett. **52**, 364 (1984).

[18] M. Takahashi and M. Suzuki, Prog. Theor. Phys. **48**, 2187 (1972).

[19] A. M. Tsvelik, Phys. Rev. B **52**, 4366 (1995).

[20] V. J. Emery and S. A. Kivelson, in *Fundamental Problems in Statistical Mechanics VIII*, edited by H. van Beijeren and M. H. Ernst (Elsevier, Amsterdam, 1994).

[21] N. Andrei, in *Series on Modern Condensed Matter Physics—Vol. 6*, Lecture Notes of ICTP Summer Course, September 1992, edited by S. Lundquist, G. Morandi, and Yu Lu (World Scientific, Singapore, 1993), pp. 458–551.

[22] To obtain the prefactor in Eq. (5) one has to study the Toulouse limit of Ref. 19.

[23] $c_{\rm bulk}$ in Eq. (15) denotes the *total* specific heat of the electron gas, which cannot be obtained from the fused BA equations, that capture only contributions from the $\mathrm{SU}(2)_f$ spin sector (see, e.g., Ref. 34).

[24] H.-B. Pang and D. L. Cox, Phys. Rev. B **44**, 9454 (1991).

[25] G. Zaránd and J. von Delft, Phys. Rev. B **61**, 6918 (2000).

[26] J. W. Ye, Phys. Rev. Lett. **77**, 3224 (1996).

[27] K. Vladár, A. Zawadowski, and G. T. Zimànyi, Phys. Rev. B **37**, 2001 (1988); **37**, 2015 (1988).

[28] V. J. Emery and S. Kivelson, Phys. Rev. B **46**, 10 812 (1992); A. M. Sengupta and A. Georges, *ibid.* **49**, 10 020 (1994).

[29] By Eq. (18) the rescaling of g exactly compensates the rescaling of h, and therefore the BA equations with $g=1$ and $g'=1$ determine directly the impurity free energy.

[30] Jan von Delft and Herbert Schoeller, Ann. Phys. (Leipzig) **7**, 225 (1998).

[31] Note, however, that the corresponding global impurity Wilson ratio for the $f=1$ case is independent of both the magnetic field and the anisotropy, and takes the usual value $R_{\rm glob}^{{\rm imp},f=1}=2$ (Refs. 12 and 11).

[32] P. D. Sacramento and P. Schlottman, Phys. Rev. B **43**, 13 294 (1991).

[33] M. Sassetti and U. Weiss, Phys. Rev. Lett. **65**, 2262 (1990).

[34] N. Andrei, M. Douglas, and A. Jerez, Phys. Rev. B **58**, 7619 (1998).