Orthogonal basis for the energy eigenfunctions of the Chern-Simons matrix model

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We study the spectrum of the Chern-Simons matrix model and identify an orthogonal set of states. The connection to the spectrum of the Calogero model is discussed.

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I. INTRODUCTION

Recently, Susskind[1] proposed a description of quantum Hall effect in terms of a noncommutative $U(1)$ Chern-Simons theory. The fields of this theory are infinite matrices corresponding to an infinite number of electrons confined in the lowest Landau level. Polychronakos, later, proposed[2] a finite matrix model as a regularized version of the noncommutative Chern-Simons theory in order to describe systems of finite many electrons. Although the proposed matrix model seems to reproduce the basic features of the quantum Hall droplets, a precise relation between the matrix model spectrum and the QHE as described by Laughlin wave functions is lacking.

A formal mapping between the states of the matrix model and Laughlin states as presented in Ref. 3 seems to be nonunitary,[4] while coherent state representations of the matrix model states produce wave functions with a short distance behavior which does not agree with that of Laughlin.[5]

On the other hand, the same matrix model was introduced by Polychronakos[6] as being equivalent to the Calogero model,[7] a one-dimensional system of particles in an external harmonic oscillator potential with mutual inverse-square interactions.

In this paper we analyze the spectrum of the matrix model and present a relatively simple way to identify an orthogonal basis of states. In doing so we make use of known properties of the energy eigenfunctions of the Calogero model. The paper is organized as follows. In Sec. II we briefly review the Chern-Simons finite matrix model and its spectrum. In Sec. III we analyze the eigenvalue problem and identify an orthogonal basis for the energy eigenstates. The relation to the equivalent eigenvalue problem of the Calogero model is discussed in Sec. IV and the Appendix.

II. CHERN-SIMONS MATRIX MODEL

The action describing the Chern-Simons matrix model (for clarification we would like to mention that Smolin has introduced a matrix model, also called matrix Chern-Simons theory.[8] Although there are some common features, the two models are different.) is given by[2]

$$S=\int dt\frac{B}{2}\mathrm{Tr}\{\epsilon_{ab}(\dot{X}_a+i[A_0,X_a])X_b+2\theta A_0-\omega X_a^2\}$$
$$+\Psi^\dagger(i\dot{\Psi}-A_0\Psi), \tag{1}$$

where X_a, $a=1,2$ are $N\times N$ matrices and Ψ is a complex N-vector that transforms in the fundamental of the gauge group $U(N)$,

$$X_a\to UX_aU^{-1},\quad \Psi\to U\Psi. \tag{2}$$

The A_0 equation of motion implies the constraint

$$G\equiv -iB[X_1,X_2]+\Psi\Psi^\dagger-B\theta=0. \tag{3}$$

The trace of this equation gives

$$\Psi^\dagger\Psi=NB\theta. \tag{4}$$

Upon quantization the matrix elements of X_a and the components of Ψ become operators, obeying the following commutation relations

$$[\Psi_i,\Psi_j^\dagger]=\delta_{ij},$$
$$[(X_1)_{ij},(X_2)_{kl}]=\frac{i}{B}\delta_{il}\delta_{jk}. \tag{5}$$

The Hamiltonian is

$$H=\omega\left(\frac{N^2}{2}+\sum A_{ij}^\dagger A_{ji}\right), \tag{6}$$

where $A=\sqrt{(B/2)}(X_1+iX_2)$. The system contains $N(N+1)$ oscillators coupled by the constraint Eq. (3). As explained in Ref. 2 upon quantization, the operator G becomes the generator of unitary rotations of both X_a and Ψ. The trace part Eq. (4) demands that $NB\theta$, being the number operator for Ψ's, is quantized to an integer. The traceless part of the constraint demands the physical states to be singlets of $SU(N)$.

Since the $A_{ij}^\dagger$ transform in the adjoint and the $\Psi_i^\dagger$ transform in the fundamental representation of $SU(N)$, a purely group theoretical argument, implies that a physical state being a singlet has to contain Nl $\Psi^\dagger$'s, where l is an integer. This leads to the quantization of $B\theta=l$.

Explicit expressions for the states were written down in Ref. 3. The ground state being an $SU(N)$ singlet with the lowest number of $A^\dagger$'s is of the form

$$|\Psi_{gr}\rangle=[\epsilon^{i_1\cdots i_N}\Psi_{i_1}^\dagger(\Psi^\dagger A^\dagger)_{i_2}\dots(\Psi^\dagger A^{\dagger N-1})_{i_N}]^l|0\rangle, \tag{7}$$

where $|0\rangle$ is annihilated by A's and Ψ's, while the excited states can be written as

$$|\Psi_{\rm exc}\rangle=\prod_{i=1}^{N}(\mathrm{Tr}\,A^{\dagger i})^{c_i}[\epsilon^{i_1\cdots i_N}\Psi^{\dagger}_{i_1}(\Psi^{\dagger}A^{\dagger})_{i_2}\cdots(\Psi^{\dagger}A^{\dagger N-1})_{i_N}]^{l}|0\rangle. \tag{8}$$

The states in Eq. (8) have energy $\omega((N^2/2)+l[N(N-1)/2]+\epsilon)$, where $\epsilon=\Sigma_i i c_i$. They are degenerate and the degeneracy is given by the number of partitions of ϵ.

The main purpose of this paper is to identify an orthogonal basis for the states in Eq. (8).

III. ENERGY EIGENFUNCTIONS, ORTHOGONAL BASIS

As we shall see later, it is convenient to work in the X-representation. We define the state $|X,\phi\rangle$ such that

$$\hat{X}_1|X,\phi\rangle=X|X,\phi\rangle,\quad \Psi|X,\phi\rangle=\phi|X,\phi\rangle. \tag{9}$$

We normalize the state such that the completeness relation is given by

$$\int |X,\phi\rangle e^{-\bar{\phi}\phi}d\phi\, d\bar{\phi}\prod_{ij} dX_{ij}\langle X,\phi|=1. \tag{10}$$

In the X-representation the wave function corresponding to a particular state of the theory is $\Phi(X,\bar{\phi})=\langle X,\phi|\mathrm{state}\rangle$. In particular the wave function corresponding to the ground state Eq. (7) is of the form[5]

$$\Phi_{\rm gr}(X,\bar{\phi})=[\epsilon^{i_1\cdots i_N}\bar{\phi}_{i_1}(\bar{\phi}A^{\dagger})_{i_2}\cdots(\bar{\phi}A^{\dagger N-1})_{i_N}]^{l}\times e^{-(B/2)\mathrm{Tr}\,X^2}, \tag{11}$$

where

$$A^{\dagger}_{ij}=\sqrt{\frac{B}{2}}\left(X_{ij}-\frac{1}{B}\frac{\partial}{\partial X_{ji}}\right). \tag{12}$$

Since Eq. (11) is completely antisymmetric in the i_n-indices, the differential operator $(\partial/\partial X_{ji})$ produces a nonzero contribution only if it acts on the $e^{-(B/2)\mathrm{Tr}X^2}$ factor. We then have that

$$\Phi_{\rm gr}(X,\bar{\phi})=(\sqrt{2B})^{lN(N-1)/2}[\epsilon^{i_1\cdots i_N}\bar{\phi}_{i_1}(\bar{\phi}X)_{i_2}\cdots(\bar{\phi}X^{N-1})_{i_N}]^{l}e^{-(B/2)\mathrm{Tr}\,X^2}. \tag{13}$$

The wave function corresponding to the excited state Eq. (8) can be written as a linear combination of wave functions of the form $(\sqrt{2B})^{\Sigma i n_i}\Pi_{i=1}^{N}(\mathrm{Tr}\,X^i)^{n_i}\Phi_{\rm gr}(X,\bar{\phi})$.

Given the constraint G, any physical wave function has to be a function of $SU(N)$ singlets made out of the Hermitian matrix X and the vector $\bar{\phi}$. There are two types of such invariants one can construct:

$$S_n(\sqrt{2B}X)=(\sqrt{2B})^{n}\mathrm{Tr}\,X^{n},\quad n=1,\ldots,N,$$
$$\Xi(\sqrt{2B}X,\bar{\phi})=(\sqrt{2B})^{N(N-1)/2}\epsilon^{i_1\cdots i_N}\bar{\phi}_{i_1}(\bar{\phi}X)_{i_2}\cdots(\bar{\phi}X^{N-1})_{i_N}. \tag{14}$$

These can be thought of as $N+1$ independent collective variables. (For an $N\times N$ matrix X, the Cayley-Hamilton theorem expressex X^N as a linear function of X^n, $n=1,\ldots,N-1$ with coefficients which are symmetric functions of the eigenvalues of X. Therefore all other invariants which involve $\bar{\phi}$'s are reduced to Ξ^l times a function of S_n.)

Any physical wave function has the general form

$$\Phi=f(S_n)\Xi^{l}e^{-(B/2)\mathrm{Tr}\,X^2}. \tag{15}$$

In the X-representation the Hamiltonian can be written as

$$H=\frac{\omega}{2}\left[-\frac{1}{B}\frac{\partial^2}{\partial X_{ij}\partial X_{ji}}+B\,\mathrm{Tr}\,X^2\right]. \tag{16}$$

We want to solve the eigenvalue problem

$$H\Phi=E\Phi, \tag{17}$$

where H is given by Eq. (16) and Φ is as in Eq. (15). Doing a simple similarity transformation we get

$$\tilde{H}f(S_n)\ \Xi^{l}=Ef(S_n)\ \Xi^{l}, \tag{18}$$

where

$$\tilde{H}=e^{(B/2)\mathrm{Tr}\,X^2}He^{-(B/2)\mathrm{Tr}\,X^2}=\omega\sum_{ij}X_{ij}\frac{\partial}{\partial X_{ij}}+\frac{\omega}{2}N^2-\frac{\omega}{2B}\sum_{ij}\frac{\partial^2}{\partial X_{ij}\partial X_{ji}}. \tag{19}$$

The Hamiltonian $\tilde{H}$ can be written as a sum of two terms

$$\tilde{H}=\tilde{H}_0+\tilde{H}_{-2}, \tag{20}$$

where

$$\tilde{H}_0=\omega\sum_{ij}X_{ij}\frac{\partial}{\partial X_{ij}}+\frac{\omega}{2}N^2,$$
$$\tilde{H}_{-2}=-\frac{\omega}{2B}\sum_{ij}\frac{\partial^2}{\partial X_{ij}\partial X_{ji}}. \tag{21}$$

Since the operator $\tilde{H}_0$ essentially counts the number of X's, one can easily check that

$$[\tilde{H}_0,\tilde{H}_{-2}]=-2\tilde{H}_{-2}. \tag{22}$$

In other words,

$$\tilde{H}=e^{(\tilde{H}_{-2}/2\omega)}\tilde{H}_0e^{-(\tilde{H}_{-2}/2\omega)}. \tag{23}$$

This implies that if P_k is an eigenfunction of $\tilde{H}_0$, then $e^{\tilde{H}_{-2}/2\omega}P_k$ is an eigenstate of the Hamiltonian $\tilde{H}$. One can easily see that

$$\tilde{H}_0 P_k = \left(k + \omega \frac{N^2}{2} \right) P_k ,$$

$$\tilde{H} e^{(\tilde{H}_{-2}/2\omega)} P_k = \left(k + \omega \frac{N^2}{2} \right) e^{(\tilde{H}_{-2}/2\omega)} P_k . \tag{24}$$

Using Eqs. (18) and (21), we see that in our case P_k is of the form

$$P_k = J_{\{\lambda\}}(\sqrt{2B}X)\Xi^l(\sqrt{2B}X,\overline{\phi}), \tag{25}$$

where $J_{\{\lambda\}}$ is a homogeneous polynomial of the form (the notation will be justified later)

$$J_{\{\lambda\}}(\sqrt{2B}X) = \sum_{\{n_i\}} a(\{n_i\}) \prod_i (\mathrm{Tr}(\sqrt{2B}X)^i)^{n_i}, \tag{26}$$

such that $\Sigma_i i n_i + l[N(N-1)/2] = k$.

Going back to the original eigenvalue problem, the energy eigenfunctions are of the form

$$\Phi = e^{-(B/2)\mathrm{Tr}\,X^2} e^{(\tilde{H}_{-2}/2\omega)}(J_{\{\lambda\}}\Xi^l). \tag{27}$$

Since

$$e^{-(B/2)\mathrm{Tr}\,X^2} e^{(\tilde{H}_{-2}/2\omega)} X_{ij} e^{-(\tilde{H}_{-2}/2\omega)} e^{(B/2)\mathrm{Tr}\,X^2} = X_{ij} - \frac{1}{B}\frac{\partial}{\partial X_{ji}}, \tag{28}$$

Φ can be written as

$$\begin{aligned}\Phi &= J_{\{\lambda\}}\left[\sqrt{\frac{B}{2}}\left(X_{ij} - \frac{1}{B}\frac{\partial}{\partial X_{ji}} \right)\right] \\ &\times \Xi^l \left[\sqrt{\frac{B}{2}}\left(X_{ij} - \frac{1}{B}\frac{\partial}{\partial X_{ji}} \right), \overline{\phi} \right] e^{-(B/2)\mathrm{Tr}\,X^2} \\ &= J_{\{\lambda\}}(A^\dagger)\Xi^l(A^\dagger,\overline{\phi}) e^{-(B/2)\mathrm{Tr}\,X^2},\end{aligned} \tag{29}$$

where $A^\dagger$ is given in Eq. (12).

There are several basis sets for the polynomials $J_{\{\lambda\}}$. There is a particular one which is orthogonal. This corresponds to choosing $J_{\{\lambda\}}$'s to be the Jack polynomials.[9–11] Although, in principle, this can be proven purely within the context of the matrix model itself, an easier proof can be given indirectly by first relating the energy eigenfunctions of the matrix model to the energy eigenfunctions of the Calogero model and then using well known properties of the Calogero eigenfunctions.[13–17] This is shown in the following section.

IV. RELATION TO CALOGERO MODEL

X being a Hermitian matrix, it can be diagonalized by a unitary transformation

$$X = U x U^{-1}, \quad x_{ij} = x_i \delta_{ij} . \tag{30}$$

The relation between the matrix model and the Calogero model is achieved by identifying the eigenvalues x_i with the one-dimensional particle coordinates of the Calogero system.

One can show that the Laplacian of the matrix model can be written as

$$\sum_{ij} \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ji}} = \frac{1}{\Delta}\sum_k \frac{\partial^2}{\partial x_k^2}\Delta - \sum_{k\neq l}\frac{J^{kl}J^{lk}}{(x_k-x_l)^2}, \tag{31}$$

where Δ is the Vandermonde determinant defined by

$$\Delta \equiv \det(x_i^{N-j}) = \epsilon^{k_1 k_2 \cdots k_N} x_{k_1}^0 x_{k_2}^1 \cdots x_{k_N}^{N-1} = \prod_{k<l}(x_k - x_l), \tag{32}$$

and J^{ij} is an operator with the following action on U

$$[J^{kl}, U_{ij}] = U_{ik}\delta_{lj}, \quad [J^{kl}, U_{ij}^{-1}] = -U_{lj}^{-1}\delta_{ik} . \tag{33}$$

For completeness we show the detailed derivation of Eq. (31) in the Appendix.

In writing down the eigenvalue equation for H expressed in terms of the eigenvalues x_i and the angular variables U, we notice that the term $\Sigma_{k\neq l}[J^{kl}J^{lk}/(x_k-x_l)^2]$ acts only on the Ξ dependence of the wave function Φ in Eq. (15). Using the particular parametrization Eq. (30), the invariants in Eq. (14) can be written as

$$S_n(\sqrt{2B}X) = (\sqrt{2B})^n \sum_{i=1}^N x_i^n ,$$

$$\begin{aligned}\Xi(\sqrt{2B}X,\overline{\phi}) &= (\sqrt{2B})^{N(N-1)/2}\epsilon^{i_1\cdots i_N}\overline{\phi}_{i_1}(\overline{\phi}X)_{i_2} \\ &\quad\times(\overline{\phi}X^2)_{i_3}\ldots(\overline{\phi}X^{N-1})_{i_N} \\ &= (\sqrt{2B})^{N(N-1)/2}\epsilon^{i_1\cdots i_N}\overline{\phi}_{i_1}(\overline{\phi}UxU^{-1})_{i_2} \\ &\quad\times(\overline{\phi}Ux^2U^{-1})_{i_3}\ldots(\overline{\phi}Ux^{N-1}U^{-1})_{i_N} \\ &= (\sqrt{2B})^{N(N-1)/2}\det(U^{-1})\epsilon^{k_1\cdots k_N}(\overline{\phi}U)_{k_1} \\ &\quad\times(\overline{\phi}U)_{k_2}x_{k_2}\ldots(\overline{\phi}U)_{k_N}x_{k_N}^{N-1} \\ &= (\sqrt{2B})^{N(N-1)/2}\prod_{i<j}(x_i-x_j)\prod_{i=1}^N(\overline{\phi}U)_i .\end{aligned} \tag{34}$$

This implies

$$\begin{aligned}J^{km}J^{mk}\ \Xi^l &= (\sqrt{2B})^{lN(N-1)/2}\Delta^l J^{km}J^{mk}\prod_{i=1}^N(\overline{\phi}U)_i^l \\ &= (\sqrt{2B})^{lN(N-1)/2}\Delta^l J^{km}\prod_{i\neq k,m}(\overline{\phi}U)_i^l\, l(\overline{\phi}U)_k^{l-1} \\ &\quad\times(\overline{\phi}U)_m^{l+1} = l(l+1)\Xi^l.\end{aligned} \tag{35}$$

The Hamiltonian acting on the space of physical wave functions Eq. (15) can therefore be written as

$$H=\frac{\omega}{2B}\left[-\frac{1}{\Delta}\sum_i \frac{\partial^2}{\partial x_i^2}\Delta+\sum_{i\neq j}\frac{l(l+1)}{(x_i-x_j)^2}+B^2\sum_i x_i^2\right]. \tag{36}$$

The expression $\Delta H\Delta^{-1}$ coincides with the Hamiltonian of the Calogero model. This actually implies that $\Delta e^{-(B/2)\mathrm{Tr}\,X^2}e^{(\tilde{H}_{-2}/2\omega)}J_{\{\lambda\}}(\sqrt{2B}X)\Xi^l(\sqrt{2B}X,\bar{\phi})$ are energy eigenfunctions of the Calogero model.

Using Eqs. (31), (34), (35) one can show that

$$\Delta e^{-(B/2)\mathrm{Tr}\,X^2}e^{(\tilde{H}_{-2}/2\omega)}J_{\{\lambda\}}(\sqrt{2B}X)\Xi^l(\sqrt{2B}X,\phi) \sim\Delta^{l+1}e^{-(B/2)\mathrm{Tr}\,X^2}e^{-(\hat{O}_L/4B)}J_{\{\lambda\}}(\sqrt{2B}x_i), \tag{37}$$

where

$$\hat{O}_L=\sum_i \frac{\partial^2}{\partial x_i^2}+(l+1)\sum_{i\neq j}\frac{1}{x_i-x_j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right). \tag{38}$$

Comparing Eq. (37) to the orthogonal basis of the energy eigenfunctions of the Calogero model,[12–17] we conclude that the polynomials $J_{\{\lambda\}}$ ought to be the symmetric Jack polynomials. The inhomogeneous polynomials $e^{-(\hat{O}_L/4B)}J_{\{\lambda\}}(\sqrt{2B}x_i)$ are the symmetric Hi-Jack polynomials[14–17] which provide an orthogonal basis for the Calogero model with the integration measure[15,16]

$$\Delta^{2l+2}e^{-B\Sigma x_i^2}\prod_i dx_i. \tag{39}$$

The symmetric Jack polynomials $J_{\{\lambda\}}$ of degree λ are defined[9–11] in terms of the monomial functions $\Pi_i x_i^{\lambda_i}$, where $\lambda=\Sigma_i\lambda_i$. Let $\{\lambda\}$ indicate the partitions $\{\lambda_1,\ldots,\lambda_N\}$. To a partition $\{\lambda\}$ we associate the symmetric monomial function $m_{\{\lambda\}}=\Sigma_P\Pi_i x_i^{\lambda_i}$, where the sum is over all distinct permutations P. For example $m_{\{2,1\}}=\Sigma_{i,j}x_i^2x_j$ for $i\neq j$. $m_{\{\lambda\}}(x_1,x_2,\ldots,x_N)=0$ if N is smaller than the number of parts of the partition $\{\lambda\}$. The symmetric Jack polynomials have the following expansion in terms of $m_{\{\lambda\}}$'s:

$$J_{\{\lambda\}}(x_i)=m_{\{\lambda\}}(x_i)+\sum_{\{\mu\}<\{\lambda\}}v_{\{\mu\lambda\}}m_{\{\mu\}}(x_i). \tag{40}$$

$v_{\{\mu\lambda\}}$ are some coefficients which depend on l and $\{\mu\}<\{\lambda\}$ defines a partial ordering such that

$$\{\mu\}<\{\lambda\}\leftrightarrow\mu=\lambda \ \text{ and } \sum_{i=1}^{j}\mu_i<\sum_{i=1}^{j}\lambda_i$$

$$\text{for all } j=1,2,\ldots,N. \tag{41}$$

Since there is no generic formula, we write down the first few symmetric Jack polynomials, relevant for the construction of the first, second, and third excited states.

$$J_{\{0\}}=1,\quad J_{\{1\}}=m_{\{1\}},$$

$$J_{\{2\}}=m_{\{2\}}+\frac{2(l+1)}{l+2}m_{\{1,1\}},$$

$$J_{\{1,1,\ldots,1\}}=m_{\{1,1,\ldots,1\}},$$

$$J_{\{2,1\}}=m_{\{2,1\}}+\frac{6(l+1)}{2l+3}m_{\{1,1,1\}},$$

$$J_{\{3\}}=m_{\{3\}}+\frac{3(l+1)}{l+3}m_{\{2,1\}}+\frac{6(l+1)^2}{(l+2)(l+3)}m_{\{1,1,1\}}. \tag{42}$$

Using these one can explicitly check that the corresponding Hi-Jack polynomials $e^{-(\hat{O}_L/4B)}J_{\{\lambda\}}$ are orthogonal with the integration measure Eq. (39).

Since $J_{\{\lambda\}}$'s are symmetric, they can also be expressed in terms of $\Pi_i(S_i)^{n_i}=\Pi_i(\mathrm{Tr}\,X^i)^{n_i}$ where $\Sigma_i i n_i=\lambda$. It is this dependence which is implied in $J_{\{\lambda\}}(\sqrt{2B}X)$ in Eq. (26) and in $J_{\{\lambda\}}(A^\dagger)$ in Eq. (29).

Going back to the matrix model eigenfunctions Eqs. (29) and recalling that $\Pi_{ij}\,d[X_{ij}]=\Delta^2[dU]\Pi_i\,dx_i$, we conclude that the states

$$\Phi_{\{\lambda\}}=J_{\{\lambda\}}(A^\dagger)\Xi^l(A^\dagger,\phi)e^{-(B/2)\mathrm{Tr}\,X^2}, \tag{43}$$

where $J_{\{\lambda\}}$'s are the symmetric Jack polynomials, provide an orthogonal basis for the matrix energy eigenfunctions

$$\int \Phi^*_{\{\lambda\}}\Phi_{\{\lambda'\}}\prod_{ij}dX_{ij}d\phi\,d\bar{\phi}\,e^{-\phi\bar{\phi}}=0 \quad \text{for } \{\lambda\}\neq\{\lambda'\}. \tag{44}$$

Using now Eq. (10), we can write an orthogonality relation for the states of the matrix model independent of representation, namely

$$\langle\Psi_{\{\lambda\}}|\Psi_{\{\lambda'\}}\rangle=0 \quad \text{for } \{\lambda\}\neq\{\lambda'\}, \tag{45}$$

where

$$|\Psi_{\{\lambda\}}\rangle =J_{\{\lambda\}}(A^\dagger)[\epsilon^{i_1\cdots i_N}\Psi^\dagger_{i_1}(\Psi^\dagger A^\dagger)_{i_2}\ldots(\Psi^\dagger A^{\dagger N-1})_{i_N}]^l|0\rangle.$$

The use of the X representation and the resulting connection to the Calogero model was very helpful in identifying the Jack polynomial dependence of an orthogonal basis for the excited states, but the final result Eq. (45) is independent of representation.

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APPENDIX

Using the "polar" decomposition Eq. (30) for X we find

$$dX = U(dx + [U^{-1}dU, x])U^{-1},$$

$$dX_{ij} = U_{ik}U^{-1}_{kj}dx_k - U_{ik}U^{-1}_{lj}(x_k - x_l)(U^{-1}dU)_{kl}. \tag{A1}$$

Using the parametrization $U = e^{i\Sigma_\alpha t_\alpha \theta_\alpha}$ we get

$$(U^{-1}dU)_{kl} = \left(U^{-1}\frac{\partial U}{\partial \theta_\alpha}\right)_{kl} d\theta_\alpha \equiv e^\alpha_{kl}(\theta)d\theta_\alpha. \tag{A2}$$

Thus

$$dx_k = (U^{-1}dXU)_{kk};$$

$$(U^{-1}dU)_{kl} = -\frac{(U^{-1}dXU)_{kl}}{x_k - x_l} \quad \text{for} \quad k \neq l. \tag{A3}$$

Using this we find

$$\frac{\partial}{\partial X_{ij}} = \sum_k U^{-1}_{ki}U_{jk}\frac{\partial}{\partial x_k} - \sum_{k\neq l}\frac{U^{-1}_{ki}U_{jl}}{(x_k - x_l)}J^{kl}, \tag{A4}$$

where

$$J^{kl} = \sum_\alpha e^{kl}_\alpha \frac{\partial}{\partial \theta_\alpha}, \tag{A5}$$

and e^{kl}_α is the inverse of e^α_{kl}, such that

$$\sum_{kl} e^{kl}_\alpha(\theta)e^\beta_{kl}(\theta) = \delta_{\alpha\beta},$$

$$\sum_p e^{kl}_\alpha(\theta)e^\alpha_{k'l'}(\theta) = \delta_{kk'}\delta_{ll'}. \tag{A6}$$

Expression (A5) implies that the action of J^{kl} on U is as follows:

$$[J^{kl}, U_{ij}] = U_{ik}\delta_{lj}. \tag{A7}$$

Further

$$[J^{kl}, J^{mn}] = J^{kn}\delta_{lm} - J^{ml}\delta_{kn}. \tag{A8}$$

The matrix Laplacian can now be rewritten as

$$\begin{aligned}\sum_{ij}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ji}} = \sum_{ij}\Bigg[&\sum_k U^{-1}_{ki}U_{jk}\frac{\partial}{\partial x_k}\sum_{k'} U^{-1}_{k'j}U_{ik'}\frac{\partial}{\partial x'_k} \\ &- \sum_k U^{-1}_{ki}U_{jk}\frac{\partial}{\partial x_k}\sum_{k'\neq l'}\frac{U^{-1}_{k'j}U_{il'}}{(x'_k - x'_l)}J^{k'l'} \\ &- \sum_{k\neq l}\frac{U^{-1}_{ki}U_{jl}}{(x_k - x_l)}J^{kl}\sum_{k'} U^{-1}_{k'j}U_{ik'}\frac{\partial}{\partial x'_k} \\ &+ \sum_{k\neq l}\frac{U^{-1}_{ki}U_{jl}}{(x_k - x_l)}J^{kl}\sum_{k'\neq l'}\frac{U^{-1}_{k'j}U_{il'}}{(x'_k - x'_l)}J^{k'l'}\Bigg].\end{aligned} \tag{A9}$$

The first term is $\Sigma_k(\partial^2/\partial x_k^2)$, the second term is zero, the third is $\Sigma_{k\neq l}[1/(x_k - x_l)][(\partial/\partial x_k) - (\partial/\partial x_l)]$, and the last is $-\Sigma_{k\neq l}[J^{kl}J^{lk}/(x_k - x_l)^2]$. Thus we obtain

$$\sum_{ij}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X_{ji}} = \frac{1}{\Delta}\sum_k \frac{\partial^2}{\partial x_k^2}\Delta - \sum_{k\neq l}\frac{J^{kl}J^{lk}}{(x_k - x_l)^2}, \tag{A10}$$

where we used

$$\sum_k \frac{\partial^2}{\partial x_k^2} + \sum_{k\neq l}\frac{1}{(x_k - x_l)}\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_l}\right) = \frac{1}{\Delta}\sum_k \frac{\partial^2}{\partial x_k^2}\Delta \tag{A11}$$

and $\Delta = \Pi_{k<l}(x_k - x_l)$.

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