Interlayer exchange interactions, SU(4) soft waves, and skyrmions in bilayer quantum Hall ferromagnets

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The exchange Coulomb interaction is the driving force for quantum coherence in quantum Hall systems. We construct a microscopic Landau-site Hamiltonian for the exchange interaction in bilayer quantum Hall ferromagnets, which is characterized by the SU(4) isospin structure. By taking a continuous limit, the Hamiltonian gives rise to the SU(4) nonlinear sigma model in the von Neumann–lattice formulation. The ground-state energy is evaluated at filling factors $\nu=1,2,3,4$. It is shown at $\nu=1$ that there are three independent soft waves, where only one soft wave is responsible for the coherent tunneling of electrons between the two layers. It is also shown at $\nu=1$ that there are three independent skyrmion states apart from the translational degree of freedom. They are CP^3 skyrmions enjoying the spin-charge entanglement confined within the lowest Landau level.

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I. INTRODUCTION

Quantum coherence is a new aspect of quantum Hall (QH) systems. Electron spins are polarized spontaneously due to the exchange Coulomb interaction rather than compulsively by the Zeeman effect. Skyrmions arise as coherent excitations,[1] which have been observed experimentally[2–4] as quasiparticles. Intriguingly, an interlayer coherence may develop spontaneously between the two layers and lead to Josephson-like phenomena in bilayer QH (BLQH) systems.[5] Recent experimental results[6] on tunnelling current may well be interpreted as the dc-Josephson current[7] though still controversial.[8] We expect even the SU(4) quantum coherence due to the spin and layer degrees of freedom.[9] The driving force of quantum coherence is the Coulomb exchange interaction.[10,11] The exchange Coulomb interaction has also been argued to create a new phase, the canted antiferromagnet,[12–14] in the BLQH system at the filling factor $\nu=2$.

In this paper we analyze the exchange Coulomb interaction to explore the SU(4) coherence in BLQH ferromagnets. We are concerned about electrons confined to the lowest Landau level, where the electron position is solely specified by the guiding center (X,Y) obeying

$$[X,Y]=-il_B^2. \tag{1.1}$$

This brings in the noncommutative W_∞ algebra[15] as the basic symmetry of the QH system. It implies that the electron position cannot be localized to a point within the lowest Landau level, and hence the system cannot be described by local field theory. We construct an effective field theory to describe physics whose scale is larger than the magnetic length $l_B \equiv \sqrt{\hbar/eB}$.

The effective Hamiltonian governing the SU(2) coherence has been derived[10,11] by making a derivative expansion of the Coulomb energy of spin or pseudospin textures at $\nu=1$, where the spin stiffness J is explicitly calculated as

$$J=\frac{1}{16\sqrt{2\pi}}\frac{e^2}{4\pi\varepsilon l_B}. \tag{1.2}$$

It is a straightforward but complicated task to generalize the SU(2) scheme to the SU(4) scheme, because the SU(4) extension of the W_∞ algebra is considerably complicated than the SU(2) extension.[11]

We overcome the problem by employing an alternative formulation. Namely, we construct a Landau-site Hamiltonian by expanding the electron field operator in terms of the one-body wave functions of electrons confined to the lowest Landau level. Then, the exchange Coulomb interaction emerges just as in ferromagnets. An effective Hamiltonian is derived by taking a continuum limit in the von Neumann–lattice formulation, where we substitute the spin stiffness (1.2) for the exchange integral. This is a consistent procedure in the context of the SU(2) coherence. We generalize this procedure to study the SU(4) coherence. In the SU(4)-invariant limit the effective Hamiltonian is given by the SU(4) nonlinear sigma model

$$H_X^{\rm eff}=2J\sum_{a=1}^{15}\int d^2x[\partial_k T_a(\mathbf{x})]^2, \tag{1.3}$$

where $T_a(\mathbf{x})$ is the isospin field normalized as $\Sigma_a T_a T_a = 3/8$ at $\nu=1$. The present approach allows us to analyze QH states at any filling factor. For the sake of simplicity, we discuss only integer QH states, though fractional QH states are treated similarly in the framework of the composite fermion theory.[16]

It is our main purpose to make a thorough investigation of soft waves and skyrmion excitations supported by the exchange Hamiltonian, though some of them have been known previously.[9,17] We examine carefully what are the dynamical fields in the BLQH system. There are three degenerate soft waves in the SU(4)-invariant limit, among which only one soft wave is responsible for the coherent tunneling of electrons between the two layers. The soft modes are Goldstone modes associated with spontaneous breakdown of the SU(4)

isospin symmetry. Actually, the degeneracy is resolved by the Zeeman effect and the tunneling interaction. Namely, the SU(4) symmetry is broken explicitly but softly by these interactions, and Goldstone modes turn into pseudo-Goldstone modes with gaps. It is also shown at $\nu=1$ that there are three independent skyrmion states apart from the translational degree of freedom. They are CP^3 skyrmions enjoying the spin-charge entanglement confined within the lowest Landau level.

II. QUANTUM HALL FERROMAGNETS

To elucidate quantum coherence we start with monolayer QH systems. Electrons make cyclotron motions under perpendicular magnetic field B. The number of flux quanta passing through the system is $N_\Phi \equiv BS/\Phi_D$, where S is the area and $\Phi_D = 2\pi\hbar/e$ is the flux quantum. There are N_Φ electron states per one Landau level by neglecting the spin degree of freedom, each of which is associated with one flux quantum. We call it the Landau site. One Landau site occupies the area $S/N_\Phi = 2\pi l_B^2$, and may accommodate two electrons with up and down spins. The filling factor is $\nu = N/N_\Phi$ with N the total number of electrons. We are concerned about physics taking place in the lowest Landau level.

The microscopic Hamiltonian is a sum of the Coulomb term and the Zeeman term

$$H_{\rm C} = \frac{1}{2}\int d^2x d^2y V(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y}), \tag{2.1}$$

$$H_{\rm Z} = -\frac{1}{2}\Delta_{\rm Z}\int d^2x[\rho^{\uparrow}(\mathbf{x}) - \rho^{\downarrow}(\mathbf{x})], \tag{2.2}$$

where $V(\mathbf{x}-\mathbf{y}) = e^2/4\pi\varepsilon|\mathbf{x}-\mathbf{y}|$ is the Coulomb potential, $\rho^\sigma(\mathbf{x}) = \psi^{\sigma\dagger}(\mathbf{x})\psi^\sigma(\mathbf{x})$ is the electron density with the spin index $\sigma = \uparrow, \downarrow$, $\rho(\mathbf{x}) = \rho^{\uparrow}(\mathbf{x}) + \rho^{\downarrow}(\mathbf{x})$, and $\Delta_{\rm Z} = |g^*\mu_B B|$ is the Zeeman gap with g^* the magnetic g-factor and μ_B the Bohr magneton.

We expand the electron field operator in terms of the one-body wave functions $\varphi_i(\mathbf{x})$ in the lowest Landau level

$$\psi^\sigma(\mathbf{x}) \equiv \sum_{i=1}^{N_\Phi} c_\sigma(i)\varphi_i(\mathbf{x}), \tag{2.3}$$

where $c_\sigma(i)$ is the annihilation operator of the up-spin ($\sigma = \uparrow$) or down-spin ($\sigma = \downarrow$) electron at the Landau site i,

$$\{c_\sigma(i), c_\tau^\dagger(j)\} = \delta_{ij}\delta_{\sigma\tau},$$

$$\{c_\sigma(i), c_\tau(j)\} = \{c_\sigma^\dagger(i), c_\tau^\dagger(j)\} = 0. \tag{2.4}$$

As is well known,[18,19] it is impossible to choose an orthonormal complete set of one-body wave functions $\varphi_i(\mathbf{x})$ in the expansion (2.3). Consequently, the electron field $\psi^\sigma(\mathbf{x})$ does not satisfy a standard canonical anticommutation relation $\{\psi^\sigma(\mathbf{x}), \psi^{\sigma\dagger}(\mathbf{y})\} \neq \delta(\mathbf{x}-\mathbf{y})$, implying that an electron cannot be localized to a point within the lowest Landau level. Nevertheless, roughly speaking, an electron can be localized into a Landau site, which has the area $2\pi l_B^2$. Thus, it is reasonable that QH effects are described by the Landau-site Hamiltonian.

Substituting Eq. (2.3) into Eq. (2.1), we derive the direct and exchange Coulomb energies

$$H_D = \frac{1}{2}\sum_{\langle i,j\rangle} U_{ij} n(i) n(j), \tag{2.5a}$$

$$H_X = 2\sum_{\langle i,j\rangle}\sum_{\sigma,\tau} J_{ij} c_\sigma^\dagger(i) c_\tau^\dagger(j) c_\tau(i) c_\sigma(j), \tag{2.5b}$$

where $n(i) \equiv \Sigma_\sigma c_\sigma^\dagger(i) c_\sigma(i)$ is the electron number at site i: U_{ij} and J_{ij} are the direct and exchange integrals

$$U_{ij} = \int d^2x d^2y \varphi_i^*(\mathbf{x})\varphi_j^*(\mathbf{y}) V(\mathbf{x}-\mathbf{y})\varphi_i(\mathbf{x})\varphi_j(\mathbf{y}), \tag{2.6a}$$

$$J_{ij} = \frac{1}{2}\int d^2x d^2y \varphi_i^*(\mathbf{x})\varphi_j^*(\mathbf{y}) V(\mathbf{x}-\mathbf{y})\varphi_i(\mathbf{y})\varphi_j(\mathbf{x}). \tag{2.6b}$$

These integrals are convergent because the wave function $\varphi_i(\mathbf{x})$ is "localized" within one Landau site i with area $2\pi l_B^2$. The sum $\Sigma_{\langle i,j\rangle}$ runs over all spin pairs ($i \neq j$) just once.

The spin $\mathbf{S}(i)$ is defined at each site i by

$$S_a = (c_\uparrow^\dagger, c_\downarrow^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} \tag{2.7}$$

with τ_a the Pauli matrix. Using the algebraic relation

$$\sum_{\sigma,\tau} c_\sigma^\dagger(i) c_\tau^\dagger(j) c_\tau(i) c_\sigma(j) = -2\mathbf{S}(i)\cdot\mathbf{S}(j) - \frac{1}{2}n(i)n(j), \tag{2.8}$$

we rewrite the exchange term as

$$H_X = -4\sum_{\langle i,j\rangle} J_{ij}\mathbf{S}(i)\cdot\mathbf{S}(j) - \sum_{\langle i,j\rangle} J_{ij} n(i) n(j). \tag{2.9}$$

The Hamiltonian has the global O(3) symmetry: It is invariant when all spins are rotated simultaneously.

At $\nu=1$, in the absence of the Zeeman effect (2.2), the spin direction is determined spontaneously to minimize the exchange energy. The product $\mathbf{S}(i)\cdot\mathbf{S}(j)$ takes the maximum value $\mathbf{S}(i)\cdot\mathbf{S}(j) = 1/4$, when $\mathbf{S}(i) = \mathbf{S}(j)$. Hence, provided $J_{ij} > 0$, all spins are spontaneously polarized to minimize the exchange energy, where the direction of polarization is arbitrary: $\mathbf{S}(i) = \mathbf{S}$ for all points i but the direction of $\mathbf{S}$ is arbitrary. Actually, the direction of the polarization is the z axis due to the Zeeman effect however small it may be. The exchange interaction contributes to the ground-state energy by

$$\langle H_X\rangle_{\rm g} = -2\sum_{\langle i,j\rangle} J_{ij} = -N_\Phi\sum_j J_{ij} = -\frac{1}{2}\varepsilon_X N, \tag{2.10}$$

where

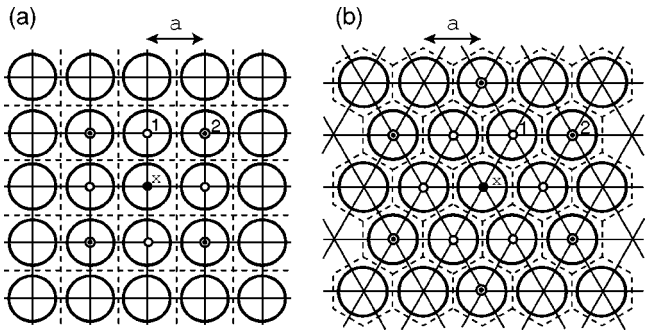

FIG. 1. An electron makes a cyclotron motion occupying an area $2\pi l_B^2$ and avoiding all others. Spin-polarized electrons fill the lowest Landau levelat the filling factor $\nu=1$. Their configuration is represented by a von Neumann lattice with the lattice point identified with the center of the cyclotron motion. Lattice points in the nearest neighborhood of the point **x** are designated by open circles numbered by 1. Lattice points in the second nearest neighborhood are designated by double circles numbered by 2. A square lattice (a) and a triangular lattice (b) are examples of von Neumann lattices.

$$\varepsilon_X \equiv 2\sum_j J_{ij} \tag{2.11}$$

with i fixed arbitrarily; the sum runs over all sites for $J_{ij} \neq 0$. It is clear that the loss of the exchange energy is ε_X when one electron is removed from filled Landau sites. This is equal to the energy necessary to flip one spin in the system (2.9). At $\nu=2$ we obtain $\langle H_X\rangle_g = -\varepsilon_X N_\Phi = -\frac{1}{2}\varepsilon_X N$ since $\mathbf{S}(i)=0$ and $n(i)=2$ in Eq. (2.9).

III. VON NEUMANN LATTICE

Because the QH system is robust against density fluctuations, the direct Coulomb term (2.5a) is irrelevant as far as perturbative fluctuations are concerned. We wish to analyze the exchange interaction (2.9). As we have stated, the exchange integral (2.6b) is convergent. Recall that we have expanded the electron field in terms of one-body wave functions $\varphi_i(\mathbf{x})$ as in Eq. (2.3). The index i may represent the angular momentum in the symmetric gauge or the linear momentum in the Landau gauge. When we evaluate the exchange integral either in the symmetric or Landau gauges, we find a large contribution from a spin pair $\langle i,j\rangle$ even if they are not in the nearest neighborhood of each other. Furthermore, it is not clear at all how the rotational and translational symmetries are recovered in these gauges when the continuum limit is taken: see Ref. 17, for instance.

It is most convenient for us to use a set of one-body wave functions $\varphi_i(\mathbf{x})$ in Eq. (2.3) so that the index i runs over a lattice such as a square lattice or a triangular lattice (Fig. 1) with the lattice point being the center of the cyclotron motion. We can construct such a lattice with the use of the coherent-state representation.

We adopt the symmetric gauge, where $A_x=\frac{1}{2}By$ and $A_y=-\frac{1}{2}Bx$. The angular momentum is given by $L=\hbar b^\dagger b$ in the lowest Landau level with

$$b \equiv \frac{1}{\sqrt{2}l_B}(X-iY) = \frac{1}{\sqrt{2}}\left(z^* + \frac{\partial}{\partial z}\right),$$

$$b^\dagger \equiv \frac{1}{\sqrt{2}l_B}(X+iY) = \frac{1}{\sqrt{2}}\left(z - \frac{\partial}{\partial z^*}\right), \tag{3.1}$$

where $z=(x+iy)/2l_B$. We introduce an eigenstate of the angular-momentum lowering operator b,

$$b|\beta\rangle = \beta|\beta\rangle. \tag{3.2}$$

Because b is an annihilation operator, the state $|\beta\rangle$ is a coherent state by definition, and is given by

$$|\beta\rangle \equiv e^{\beta b^\dagger - \beta^* b}|0\rangle = e^{-|\beta|^2/2}e^{\beta b^\dagger}|0\rangle, \tag{3.3}$$

where $|0\rangle$ is the angular-momentum zero state obeying $b|0\rangle=0$. The wave function $\varphi_\beta(\mathbf{x})=\langle\mathbf{x}|0\rangle$ is calculated as

$$\varphi_\beta(\mathbf{x}) = \frac{1}{\sqrt{2\pi l_B^2}}\exp\left(-\left|z-\frac{1}{\sqrt{2}}\beta\right|^2 + \frac{i(y\beta_{\mathrm{Re}}+x\beta_{\mathrm{Im}})}{\sqrt{2}l_B}\right), \tag{3.4}$$

where $\beta=\beta_{\mathrm{Re}}+i\beta_{\mathrm{Im}}$. It describes an electron localized around the point $z=\beta/\sqrt{2}$.

The coherent state has the minimum uncertainty subject to the Heisenberg uncertainty relation associated with the noncommutativity (1.1) between the coordinates X and Y. The state $|\beta\rangle$ corresponds to the classical state describing a cyclotron motion around the point

$$x=\sqrt{2}l_B\beta_{\mathrm{Re}}, \quad y=-\sqrt{2}l_B\beta_{\mathrm{Im}}, \tag{3.5}$$

as follows from Eqs. (3.1) and (3.2). Because β is an arbitrary complex number, an electron may be localized around any point.

We consider the QH state at $\nu=1$. The system is filled up with electrons each of which occupies an area $2\pi l_B^2$. It is reasonable to put electrons on a lattice with the unit cell area $2\pi l_B^2$. Such a lattice is nothing but a von Neumann lattice.[20–24] The states on a von Neumann lattice form a minimum complete set[21,22] in the lowest Landau level. Thus, we may expand the electron field in terms of coherent states $\varphi_i(\mathbf{x})$ as in Eq. (2.3), where i runs over all lattice points.

We consider a square lattice for simplicity [Fig. 1(a)]. Lattice points are given by $\beta_{mn}=\sqrt{\pi}(m+in)$ or

$$X_m = \sqrt{2\pi}l_B m, \quad Y_n = -\sqrt{2\pi}l_B n, \tag{3.6}$$

so that the unit cell area is $2\pi l_B^2$. States are given by

$$|X_m, Y_n\rangle = \exp[-\pi(m^2+n^2)]\exp[\sqrt{\pi}(m+in)b^\dagger]|0\rangle. \tag{3.7}$$

They are not orthogonal,

$$\langle X_{m'}, Y_{n'}|X_m, Y_n\rangle = \exp\left(-\frac{\pi}{2}[(m-m')^2+(n-n')^2]\right). \tag{3.8}$$

The wave function (3.4) reads

$$\varphi_{mn}(\mathbf{x})=\langle \mathbf{x}|X_m,Y_n\rangle=\frac{1}{\sqrt{2\pi l_B^2}}\exp\left(-\left|z-\frac{1}{\sqrt{2}}\beta_{mn}\right|^2\right)$$
$$\times\exp\left(\frac{i\sqrt{\pi}}{\sqrt{2}l_B}(ym+xn)\right), \tag{3.9}$$

which describes an electron localized around the lattice point $(x,y)=(X_m,Y_n)$.

IV. CONTINUUM LIMIT

In the von Neumann–lattice formulation it is straightforward to take the field-theoretical limit of the exchange energy (2.9), by letting the lattice spacing a vanish just as in a lattice model for ferromagnets. The resulting Hamiltonian describes correctly physical phenomena whose typical size is much larger than the spacing a.

We first analyze the nearest-neighbor terms, for which we set $J_{ij}\equiv J_1$. Let the lattice points be specified by lattice vectors $\mathbf{a}^\alpha$ with $\Sigma_\alpha \mathbf{a}^\alpha=0$. We expand the spin product as

$$\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j=\frac{1}{2}\sum_{\mathbf{x}}\sum_{\alpha}\mathbf{S}(\mathbf{x})\cdot\mathbf{S}(\mathbf{x}+\mathbf{a}^\alpha)$$
$$\simeq\frac{1}{2}\sum_{\mathbf{x}}\sum_{\alpha}\left[\mathbf{S}(\mathbf{x})^2-\frac{1}{2}a_i^\alpha a_j^\alpha\partial_i\mathbf{S}(\mathbf{x})\cdot\partial_j\mathbf{S}(\mathbf{x})\right], \tag{4.1}$$

where a partial integration was made. The exchange Hamiltonian (2.9) yields

$$H_X^{\rm eff}\simeq J_1\left(\sum_\alpha a_i^\alpha a_j^\alpha\right)\sum_{\mathbf{x}}\partial_i\mathbf{S}(\mathbf{x})\cdot\partial_i\mathbf{S}(\mathbf{x}) \tag{4.2}$$

as the lowest order term in the derivative expansion. The ground-state energy is given by Eq. (2.10) with $\varepsilon_X=2J_1\Sigma_\alpha$. We next analyze the second nearest-neighbor terms with the lattice vectors $\mathbf{b}^\beta$, for which we set $J_{ij}\equiv J_2$. We obtain the same formula as Eq. (4.2) with the replacement of $\mathbf{a}^\beta$ by $\mathbf{b}^\beta$. Any lattice points can be treated in the same way.

We explicitly consider a square lattice [Fig. 1(a)] as a simplest von Neumann lattice, where $\Sigma_{\alpha=1}^4 a_i^\alpha a_j^\alpha=2a^2\delta_{ij}$ and $\Sigma_x=a^{-2}\int d^2x$. Hence, Eq. (4.2) amounts to the O(3) nonlinear sigma model

$$H_X^{\rm eff}=2J\int d^2x\,\partial_i\mathbf{S}(\mathbf{x})\cdot\partial_i\mathbf{S}(\mathbf{x}), \tag{4.3}$$

where $J=J_1$ and $h_X=8J_1$ in the ground-state energy (2.10) for the nearest-neighbor terms. It is easy to see that $J=J_1+2J_2+4J_3+\cdots$ and $\varepsilon_X=8J_1+8J_2+8J_3+\cdots$ by taking into account all lattice points; the series would converge rapidly. When we adopt another lattice such as the triangular lattice [Fig. 1(b)] and take the continuum limit, we reproduce the same effective Hamiltonian (4.3) together with the ground-state energy (2.10) but with different definitions of J and ε_X in terms of the exchange integrals J_{ij}.

We determine the parameters J and ε_X as follows. The effective Hamiltonian (4.3) was first proposed to study skyrmion excitations,[1] where the spin stiffness J was identified[25] with

$$J=\frac{1}{16\sqrt{2\pi}}\frac{e^2}{4\pi\varepsilon l_B}. \tag{4.4}$$

The formula has been verified by evaluating explicitly the energy of a spin texture.[10,11] We next estimate the parameter ε_X, by substituting the skyrmion configuration

$$S_x=\frac{\kappa x}{r^2+\kappa^2},\quad S_y=-\frac{\kappa y}{r^2+\kappa^2},\quad S_z=\frac{1}{2}\frac{r^2-\kappa^2}{r^2+\kappa^2} \tag{4.5}$$

into the nonlinear sigma model (4.3). One skyrmion increases the exchange energy[1] by

$$\langle H_X\rangle_{\rm sky}=4\pi J, \tag{4.6}$$

which is independent of the skyrmion size κ in Eq. (4.5). In its small-size limit ($\kappa\to 0$) the skyrmion is reduced to a hole.[26] The resultant system is the QH system from which one electron has been removed. It corresponds to the loss of the exchange energy ε_X in ferromagnets when one electron is removed. Hence, $\varepsilon_X=4\pi J$.

Consequently, the effective Hamiltonian is given by Eq. (4.3), which is appropriate to analyze phenomena whose scale is larger than the magnetic length l_B. The ground-state exchange energy is given by

$$\langle H_X\rangle_{\rm g}=-2\pi JN, \tag{4.7}$$

together with the spin stiffness (4.4). The effective Hamiltonian (4.3) describes the spin wave in the QH ferromagnet at $\nu=1$. The spin wave is a Goldstone mode associated with the global O(3) symmetry spontaneously broken. Due to the Zeeman effect the Goldstone mode acquires a gap and the coherent length is made finite, see Eq. (8.29a).

V. BILAYER QUANTUM HALL FERROMAGNETS

We generalize arguments to analyze electrons in the lowest Landau level in BLQH systems. The SU(2) pseudospin structure is introduced by assigning up (down) pseudospin to the front (back) layer. One Landau site contains four electron states in the lowest Landau level, which are distinguished by the SU(4) isospin index $\sigma=f\uparrow,f\downarrow,b\uparrow,b\downarrow$. For instance, $\sigma=f\uparrow$ implies that the electron is in the front layer and its spin is up. The group SU(4) is generated by the Hermitian, traceless, 4×4 matrices. There are (4^2-1) independent matrices. We take a standard basis,[27] λ_a, $a=1,2,\ldots,15$, normalized as $\mathrm{Tr}(\lambda_a\lambda_b)=2\delta_{ab}$. They are the generalization of the Pauli matrices.

We decompose the microscopic Coulomb interaction into two terms

$$H_C^+=\frac{1}{2}\int d^2x d^2y V_+(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y}), \tag{5.1a}$$

$$H_C^- = \frac{1}{2}\int d^2x d^2y V_-(\mathbf{x}-\mathbf{y})\Delta\rho(\mathbf{x})\Delta\rho(\mathbf{y}), \quad (5.1b)$$

where H_C^+ depends on the total density $\rho(\mathbf{x})$, and H_C^- on the density difference $\Delta\rho(\mathbf{x})$ between the front and back layers

$$\Delta\rho(\mathbf{x}) = \rho^{f\uparrow}(\mathbf{x}) + \rho^{f\downarrow}(\mathbf{x}) - \rho^{b\uparrow}(\mathbf{x}) - \rho^{b\downarrow}(\mathbf{x}). \quad (5.2)$$

The Coulomb term H_C^+ is invariant under the SU(4) transformation.

The electron field $\psi^\sigma(\mathbf{x})$ is expanded as in Eq. (2.3),

$$\psi^\sigma(\mathbf{x}) \equiv \sum_{i=1}^{N_\Phi} c_\sigma(i)\varphi_i(\mathbf{x}), \quad (5.3)$$

where $c_\sigma(i)$ is the annihilation operator of the electron with isospin σ at site i. Substituting the expansion (5.3) into the Coulomb term (5.1), we extract the direct and exchange Coulomb terms. Because the QH system is robust against density fluctuations, the direct Coulomb term arising from the SU(4)-invariant term (5.1a) is irrelevant as far as perturbative fluctuations are concerned. The direct term from the SU(4)-noninvariant term (5.1b) is

$$H_{\rm cap} = \varepsilon_{\rm cap}\sum_{i=1}^{N_\Phi} P_z(i)P_z(i), \quad (5.4)$$

where $P_z = P_z^\uparrow + P_z^\downarrow$ at each site and

$$\varepsilon_{\rm cap} = \frac{e^2}{4\pi\varepsilon l_B}\sqrt{\frac{\pi}{2}}(1 - e^{d^2/2l_B^2}\{1 - {\rm erf}(d/\sqrt{2}l_B)\}) \quad (5.5)$$

with the error function erf(x). Here, $\mathbf{P}^\uparrow(i)$ is the SU(2) pseudospin at site i made of the two component spinor $(c_{f\uparrow}, c_{b\uparrow})$ as in Eq. (2.7). We call the SU(4)-noninvariant Coulomb interaction (5.1b) the capacitance term since $\varepsilon_{\rm cap}P_z(i)P_z(i)$ describes the capacitance energy per one Landau site.

We proceed to study the exchange Coulomb interaction. For this purpose we define the SU(4) isospin at each site i by

$$T_a = (c_{f\uparrow}^\dagger, c_{f\downarrow}^\dagger, c_{b\uparrow}^\dagger, c_{b\downarrow}^\dagger)\frac{\lambda_a}{2}\begin{pmatrix} c_{f\uparrow} \\ c_{f\downarrow} \\ c_{b\uparrow} \\ c_{b\downarrow} \end{pmatrix}. \quad (5.6)$$

Substituting the expansion (5.3) into Eq. (5.1a), and using the algebraic relation

$$\sum_{\sigma,\tau} c_\sigma^\dagger(i)c_\tau^\dagger(j)c_\tau(i)c_\sigma(j) = -2\mathbf{T}(i)\cdot\mathbf{T}(j) - \frac{1}{4}n(i)n(j), \quad (5.7)$$

we obtain the SU(4)-invariant exchange energy as

$$H_X^+ = -4\sum_{\langle i,j\rangle} J_{ij}^+\left(\mathbf{T}(i)\cdot\mathbf{T}(j) + \frac{1}{8}n(i)n(j)\right). \quad (5.8)$$

The exchange integral J_{ij}^+ is defined by Eq. (2.6b) with the use of the Coulomb potential $V_+(\mathbf{x}-\mathbf{y})$. The Hamiltonian (5.8) takes the minimum value when the product $\mathbf{T}(i)\cdot\mathbf{T}(j)$ takes the maximum value. It occurs for $\mathbf{T}(i) = \mathbf{T}(j)$, where $\mathbf{T}(i)\cdot\mathbf{T}(j) = 3/8$ at $\nu = 1$.

It is convenient to decompose the exchange term (5.8) in terms of various SU(2) components

$$\begin{aligned} H_X^+ = &-4\sum_{\langle i,j\rangle} J_{ij}^+[\mathbf{S}^f(i)\cdot\mathbf{S}^f(j) + \mathbf{S}^b(i)\cdot\mathbf{S}^b(j)]_{xy} \\ &-4\sum_{\langle i,j\rangle} J_{ij}^+[\mathbf{P}^\uparrow(i)\cdot\mathbf{P}^\uparrow(j) + \mathbf{P}^\downarrow(i)\cdot\mathbf{P}^\downarrow(j)]_{xy} \\ &-4\sum_{\langle i,j\rangle} J_{ij}^+[\mathbf{I}(i)\cdot\mathbf{I}(j) + \tilde{\mathbf{I}}(i)\cdot\tilde{\mathbf{I}}(j)]_{xy} \\ &-2\sum_{\langle i,j\rangle} J_{ij}^+\left(\sum_{\sigma=1}^{4} n^\sigma(i)n^\sigma(j)\right), \end{aligned} \quad (5.9)$$

where $[\mathbf{S}^f(i)\cdot\mathbf{S}^f(j)]_{xy} \equiv S_x^f(i)S_x^f(j) + S_y^f(i)S_y^f(j)$, etc., and

$$\begin{aligned} &S_a^f = (c_{f\uparrow}^\dagger, c_{f\downarrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{f\uparrow} \\ c_{f\downarrow}\end{pmatrix}, \quad S_a^b = (c_{b\uparrow}^\dagger, c_{b\downarrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{b\uparrow} \\ c_{b\downarrow}\end{pmatrix}, \\ &P_a^\uparrow = (c_{f\uparrow}^\dagger, c_{b\uparrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{f\uparrow} \\ c_{b\uparrow}\end{pmatrix}, \quad P_a^\downarrow = (c_{f\downarrow}^\dagger, c_{b\downarrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{f\downarrow} \\ c_{b\downarrow}\end{pmatrix}, \\ &I_a = (c_{f\uparrow}^\dagger, c_{b\downarrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{f\uparrow} \\ c_{b\downarrow}\end{pmatrix}, \quad \tilde{I}_a = (c_{f\downarrow}^\dagger, c_{b\uparrow}^\dagger)\frac{\tau_a}{2}\begin{pmatrix} c_{f\downarrow} \\ c_{b\uparrow}\end{pmatrix}. \end{aligned} \quad (5.10)$$

The exchange energy due to the SU(4)-noninvariant term (5.1b) is also evaluated. Combining them we obtain

$$\begin{aligned} H_X = &-4\sum_{\langle i,j\rangle} J_{ij}[\mathbf{S}^f(i)\cdot\mathbf{S}^f(j) + \mathbf{S}^b(i)\cdot\mathbf{S}^b(j)]_{xy} \\ &-4\sum_{\langle i,j\rangle} J_{ij}^d[\mathbf{P}^\uparrow(i)\cdot\mathbf{P}^\uparrow(j) + \mathbf{P}^\downarrow(i)\cdot\mathbf{P}^\downarrow(j)]_{xy} \\ &-4\sum_{\langle i,j\rangle} J_{ij}^d[\mathbf{I}(i)\cdot\mathbf{I}(j) + \tilde{\mathbf{I}}(i)\cdot\tilde{\mathbf{I}}(j)]_{xy} \\ &-2\sum_{\langle i,j\rangle} J_{ij}\left(\sum_{\sigma=1}^{4} n^\sigma(i)n^\sigma(j)\right), \end{aligned} \quad (5.11)$$

where $J_{ij}^d \equiv 2J_{ij}^+ - J_{ij}$.

The exchange Hamiltonian (5.11) is valid at any integer filling factor with common exchange integrals J_{ij} and J_{ij}^d. It is an operator which may act on states possessing various isospins. We may restrict the Hilbert space appropriately at a specific filling factor. We examine two special limits to see that it is reduced to the well-established results at $\nu = 1$. First, we apply a large bias voltage and move all electrons to the front layer. The resulting system is dynamically equivalent to the monolayer system with spin. Indeed, by using $S_z = (n^\uparrow - n^\downarrow)/2$ and $n = n^\uparrow + n^\downarrow$, it is easy to see that Eq. (5.11) is reduced to the exchange interaction (2.9) describing the monolayer QH ferromagnet. Second, we assume a large Zeeman effect so that all spins are forced to be polarized,

where the system describes the spin-frozen bilayer system. We now use $P_z=(n^f-n^b)/2$ and $n=n^f+n^b$ to rewrite Eq. (5.11) as

$$H_X=-4\sum_{\langle i,j\rangle}\{J_{ij}P_z(i)P_z(j)+J^d_{ij}[P_x(i)P_x(j)+P_y(i)P_y(j)]\}-\sum_{\langle i,j\rangle}J_{ij}n(i)n(j). \tag{5.12}$$

By taking the continuum limit as in Sec. IV, the effective Hamiltonian is found to be

$$H_X^{\rm eff}=2J^d\int d^2x[\partial_i P_x(\mathbf{x})\cdot\partial_i P_x(\mathbf{x})+\partial_i P_y(\mathbf{x})\cdot\partial_i P_y(\mathbf{x})]+2J\int d^2x\,\partial_i P_z(\mathbf{x})\cdot\partial_i P_z(\mathbf{x}), \tag{5.13}$$

where the pseudospin field obeys the normalization condition $\mathbf{P}(\mathbf{x})^2=1/4$ at $\nu=1$; the stiffness J is given by Eq. (4.4) and

$$J^d=\frac{1}{8}\rho_0 l_B^4\int\frac{d^2q}{2\pi}V(\mathbf{q})e^{-|\mathbf{q}|d}\mathbf{q}^2\exp\left[-\frac{l_B^2}{2}\mathbf{q}^2\right] \tag{5.14}$$

or

$$\frac{J^d}{J}=-\sqrt{\frac{2}{\pi}}\frac{d}{l_B}+\left(1+\frac{d^2}{l_B^2}\right)e^{d^2/2l_B^2}[1-\mathrm{erf}(d/\sqrt{2}l_B)]. \tag{5.15}$$

It agrees with the effective Hamiltonian obtained from the Coulomb energy of the pseudospin texture,[10,11] where $J=\rho_A$ and $J^d=\rho_E$ in their notation.

VI. U(1) GAUGE SYMMETRIES

The density imbalance $\sigma_0\equiv 2\langle P_z(i)\rangle$ between the two layers is controlled by applying a bias voltage. It affects the system via the interaction term

$$H_{\rm bias}=-eV_{\rm bias}\sum_i P_z(i). \tag{6.1}$$

The tunneling interaction is

$$H_T=-\Delta_{\rm SAS}\sum_i P_x(i), \tag{6.2}$$

where $\Delta_{\rm SAS}$ is the tunneling gap between the symmetric and antisymmetric states. The total Hamiltonian $H_{\rm tot}$ is the sum of the exchange term (5.11) and

$$H_D=\sum_{i=1}^{N_\Phi}[-\Delta_Z S_z(i)+\varepsilon_{\rm cap}P_z(i)P_z(i)-\Delta_{\rm SAS}P_x(i)-eV_{\rm bias}P_z(i)]. \tag{6.3}$$

It consists of the Zeeman term, the capacitance term, the tunneling term and the bias term.

Since one electron has four components, we may perform local U(4) transformations to the electron field. However, the Hamiltonian is not invariant under most of them. The symmetry of the direct interaction H_D is a direct product of two U(1) symmetries $\mathrm{U}^\uparrow(1)\otimes\mathrm{U}^\downarrow(1)$,

$$\begin{pmatrix}\psi^{f\uparrow}(\mathbf{x})\\ \psi^{b\uparrow}(\mathbf{x})\end{pmatrix}\to e^{i\alpha(\mathbf{x})}\begin{pmatrix}\psi^{f\uparrow}(\mathbf{x})\\ \psi^{b\uparrow}(\mathbf{x})\end{pmatrix},\qquad \begin{pmatrix}\psi^{f\downarrow}(\mathbf{x})\\ \psi^{b\downarrow}(\mathbf{x})\end{pmatrix}\to e^{i\beta(\mathbf{x})}\begin{pmatrix}\psi^{f\downarrow}(\mathbf{x})\\ \psi^{b\downarrow}(\mathbf{x})\end{pmatrix}. \tag{6.4}$$

The exchange interaction H_X breaks this into a single U(1) symmetry

$$\psi^\sigma(\mathbf{x})\to e^{i\alpha(\mathbf{x})}\psi^\sigma(\mathbf{x}). \tag{6.5}$$

This is the exact local symmetry of the total Hamiltonian. It should be emphasized, however, that there is no gapless mode because there is no propagating mode associated with it: Indeed, the kinetic term of the would-be phase field $\psi^{\sigma\dagger}(\mathbf{x})\partial_k\psi^\sigma(\mathbf{x})$, is absent in the Hamiltonian.

It is important to recognize[17] that the gauge symmetry (6.5) characterizes the genuine BLQH system. [See Eq. (8.14) why we call it the gauge symmetry.] To show this, let us consider a system where the two layers are separated sufficiently so that there are no interlayer exchange interaction ($J^d_{ij}=0$) nor the tunneling interaction ($\Delta_{\rm SAS}=0$). Then, the total Hamiltonian is invariant under two local transformations $U^f(1)$ and $U^b(1)$, which act on electrons on the two layers independently,

$$\begin{pmatrix}\psi^{f\uparrow}(\mathbf{x})\\ \psi^{f\downarrow}(\mathbf{x})\end{pmatrix}\to e^{i\alpha(\mathbf{x})}\begin{pmatrix}\psi^{f\uparrow}(\mathbf{x})\\ \psi^{f\downarrow}(\mathbf{x})\end{pmatrix},\qquad \begin{pmatrix}\psi^{b\uparrow}(\mathbf{x})\\ \psi^{b\downarrow}(\mathbf{x})\end{pmatrix}\to e^{i\beta(\mathbf{x})}\begin{pmatrix}\psi^{b\uparrow}(\mathbf{x})\\ \psi^{b\downarrow}(\mathbf{x})\end{pmatrix}. \tag{6.6}$$

We may also consider a case without the interlayer exchange interaction ($J^d_{ij}=0$) but with the tunneling interaction ($\Delta_{\rm SAS}\neq 0$). Then, the symmetry (6.6) is broken into the symmetry (6.5). The number of U(1) gauge symmetries distinguish various bilayer systems. We come back to this observation to examine the dynamical degrees of freedom in Sec. VIII.

VII. GROUND-STATE ENERGIES

We evaluate the ground-state energy. Let us consider the case $\nu=1$. Unless $\langle P^z(i)\rangle=\pm 1/2$ electrons are not localized in one of the two layers but rather expand over the two layers. The ground state is the up-spin bonding state, which is reduced to the up-spin symmetric state in the balanced configuration with $\langle P^z(i)\rangle=0$. When $\langle P^z(i)\rangle=\sigma_0/2$, the exchange Coulomb energy reads

$$\langle H_X\rangle_{\nu=1}=-\sum_{\langle i,j\rangle}[(1+\sigma_0^2)J_{ij}+(1-\sigma_0^2)J^d_{ij}]=-2\pi(J^+ +\sigma_0^2 J^-)N_\Phi, \tag{7.1}$$

where $2J^\pm\equiv J\pm J^d$. The ground-state energy is

$$\frac{\langle H\rangle_{\nu=1}}{N_\Phi}=-\frac{1}{2}(\Delta_Z+\Delta_{SAS})+\frac{1}{4}\sigma_0^2\varepsilon_{cap}-\frac{1}{2}\sigma_0 eV_{bias}$$
$$-2\pi(J^+ +\sigma_0^2 J^-). \tag{7.2}$$

In particular, $\langle H\rangle_{\nu=1}=-\frac{1}{2}(\Delta_Z+\Delta_{SAS})N_\Phi-2\pi J^+ N_\Phi$ in the balanced configuration ($\sigma_0=0$).

We next consider the case $\nu=2$. Since two electrons exist in one Landau site, we make the composition of (pseudo)spins $\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1$. We have two types of states within the lowest Landau level: (a) three pseudospin-singlet and spin-triplet states (the spin sector) and (b) three pseudospin-triplet and spin-singlet states (the ppin sector).

The spin sector consists of $|f^\uparrow,b^\uparrow\rangle,(1/\sqrt{2})(|f^\uparrow,b^\downarrow\rangle+|f^\downarrow,b^\uparrow\rangle),|f^\downarrow,b^\downarrow\rangle$. They are the eigenstates of the total Hamiltonian within the sector. The ground state is given by $|f^\uparrow,b^\uparrow\rangle$, and the ground-state energy is

$$\langle H\rangle_{\nu=2}^{spin}=-(\Delta_Z+4\pi J)N_\Phi . \tag{7.3}$$

The state is stable only in the balanced configuration.

The ppin sector consists of $|f^\uparrow,f^\downarrow\rangle,(1/\sqrt{2})(|f^\uparrow,b^\downarrow\rangle-|f^\downarrow,b^\uparrow\rangle)$, $|b^\uparrow,b^\downarrow\rangle$. Within the sector the total Hamiltonian reads

$$H^{ppin}=N_\Phi\begin{pmatrix}\varepsilon_{cap}+eV_{bias} & -\Delta_{SAS}/\sqrt{2} & 0\\ -\Delta_{SAS}/\sqrt{2} & 0 & -\Delta_{SAS}/\sqrt{2}\\ 0 & -\Delta_{SAS}/\sqrt{2} & \varepsilon_{cap}-eV_{bias}\end{pmatrix} \tag{7.4}$$

apart from a constant exchange energy. The eigenvalue equation is easily solved in the balanced configuration with the zero bias voltage ($V_{bias}=0$). The ground state is given by

$$|g\rangle_{\nu=2}^{ppin}=-\cos\theta|S^\uparrow,S^\downarrow\rangle+\sin\theta|A^\uparrow,A^\downarrow\rangle, \tag{7.5}$$

where $\tan\theta=\varepsilon_{cap}/(2\Delta_{SAS}+\sqrt{4\Delta_{DSAS}^2+\varepsilon_{cap}^2})$. Here, $|S^\uparrow,S^\downarrow\rangle$ and $|A^\uparrow,A^\downarrow\rangle$ are the symmetric and antisymmetric states. The ground state is no longer a symmetric state unless $\varepsilon_{cap}=0$ or $d=0$. A certain amount of the antisymmetric state is necessarily mixed due to the capacitance effect. The ground-state energy is

$$\frac{\langle H\rangle_{\nu=2}^{ppin}}{N_\Phi}=\frac{1}{2}(\varepsilon_{cap}-\sqrt{4\Delta_{SAS}^2+\varepsilon_{cap}^2})-2\pi(J+J^d\cos^2 2\theta) \tag{7.6}$$

in the balanced configuration. The state is stable also in unbalanced configurations.

The (pseudo)spin composition at $\nu=3$ reads $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$, where only the doublet is allowed within the lowest Landau level. The ground state is a pseudospin doublet and spin doublet. It is essentially the same as the bilayer state at $\nu=1$. The ground state is

$$|g\rangle_{\nu=3}=\frac{1}{\sqrt{2}}(|f^\uparrow,f^\uparrow,b^\downarrow\rangle+|f^\uparrow,b^\downarrow,b^\uparrow\rangle), \tag{7.7}$$

and its energy is

$$\frac{\langle H\rangle_{\nu=3}}{N_\Phi}=-\frac{1}{2}(\Delta_Z+\Delta_{SAS})-\pi(5J+J^d) \tag{7.8}$$

in the balanced configuration. The state is stable also in unbalanced configuration.

At $\nu=4$ all the Landau sites are filled up. The ground state is pseudospin-singlet and spin-singlet. The ground state is

$$|g\rangle_{\nu=4}=|f^\uparrow,f^\downarrow,b^\uparrow,b^\downarrow\rangle \tag{7.9}$$

and its energy is

$$\langle H\rangle_{\nu=4}=-8\pi JN_\Phi . \tag{7.10}$$

The state is stable only in the balanced configuration.

In the SU(4)-invariant limit, where $d\to 0$ and $J^d\to J$, the exchange energy is reduced to a unified formula $\langle H_X\rangle=-2\pi JN$ at $\nu=1,2,3,4$. In the SU(4)-noninvariant case we note the following intriguing properties. (A) In the ''layer basis,'' where we take four independent one-body states $|f\uparrow\rangle,|f\downarrow\rangle,|b\uparrow\rangle$, and $|b\downarrow\rangle$, the exchange interaction operates only between the same isospin states, i.e.,

$$\langle H_X\rangle=-2\pi JN \tag{7.11}$$

between a pair of $|f\uparrow\rangle$'s, a pair of $|f\downarrow\rangle$'s, a pair of $|b\uparrow\rangle$'s and a pair of $|b\downarrow\rangle$'s: All others vanish. (B) In the ''SAS'' basis, where we take four independent one-body states $|S\uparrow\rangle,|S\downarrow\rangle,|A\uparrow\rangle$, and $|A\downarrow\rangle$, we naively expect that the exchange interaction operates only between the same isospin states as before, i.e.,

$$\langle H_X\rangle=-\pi(J+J^d)N \tag{7.12}$$

between a pair of $|S\uparrow\rangle$'s, a pair of $|S\downarrow\rangle$'s, a pair of $|A\uparrow\rangle$'s, and a pair of $|A\downarrow\rangle$'s: Actually there appears also an exchange interaction between different isospin states, i.e.,

$$\langle H_X\rangle=-\pi(J-J^d)N \tag{7.13}$$

between $|S\uparrow\rangle$ and $|A\uparrow\rangle$, and between $|S\downarrow\rangle$ and $|A\downarrow\rangle$: All others vanish. We recover the naive expectation in the SU(4) invariant limit.

We explain why the exchange term J^d does not appear in the layer basis but does in the SAS basis. It arises for instance from the term $\mathbf{P}^\uparrow(i)\cdot\mathbf{P}^\uparrow(j)$ in Eq. (5.11). We find

$$\langle f\uparrow|\mathbf{P}^\uparrow(i)|f\uparrow\rangle=\langle f\downarrow|\mathbf{P}^\downarrow(i)|f\downarrow\rangle=\left(0,0,\frac{1}{2}\right),$$
$$\langle b\uparrow|\mathbf{P}^\uparrow(i)|b\uparrow\rangle=\langle b\downarrow|\mathbf{P}^\downarrow(i)|b\downarrow\rangle=-\left(0,0,\frac{1}{2}\right), \tag{7.14}$$

while

$$\langle S\uparrow|\mathbf{P}^\uparrow(i)|S\uparrow\rangle=\langle S\downarrow|\mathbf{P}^\downarrow(i)|S\downarrow\rangle=\left(\frac{1}{2},0,0\right),$$
$$\langle A\uparrow|\mathbf{P}^\uparrow(i)|A\uparrow\rangle=\langle A\downarrow|\mathbf{P}^\downarrow(i)|A\downarrow\rangle=-\left(\frac{1}{2},0,0\right). \tag{7.15}$$

Because only the x and y components contributes to the exchange interaction (5.11), there is no contribution in the layer basis but there is in the SAS basis.

VIII. SU(4) SOFT WAVES

We investigate the SU(4) soft waves at $\nu=1$, which are perturbative excitations supported by the exchange interaction. To identify the dynamical degree of freedom we use the composite boson (CB) theory of quantum Hall ferromagnets[9,26] by attaching flux quanta to electrons.[28–30] The CB field $\phi^\sigma(\mathbf{x})$ is defined by making a singular phase transformation to the electron field $\psi^\sigma(\mathbf{x})$,

$$\phi^\sigma(\mathbf{x})=e^{-ie\Theta(\mathbf{x})}\psi^\sigma(\mathbf{x}), \tag{8.1}$$

where the phase field $\Theta(\mathbf{x})$ attaches one flux quantum to each electron via the relation $\varepsilon_{ij}\partial_i\partial_j\Theta(\mathbf{x})=\phi_D\rho(\mathbf{x})$. We then introduce the normalized CB field $n^\sigma(\mathbf{x})$ by

$$\phi^\sigma(\mathbf{x})=\phi(\mathbf{x})n^\sigma(\mathbf{x}), \tag{8.2}$$

so that the N-component field $n^\sigma(\mathbf{x})$ obeys the constraint

$$\mathbf{n}^\dagger(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})=\sum_\sigma n^{\sigma\dagger}(\mathbf{x})n^\sigma(\mathbf{x})=1. \tag{8.3}$$

It follows that $\phi(\mathbf{x})=\sqrt{\rho(\mathbf{x})}$. Because the QH system is robust against density fluctuations, as far as perturbative fluctuations are concerned, we may set $\rho(\mathbf{x})=\rho_0$ or

$$\phi^\sigma(\mathbf{x})=\sqrt{\rho_0}n^\sigma(\mathbf{x}), \tag{8.4}$$

where $\rho_0=N/S$ is the average electron density.

We count the number of independent fields. The field $\mathbf{n}(\mathbf{x})$ consists of four complex fields, but one real field is eliminated by the constraint (8.3). Furthermore, the U(1) phase field is not dynamical due to the gauge symmetry (6.5), or

$$n^\sigma(\mathbf{x})\to e^{i\alpha(\mathbf{x})}n^\sigma(\mathbf{x}). \tag{8.5}$$

See also Eq. (8.12). This is the only gauge symmetry in the BLQH system. Hence, it contains only three independent complex fields. Such a field is the CP^3 field.[32]

A comment is in order. But for the tunneling interaction and the interlayer exchange interaction, the symmetry group is given by Eq. (6.6), or

$$\begin{pmatrix} n^{f\uparrow}(\mathbf{x}) \\ n^{f\downarrow}(\mathbf{x}) \end{pmatrix}\to e^{i\alpha(\mathbf{x})}\begin{pmatrix} n^{f\uparrow}(\mathbf{x}) \\ n^{f\downarrow}(\mathbf{x}) \end{pmatrix},$$

$$\begin{pmatrix} n^{b\uparrow}(\mathbf{x}) \\ n^{b\downarrow}(\mathbf{x}) \end{pmatrix}\to e^{i\beta(\mathbf{x})}\begin{pmatrix} n^{b\uparrow}(\mathbf{x}) \\ n^{b\downarrow}(\mathbf{x}) \end{pmatrix}. \tag{8.6}$$

Because there exist two U(1) gauge symmetries, we have a set of two CP^1 fields rather than one CP^3 field.

We conclude that the dynamical field is the CP^3 field in the BLQH system due to the exchange interaction and the tunneling interaction. See also Sec. IX. The isospin field $\mathbf{T}$ and the CP^3 field $\mathbf{n}$ are related by

$$T_a(\mathbf{x})=\mathbf{n}^\dagger(\mathbf{x})\frac{\lambda_a}{2}\mathbf{n}(\mathbf{x}). \tag{8.7}$$

Though there are 15 isospin components, only 6 of them are independent.

Let us first analyze the exchange Hamiltonian in the SU(4)-invariant limit, where the exchange interaction (5.8) yields a nonlinear sigma model

$$H_X^{\rm eff}=2J^+\sum_{a=1}^{15}\int d^2x[\partial_k T_a(\mathbf{x})]^2. \tag{8.8}$$

The SU(4) isospin field obeys the normalization condition $\mathbf{T}(\mathbf{x})^2=3/8$ at $\nu=1$.

Since the independent fields are the CP^3 fields $n^\sigma(\mathbf{x})$, we rewrite the exchange Hamiltonian (8.8) in terms of them. Let us define

$$\mathcal{T}(\mathbf{x})=\sum_a T^a(\mathbf{x})\frac{\lambda_a}{2}. \tag{8.9}$$

We then have

$$\mathcal{T}^{\alpha\beta}(\mathbf{x})=-\frac{1}{2N}[\delta^{\alpha\beta}-Nn^{\beta\dagger}(\mathbf{x})n^\alpha(\mathbf{x})] \tag{8.10}$$

with $N=4$ for the SU(4) isospin field. Using this, it is straightforward to derive[31] from Eq. (8.8) that

$$H_X^{\rm eff}=2J^+\int d^2x\{(\partial_j\mathbf{n}^\dagger\cdot\partial_j\mathbf{n})-(\mathbf{n}^\dagger\cdot\partial_j\mathbf{n})(\partial_j\mathbf{n}^\dagger\cdot\mathbf{n})\}. \tag{8.11}$$

This Hamiltonian has the U(1) gauge symmetry (8.5). To see this more explicitly, we rewrite it as[31,32]

$$H_X^{\rm eff}=2J^+\int d^2x(\partial_j\mathbf{n}^\dagger+iK_j\mathbf{n}^\dagger)\cdot(\partial_j\mathbf{n}-iK_j\mathbf{n}), \tag{8.12}$$

with

$$K_\mu(\mathbf{x})=-i\mathbf{n}^\dagger(\mathbf{x})\partial_\mu\mathbf{n}(\mathbf{x}). \tag{8.13}$$

The Hamiltonian (8.12) is invariant under the gauge tranformation

$$n^\sigma(\mathbf{x})\to e^{i\alpha(\mathbf{x})}n^\sigma(\mathbf{x}),\quad K_\mu\to K_\mu+\partial_\mu\alpha(\mathbf{x}). \tag{8.14}$$

Here, the field K_μ is not a dynamical field,[32] since it is an auxiliary field defined by Eq. (8.13).

We study small fluctuations of the SU(4) soft waves in the balanced configuration with no bias voltage ($V_{\rm bias}=0$). The ground state is an up-spin symmetric state at $\nu=1$. It is convenient to use the SAS basis rather than the layer basis. The ground state is given by

$$(n^{S\uparrow},n^{S\downarrow},n^{A\uparrow},n^{A\downarrow})=(1,0,0,0). \tag{8.15}$$

We expand the CP^3 field up to the first order of fluctuation fields (Fig. 2),

$$(n^{S\uparrow},n^{S\downarrow},n^{A\uparrow},n^{A\downarrow})\simeq(1,\zeta_1,\zeta_2,\zeta_3), \tag{8.16}$$

where

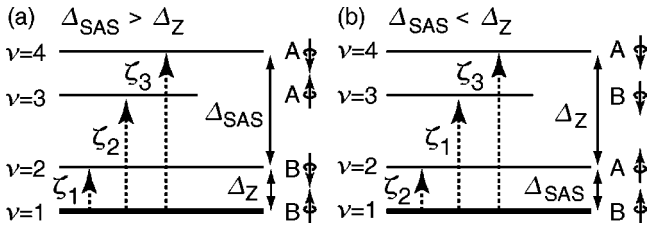

FIG. 2. The lowest Landau level contains four energy levels corresponding to the two layers and the two spin states. They are represented (a) for $\Delta_{SAS} > \Delta_Z$ and (b) for $\Delta_{SAS} < \Delta_Z$. The lowest-energy level consists of up-spin bonding states, and is filled at $\nu = 1$. Small fluctuations are Goldstone modes ζ_1, ζ_2, and ζ_3.

$$\zeta_i(\mathbf{x}) = \frac{1}{2}[\sigma_i(\mathbf{x}) + i\,\vartheta_i(\mathbf{x})]. \tag{8.17}$$

They are canonical fields obeying [see Eq. (8.4)]

$$[\zeta_i(\mathbf{x}), \zeta_j^\dagger(\mathbf{y})] = \rho_0^{-1}\delta_{ij}\delta(\mathbf{x}-\mathbf{y}). \tag{8.18}$$

It is manifest that $\rho_0\sigma_i(\mathbf{x})$ denotes the number density excited from the ground state $|S\uparrow\rangle$ to the ith level designated by Eq. (8.16). The field $\vartheta_i(\mathbf{x})$ is the conjugate phase variable.

We expand the exchange interaction (8.12) up to the second order

$$\mathcal{H}_X^{\text{eff}} = 2J^+ \sum_{i=1}^{3} \partial_k \zeta_i^\dagger(\mathbf{x}) \partial_k \zeta_i(\mathbf{x})$$
$$= \frac{J^+}{2} \sum_{i=1}^{3} \{(\partial_k \sigma_i)^2 + (\partial_k \vartheta_i)^2\}. \tag{8.19}$$

This Hamiltonian describes three Goldstone modes associated with spontaneous symmetry breakdown of the SU(4) isospin symmetry.

Actually, the SU(4) symmetry is broken explicitly but softly by various direct interactions. Important SU(2) operators are

$$S_z \simeq -\frac{1}{4}(\sigma_1^2 + \vartheta_1^2 + \sigma_3^2 + \vartheta_3^2) + \frac{1}{2}, \tag{8.20}$$

$$P_z = \frac{1}{2}\sigma_2, \tag{8.21}$$

$$P_x \simeq -\frac{1}{4}(\sigma_2^2 + \vartheta_2^2 + \sigma_3^2 + \vartheta_3^2) + \frac{1}{2}, \tag{8.22}$$

up to the second order of fluctuation fields. Note that Eq. (8.21) is an exact formula. Direct interaction terms read

$$\mathcal{H}_C = \frac{\varepsilon_{\text{cap}}\rho_0}{4}\sigma_2^2, \tag{8.23}$$

$$\mathcal{H}_Z = \frac{\Delta_Z \rho_0}{4}(\sigma_1^2 + \vartheta_1^2 + \sigma_3^2 + \vartheta_3^2), \tag{8.24}$$

$$\mathcal{H}_T = \frac{\Delta_{SAS}\rho_0}{4}(\sigma_2^2 + \vartheta_2^2 + \sigma_3^2 + \vartheta_3^2). \tag{8.25}$$

Taking into account the SU(4)-noninvariant exchange interaction as well, we find that the effective Hamiltonian is decomposed into three independent modes $\mathcal{H} = \mathcal{H}_{\text{spin}} + \mathcal{H}_{\text{ppin}} + \mathcal{H}_{\text{ipin}}$, where

$$\mathcal{H}_{\text{spin}} = \frac{J^+}{2}\{(\partial_k \sigma_1)^2 + (\partial_k \vartheta_1)^2\} + \frac{\Delta_Z \rho_0}{4}(\sigma_1^2 + \vartheta_1^2), \tag{8.26}$$

$$\mathcal{H}_{\text{ppin}} = \frac{J}{2}(\partial_k \sigma_2)^2 + \frac{J^d}{2}(\partial_k \vartheta_2)^2 + \frac{\varepsilon_{\text{cap}}\rho_0}{4}\sigma_2^2 + \frac{\Delta_{SAS}\rho_0}{4}(\sigma_2^2 + \vartheta_2^2), \tag{8.27}$$

and

$$\mathcal{H}_{\text{ipin}} = \frac{J^+}{2}\{(\partial_k \sigma_3)^2 + (\partial_k \vartheta_3)^2\} - \frac{(\Delta_Z + \Delta_{SAS})\rho_0}{4}(\sigma_3^2 + \vartheta_3^2). \tag{8.28}$$

They describe three independent soft waves, which are pseudo-Goldstone modes by acquiring gaps. Equations (8.26) and (8.27) agree with the results[10,11] derived previously for the spin wave and the pseudospin wave (which we call the ppin wave), respectively. The group SU(4) is more than $SU(2) \otimes SU(2)$. Equation (8.28) is the Hamiltonian obtained newly for the SU(4) component missed in the $SU(2) \otimes SU(2)$ component, which we call the ipin mode. It is notable that the exchange interactions for the spin and ipin modes are solely determined by the SU(4)-invariant Coulomb interaction (5.1a). The SU(4)-noninvariant Coulomb interaction contributes only to the interaction Hamiltonian (8.27) of the ppin mode. This is because the noninvariant term (5.1b) involves only the density difference $\Delta\rho$ or the ppin mode $\sigma_2 = 2P_z$.

The coherence lengths (correlation lengths) are not infinite because the soft modes are gapful. They are

$$\xi_{\text{spin}} = l_B \sqrt{\frac{4\pi J^+}{\Delta_Z}}, \tag{8.29a}$$

$$\xi_{\text{ipin}} = l_B \sqrt{\frac{4\pi J^+}{\Delta_Z + \Delta_{SAS}}} \tag{8.29b}$$

for the spin and ipin modes. The ground state of the ppin mode (8.27) is a squeezed state,[33] where the coherence lengths are different between the conjugate variables σ_2 and ϑ_2,

$$\xi_{\text{ppin}}^{\vartheta} = l_B \sqrt{\frac{4\pi J^d}{\Delta_{SAS}}}, \tag{8.30a}$$

$$\xi_{\text{ppin}}^{\sigma} = l_B \sqrt{\frac{4\pi J}{\varepsilon_{\text{cap}} + \Delta_{SAS}}}. \tag{8.30b}$$

It is notable that $\xi^{\vartheta}_{\rm ppin}$ is very large when $\Delta_{\rm SAS}$ is very small. However, $\xi^{\sigma}_{\rm ppin}$ is quite small since $\varepsilon_{\rm cap}$ is quite large in actual samples.

It is important that the bias voltage $V_{\rm bias}$ couples only with the ppin wave. The ipin wave connects the symmetric state with the antisymmetric state and requires the tunneling gap for its excitation, but it is insensible to the density difference between the two layers. This is a direct consequence of the formula (8.21). Furthermore, it is easy to check that the electromagnetic field couples only with the ppin mode because it does not affect the spin. Consequently, the pseudospin wave is only the one that is responsible to the coherent tunneling in the BLQH system. The mode has been argued[5] to lead to the Josephson effect with charge e.

IX. CP^3 SKYRMIONS

Provided the Zeeman effect is small enough, charged excitations are skyrmions in monolayer QH ferromagnets.[1] They are topological solitons in the O(3) nonlinear sigma model (4.3). It should be emphasized that the existence of skyrmions is based on the topological reasoning. It is argued as follows. The dynamical field of the nonlinear sigma model is the O(3) spin field $\mathbf{S}(\mathbf{x})$. Since it takes value in the two-sphere S^2, the topological stability is guaranteed based on the theorem $\pi_2(S^2)=Z$ implying that the second homotopy class of S^2 is the set of integers $Z=\{0,\pm 1,\pm 2,\dots\}$. The theorem is rephrased as $\pi_2(CP^1)=Z$. We now argue that skyrmions arise based on the theorem $\pi_2(CP^3)=Z$ in the BLQH system with the SU(4) coherence.

We consider a generic excitation in SU(N) QH ferromagnets at $\nu=1$. Here, $N=2$ in monolayer QH ferromagnets and $N=4$ in BLQH ferromagnets. It can be proved[9,26] that any excitation confined to the lowest Landau level is expressed in terms of the CB field (8.2) as

$$\phi^{\sigma}(\mathbf{x})=\sqrt{\rho(\mathbf{x})}n^{\sigma}(\mathbf{x})=e^{\mathcal{A}(\mathbf{x})}\omega^{\sigma}(z), \tag{9.1}$$

where $\omega(z)$ is an arbitrary analytic function, and $\mathcal{A}(\mathbf{x})$ is an auxiliary field obeying

$$\boldsymbol{\nabla}^2\mathcal{A}(\mathbf{x})=2\pi[\rho(\mathbf{x})-\rho_0]. \tag{9.2}$$

The holomorphicity of $\omega^{\sigma}(z)$ in Eq. (9.1) is a consequence of the requirement that the excitation is confined within the lowest Landau level.

We solve Eq. (9.1) for the CP^{N-1} field

$$n^{\sigma}(\mathbf{x})=\frac{\omega^{\sigma}(z)}{\sqrt{\sum_{\sigma}|\omega^{\sigma}(z)|^2}}. \tag{9.3}$$

Substituting Eqs. (9.1) and (9.3) into Eq. (9.2) we find

$$\frac{1}{4\pi}\boldsymbol{\nabla}^2\ln\rho(\mathbf{x})-\rho(\mathbf{x})+\rho_0=J^0_{\rm sky}(\mathbf{x}), \tag{9.4}$$

where

$$J^0_{\rm sky}(\mathbf{x})=\frac{1}{4\pi}\boldsymbol{\nabla}^2\ln\sum_{\sigma}|\omega^{\sigma}(z)|^2. \tag{9.5}$$

With the aid of the Cauchy-Riemann equation for $\omega(z)$ in Eq. (9.3), this is shown[26] to be the time component of the topological current[32] defined by

$$J^{\mu}_{\rm sky}(\mathbf{x})=\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_{\nu}K_{\lambda}(\mathbf{x}), \tag{9.6}$$

with Eq. (8.13). The topological charge is given by

$$Q_{\rm sky}=\int d^2x J^0_{\rm sky}(\mathbf{x}). \tag{9.7}$$

It is conserved trivially, $\partial_{\mu}J^{\mu}_{\rm sky}(\mathbf{x})=0$.

Equation (9.3) is the generic formula for skyrmions.[32] Equation (9.4) implies that the density modulation $\delta\rho(\mathbf{x})\equiv\rho(\mathbf{x})-\rho_0$ is induced around a skyrmion. It follows from Eq. (9.4) that

$$\int d^2x[\rho(\mathbf{x})-\rho_0]=-\int d^2x J^0_{\rm sky}(\mathbf{x})=-Q_{\rm sky}=-1, \tag{9.8}$$

as implies that one skyrmion removes one electron.

The key of the topological stability is whether the skyrmion configuration (9.1) can be brought into the ground-state configuration by a continuous deformation of the CB field. First, the CP^{N-1} field (9.3) with $Q_{\rm sky}\neq 0$ cannot be deformed continuously into the ground-state value based on the topological theorem $\pi_2(CP^{N-1})=Z$. Second, the density $\rho(\mathbf{x})$ cannot be deformed continuously into the ground-state value ρ_0 because in the midstream of this deformation the field configuration escapes the lowest Landau level. Indeed, we have shown that $\rho(\mathbf{x})$ should obey the soliton equation (9.4) as far as it is confined within the lowest Landau level. Consequently, skyrmions are stable in QH systems because $\pi_2(CP^{N-1})=Z$ and the QH system is robust against density fluctuations.

The topological charge (9.7) is determined by the highest power of $\omega^{\sigma}(z)$. We find $Q_{\rm sky}=n$ if $\omega^{\sigma}(z)\to a^{\sigma}z^n$ with $\Sigma_{\sigma}|a^{\sigma}|^2\neq 0$. The lightest skyrmion has the topological charge $Q_{\rm sky}=1$. It is given by the choice of $\omega^{\sigma}(z)=a^{\sigma}z+b^{\sigma}$ with $\mathbf{a}\cdot\mathbf{b}=0$ in Eq. (9.3). The skyrmion field (9.3) transforms under the action of SU(N). Since it is specified by two parameters $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a}\cdot\mathbf{b}=0$, there are $N(N-1)$ skyrmion states apart from the translational degree of freedom. If the energy is solely determined by the nonlinear sigma model (4.3) or (8.8), all these states are degenerate with the energy given by Eq. (4.6). When the skyrmion is required to approach a specific ground state asymptotically, the parameter $\mathbf{a}$ is fixed, and hence there are $N-1$ degenerate skyrmion states. This is physically reasonable since there exists one ground state and $N-1$ excitation states in the lowest Landau level (Fig. 2).

Let us review skyrmions in monolayer QH ferromagnets ($N=2$). The skyrmion is required to approach the spin polarized ground state $\mathbf{S}=(0,0,1/2)$ asymptotically, and there is no degeneracy since $N-1=1$. The skyrmion configuration

(4.5) is uniquely given by $\boldsymbol{\omega}=(z,\kappa)$ in terms of the CP^1 field. It gives Eq. (4.5) via $S^a=\frac{1}{2}\mathbf{n}^\dagger\tau_a\mathbf{n}$.

We study skyrmions in BLQH systems ($N=4$). The skyrmion is required to approach the ground state (8.15) asymptotically, and there are three degenerate states since $N-1=3$. Typical three skyrmions are given by

$$\boldsymbol{\omega}_{\rm spin}=(z,\kappa,0,0),$$

$$\boldsymbol{\omega}_{\rm ppin}=(z,0,\kappa,0),$$

$$\boldsymbol{\omega}_{\rm ipin}=(z,0,0,\kappa), \tag{9.9}$$

which we call the spin skyrmion, the ppin skyrmion and the ipin skyrmion, respectively. They are essentially SU(2) skyrmions embedded in the SU(4) theory.

The degeneracy of these three types of skyrmions is resolved by the Zeeman effect and the tunneling interaction. Estimation of their excitation energies is straightforward[9] and compared with experimental data.[34,35] As is obvious in Fig. 2, it depends on a competition between the Zeeman effect and the tunneling interaction whether spin skyrmions or ppin skyrmions are excited thermally.

X. DISCUSSION

We have derived the Landau-site Hamiltonian (5.11) for the exchange interaction in BLQH systems. It is valid at any integer filling factor. A field-theoretical Hamiltonian is constructed from it based on the von Neumann–lattice formulation. We may use it to analyze phenomena whose scale is larger than the magnetic length l_B. We have analyzed carefully BLQH states at $\nu=1$. The dynamical field is the CP^3 field because of the U(1) gauge symmetry inherent in the system. We have found that there are three soft waves and three skyrmions. They are excitations from the ground state to three excitation levels (Fig. 2) in the lowest Landau level.

Though there are three types of skyrmions, only the lightest skyrmions are excited thermally. They are spin skyrmions when the Zeeman gap is small enough compared with the tunneling gap, while they are ppin skyrmions when the tunneling gap is small enough compared with the Zeeman gap.

It is interesting to apply the present results to BLQH systems at $\nu=2$. In a forthcoming paper we would analyze the predicted canted antiferromagnetic phase.[12–14] We would also examine a prediction that one skyrmion is composed of two skyrmions,[9] which seems to have some experimental supports.[34,35]

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