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Synchronization of overdamped Josephson junctions shunted by a superconducting resonator

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Synchronization was investigated numerically as well as by the method of slowly varying amplitudes for two Josephson junctions with McCumber parameters near 1 shunted by a superconductor and a dividing capacitor building up a resonator for ac currents. Because of the current resonance in the system the synchronizing ac current increases and provides phase locking up to 15% spread of critical currents. Thresholds of the phase-locked state at small as well as at large values of system parameters were explained in the developed model. Conditions of forming the phase-locked state are investigated in the many-junction array in which ac currents between neighbor junctions are formed through the superconducting shunt with a divided capacitance. It is shown that at certain conditions the total phase locking of all junctions switched into the voltage state appears when their quantity exceeds some critical value.

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I. INTRODUCTION

The mutual phase locking phenomena in arrays of Josephson junctions have been intensively studied as theoretically as experimentally with regard to their applications as tunable submillimeter oscillators.[1–10] It was shown that linear arrays of junctions connected by the feedback load are convenient objects for synchronization and satisfy all needed conditions.[1] The feedback must provide an ac current which is large enough to synchronize junctions with some spread of critical currents. For example, the load consisted of a normal resistance and an inductance is able to synchronize radiation of junctions with 8–10% spread of critical currents.[8] However, it is hard to obtain small values of spread in experiments.[9] There are several solutions of the feedback problem which lead to phase locking at higher spreads of critical currents. One of them is the use of Josephson junctions in the feedback superconducting loop (so-called many-junction superconducting quantum interference device (SQUID) or multijunction superconducting loop).[10,11] It was shown[11,12] that due to the circulating current in the system there is the stable in-phase solution of dynamic equations which can provide the phase locking up to 15% spread of critical currents. Experiments on the systems made of high-temperature superconductors (HTSC's) with values of McCumber parameters of junctions near 1 proved the ability of such a feedback to provide synchronization of all junctions in the circuit.[12] However, there is a strong dependence of the interval of currents in which the phase-locked state is observed on the external magnetic field.[10] Another type of feedback was applied for the system of junctions with high values of McCumber parameters.[2,3,13,14] The system was shunted by a superconducting shunt and a dividing capacitance. The common resonance mode for all junctions was obtained by means of such a shunting. The current resonance enhances the amplitude of the common ac current through junctions. Experiments on synchronization of two PbInAu-oxide-PbSn junctions with high values of McCumber parameters showed the validity of the developed model.[13] Accordingly to theoretical predictions (obtained by modification of the method of slowly varying amplitudes for junctions with high McCumber parameter) the phase-locked state can exist in the system with such a feedback up to $\sim$15% spread of critical currents.[14] Recently, there are experiments on synchronization of two-dimensional arrays of underdamped Nb/Al/AlO_x/Nb Josephson tunnel junctions with a superconductor plane above (or under) them.[2,3] Numerical investigations showed that arrays of underdamped junctions can be successfully synchronized in such a scheme.[4]

Because of promising results of both theoretical and experimental investigations of phase locking in systems of underdamped junctions with the common resonance mode, it is necessary to expand investigations to arrays of junctions with small values of McCumber parameters (overdamped junctions) such as SNS junctions (here *S* is for a superconductor and *N* is for a normal metal) or most of the HTSC Josephson junctions. Analytical results obtained in Ref. 14 are not valid in such systems and we apply the method of slowly varying amplitudes (SVA's) for overdamped junctions[1,15] which we expanded for a limited range of capacitances of junctions.[16] In the present paper we report about as analytical as numerical investigations of synchronization of two Josephson junctions with small McCumber parameters ($\beta_c \sim 1$). We discuss a feedback consisting of a superconducting shunt and a dividing capacitance. We show that this feedback being the *sequential* resonance contour forms the *parallel* resonance contour together with the capacitances of junctions in the array and provides the current resonance in the circuit. Changing the parameter of the dividing capacitor and the inductance of the loop formed by the shunt and the array we can tune the quality factor of the parallel resonance contour. Because of the resonance the ac current through the junctions increases and the system can be in the phase-locked state even at large spreads of critical currents. To explain the particularities of the phase locking behavior we apply the SVA method. Using this method we also explain the thresholds of the phase-locked state at small as well as large values of the system parameters.

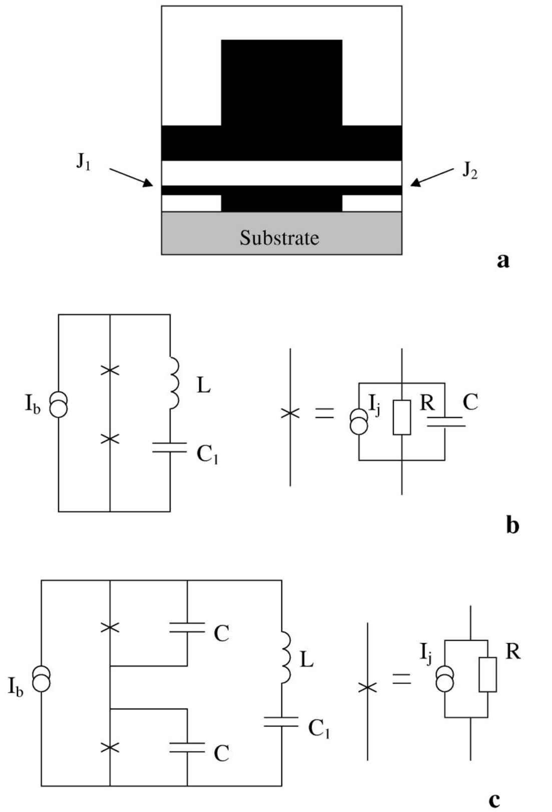

FIG. 1. (a) Scheme of the system of two Josephson junctions (shown by arrows) with a superconducting shunt resonator (black area). (b) The equivalent circuit of the system. (c) The equivalent scheme of the system used within the SVA model.

We apply the SVA method also to the consideration of the quasi-infinite system consisted of a many-junction array and a superconducting shunt (with a dividing capacitance) which is placed close enough to the array to form ac currents between neighbor junctions through the plane. In recent experiments[3] analogous systems were investigated experimentally with two-dimensional arrays of SIS junctions and a threshold of emitted power was found when the part of rows of junctions were switched into the voltage state. We show that in such a system the interaction between junctions decays with the distance from the junctions switched into the voltage state (active junctions). We show that if the distance between active junctions is large then they oscillate antiphase but when the distance decreases to some critical value the active junctions start to oscillate in phase. Because of antiphase oscillations the ac power across the array is almost zero till the concentration of active junctions becomes so large that the distance between them can provide the in-phase synchronization. We show finally that there are configurations of the active junctions which lead to the threshold of the total phase locking of all active junctions. We support our SVA consideration with the numerical calculations of the phase dynamic.

II. THE MODEL

The scheme of one possible kind of such a shunted array is shown in Fig. 1(a). Two Josephson junctions have no Josephson connection with the shunt made of a superconductor film. The layer of insulator between each junction and the shunt is so wide that there are not any supercurrents between the array and the shunt. Because of this condition there is no quantization of magnetic flux inside the loop formed by junctions, insulator layers and the shunt.[1] Layers of insulator have large resistances $R_l \gg R_a$ (R_a is the averaged resistance of the Josephson junction), so we can neglect the normal conductivity of those layers in comparison to the Josephson junctions. Each layer has the capacitance C_2. The loop formed by the array of junctions and the shunt has the inductance L. It is assumed that in this contour the active resistance due to the surface impedance of the superconducting film and the resistivity of the insulator layers is negligible low against the resistance of the single junction (the case in which the active resistance is the same order of magnitude as the resistance of the junction is described in Refs. 1, 16, and 17; in this case the highest spread of critical currents is only 4–5%). The equivalent scheme of the shunted array is shown in Fig. 1(b) (here $C_1 = C_2/2$). Note that due to the resistances of overdamped junctions the whole contour has a low quality factor $Q \leqslant 10$ at reasonable values of inductances and capacitances. The junctions have different critical currents I_{c1} and $I_{c2} I_{c1} = (1+\delta) I_{ca}$ and $I_{c2} = (1-\delta) I_{ca}$, where δ is the spread of critical currents and $I_{ca} = (I_{c1} + I_{c2})/2$ is the averaged critical current. We suppose spread in normal resistances of junctions R_k, however, keeping for simplicity the product $I_{c,k} R_k$ constant for all junctions. The dynamic behavior of the Josephson junctions can be modeled using the resistively shunted junction (RSJ) model.[1] Within this model the dynamic equations are

$$[1-(-1)^k \delta][\beta_C \ddot{\varphi}_k(\tau) + \dot{\varphi}_k(\tau) + \sin(\varphi_k)] = \overline{i} - \dot{q}(\tau),$$

$$\ddot{q} + \frac{1}{\beta_{C1}\beta_L} q = \frac{1}{\beta_L} \sum_{k=1}^{2} \dot{\varphi}_k, \quad k = 1,2, \tag{1}$$

where k is the number of the junction, φ_k is the phase difference across kth junction, $\dot{\varphi}_k$ and $\ddot{\varphi}_k$ are the first and the second derivatives of the phase difference used with respect to dimensionless time $\tau = 2\pi R_a I_{ca} t/\Phi_0$, $\overline{i}$ is the bias current (all small letters here and further denote normalized units (see Ref. 1), q is the charge on the capacitance of the load, impedances are normalized with respect to the averaged resistance of junctions R_a, i_L is the current flowing through the shunt, $\beta_C = 2\pi I_{ca} R_a^2 C/\Phi_0$ is the McCumber parameter for the averaged critical current, Φ_0 is the quantum of magnetic flux, $\beta_{C_1} = 2\pi I_{ca} R_a^2 C_1/\Phi_0$ is the normalized capacitance C_1, and $\beta_L = 2\pi I_{ca} L/\Phi_0$ is the inductance parameter of the system. Voltages across junctions measured in units of $I_a R_a$ are $v_k = \dot{\varphi}_k$.

To investigate the particularities of phase locking, it is more convenient to consider the system by the method of slowly varying amplitudes (SVA).[15,1] With this purpose we can formally attach the capacitance of each junction to the external load [see Fig. 1(c)] and consider the junction as one without capacitance. The kth junction consists of the source of the ac current with the amplitude I_{ck} and the resistance R_k within the approximation of the RSJ model. For simplicity let us consider at first a case of junctions with negligible small difference of critical currents. Because we formally attached the capacitances of junctions to the shunt, the phase dynamics of junctions can be described by the system of first order differential equations[17]

$$\dot{\varphi}_k+\sin(\varphi_k)=\overline{i}+\tilde{i}_k,\quad \tilde{i}_k=\sum_{k'=1}^{2} y_{kk'}v_{k'},\quad k=1,2, \tag{2}$$

where $\tilde{i}_k$ is the ac current flowing through the kth junction in the circuit which is formed with junctions and the load consisting of the shunt impedance and capacitances of both junctions, $y_{kk'}$ are the connection coefficients which are equal to the amplitudes of the ac current in the kth junction when there is one unit of voltage on the k'th junction. We can solve the system (2) of the first order differential equations by means of the SVA to obtain the interval of voltages in which junctions oscillate coherently (the locking interval Δ). The approximations of this method are the small differences between frequencies of junctions generation with respect to their absolute values and small amplitudes of the ac currents $\tilde{i}_k$ in comparison to $\overline{i}$ that let us to consider currents $\tilde{i}_k$ as perturbations and expand the phase differences across junctions in series on the small parameter of the amplitude of this perturbation. Accordingly to the SVA method we consider the first harmonic of the Fourier series of the voltage on the kth junction $v_k=\hat{v}_k+\mathrm{Re}(\varepsilon_1 e^{j\theta_k})$, where the caret represents averaging over fast processes such as the Josephson generation,[1] ε_1 is the Fourier coefficient, and the averaged phases θ_k are determined from the relations $\dot{\theta}_k=\hat{v}_k$. To obtain the values θ_k we have to solve the system of reduced equations[17]

$$\dot{\theta}_k=\overline{v}_k^A+\frac{a}{2}\mathrm{Re}\sum_{k'} y_{kk'}e^{j(\theta_{k'}-\theta_k)},\quad k=1,2, \tag{3}$$

where $\overline{v}_k^A$ is the full averaged voltage of the single noninteracting kth junction, $a=\varepsilon_1 r_d^A/\overline{i}$, $r_d^A=\overline{i}/\overline{v}^A$, and $\overline{v}^A$ is the voltage across the noninteracting junction with the averaged critical current. Solving the system (3) we obtain[1,8]

$$\dot{\eta}=a\,\mathrm{Im}(y_{12})\sin(\eta)+(\overline{v}_1^A-\overline{v}_2^A), \tag{4}$$

where $\eta=\theta_1-\theta_2$. This equation has the solution $\eta=\text{const.}$ which describes synchronization of radiation in the interval

$$\mu\equiv|\overline{v}_1^A-\overline{v}_2^A|=F(\overline{v})|\sin(\eta)|, \tag{5}$$

where $F=a|\,\mathrm{Im}(y_{12})|$ Supposing a small spread of critical currents (so that $\mu\ll\overline{v}$) within the RSJ approximation the relation $\mu=|\sqrt{(I/I_{c1})^2-1}-\sqrt{(I/I_{c2})^2-1}|$ is valid (here I is the bias current). The right side of Eq. (5) contains the dependence $F(\overline{v})$ obtained for the voltage corresponding to the averaged critical current $\overline{v}=\sqrt{(I/I_{ca})^2-1}\equiv\sqrt{\overline{i}^2-1}$. The coefficient a can be written as $a=2/(\overline{v}^2+\sqrt{\overline{v}^2+1})$ in this case.[1] At $\eta=\pm\pi/2$, $|\sin\eta|=1$ Eq. (5) can be solved numerically for the different values of the spread of critical currents δ. The ranges of the phase-locked state Δ at given δ are between the points of the intersection of the curves $F(\overline{v})$ and $\mu(\overline{v})$ [see Figs. 2(a) and 2(b)].

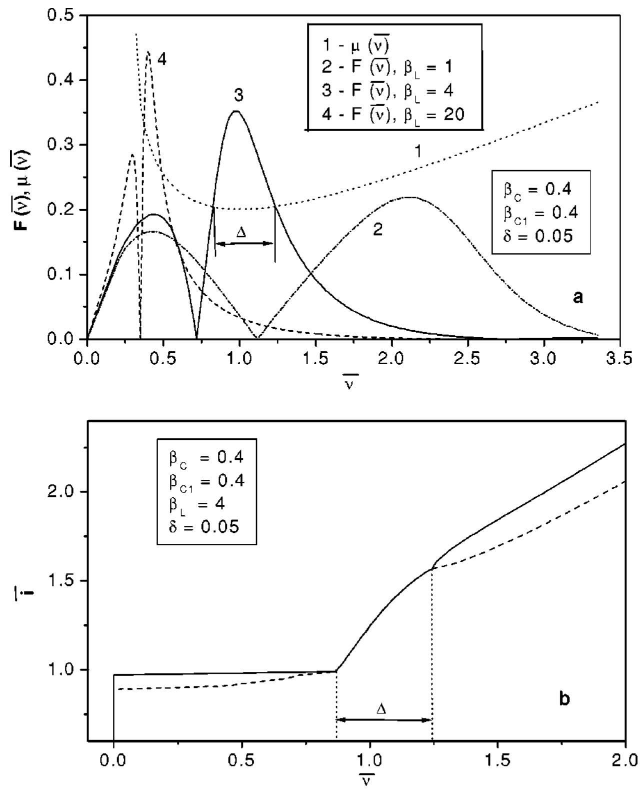

FIG. 2. (a) Dependencies $F(\overline{v})$ at different values of β_L. The dependence $\mu(\overline{v})$ at $\delta=0.05$ is shown by a dotted line. (b) IV characteristics of the two feedbacked junctions $\beta_C=0.4$, $\beta_{C_1}=0.4$, $\beta_L=4$, and $\delta=0.05$.

The limitations of the SVA method for finite values of δ and β_C follow from the suppositions about the small difference of junctions frequencies $\mu\ll\overline{v}$ and the small values of the high frequency part in the first term of the phase difference expansion on the amplitude of the ac current[1,16] $|\tilde{\varphi}|\ll 1$. The numerical evaluations give the values of limitations[16] $\delta\le 0.04$–0.05, $\beta_C=0.1$–0.5. For the values of δ and β_C which exceed these limitations the SVA method is not available and to obtain the locking interval Eq. (1) have to be solved numerically.

III. RESULTS AND DISCUSSION

Using the method of contour currents we found the imaginary part of the coefficients y_{12}:

$$\mathrm{Im}\,y_{12}=\frac{\overline{\nu}^3\beta_C^2\beta_L-\overline{\nu}[\beta_C(2+\alpha)+\beta_L]+\dfrac{\alpha}{\overline{\nu}\beta_C}}{4[\overline{\nu}^2\beta_C\beta_L-(1+\alpha)]^2+\left[\overline{\nu}^3\beta_C^2\beta_L-\overline{\nu}[\beta_C(2+\alpha)+\beta_L]+\dfrac{\alpha}{\overline{\nu}\beta_C}\right]^2}, \tag{6}$$

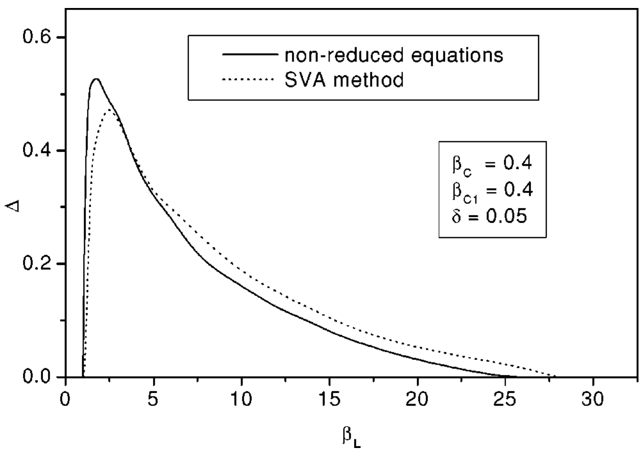

FIG. 3. Dependencies of the locking interval Δ on the parameter β_L. The solid line represents data obtained from the dynamic equations (1), dotted line represents data obtained by the SVA approximation.

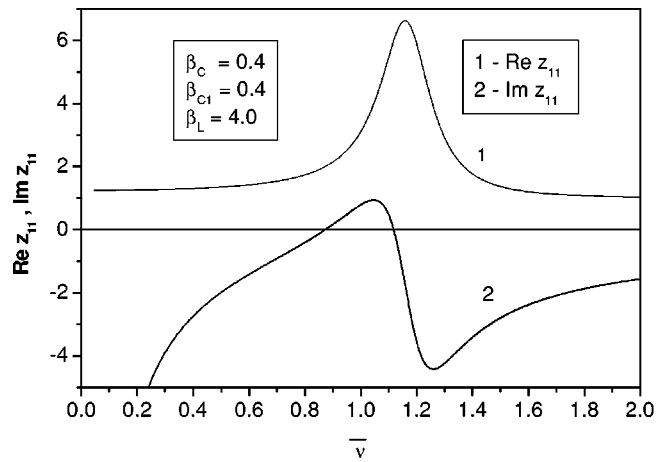

FIG. 4. Dependencies of the real and imaginary parts of the system impedance on the averaged voltage $\overline{v}$.

where $\alpha=\beta_C/\beta_{C_1}$. It was shown[16] that the SVA approximation for the inductance parameter of the loop is valid for $\beta_L \leqslant 15$.

We solved Eq. (5) numerically for $\eta=\pm\pi/2,|\sin\eta|=1$ using the expression (6) for different values of parameters β_C, β_{C_1}, β_L, and δ within the interval of the validity of the SVA method. For the comparison we also calculated locking intervals Δ by means of the numerical solution of dynamic equations (1) using the Runge-Kutta method. For the detailed analyzes of the system by means of the SVA model we choose values of parameters $\beta_C=0.4$, $\beta_{C_1}=0.4$, and $\beta_L=4$ which are in the range of validity of the SVA approximations[16] and allow one to obtain the locking interval near the voltage $\overline{\nu}=1$. Examples of the dependencies $F(\overline{\nu})$ at values of parameters which are close to chosen are shown in Fig. 2(a). From Fig. 3 one can see that the SVA approximation for the $\Delta(\beta_L)$ dependencies are in a good agreement with those obtained by means of the numerical solution of Eqs. (1).

The values of $F(\overline{\nu})$ for the investigated system are as much as fifty percent higher than those obtained for the load consisting of a resistance and an inductance.[16] To explain this enhance let us consider the electrical properties of the system in detail. If there is voltage across one junction then the current circuit corresponds to a parallel resonance contour, one current branch of each is formed by the capacitance of the junction and another branch is formed by the inductance L, capacitance C_1, and the other junction (LC_1J branch). Dependencies of real and imaginary parts of the contour impedance on the frequency (voltage) are shown in Fig. 4. The behavior of Im(z_{11}) shows that there are two resonances in the circuit. The first resonance is the voltage resonance which corresponds to the equality of positive and negative parts of the impedance of the LC_1J branch ($\overline{\nu_1} \approx \sqrt{1/(\beta_L\beta_{C_1}-\beta_C\beta_{C_1})}$ for $\overline{\nu_1}\beta_C \ll 1$; $\overline{\nu_1}\approx 0.83$ in the example of Fig. 4). The second resonance at $\overline{\nu_2}=1.12$ is the current resonance in the parallel contour formed by both the LC_1J branch and the capacitance of the junction. The behavior of the impedance near the resonance frequency $\overline{\nu_2} = \sqrt{(\beta_C+\beta_{C_1})/\beta_L\beta_C\beta_{C_1}}$ ($\overline{\nu_2}=1.12$ for the parameter given in Fig. 4) is typical for a parallel contour with a low quality factor.[18] We can estimate the quality factor Q in Fig. 4 using the expression $\Delta'/\overline{\nu_2}=1/Q$ (here Δ' is an interval of voltages where Re z_{11} decreases on the factor $\sqrt{2}$) and get $Q \approx 6.6$. The current resonance specifies the circulating current in the contour. This circulating current synchronizes the phase dynamics of both junctions. The main maxima of $F(\overline{\nu})$ are disposed at the left side of the resonance voltage $\overline{\nu_2}$ [for example, $\overline{\nu}_{\max}=1.0$ in Fig. 2(a), curve 3]. Such a behavior of $\overline{\nu}_{\max}$ can be explained within the frames of the general consideration of the system behavior near the resonance.[1] It is known[1] that the solutions of the dynamic equation have smaller stability on the right part of the current resonance than on the left part and can become unstable at some values of parameters within the right part of the resonance. The maxima of $F(\overline{\nu})$ dependencies are shifted to low voltages together with a shift of the resonance voltage $\overline{\nu_2}$. Thus, the origin of the $F(\overline{\nu})$ increase is the enhancement of the ac current in the load due to the current resonance.

To prove the stability of the in-phase solution of the dynamic equation (1) in the region of the current resonance we performed the stability analyzes of these in-phase solutions for different values of parameters β_C, β_{C_1}, β_L, and $\delta=0$. The procedure of the stability analysis is described in details elsewhere.[19,12] In this procedure the real parts of Floquet exponents λ_k are obtained for the perturbations of in-phase solutions of the differential Eqs. (1). If both the values of Re λ_k are negative, the in-phase solution is stable and any perturbation of this solution decays. On the contrary, if no or even one of the Re λ_k is positive, the in-phase solution is unstable and any perturbation increases with time. When Re $\lambda_k<0$, the in-phase solution with larger absolute value of Re λ_k is more stable because perturbations decay faster if the absolute value of Re λ_k is large. The dependence of the largest of the two values of Re λ_k on the averaged voltage is shown in Fig. 5. Because the capacitive load (the capacitance of the junctions together with the capacitance C_1) gives a negative slope in the IV characteristic[1] at voltages $\overline{\nu}\leqslant 0.83$,

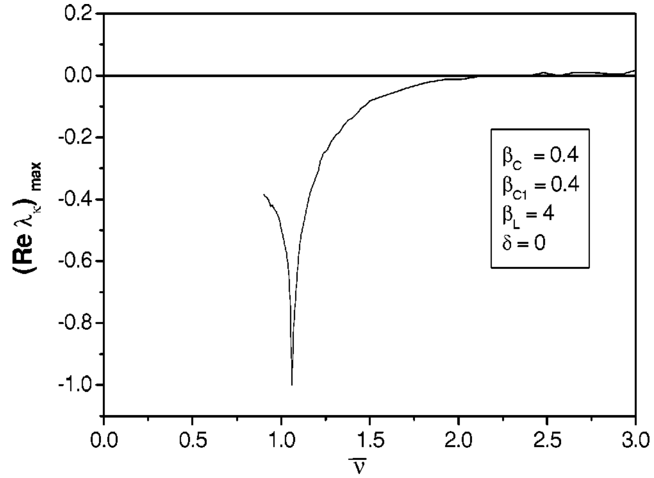

FIG. 5. The dependence of the largest real part of the Floquet exponent on the average voltage $\overline{\upsilon}$.

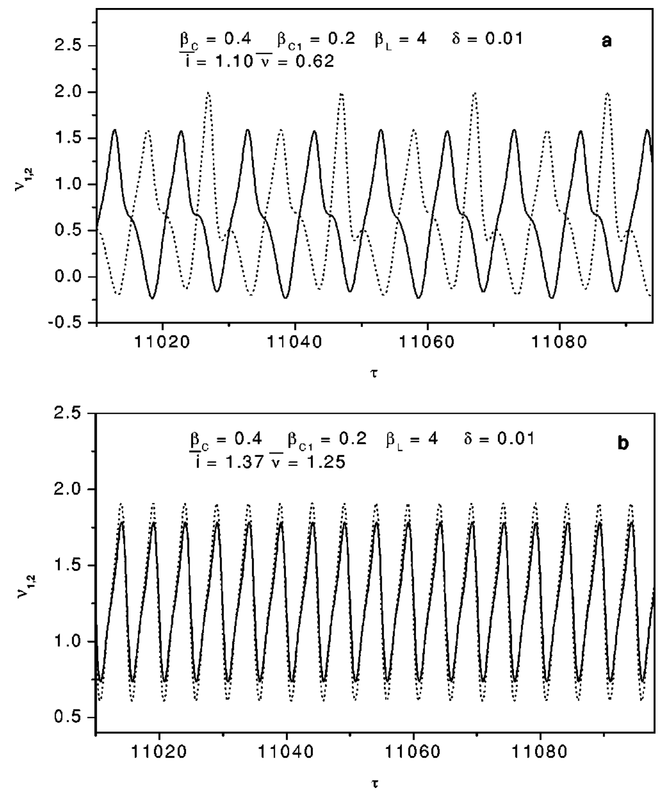

FIG. 6. Time dependencies of voltages across junctions at (a) $\overline{i}=1.10$ and (b) $\overline{i}=1.37$.

there are not any stable solution of dynamic equations in this region and we can not obtain Floquet exponents at $\overline{\nu} \leqslant 0.83$. The *IV* characteristic in this region has a jump from $\overline{\nu}=0$ to $\overline{\nu}\approx 0.83$ [see Fig. 2(b)]. The *IV* characteristics calculated within the SVA approximation [Eq. (3)] have such a jump, too. On the contrary, the function $F(\overline{\nu})$ determines only the amplitude of the locking voltage [but not the *IV* characteristic, see Eqs. (4), (5)] and can be calculated in the whole region of voltages. The voltage which corresponds to the first maximum of $F(\overline{\nu})$ [see Fig. 2(a), curve 3] is placed entirely within the ranges of the above mentioned jump in the *IV* characteristic and therefore this maximum does not manifest itself at the dependence $(\mathrm{Re}\,\lambda_k)_{\max}$ on voltage. The dependence of $(\mathrm{Re}\,\lambda_k)_{\max}$ on the averaged voltage has a deep minimum at the voltage which corresponds to the second $F(\overline{\nu})$ maximum [see Fig. 2(a), curve 3]. Such a behavior signifies the largest stability against perturbations (in connection with the spread of critical currents, for example) at this voltage. We can make a conclusion that the in-phase solutions of dynamic Eqs. (1) are very stable at the main maximum of $F(\overline{\nu})$ dependencies. Note that as far as we know there is no quantitative theory which allows one to connect the stability against perturbations with the particularities of the $F(\overline{\nu})$ function, such as the connection between the minimum of $(\mathrm{Re}\,\lambda_k)_{\max}$ and the maximum of $F(\overline{\nu})$. Another particularity of the $(\mathrm{Re}\,\lambda_k)_{\max}$ dependence consists in the zero crossing of this function from negative to positive within the region $\overline{\nu}\sim 2.3$–2.6. We can see that this region corresponds to the decay of $F(\overline{\nu})$ to almost zero values [Fig. 2(a), curve 3]. Accordingly to the SVA approximation, at these voltages a locking interval Δ exists only at negligible small spreads of critical currents. The stability analysis proves this conclusion because the change of the sign of $(\mathrm{Re}\,\lambda_k)_{\max}$ means the loss of stability of the in-phase solution. Thus, we can state the connection between the behavior of Floquet exponents and the behavior of the function $F(\overline{\nu})$.

The curves for $F(\overline{\nu})$ have two maxima. The first maximum appears at low frequencies and is formed by the positive values of $\mathrm{Im}(y_{12})$. The second maximum at higher frequencies is formed by negative values of $\mathrm{Im}(y_{12})$. Accordingly to the theory of phase locking at positive values of $\mathrm{Im}(y_{12})$ junctions oscillate antiphase and at negative values of $\mathrm{Im}(y_{12})$ junctions oscillate in phase.[1,8] If the parameter δ is small ($\delta < 0.03$–0.04) there are two intervals of the phase-locked state in the system: antiphase locking interval at small voltages and in-phase locking interval at higher voltages. To demonstrate this, we chose the system parameters in such a way that the initial jump of voltages on the *IV* characteristic is small enough to show the behavior of *IV* characteristics at the first maximum of the function $F(\overline{\nu})$ and solved Eqs. (1). Dependencies of the voltages across both junctions on time are shown in Fig. 6 at very small spread $\delta=0.01$ and at bias currents $\overline{i}=1.10$ ($\overline{\nu}=0.62$) and $\overline{i}=1.37$ ($\overline{\nu}=1.25$), the tops of maxima in $F(\overline{\nu})$. Note that at $\delta=0$ both junctions oscillate coherently. One can see that for a spread $\delta=0.01$ at $\overline{i}=1.10$ voltages oscillate antiphase but in phase at $\overline{i}=1.37$. This coincidence of numerical calculations with the SVA approximation proves once more the validity of SVA method. Because the in-phase maximum is higher than that for anti-phase, only one locking interval of in-phase oscillations remains with increasing of spread of critical currents.

The influence of the parameter β_L reveals in increase of the $F(\overline{\nu})$ maxima, while decreasing of the curve width and shifting the maxima to lower voltages. The same behavior of β_L is noted for the system of two Josephson junctions with a shunt consisting of an inductance and a resistance.[16] This behavior predicts the existence of two thresholds of phase locking. The first threshold appears at small β_L when the

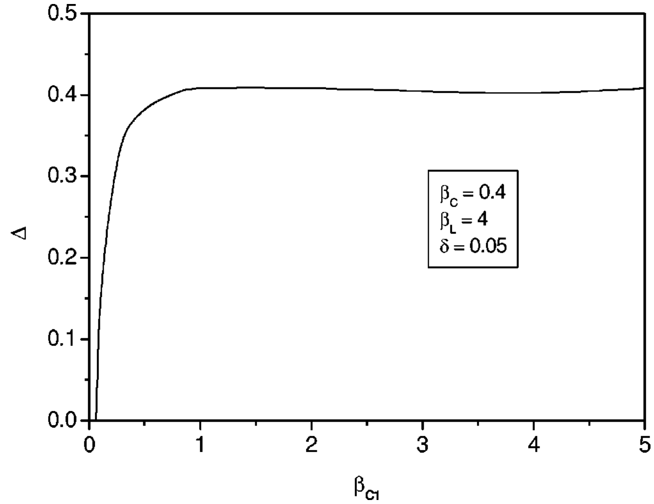

FIG. 7. The dependence of the locking interval Δ on the parameter β_{C_1}.

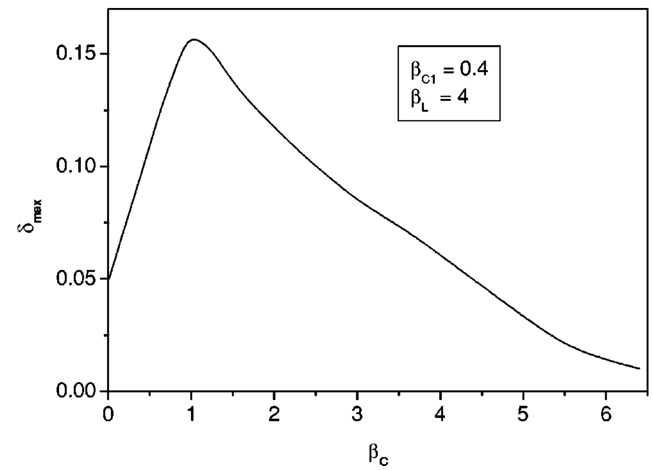

FIG. 8. The dependence of the maximal spread of critical current $\delta_{\max}$ on the junction parameter β_C.

maxima of $F(\overline{\nu})$ become larger than values of $\mu(\overline{\nu})$. The second threshold exists at large β_L when values $\mu(\overline{\nu})$ become so high that the maxima of $F(\overline{\nu})$ appear again beneath them [see Fig. 2(a) and Fig. 3].

The behavior of $F(\overline{\nu})$ with the change of the parameter β_C (within the limitations of the SVA model, i.e., at $\beta_C \leqslant 0.5$) is analogous to that described above. When the parameter β_{C_1} is increased the height of $F(\overline{\nu})$ maxima also increases as well as the shift of them to lower frequencies and the existence of both thresholds of phase locking is also found similar to the previous cases but the width of the $F(\overline{\nu})$ maxima in this case increases with increasing of β_{C_1}. Though the top of $F(\overline{\nu})$ maximum can be shifted beneath the curve $\mu(\overline{\nu})$, the part of the broad maximum remains under that curve. This leads to the very slow decay of the locking interval Δ with increase of β_{C_1} (Fig. 7). We calculated the dependence $\Delta(\beta_{C_1})$ at large β_{C_1} and found that $\Delta=0.24$ at $\beta_{C_1}=100$.

We investigated the highest spread of critical currents $\delta_{\max}$ at which synchronization exists at different values of parameters β_L, β_{C_1}, and β_C. The optimal values of parameters β_{C_1} and β_L are $\beta_{C_1}\sim 0.4$–0.6, $\beta_L\sim 3$–5. We solved Eqs. (1) numerically using these optimal values and found the dependence $\delta_{\max}(\beta_C)$ (Fig. 8). The minimal value of β_C was taken $\beta_C=0.01$ in our calculations. This dependence has a maximum at $\beta_C=1$ and the highest value of $\delta_{\max}$ exceeds 0.15. The evaluation of the highest spread of critical currents within the SVA approximation is as follows:[1,8] $\delta_{\max}\approx\sqrt{\overline{\nu}/i(i+\overline{\nu})}|\mathrm{Im}(y_{12})|$. Though the value $\beta_C=1$ is out of the range of the SVA approximations, we can roughly estimate $\delta_{\max}$ using this expression and Eq. (6): $\delta_{\max}\approx 0.13$. Despite this estimation of $\delta_{\max}$ is out of ranges of the SVA approximation, it shows that at $\beta_C=1$ the highest spread of critical currents increases. Furthermore, this value is close to the obtained numerically value $\delta_{\max}=0.15$. Such a value of the highest spread of critical currents coincides also with predictions[14] for systems of underdamped junctions with LC load and the high enough quality factor of the resonator $Q\sim 30$. We can see that at optimal values of the system parameters the highest spread is the same for the system of overdamped junctions but with lower values of the quality factor (because the synchronizing current $\tilde{i}_k$ flowing through each junction can reach the same high value at optimal parameters).

IV. SOME ASPECTS OF EXPERIMENTAL REALIZATION

Though this paper is devoted to theoretical analysis of synchronization of a Josephson junction array with a superconducting resonator we like to discuss some aspects of experimental realization of such system corresponding to our analysis. The experimental investigations of systems with the superconducting shunts are more advanced now than the understanding of the phase locking mechanisms in these systems.[2,3,20,21] Recently, there are experimental investigations which prove the usefulness of the superconducting shunt for phase locking in SIS and SINIS (here N is for a normal metal) arrays of Josephson junctions[2,3,20] and applications of this effect.[21] For example, in the two-dimensional array of junctions with a superconducting ground plane (divided by a layer of insulator from the array), the dc-to-ac power conversion efficiency is one order of magnitude higher than in other Josephson junction arrays.[3] An one-dimensional array of overdamped SINIS junctions integrated into the low-impedance superconducting microstripline was used for a programmable Josephson voltage standard due to internal phase locking of junctions.[20,21] Together with the successful applications of these systems the new physical effect of a threshold of the ac power was discovered in systems with the resonator.[3] Thus, a question of optimization of these circuits is strongly connected with understanding of the origin of phase locking.

Because the resonance contour is used for obtaining of phase locking the drawback of the proposed scheme is the limited tunability of the scheme to the necessary range of frequencies. To satisfy the condition of obtaining of the widest locking interval (i.e. the highest range of frequencies at which junctions radiate coherently) all the parameters of the circuit should be close to the calculated values (β_{C_1} ~ 0.4–0.6, $\beta_L \sim 3$–5, and $\beta_C \sim 0.8$–1.3). The array of overdamped junctions has the advantage in this point in comparison with the array of underdamped junctions because any resonance system based on the overdamped junctions has a low quality factor due to the low resistances of junctions. The resonance is broad in this case and deviations of parameters in the above mentioned intervals can provide phase locking in the system.

The considered system of junctions with a low-capacity resonator can be realized experimentally in arrays of SINIS overdamped junctions[20,21] or HTSC grain boundary Josephson junctions because the McCumber parameter of many HTSC Josephson junctions satisfies the optimal conditions $\beta_C \sim 1$. For high-T_C superconductors the system can be fabricated using HTSC technology in vertical geometry [as it is shown in Fig. 1(a)] or in planar geometry using deep step-edge junctions[10] and thick layers of HTSC superconductors to provide a good capacitive connection. To satisfy the optimal conditions ($\beta_{C_1} \leqslant 1$) the value of capacitor C_1 should be $C_1 \sim 1$ pF. The resistance of the insulator inside the capacitor should exceed the normal resistance of Josephson junctions as much as 100 times and the thickness of this insulator should provide a negligible small supercurrent between the array of junctions and the superconducting shunt. Within these limitations the proper material of the insulator and its thickness can be chosen. Note that one can fabricate two capacitors on the edges of the array and connect them in succession [as it is shown in Fig. 1(a)] or prepare only one capacitor and connect the other end of the array with the superconducting shunt. The capacitors should have small geometrical dimensions to provide the inductance parameter of the loop $\beta_L \sim 3$.

V. THRESHOLD OF PHASE LOCKING IN SHUNTED MANY-JUNCTION ARRAYS

Recent investigations of two dimensional arrays of Josephson junctions connected capacitively to a superconducting ground plane showed that the emitted ac power has a sharp increase from almost zero values to finite values (a threshold) if the amount of junctions activated into the voltage state (active junctions) exceeds some critical value.[3] In this section we demonstrate a mechanism of synchronization which leads to a threshold of emitted power and is valid both for underdamped and overdamped junctions.

We consider a infinite one-dimensional array of junctions with a superconducting plane and a dividing layer of insulator. If the superconducting plane is placed close enough to the array, ac current circuits are formed between neighbor junctions. The electrical scheme of the ac circuit of such system is shown in Fig. 9 (it is supposed that the dc bias current flows through the line of junctions). It follows from Eq. (2) that synchronization of active junctions in the system appears due to the ac current $\tilde{\iota}_k$ produced by each active junction and flowing through other active junctions. Coefficients $y_{kk'}$ which determine this ac current have different characteristic distances of the decay (or distances of the interaction with other junctions) in different systems.[1,17] For

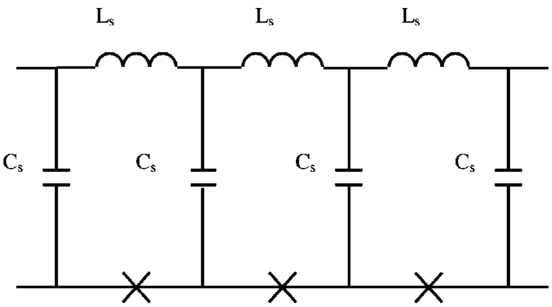

FIG. 9. (a) The ac circuit scheme of the array of four junctions with a superconducting resonator strip line which provides currents between neighbor junctions through the plane, L_s is the inductance of the strip line between junctions and C_s are capacitance of the insulator layer between the array and the strip line.

the infinite system shown in Fig. 9 coefficients $y_{kk'}$ are given with[17]

$$y_{kk'} = \frac{1}{2\sqrt{-z_1(z_2+z_j)}} e^{-|k-k'|\sqrt{(z_2+z_j)/z_1}}, \tag{7}$$

where z_1, z_2, and z_j are impedances of the capacitance C_s, the inductance L_s and the junction, respectively. The absolute value of the difference between numbers of junctions $|k - k'|$ we call "the distance" in the following consideration.

We suppose that there are only two active junctions and all other junctions are in the superconducting (passive) state. At first we assume that junctions are nearly identical, i.e., $\delta \to 0$. Accordingly to Eq. (5), at $\eta = \pm \pi/2, |\sin \eta| = 1$ the ranges of synchronization for the given bias current are given by zero points of the function F. We calculated the imaginary part of the connection coefficient $y_{kk'}$ using Eq. (7) [Fig. 10(a), solid line]. For the calculation of impedances we normalized the capacitance C_s and the inductance L_s: $\beta_{Cs} = 2\pi I_{ca} R_a^2 C_s/\Phi_0$, $\beta_{Ls} = 2\pi I_{ca} L_s/\Phi_0$. The sign of the function $\mathrm{Im}(y_{kk'}) = f(|k-k'|)$ is negative at $|k-k'| \leqslant 3$ and positive at $|k-k'| > 3$. It means that active junctions oscillate in phase if the distance between them does not exceed 3 and they oscillate antiphase if the distance exceeds 3. Thus, there is a certain critical distance r_c between active junctions to obtain their in-phase synchronization ($r_c = 3$ in this case). If we consider the infinite chain of junctions in which active junctions are distributed uniformly at some distance from each other, they do not oscillate in phase with each other until the distance between neighbors is larger than r_c. The distance r_c is reached at a certain quantity of active junctions $N_a = N/r_c$ that gives a critical concentration of active junctions $x_{cr} = N_a/N = 1/r_c$ ($x_{cr} = \frac{1}{3}$ in our case). Accordingly to Eq. (5), if there is some spread of critical currents of junctions, the distance at which the in-phase locked state is observed at given δ and $\overline{\nu}$ is between the points of the intersection of the curves $F(|k-k'|)$ and $\mu = \mathrm{const}$.

To check our consideration we made numerical simulations of the quasi-infinite chain of junctions with a superconducting plane shown in Fig. 9. The dynamic equations in Eq. (1) get the extended form

$$[1-(-1)^k \delta][\beta_C \ddot{\varphi}_k(\tau) + \dot{\varphi}_k(\tau) + \sin(\varphi_k)] = \overline{\iota} - \dot{q}_k(\tau),$$

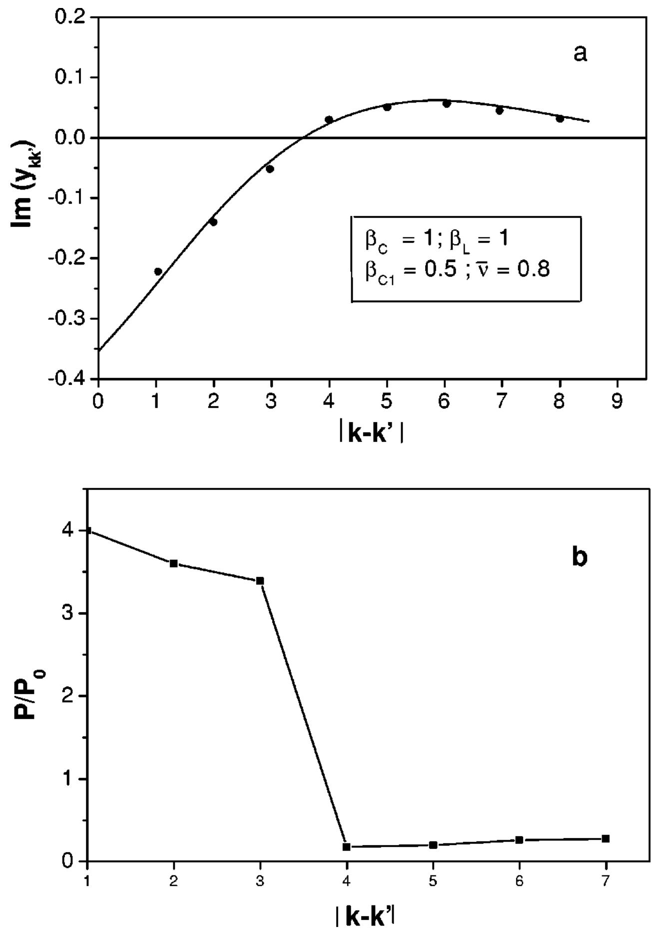

FIG. 10. (a) The dependence of $|\mathrm{Im}(y_{kk'})|$ on the distance between the junctions $|k-k'|$. The solid line represents the SVA approximation, solid circles obtained from the solution of dynamic equations. The parameters of the calculations are $\beta_C=1$, $\beta_{C_s}=0.5$, $\beta_{Ls}=1$, $\overline{i}=1.10$, (b) The dependence of the normalized ac power across the array of two junctions P/P_0 on the distance between junctions. $\delta=0.01$, other parameters are the same as in (a).

$$\beta_{Ls}\ddot{q}_k+\frac{1}{\beta_{Cs}}(2q_k-q_{k-1}-q_{k+1})=\dot{\varphi}_k,\quad k=1,2, \tag{8}$$

where q_k is the charge on the capacitance of the kth junction.

We investigated the imaginary part of the connection coefficients and the total ac power of two active junctions placed on different distance from each other. The evaluation of $|\mathrm{Im}(y_{kk'})|$ within the SVA approximation is as follows:[1,8] $|\mathrm{Im}(y_{kk'})|\approx[\overline{\imath(\imath+\nu)}/\overline{\nu}]\delta_{\max}$, where $\delta_{\max}$ is the highest spread of critical currents at which synchronization exists when the first junction is at kth position and the second junction is at the k'th position. To know the sign of the values $\mathrm{Im}(y_{kk'})$ we checked the in-phase or antiphase behavior of voltage oscillations. The calculated values of the $\mathrm{Im}(y_{kk'})$ [shown by solid circles in Fig. 10(a)] are in a good agreement with those obtained by means of the SVA approximation.

We consider the cluster of two active junctions placed on different distance from each other in the middle of the array of 30 passive junctions. Such disposition of the cluster of active junctions allows us to avoid the effects of reflections from the ends of the array. The ac currents produced by active junctions in the cluster decay at edges of the array that provides the condition of the quasi-infinite system for which the Eq. (7) is fitted. We set values of critical currents of passive junctions $i_{c,\mathrm{pas}}=1.3$ while critical currents of active junctions were distributed around the averaged value $i_{c,\mathrm{act}}=1$ with the spread δ.

The ac power P across the whole array normalized on the ac power of the single active junction P_0 placed in the middle of the array of passive junctions is as follows:

$$\frac{P}{P_0}=\frac{\overline{\overline{\left(\sum_{n=1}^{N}(\dot{\varphi}_n-\overline{\nu_n})\right)^2}}}{\overline{\overline{(\dot{\varphi}_1-\overline{\nu_1})^2}}},\quad n=1,\ldots,N, \tag{9}$$

where n is the number of the junction, N is the total quantity of junctions (active and passive), $\dot{\varphi}_n$ is the time-dependent voltage across the nth junction, $\overline{\nu_n}$ is the averaged dc voltage across the nth junction, $\dot{\varphi}_1$ and $\overline{\nu_1}$ are the time-dependent voltage and the averaged dc voltage across the single active junction which is placed in the middle of the array of passive junctions, two lines above expressions represent averaging over time and over the random value of critical currents distributed uniformly within the interval $\{1-\delta,1+\delta\}$. If active junctions are synchronized in phase with zero phase shift then $P/P_0=N_a^2\overline{\overline{(\dot{\varphi}_1-\overline{\nu_1})}}^2/\overline{\overline{(\dot{\varphi}_1-\overline{\nu_1})}}^2=N_a^2$, where N_a is the amount of active junctions in the cluster.

The normalized ac power across the array of two active junctions as a function of the distance between them is shown in Fig. 10(b). We can see that P/P_0 changes from 4 to $\sim$3.5 if the distance increases to 3 (in-phase oscillations) and becomes almost zero if $|k-k'|>4$ (antiphase oscillations). These results are also in full agreement with predictions of the SVA approximation.

We investigated the ac power across a cluster of 11 junctions placed in the middle of the array of 30 junctions. Active junctions were uniformly distributed along the cluster. To investigate the influence of the distance between active junctions on the ac power across the array we consider only the set of configurations of active junctions in which they distributed uniformly along the cluster, so we do not average P/P_0 over different configurations of active junctions inside the cluster. The relation P/P_0 as a function of the amount of active junctions N_a is plotted in Fig. 11. The value of P/P_0 is insufficient until $N_a<5$ (antiphase oscillations of junctions), then P/P_0 increases slowly until $N_a=6$ and then increases sharply at $N_a=7$. The further increase of N_a leads to the increase of P/P_0 as a square of their quantity N_a that is characteristic of the phase coherence of all active junctions.[1] We can note two particularities in Fig. 11: the increase of P/P_0 from almost zero values at $N_a>4$ and the beginning of the total in-phase radiation of all junctions at $N_a=7$. These features of $P/P_0=f(N_a)$ behavior can be explained in the ranges of the SVA approximation. When there are few active junctions in the array, they are placed far from each other and oscillate antiphase [see Figs. 10(a) and 10(b)]. The ac power across the array is therefore insufficient. The critical distance for the array with $\delta=0.01$ is $r_c=2$. If active junctions are

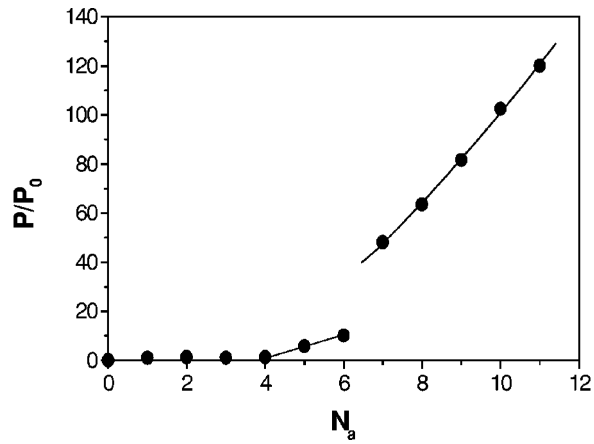

FIG. 11. The dependence P/P_0 on the quantity of active junctions N_a. Parameters of calculations: $\beta_C=1$, $\beta_{C_s}=0.5$, $\beta_{Ls}=1$, $\overline{i}=1.10$, and $\delta=0.01$.

distributed regularly along the cluster, at $N_a=5$ there appear configurations in which active junctions are placed at the distance r_c from each other, several active junctions are synchronized in phase and others are not synchronized with them. At $N_a=6$ junctions oscillate in phase with their first neighbors and antiphase with second neighbors. The addition of one more active junction ($N_a=7$) into the center of the cluster creates the nucleous of the several junctions which oscillate in phase and pull all active junctions into the phase-locked state. Note that if the seventh active junction is added to the one of the ends of the cluster, the state in which only the part of junctions is synchronized still exists and the ac power across the array is not so large (but the total phase locking appears if an eighth active junction is added in the middle of the cluster). Thus, from the consideration of Fig. 11 we can make the conclusion that in the system with many active junctions there are configurations which lead to the threshold of the ac power.

VI. CONCLUSIONS

In the first part, we investigated theoretically the system of two Josephson junctions with McCumber parameters $\beta_C \sim 1$ loaded by a superconducting shunt and a capacitance. We analyzed the dependence of voltages at which junctions oscillate coherently (the locking interval) on parameters of the system by both the method of slowly varying amplitudes (SVA's) in the available ranges of parameters as well as by solving numerically the system of dynamic equations. Dependencies of the locking interval on different parameters obtained within SVA approximation are in a good agreement with those obtained by means of solution of dynamic equations. The analytic result give a contribution to better understanding of the mechanism of phase locking and dynamic behavior of the system.

The load consisted of inductance and capacitance provides the current resonance in the system and the circulating current through junctions. This circulating current synchronizes the oscillations of both junctions. We analyzed the behavior of real and imaginary parts of the circuit impedance and showed that the maxima of the "locking voltage" F appear at the left side of the resonance voltage $\overline{\nu_2}$. We analyze the stability of the in-phase solutions of dynamic equations and found that solutions in the vicinity of the resonance at the maximum of F have the highest stability.

Due to the circulating current the values of F are as much as fifty percent higher than those obtained for the load consisting of a resistance and an inductance thus provides the phase locking up to 15% spread of critical currents. We showed that a superconducting resonator shunt gives advantages for synchronization of Josephson junctions in comparison with the traditional shunts made of normal conductors. We found the optimal values of McCumber parameters β_C, parameters of the load capacitance β_{C_1}, and inductance parameters of the system β_L for phase locking.

There are thresholds of the phase-locked state at small as well as large values of system parameters. By means of the SVA method we found that the origins of these thresholds are an insufficiency of the "locking voltage" F to get phase locking at these values of parameters. We finally discussed aspects of realization of this promising system, particularly, using HTSC technology. Because the resonance contour is used as a load, the system has to be tuned to the required frequencies of radiation. Parameters of the circuit should be close to calculated values ($\beta_{C_1}\sim 0.4-0.6, \beta_L \sim 3-5, \beta_C \sim 0.8-1.3$) to get a wider the range of frequencies at which junctions radiate coherently. We can consider the present system as an alternative to the many-junction SQUID (which also gives phase locking up to 15% spread of critical currents) but is sensitive to external magnetic fields because of ist closed superconducting loop.

In the second part, we applied the SVA method to the consideration of the quasi-infinite system of a many-junction array and a superconducting shunt which is placed close enough to the array to form ac currents between neighbor junctions through the plane. In such a system the connection coefficients decay with the distance from the junctions which are into the voltage state (active junctions). If active junctions are placed far from each other, they oscillate antiphase and the total ac power of radiation is almost zero. We show that in-phase synchronization of active junctions appears if they are placed at some critical distance from each other. To provide this critical distance the concentration of active junctions must be high enough depending on the spread parameter δ. We checked these predictions of the SVA approximation by numerical solution of the dynamic equation. We showed that if junctions are distributed uniformly along the chain then there is in-phase locking of some active junctions when the quantity of active junctions exceeds the critical value. We also showed that there are configurations at which the total phase locking of all active junctions appears after the addition of one more active junction to the cluster. This addition leads to the threshold of ac power. We believe that this mechanism is valid for two-dimensional arrays, too, and gives an explanation of the experimental observed threshold in the number of rows.[3]

ACKNOWLEDGMENTS

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[1] K. K. Likharev, *Dynamics of Josephson Junctions and Circuits* (Gordon and Breach, Philadelphia, 1991).
[2] A. B. Cawthorne, P. Barbara, S. V. Shitov, C. J. Lobb, K. Wiesenfeld, and A. Zangwill, Phys. Rev. B **60**, 7575 (1999).
[3] P. Barbara, A. B. Cawthorne, S. V. Shitov, and C. J. Lobb, Phys. Rev. Lett. **82**, 1963 (1999).
[4] Giovanni Filatrella, Niels Falsing Pedersen, and Kurt Wiesenfeld, Phys. Rev. E **61**, 2513 (2000).
[5] R. Kleiner and P. Müller, Physica C **293**, 156 (1997).
[6] B. R. Trees and E. B. Harris, Phys. Rev. B **57**, 5425 (1998).
[7] I. P. Nevirkovets, Appl. Supercond. **5**, 291 (1998).
[8] L. S. Kuz'min, K. K. Likharev, and G. A. Ovsyannikov, Radio Eng. Electron. Phys. **26**, 102 (1981).
[9] M. Darula, H. Kohlstedt, L. Amatuni, and A. M. Klushin, Inst. Phys. Conf. Ser. **158**, 599 (1997).
[10] M. Darula, S. Beuven, M. Siegel, A. Darulova, and P. Seidel, Appl. Phys. Lett. **67**, 1618 (1995).
[11] A. N. Grib, P. Seidel, and M. Darula, J. Low Temp. Phys. **112**, 323 (1998).
[12] M. Darula, S. Beuven, M. Siegel, P. Seidel, and A. Darulova, Europhys. Lett. **33**, 41 (1996).
[13] L. S. Kuz'min, Ya. S. Levenberg, K. K. Likharev, E. V. Nikitina, and E. S. Soldatov, Radiotekh. Elektron. (Moscow) **34**, 419 (1989).
[14] V. K. Kornev, K. K. Likharev, and E. S. Soldatov, Radiotekh. Elektron. (Moscow) **32**, 1521 (1987).
[15] E. D. Thompson, J. Appl. Phys. **44**, 5587 (1973).
[16] A. N. Grib, J. Scherbel, and P. Seidel, Phys. Lett. A **266**, 76 (2000).
[17] G. A. Ovsyannikov, L. S. Kuz'min, and K. K. Likharev, Radio Eng. Electron. Phys. **27**, 142 (1982).
[18] H. J. Pain, *The Physics of Vibrations and Waves* (Wiley, London, 1976).
[19] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988).
[20] H. Schulze, F. Müller, R. Behr, J. Kohlmann, J. Niemeyer, and D. Balashov, IEEE Trans. Appl. Supercond. **9**, 4241 (1999).
[21] H. Schulze, I. V. Krasnopolin, J. Kohlmann, R. Behr, F. Müller, and J. Niemeyer, Inst. Phys. Conf. Ser. **167**, 761 (1999).