Electronic transport in strongly anisotropic disordered systems: Model for the random matrix theory with noninteger β

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We study numerically electronic transport in strongly anisotropic weakly disordered two-dimensional systems. We find that the conductance distribution is Gaussian. The conductance fluctuations increase when anisotropy becomes stronger. The statistics of the transport parameters can be interpreted by the random matrix theory with a noninteger symmetry parameter β. Our results are in a good agreement with recent theoretical work of K. A. Muttalib and J. R. Klauder [Phys. Rev. Lett. **82**, 4272 (1999)].

It is generally accepted that the electronic transport in weakly disordered metallic systems is successfully described by the random matrix theory (RMT),[1] and the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation.[2] Both theories predict a Gaussian distribution of the conductance and provides us with the exact value of the conductance fluctuations in agreement with data obtained by diagrammatic expansion.[3] The Landauer formula for the conductance, $g = \mathrm{Tr}\, t^\dagger t$ enables us to express g in terms of eigenvalues of the transmission matrix $t^\dagger t$:

$$g = \sum_i^N \cosh^{-2}(z_i/2), \qquad (1)$$

where N is the number of open channels.

In the limit $N \gg 1$, RMT proposed a common probability distribution of parameters z,

$$P(z) = \exp - \mathcal{H}(z)/\beta \qquad (2)$$

with

$$\mathcal{H}(z) = \sum_i^N \frac{k}{2} z_i^2 - \sum_i^N J(z_i) - \beta \sum_{i<j}^N u(z_i, z_j). \qquad (3)$$

In RMT, physical properties of the sample are specified only by two parameters: $\beta = 1, 2, 4$ for the orthogonal, unitary, and symplectic symmetry of the model, respectively, and the mean free path l, which determines, together with the system length L_x, the strength of one-particle potential: $k = lN/L_x$. RMT describes successfully the transport properties of weakly disordered quasi-one-dimensional (Q1D) systems. It could be applied also to squares[4] or cubes[5] if the length L_x of the system fulfills the relations

$$l \ll L_x < \xi \qquad (4)$$

with ξ being the localization length. The absence of any other parameters reveals the universal transport properties of weakly disordered systems. In particular, the variance of the conductance is universal, and depends only on the symmetry and the shape of the sample.[3,6,7]

Recently, Muttalib and Klauder[8] showed that the requirement of the large system length is not necessary for the derivation of the DMPK equation. In their DMPK equation the parameter β depends on the statistical properties of the model. This allows β to possess any positive value. As supposed, β converges to unity when the system length increases.

In this paper we present a physical realization of the theoretical model proposed in Ref. 8. We calculate the conductance and the statistics of the parameters z for weakly disordered strongly anisotropic two-dimensional systems and show that their transport properties can be described within the framework of the RMT with the symmetry parameter β smaller than 1.

Our model is defined by two-dimensional (2D) anisotropic Anderson Hamiltonian

$$H = \sum_{ij} \varepsilon_{ij} |ij\rangle\langle ij| + \sum_i |ij\rangle\langle i+1\,j| + t \sum_j |ij\rangle\langle ij+1|, \qquad (5)$$

where $i \leq L_x$ ($j \leq L_y$) numerates sites in the x (y) direction, respectively. Hard wall boundary conditions are considered and $E_F = 0$, then $N = L_y$. Random energies ε_{ij} are distributed uniformly between $-W/2$ and $W/2$. We put $W = 2$ throughout the paper. Then the localization length increased from $\xi \sim 25$ for the one-dimensional (1D) chain ($t = 0$) (Ref. 9) to $\sim 10^6$ for the 2D isotropic systems[10] while the mean free path decreases from $l \sim 25$ to $l \sim 4$ (Ref. 4) in the same range of t. As a typical size of our samples varies between 20 and 100, we expect to find metallic behavior even for strong anisotropy.

We found that the conductance distribution is Gaussian for each value of t. As an example, Fig. 1 presents $P(g)$ for systems with $t = 0.05$ and $t = 1$. The inset of Fig. 1 shows that the mean conductance is always larger than 1. The variance var $g = \langle g^2 \rangle - \langle g \rangle^2$ increases as t decreases. The system size dependence of the mean conductance is presented in Fig. 2 for both the Q1D and the square samples. For $t \geq t_c \approx 0.2$, the mean conductance decreases as $\langle g \rangle \propto L_y/L_x$ in the Q1D case and is almost system-size independent for the squares. This confirms that these systems are in the diffusive regime. For a stronger anisotropy, $t < t_c$, the diffusive regime takes place for much smaller systems ($30 \leq L \leq 50$ for $t = 0.1$).
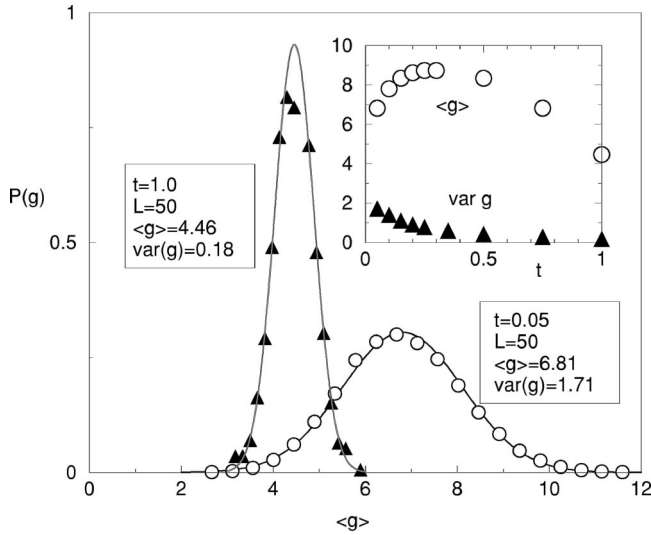
FIG. 1. The distribution of the conductance for $t=0.05$ and $t=1$. Inset: The t dependence of $\langle g \rangle$ and var g for squares 50×50.

The analytical expression for var g derived by Stone et al.[3] states that the anisotropy influences the variance of the conductance only in the combination with the size of the system:

$$\text{var } g = f\left(\frac{L_x}{L_y}\sqrt{t}\right). \tag{6}$$

Figure 3 shows that for $L_x/L_y\sqrt{t}>2$ and $t\geq t_c$, var g reaches the universal value $=2/15$. For the squares, var g is independent on the system size for $t>0.1$.

The increase of var g for $t<1$ is in a qualitative agreement with the universal relation for the conductance fluctuation,[3,8]

$$\text{var } g \sim \beta^{-1}, \tag{7}$$

provided that $\beta<1$ when $t<1$. It seems straightforward to compare numerical data for var g with Eq. (7) and calculate $\beta=\beta(t)$. However, detailed numerical analyses have shown that relation (7) underestimates numerical data if the disorder is weak.[11,7] Therefore we prefer to estimate $\beta(t)$ from the statistics of the parameters z for anisotropic square samples. The quantity of interest is the probability distribution $P(s)$ of normalized differences[12]
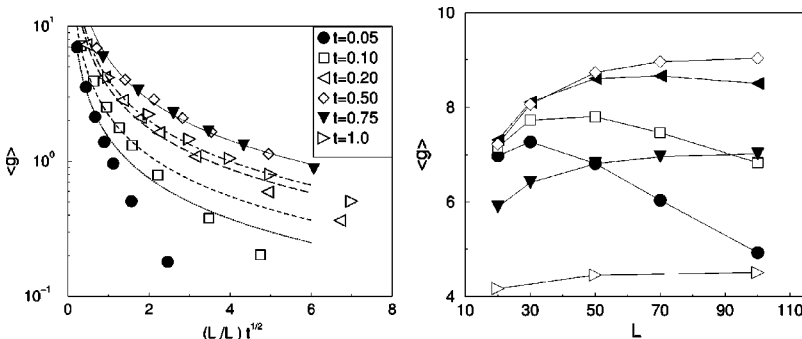
$$s=(z_{i+1}-z_i)/\langle z_{i+1}-z_i \rangle. \tag{8}$$

Figure 4 shows the nonanalytical behavior

$$P(s)\sim s^{\beta(t)} \quad \text{for} \quad s\ll1. \tag{9}$$

The t-dependent exponent $\beta=\beta(t)$ (inset of Fig. 4) is calculated from the logarithmic fit

$$\ln P(\ln s)=[1+\beta(t)]\ln s. \tag{10}$$

Relation (9) with $\beta=1$, 2, or 4 is well known from the RMT. The small s behavior is determined by the symmetry of the model. Here, however, β is given by statistical properties of the system. In the limit $t\rightarrow0$ our system dissociates to a set of independent chains, each of them is characterized by its z. Hence z are statistically independent variables and the distribution of their difference should be Poissonian: $P(\ln s)\propto\ln s$ and $\beta(t=0)=0$. The exponent $\beta(t)$ increases as t increases and the distribution $P(s)$ converges to the Wigner surmise (WS) as $t\rightarrow1$: $P(\ln s)\propto2\ln s$ as expected.

Contrary to $P(s)$, the distribution $P(z_1)$ of the smallest z_1 is similar to WS for any t. This is consistent with RMT which states that the distribution of z_1 is β independent.[1]

To test the applicability of RMT with noninteger β to anisotropic systems, we studied the spectra of z for squares (Fig. 5). Assuming that the distribution $P(z)$ has the form (2) and (3), we can find the most probable values $\tilde{z}$ of the parameters z from the system of nonlinear equations,

$$\partial\mathcal{H}/\partial z_i|_{z_i=\tilde{z}_i}=0. \tag{11}$$

In the limit of small z the "interaction" and "Jacobian" terms in Eq. (3) can be approximated as[1]

$$u(z_i,z_j)=\ln|z_i^2-z_j^2| \quad \text{and} \quad J(z)=\ln z. \tag{12}$$

System (11) is then exactly solvable.[13] After some algebra we find[14]

$$\tilde{z}_i\sim j_\alpha(i), \quad \alpha=\frac{1}{\beta}-1, \tag{13}$$

where $j_\alpha(i)$ is the ith zero of the Bessel function J_α. From Eq. (13) we can express the ratio

$$\frac{\tilde{z}_{i+1}}{\tilde{z}_i}=\frac{j_\alpha(i+1)}{j_\alpha(i)} \tag{14}$$

which depends only on β.[15]

FIG. 2. System size dependence of the mean conductance $\langle g \rangle$. Left: for the quasi-one-dimensional systems $L_x\times L_y$. The width of the system is $L_y=10$. Lines represent the relation $\langle g \rangle\sim a\times L_x/L_y$ which is characteristic for the diffusive regime. Right: for squares $L\times L$. Increase of the mean conductance for small L indicates a ballistic regime, decrease for large L is due to the crossover to the localized regime.
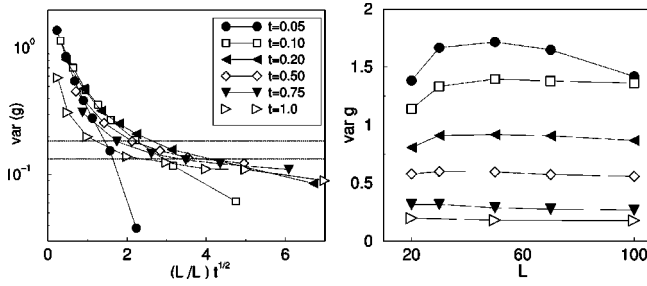
FIG. 3. Left: var g as a function of the system length L_x. The width of the system $L_y = 10$. Dotted lines show universal values of var g for squares (0.185) and Q1D systems (0.133) (Ref. 3). The increase of the system length causes the transition to localized regime with decrease of var g. For $t = 1$, also data for shorter systems 40×10 and 20×10 are present to show an increase of the conductance fluctuations. Right: var g as a function of the system size for squares $L \times L$.

The derivation of Eq. (14) holds for any value of β. Using the values of $\beta(t)$ presented in Fig. 4 we calculated the ratio $\widetilde{z}_{i+1}/\widetilde{z}_i$ and compared it with numerical data. As it is shown in inset of Fig. 5, the agreement is very good for $\alpha < 2$, which corresponds to $t \geq t_c = 0.2$.

In Fig. 6 we show that the exponent β depends also on the shape of the system. β converges toward 1 as the length L_x increases. This is consistent with Ref. 8. For $t = 0.2$ we found that the distribution $P(s)$ is WS for $L_x/L_y \approx 8$. For this system length the system is still in the metallic regime: the mean conductance $\langle g \rangle \sim 1$, and the RMT with $\beta = 1$ is applicable to describe its properties. Of course, further increase of the
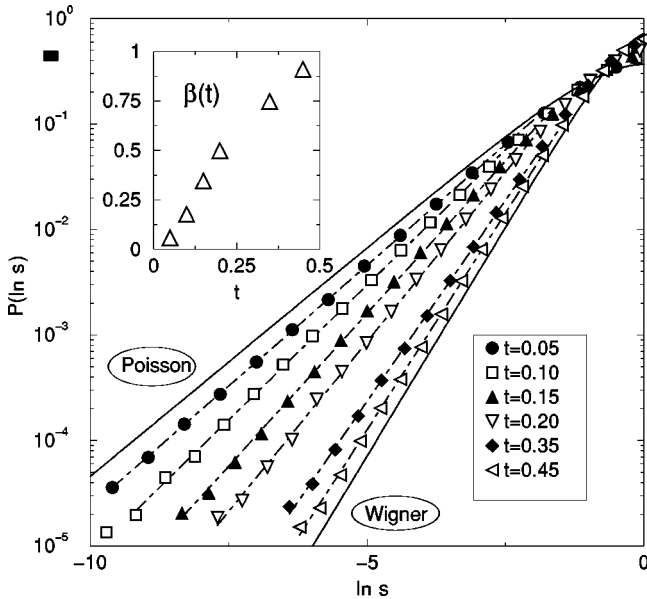

FIG. 5. Spectrum of zs for small anisotropy parameters. Note the common crossing point of spectra for different t. Inset: comparison of ratios of the most probable values, z_{i+1}/z_i for $i = 1, 2, 3$ (open symbols) with theoretical prediction $j_\alpha(i+1)/j_\alpha(i)$ (full symbols). Parameter $\alpha = \beta^{-1} - 1$ [Eq. (13)].

system length causes a decrease of the mean conductance and $P(s)$ becomes Gaussian. A qualitatively similar behavior can be found for any $t > 0.2$.

For $t = 0.05$, $P(s)$ reaches WS only for $L_x/L_y \approx 36$. However, the coincidence of $P(s)$ with WS does not indicate the metallic behavior. The mean conductance is $\langle g \rangle \sim 10^{-2}$. Thus we have an interesting paradox: the strongly anisotropic system exhibits the metallic behavior with a distribution $P(s)$ very close to Poissonian distribution. By increasing the system length we obtain an insulating regime in which $P(s)$ becomes WS.

In conclusion, we have presented numerical data for the strongly anisotropic weakly disordered systems. For $t \geq t_c \approx 0.2$ we found the metallic behavior with the mean conductance independent on the system size (for size $L < 100$). The distribution of the conductance is Gaussian. We found that the anisotropy causes the increase of the var g. We analyzed also the spectrum of the parameters z. We found that the shape of the distribution $P(s)$ of the normalized difference s [Eq. (8)] depends on the anisotropy. We interpret these results by the random matrix theory in which the "symmetry parameter" β depends on the anisotropy and can possess any positive value. From such RMT we derived the analytical formula for the spectrum of z which agrees very well with numerical data.

The assumption that β could be noninteger corresponds with the theoretical prediction of Muttalib and Klauder.[8] In their theory, the DMPK equation can be generalized to the description of shorter systems. The parameter β becomes then a function of mutual correlations of eigenvalues and eigenvectors of the transfer matrix. In agreement with Ref. 8, we found that β depends on the length of the system and converges to 1 when the system length increases.

Another, more formal interpretation of RMT with noninteger β is based on the Coulomb gas analogy:[4] the probabil-

FIG. 4. Probability density $P(\ln s)$ for different anisotropy t of the system. s is the (normalized) difference $z_{i+1} - z_i$. Solid lines are Wigner surmise $W_1(s) = (\pi/2)s \exp{-(\pi/4)s^2}$ and Poisson distribution e^{-s}. The size of the samples $L_x = L_y = 50$. Statistical ensembles of $N_{\text{stat}} = 10^5$ samples have been considered. Dot-dashed lines represent fits (10). Inset: t dependence of exponent β.
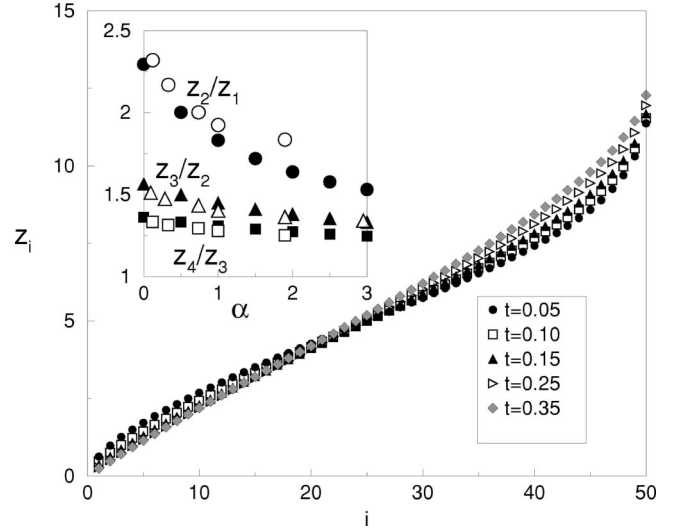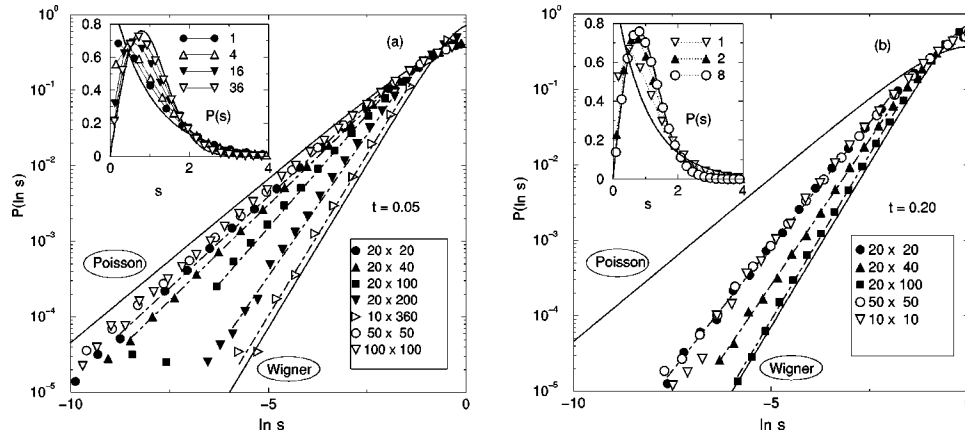
FIG. 6. Change of $P(\ln s)$ with the length of the system. For a longer system, the exponent β increases and $P(s)$ converges to Wigner surmise. Insets show $P(s)$ in linear scale for some ratio L_x/L_y. (a) $t=0.05$: $P(s)$ achieves Wigner surmises for $L_x/L_y>36$. Comparison with Fig. 2 shows that the conductance of such long system is small. (b) $t=0.2$. $P(s)$ has a form of Wigner surmises already for $L_x/L_y=8$, when the system is still in a metallic state, $\langle g \rangle \sim 1$. We present also $P(s)$ for square samples of various size to show that exponent β is system-size independent although $\langle g \rangle$ is not constant (see Fig. 2).

ity distribution (2) is formally equivalent to the statistical weight of the classical system of charged interacting particles in one dimension. z determines the position of the particles which interact via interaction $\beta u(z_i,z_j)$. The parameter β represents the strength of interaction. The anisotropy parameter t tunes this interaction. The limit $t=0$ represents the system of noninteracting particles with $\beta=0$. This effect of weak interaction must not be confused with the decrease of the interaction which appears in the isotropic Q1D system. In the last phenomena the confining one-particle potential $kz^2/2$ becomes weaker as the length of the system increases. This enables an increase of the mutual distance $z_{i+1}-z_i$ between particles. The effect of the interaction (which is a function of the particle distance) is therefore less important than in the metallic (short) systems. This does not affect the value of the interaction constant β.

The critical value of anisotropy $t_c=0.2$ appears frequently throughout the paper. For stronger anisotropy (smaller t), no diffusive regime exists. The actual value of t_c is, of course, determined by our choice of the strength of disorder and is expected to be smaller for smaller W.

We suppose that the anisotropic model discussed in this paper represents the physical system to which the generalized DMPK equation of Muttalib and Klauder[8] can be applied.

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[15] In Ref. 14 we applied relation (14) to orthogonal ($\beta=1$) and symplectic ($\beta=4$) systems to describe the exact form of the spectrum of z. In particular, formula (14) predicts that $z_2/z_1 \approx 2.3$ for the orthogonal and ≈ 4.94 for the symplectic ensembles. This agrees with numerical data.