Exact renormalization of the random transverse-field Ising spin chain in the strongly ordered and strongly disordered Griffiths phases

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Fisher's [Phys. Rev. B **51**, 6411 (1995)] real-space renormalization-group (RG) treatment of random transverse-field Ising spin chains is extended into the strongly ordered and strongly disordered Griffiths phases, and asymptotically exact results are obtained. In the noncritical region the asymmetry of the renormalization of the couplings and the transverse fields is related to a nonlinear quantum control parameter Δ, which is a natural measure of the distance from the quantum critical point. Δ, which is found to stay invariant along the RG trajectories, and has been expressed by the initial disorder distributions, stands in the singularity exponents of different physical quantities (magnetization, susceptibility, specific heat, etc.), which are exactly calculated. In this way we have observed a weak-universality scenario: the Griffiths-McCoy singularities do not depend on the form of the disorder, provided the nonlinear quantum control parameter has the same value. The exact scaling function of the magnetization with a small applied magnetic field is calculated, and the critical point magnetization singularity is determined in a simple, direct way.

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I. INTRODUCTION

In statistical physics and in the theory of interacting many-body systems, exact solutions are of great importance, especially in the vicinity of singular points, such as at phase transitions. They provide physical insights into cooperative processes, and their results could be used as testing grounds for different approximations and numerical methods. Exact solutions for models with short-range interactions and in the presence of quenched randomness are scarce, which greatly hampered our understanding of collective phenomena in disordered systems. At present, remarkable examples of exactly soluble problems in the above class include the critical behavior of low-dimensional random quantum systems.[1] Here the interplay of quenched disorder, quantum fluctuations, and correlations can be systematically studied within a real-space renormalization group (RG) scheme, which is expected to lead to asymptotically exact results, at least for strong enough disorder. The prototype of such types of random quantum systems is the random transverse-field Ising spin chain (RTIC) for which perhaps the most detailed analytical and numerical information is available, as far as the random quantum critical behavior is concerned. The RTIC is defined by the Hamiltonian

$$\hat{H} = -\sum_l J_l \sigma_l^x \sigma_{l+1}^x - \sum_l h_l \sigma_l^z - H \sum_l \sigma_l^x, \qquad (1.1)$$

in terms of the $\sigma_l^{x,z}$ Pauli matrices at site l, whereas the transverse fields $h_l > 0$ and the couplings $J_l > 0$ are independent random variables taken from the (initial) distributions $P_{\rm in}(h)dh$ and $R_{\rm in}(J)dJ$, respectively. The (linear) quantum control parameter of the model is defined as[2]

$$\delta = \frac{[\ln h]_{\rm av} - [\ln J]_{\rm av}}{{\rm var}[\ln h] + {\rm var}[\ln J]}. \qquad (1.2)$$

where ${\rm var}(x)$ is the variance of x, and here and in the following we use $[\cdots]_{\rm av}$ to denote an averaging over quenched disorder. The quantum critical point at $\delta = 0$ separates the ferromagnetic ($\delta < 0$) and paramagnetic ($\delta > 0$) phases.

Some previously known exact results for the surface magnetization[3,4] and typical correlations[5] were greatly extended by Fisher[6] using the strong disorder RG method, originally introduced by Ma, Dasgupta, and Hu.[7] Fisher showed that at the critical point the distribution functions of the couplings and the transverse fields broaden without limits as the energy scale Ω defined by the strongest bond or transverse field, is lowered. Therefore, as the fixed point of the transformation with $\Omega = 0$ is approached, the disorder becomes stronger and stronger, so that in this, so-called, infinite randomness fixed point (IRFP) the ratio of typical couplings and transverse fields at neighboring sites is either zero or tends to infinity. As a consequence in the IRFP the RG transformation becomes asymptotically exact and the fixed-point RG equations for the RTIC can be solved, to a large extent, analytically. From the RG treatment and from other analytical results,[8] we have a clear physical picture of the origin of the critical behavior of the RTIC, which is most probably also relevant for other low-dimensional random quantum critical systems, having an IRFP. The critical properties of these systems are determined by so-called *rare events*, which are realizations occurring with a vanishing probability, but which dominate the *average* properties. Conversely, *typical realizations*, which appear with probability 1, make a vanishing contribution to the average critical quantities. The mathematical origin of the solubility of these models is connected with the above observation, since it is enough only to deal with the rare events, and an overwhelming part of the realizations is irrelevant in respect to the average critical properties.

In the following we list the existing results about the singular behavior of the RTIC. At $T = 0$ and without an external

field $H=0$, the average magnetization of the surface spin m_s, asymptotically vanishes as $m_s \sim (-\delta)^{\beta_s}$, with an exponent[3,4]

$$\beta_s = 1, \tag{1.3}$$

whereas the same behavior for the bulk magnetization m, involves the bulk exponent[6] β:

$$\beta = 2 - \Phi, \quad \Phi = \frac{1+\sqrt{5}}{2}. \tag{1.4}$$

Average correlations $G(r) = [\langle \sigma_l^x \sigma_{l+r}^x \rangle]_{\rm av}$, outside the critical point, decay exponentially, and the correlation length ξ asymptotically diverges in the vicinity of the critical point as[6]

$$\xi \sim |\delta|^{-\nu}, \quad \nu = 2. \tag{1.5}$$

On the other hand, the *typical* correlation length ξ_{typ}, defined through the relation $[\ln(\langle \sigma_l^x \sigma_{l+r}^x \rangle)]_{\rm av} \sim -r/\xi_{typ}$ involves another exponent[5] $\nu_{typ}=1$. At the critical point average correlations decay as a power, $G(r) \sim r^{-2x}$, and the scaling dimension of the bulk magnetization x satisfies the scaling relation $x = \beta/\nu$. Similarly, for end-to-end critical correlations the corresponding scaling dimension x_s is expressed by $x_s = \beta_s/\nu$.

In a quantum system statical and dynamical correlations are inherently related. In the RTIC at the critical point the dynamical scaling is strongly anisotropic, and the relevant (imaginary) time scale—the relaxation time t_r—is related to the length scale ξ as

$$\ln t_r \sim \xi^{1/2}. \tag{1.6}$$

In his RG study Fisher also considered the *weakly ordered and weakly disordered* Griffiths phases,[9] which are situated in the vicinity of the critical point. He found an anisotropic scaling relation $t_r \sim \xi^{1/2|\delta|}$, which can be interpreted as a δ-dependent dynamical exponent z, which in leading order is given by

$$z \approx \frac{1}{2|\delta|}, \quad |\delta| \ll 1. \tag{1.7}$$

In the presence of a small external field, $H \ll 1$, and in the vicinity of the critical point, $|\delta| \ll 1$, but with a finite combination of, $\gamma = \delta \ln(H) = O(1)$, Fisher obtained the exact scaling function of the magnetization of the form

$$m(\delta, H) \approx m_0 [\ln(H_0/H)]^{\Phi-2} \tilde{m}[\delta \ln(H_0/H)], \tag{1.8}$$

where m_0 and H_0 are dimensional constants, and $\tilde{m}(\gamma)$ is given in Ref. 6.

In the present paper we extend the RG treatment of Fisher into the entire Griffiths region. Which for some type of initial disorder distribution could cover the entire off-critical region, $0 < |\delta| < \infty$. Our analytical results in the Griffiths phases are asymptotically exact in the same sense as argued by Fisher at the critical point. Here we summarize our main findings. A short account of our results, especially about the renormalization of couplings and transverse fields, was given in Ref. 10.

The asymmetry in the renormalization of the couplings and the transverse fields is related to a *nonlinear* quantum control parameter Δ, which is defined as the root of the following equation:

$$\left[\left(\frac{J^2}{h^2} \right)^{\Delta} \right]_{\rm av} = 1.. \tag{1.9}$$

We have shown that Δ remains invariant along the RG trajectories (i.e. as the energy scale Ω is lowered), and so can be expressed by initial disorder distributions. Note that for very different disorder distributions one might have the same Δ, provided the distributions have the same form of asymmetry. The dynamical exponent z is simply expressed by the non-linear quantum control parameter as

$$z = \frac{1}{2|\Delta|}, \tag{1.10}$$

which is an exact relation in the entire Griffiths region. We have calculated the singular behavior of different physical quantities (magnetization, susceptibility, specific heat, etc.), and the singularity exponents are all expressed by the nonlinear quantum control parameter Δ. In this way we have demonstrated a weak-universality scenario: details of the disorder distributions are irrelevant for the Griffiths-McCoy singularities, provided the nonlinear quantum control parameter has the same value.

In the vicinity of the random quantum critical point, the nonlinear and linear quantum control parameters are asymptotically identical:

$$\Delta = \delta + O(\delta^2); \tag{1.11}$$

consequently in the linear δ limit, from our formulas we can recover Fisher's results[6] about *weakly* disordered and *weakly* ordered Griffiths phases. From our results one can also obtain the scaling function of the magnetization as a function of a small applied magnetic field H, which in the paramagnetic phase is given in the functional form

$$m(\Delta, H) = m_D \left(\frac{H}{H_D} \right)^{2\Delta} \tilde{m} \left[\frac{\Delta}{\Delta_D}, \left(\frac{H}{H_D} \right)^{2\Delta} \right], \tag{1.12}$$

where m_D, H_D and Δ_D are nonuniversal dimensional constants. Here Δ_D is a nonuniversal parameter, which depends on the details of the distribution of the disorder. At the critical point, $\Delta = 0$, our result in Eq. (1.12) goes over Fisher's result in Eq. (1.8).

The structure of the paper is the following. The RG equations and their solution for the fixed-point distribution of the couplings and the transverse fields are presented in Sec. II. Renormalizations of lengths and magnetic moments are

given in Sec. III. The scaling behavior of different thermo dynamic quantities in the presence of a finite external mag netic field or at small, but nonzero, temperature is calculated in Sec. IV. We conclude our paper in Sec. V with a discussion of a possible extension of our results to other problems. Some detailed calculations about the distribution function of lengths are given in the Appendix.

II. RENORMALIZATION OF COUPLINGS AND TRANSVERSE FIELDS

Here we consider the RTIC as given by the Hamiltonian in Eq. (1.1) without external magnetic field, i.e., $H=0$. To a spin at lattice site l, we assign a magnetic moment μ_l and a length l_l^s, while the transverse field acting on this spin is denoted by h_l. Similarly to the lth bond, connecting lattice sites l and $l+1$, we assign a length l_l^b, and the associated coupling is denoted by J_l. In the initial situation $l_l^s=l_l^b=1/2$, $\mu_l=1$ and the couplings and fields are taken from the initial distributions, $P_{\rm in}(h)dh$ and $R_{\rm in}(J)dJ$, respectively.

During renormalization the strongest term in the Hamiltonian, coupling, or transverse field, of strength Ω, is successively decimated out and the neighboring transverse-fields or couplings are replaced by weaker ones, which are generated by a second-order perturbation calculation. If the strongest term is a coupling, say $\Omega=J_l$, then the two spins connected by J_l flip coherently in a longitudinal field; thus they act as an effective, composite spin having the renormalized parameters

$$\tilde{h}=\frac{h_l h_{l+1}}{J_l}, \quad \tilde{l}^b=l_l^s+l_l^b+l_{l+1}^s, \quad \tilde{\mu}=\mu_l+\mu_{l+1}. \tag{2.1}$$

On the other hand, if the strongest term in the Hamiltonian is a transverse field, say, $\Omega=h_l$, then the state of this spin in a small longitudinal field is practically unchanged, thus its contribution to the susceptibility is negligible. Consequently, from a magnetic point of view this spin can be decimated out, and the renormalized parameters of the effective bond connecting sites $l-1$ and $l+1$ are given by

$$\tilde{J}=\frac{J_{l-1}J_l}{h_l}, \quad \tilde{l}^s=l_{l-1}^b+l_l^s+l_l^b. \tag{2.2}$$

Note that the decimation equations for $\tilde{h}$ and $\tilde{J}$ are related through duality.

During renormalization the energy scale is reduced and joint distribution functions, such as the for the spins, $P(h,l^s,\mu;\Omega)dhdl^s d\mu$, and those for the bonds, $R_l(J,l^b;\Omega)dJdl^b$, are also Ω dependent. Generally we deal with the following reduced distribution functions:

$$P_0(h,\Omega)=\int\int P(h,l^s,\mu;\Omega)dl^s d\mu,$$

$$P_l(h,l^s,\Omega)=\int P(h,l^s,\mu;\Omega)d\mu$$

$$P_\mu(h,\mu,\Omega)=\int P(h,l^s,\mu;\Omega)dl^s$$

$$R_0(J,\Omega)=\int R_l(J,l^b;\Omega)dl^b, \tag{2.3}$$

all of which are normalized. In this section we consider the distribution of transverse fields and couplings, so that we work with $P_0(h,\Omega)$ and $R_0(J,\Omega)$, whereas the other joint distributions, which are connected to the size of average lengths and average moments will be considered in Sec. III.

We start by calculating the variation of the distribution function of transverse fields, $dP_0(h,\Omega)$, when the energy scale is lowered by $\Omega\to\Omega-d\Omega$, which amounts to eliminating a fraction of $d\Omega[P_0(\Omega,\Omega)+R_0(\Omega,\Omega)]$ spins. Here one should take into account the fact that as a strong bond is decimated out, two original fields are also eliminated and one field is created, the strength of which is given in Eq. (2.1). Since the distribution function should also be normalized, we arrive at the equation

$$\frac{dP_0}{d\Omega}=P_0(h,\Omega)[R_0(\Omega,\Omega)-P_0(\Omega,\Omega)] - R_0(\Omega,\Omega)\int_h^\Omega dh' P_0(h',\Omega)P_0\left(\frac{h}{h'}\Omega,\Omega\right)\frac{\Omega}{h'}. \tag{2.4}$$

One can similarly derive the evaluation equation of the coupling distribution,

$$\frac{dR_0}{d\Omega}=R_0(J,\Omega)[P_0(\Omega,\Omega)-R_0(\Omega,\Omega)] - P_0(\Omega,\Omega)\int_J^\Omega dJ' R_0(J',\Omega)R_0\left(\frac{J}{J'}\Omega,\Omega\right)\frac{\Omega}{J'}, \tag{2.5}$$

which follows simply from Eq. (2.4) by duality, which amounts to interchanging $h\leftrightarrow J$ and $P_0\leftrightarrow R_0$. The two integrodifferential equations in Eqs. (2.4) and (2.5) have to be supplemented by the initial conditions, represented by the distributions $P_{\rm in}(h)$ and $R_{\rm in}(J)$.

A. Fixed-point solution

A special solution to the problem in Eqs. (2.4) and (2.5) is given by the functions

$$P_0(h,\Omega)=\frac{p_0(\Omega)}{\Omega}\left(\frac{\Omega}{h}\right)^{1-p_0(\Omega)}, \tag{2.6}$$

$$R_0(J,\Omega)=\frac{r_0(\Omega)}{\Omega}\left(\frac{\Omega}{J}\right)^{1-r_0(\Omega)}; \tag{2.7}$$

thus they depend only on the values of the distributions at their edges, at $P_0(\Omega,\Omega)=p_0/\Omega$ and at $R_0(\Omega,\Omega)=r_0/\Omega$. At the end of this section we present arguments that this special solution represents the true solution of the problem at the fixed point, i.e., as $\Omega\to 0$. Later we also show how the parameters of the special solution can be related with the initial distributions $P_{\rm in}(h)$ and $R_{\rm in}(J)$.

Putting Eqs. (2.6) and (2.7) into Eq. (2.4) we obtain

$$\left[p_0 r_0-\Omega\frac{dp_0}{d\Omega}\right]\left[\ln\frac{\Omega}{h}-\frac{1}{p_0}\right]=0, \tag{2.8}$$

which leads to the ordinary differential equation

$$\frac{dp_0}{d\Gamma}=-p_0 r_0 \tag{2.9}$$

in terms of the log-energy variable $\Gamma=-\ln\Omega$. Similarly, from Eq. (2.5) for the edge parameter r_0 we obtain

$$\frac{dr_0}{d\Gamma}=-r_0 p_0. \tag{2.10}$$

Subtracting Eq. (2.9) from Eq. (2.10) we obtain that p_0 and r_0 differ from each other by a constant 2Δ:

$$p_0-r_0=2\Delta. \tag{2.11}$$

Thus in terms of the variable

$$y_0=p_0-\Delta=r_0+\Delta, \tag{2.12}$$

we obtain one differential equation:

$$\frac{dy_0}{d\Gamma}+y_0^2=\Delta^2. \tag{2.13}$$

Here we note that Δ is related to the asymmetry in the renormalization of couplings and transverse fields; its value, which can be expressed by the initial distributions, will be determined later. At the critical point, where the distributions of the transverse fields and that of the couplings evolve to the same limiting function as $\Omega\to 0$, we have $\Delta=0$. In the paramagnetic phase, where according to Eq. (1.2) transverse fields on average are stronger than the average couplings, $\Delta>0$, whereas in the ferromagnetic phase we have the opposite situation: $\Delta<0$.

At the critical point, $\Delta=0$, the solution to Eq. (2.13) is given by

$$y_0=p_0=r_0=\frac{1}{\Gamma-\Gamma_0}=\frac{1}{\ln(\Omega_0/\Omega)},\quad \delta=\Delta=0, \tag{2.14}$$

where $\Gamma_0=-\ln\Omega_0$ is a reference (log) energy scale. It is instructive to consider the distribution of the reduced log-coupling variable $\eta=-(\ln\Omega-\ln h)/\ln\Omega=-(\ln\Omega-\ln J)/\ln\Omega$, which is given from Eqs. (2.6) and (2.14) as

$$\rho(\eta)d\eta=\exp(-\eta)d\eta. \tag{2.15}$$

This is just the critical point solution of the RTIC of Fisher.[6]

The solution to Eq. (2.13) *in the off-critical region,* $\Delta\neq 0$, is given by

$$y_0=\frac{\Delta\overline{y_0}+\Delta^2\mathrm{th}[\Delta(\Gamma-\Gamma_0)]}{\Delta+\overline{y_0}\mathrm{th}[\Delta(\Gamma-\Gamma_0)]}$$
$$=|\Delta|\left(1+2\frac{\overline{y_0}-\Delta}{\overline{y_0}+\Delta}(\Omega/\Omega_0)^{2\Delta}+\cdots\right), \tag{2.16}$$

where the solution goes through the point $y_0=\overline{y_0}$ at the reference (log) energy cutoff Γ_0. The second equation in Eq.(2.16) is the approximate form of the solution close to the line of fixed points, where in terms of the original energy-scale variable $\Omega/\Omega_0\ll 1$. We note that y_0/Δ is a unique function of two dimensionless variables,

$$\frac{y_0}{\Delta}=y\left[\frac{\Delta}{\Delta_D},\left(\frac{\Omega}{\Omega_D}\right)^{2\delta}\right], \tag{2.17}$$

where $\Delta_D=\overline{y_0}$ and $\Omega_D=\Omega_0$.

In the following we relate the asymmetry parameter Δ to the properties of the initial distributions, $P_{\rm in}(h)$ and $R_{\rm in}(J)$. For this purpose we first calculate the derivative

$$\frac{d}{d\Omega}[J^\mu]_{\rm av}=R_0(\Omega,\Omega)\Omega^\mu+\int_0^\Omega\frac{dR_0(J,\Omega)}{d\Omega}J^\mu dJ$$
$$=R_0(\Omega,\Omega)\Omega^\mu+[P_0(\Omega,\Omega)-R_0(\Omega,\Omega)][J^\mu]_{\rm av}$$
$$-P_0(\Omega,\Omega)\int_0^\Omega dJJ^\mu\int_J^\Omega dJ' R\left(\frac{J\Omega}{J'}\Omega,\Omega\right)\frac{\Omega}{J'}, \tag{2.18}$$

where in the second equation we have used the RG equation in Eq. (2.5). In the last term we change the order of the integration, and thus obtain

$$\int_0^\Omega dJ'\int_0^{J'}dJJ^\mu R\left(\frac{J\Omega}{J'}\Omega,\Omega\right)\frac{\Omega}{J'}=\Omega^{-\mu}[J^\mu]_{\rm av}^2. \tag{2.19}$$

In a similar way one can evaluate $(d/d\Omega)[h^{-\mu}]_{\rm av}$, and then obtain for the average value of the following derivative:

$$\frac{d}{d\Omega}\left[\left(\frac{J}{h}\right)^\mu\right]_{\rm av}=\left\{1-\left[\left(\frac{J}{h}\right)^\mu\right]_{\rm av}\right\}\{P_0(\Omega,\Omega)\Omega^{-\mu}[J^\mu]_{\rm av}$$
$$+R_0(\Omega,\Omega)\Omega^\mu[h^{-\mu}]_{\rm av}\}. \tag{2.20}$$

Note that this quantity is vanishing for a parameter $\mu=\tilde{\mu}$, provided $[(J/h)^{\tilde{\mu}}]_{\rm av}=1$; thus $\tilde{\mu}$ defined in this way stays invariant along the RG trajectory. This relation is valid even

if the starting RG steps are approximative. At the fixed point $\Omega \to 0$, with the solution in Eqs. (2.6) and (2.7), we have evaluated the average

$$\left[\left(\frac{J^2}{h^2}\right)^{\Delta}\right]_{\rm av}=1. \tag{2.21}$$

Consequently the asymmetry parameter Δ is an invariant quantity of the RG transformation, which is then determined by the initial distributions, as shown in Eq. (1.9). In the following Δ is considered the nonlinear quantum control parameter of the RTIC.

Next we show that the RG equations in Eqs. (2.1) and (2.2) become asymptotically exact as the line of fixed points is approached, i.e., as $\Omega/\Omega_0 \to 0$. Here let us consider the disordered Griffiths phase $\Delta > 0$, the reasoning for $\Delta > 0$ follows from duality. Here the ratio of decimated bonds, Δn_J, and decimated transverse fields, Δn_h, goes to zero as $\Delta n_J/\Delta n_h = R_0(\Omega,\Omega)/P_0(\Omega,\Omega) = r_0/p_0 \sim \Omega^{2\Delta}$, thus close to the fixed point almost exclusively, transverse fields are decimated out. Then the probability $\Pr(\alpha)$, that the value of a coupling J, which is a neighbor to a decimated transverse field, is $\Omega > J > \alpha\Omega$, with $0 < \alpha < 1$ given by

$$\Pr(\alpha) \simeq \int_{\alpha\Omega}^{\Omega} R_0(J,\Omega)dJ = 1-\alpha^{r_0} \approx r_0 \ln(1/\alpha), \tag{2.22}$$

which goes to zero during iteration, since according to Eqs. (2.16) and (2.12) $r_0 = R_0(\Omega,\Omega)\Omega \to 0$. Consequently the RG transformation becomes asymptotically exact and the singularities, which are characterized by the parameter Δ as calculated by the original distributions in Eq. (2.21), are also exact.

B. Relation between energy and length scales

Next we are going to study the actual relation between the asymmetry or nonlinear quantum control parameter Δ and the Griffiths-McCoy singularities of the RTIC. For this we investigate the relation between the energy scale Ω and the length scale L_Ω, by studying the fraction of nondecimated spins, n_Ω. When the energy scale is decreased by an amount of $d\Omega$ a fraction of spins. $dn_\Omega = n_\Omega[P_0(\Omega,\Omega)+R_0(\Omega,\Omega)]$, is decimated out, so that we obtain the differential equation

$$\frac{dn_\Omega}{d\Omega} = n_\Omega[P_0(\Omega,\Omega)+R_0(\Omega,\Omega)], \tag{2.23}$$

which can be rewritten as

$$-\frac{d\ln n_\Omega}{d\ln\Omega} = -[r_0(\Omega)+p_0(\Omega)] = -2y_0(\Omega). \tag{2.24}$$

Using the solution to $y_0(\Omega)$ in Eq. (2.16) one can integrate Eq. (2.24) with the result

$$n_\Omega = \left\{ \mathrm{ch}\left[\Delta \ln\frac{\Omega_0}{\Omega}\right] + \frac{\overline{y_0}}{\Delta}\mathrm{sh}\left[\Delta\ln\frac{\Omega_0}{\Omega}\right]\right\}^{-2}. \tag{2.25}$$

To obtain the scaling form at the critical point, in Eq. (2.25) we take the limits $\Delta \to 0$ and $\Gamma = -\ln\Omega \to \infty$, with, however, $\Delta \times \Gamma \to 0$, and obtain

$$n_\Omega = \left[1+\overline{y}_0 \ln\frac{\Omega_0}{\Omega}\right]^{-2} \sim \left[\ln\frac{\Omega_0}{\Omega}\right]^{-2}, \quad \Delta = 0, \tag{2.26}$$

which could be also directly calculated from Eq. (2.25) with the critical-point solution in Eq. (2.14). Thus, from Eq. (2.26) for the typical distance between remaining spins, L_Ω, we obtain

$$L_\Omega \sim \frac{1}{n_\Omega} \sim \left[\ln\frac{\Omega_0}{\Omega}\right]^2, \quad \Delta = 0, \tag{2.27}$$

which is just the relation in Eq. (1.6), as found earlier by Fisher.[6]

In the Griffiths phases, $|\Delta| > 0$, in Eq. (2.25), one obtains $n_\Omega \sim \Omega^{2|\Delta|}$, in the limit $\Omega \to 0$. Consequently the relation between typical distance between remaining spins, $L_\Omega \sim 1/n_\Omega$, and the energy scale is given by

$$L_\Omega \simeq L_{\Omega_0}(\Delta+y_0)^2\left(\frac{2}{\Delta}\right)^2\left(\frac{\Omega_0}{\Omega}\right)^{2|\Delta|} \sim \Omega^{-2|\Delta|}. \tag{2.28}$$

Thus Δ is simply related to the dynamical exponent z,

$$z = \frac{1}{2|\Delta|}, \tag{2.29}$$

as shown in Eq. (1.10).

To obtain a relation between the nonlinear quantum control parameter Δ, in Eq. (2.21), and the linear control parameter δ, as defined in Eq. (1.2) we perform a Taylor expansion for

$$[J^{2\Delta}]_{\rm av} = 1 + 2\Delta[\ln J]_{\rm av} + 2\Delta^2[\ln J)^2]_{\rm av} + O(\Delta^3), \tag{2.30}$$

and similarly for $[h^{2\Delta}]_{\rm av}$. Putting these into Eq. (2.21) we obtain that $\Delta(\delta) = \delta + O(\delta^2)$, as shown in Eq. (1.11).

Closing this section, we argue that the special solution in Eqs. (2.6) and (2.7) is a true fixed-point solution as $\Omega \to 0$. First we refer to Fisher's results at the critical point,[6] which justifies that any nonsingular initial distribution is attracted by the special solution in Eqs. (2.6) and (2.7). Second, we consider a finite-energy scale $\Omega > 0$, when nonasymptotic solutions to Eqs. (2.4) and (2.5) are given by the special solutions in Eqs. (2.6) and (2.7) extended by nonuniversal functions $P_0'(h,\Omega)$ and $R_0'(J,\Omega)$, respectively. Inserting these nonasymptotic solutions into Eq. (2.6), the relation in Eq. (2.8) will be extended by other terms containing P_0' and R_0'. As Ω goes to zero, however, the second factor in Eq. (2.8) is diverges; therefore, the corrections become irrelevant and the relation in Eq. (2.9) will govern the fixed-point behavior. Our third argument is based on numerical solutions of Eqs. (2.4) and (2.5), which are evolving toward special solutions in Eqs. (2.6) and (2.7) for different initial distributions.

Thus we can summarize that in the Griffiths phases the RTIC is uniquely characterized by a nonuniversal quantum control parameter Δ, which is related to the dynamical ex-

ponent z through Eq.(1.10). The possible difference between two initial distributions leading to the same Δ is given by the nonuniversal parameters Ω_0 and $\overline{y_0}$, which account for the number of necessary RG steps until the fixed-point distributions in Eqs. (2.6) and (2.7) are sufficiently approached.

III. RENORMALIZATION OF LENGTHS AND MAGNETIC MOMENTS

The scaling behavior of lengths and magnetic moments during renormalization can be deduced from the joint distribution functions $P_l(h,l,\Omega)$,$R_l(J,l,\Omega)$, and $P_\mu(h,\mu,\Omega)$, as defined in Eqs. (2.3). From here on we drop the index s or b to indicate the type of the length.

A. Scaling of lengths

In the following we consider the joint distribution $P_l(h,l,\Omega)$, and write down the relevant evaluation equation when energy scale is lowered as $\Omega\to\Omega-d\Omega$. Generalizing the reasoning leading to Eq. (2.4) we obtain

$$\frac{dP_l(h,l,\Omega)}{d\Omega}=P_l(h,l,\Omega)[R_0(\Omega,\Omega)-P_0(\Omega,\Omega)] - \int_h^\Omega dh_1\frac{\Omega}{h_1}\int_0^l dl_2\int_0^{l-l_2}dl_1 R_l(\Omega,l_2,\Omega)P_l \times(h_1,l_1,\Omega)P_l\left(\frac{h}{h_1}\Omega,l-l_1-l_2,\Omega\right), \quad (3.1)$$

and similarly for the coupling distribution by interchanging $R\leftrightarrow P$ and $J_i\leftrightarrow h_i$, as in Eqs. (2.4) and (2.5). The second term on the right-hand side of Eq. (3.1) can be written as a convolution in terms of the variable $l'=l_1+l_3=l-l_2$ as

$$-\int_0^l dl' R_l(\Omega,l-l',\Omega)\int_0^{l'}dl_1\int_h^\Omega dh_1 \times\frac{\Omega}{h_1}P_l(h_1,l_1,\Omega)P_l\left(\frac{h}{h_1}\Omega,l'-l_1,\Omega\right). \quad (3.2)$$

Consequently taking the Laplace transforms

$$\tilde P_l(h,\lambda,\Omega)=\int_0^\infty e^{-l\lambda}P_l(h,l,\Omega)dl,$$

$$\tilde R_l(J,\lambda,\Omega)=\int_0^\infty e^{-l\lambda}R_l(J,l,\Omega)dl, \quad (3.3)$$

we obtain a simpler relation

$$\frac{d\tilde P_l(h,\lambda,\Omega)}{d\Omega} =\tilde P_l(h,\lambda,\Omega)[\tilde R_l(\Omega,0,\Omega)-\tilde P_l(\Omega,\lambda,\Omega)]-\tilde R_l(\Omega,\lambda,\Omega) \times\int_h^\Omega dh'\tilde P_l(h',\lambda,\Omega)\tilde P_l\left(\frac{h}{h'}\Omega,\lambda,\Omega\right)\frac{\Omega}{h'}, \quad (3.4)$$

and similarly for the coupling distribution:

$$\frac{d\tilde R_l(J,\lambda,\Omega)}{d\Omega} =\tilde R_l(J,\lambda,\Omega)[\tilde P_l(\Omega,0,\Omega)-\tilde R_l(\Omega,\lambda,\Omega)]-\tilde P_l(\Omega,\lambda,\Omega) \times\int_J^\Omega dJ'\tilde R_l(J',\lambda,\Omega)\tilde R_l\left(\frac{J}{J'}\Omega,\lambda,\Omega\right)\frac{\Omega}{J'}. \quad (3.5)$$

Note that the different λ components are separated, which makes it possible to solve the equations. For $\lambda=0$, when $\tilde P_l(h,0,\Omega)=P_0(h,\Omega)$ and $\tilde R_l(J,0,\Omega)=R_0(J,\Omega)$, the solutions are given in Eqs. (2.6) and (2.7). With this guidance we are looking for a solution for general λ in the forms

$$\tilde P_l(h,\lambda,\Omega)=\frac{\pi_l(\lambda,\Omega)}{\Omega}\left(\frac{\Omega}{h}\right)^{1-p_l(\lambda,\Omega)}, \quad (3.6)$$

$$\tilde R_l(J,\lambda,\Omega)=\frac{\rho_l(\lambda,\Omega)}{\Omega}\left(\frac{\Omega}{J}\right)^{1-r_l(\lambda,\Omega)}, \quad (3.7)$$

where now $p_l(0,\Omega)=\pi_l(0,\Omega)=p_0(\Omega)$ and $r_l(0,\Omega)=\rho_l(0,\Omega)=r_0(\Omega)$, whereas for $\lambda>0$ $p_l(\lambda,\Omega)>\pi_l(\lambda,\Omega)$ and $r_l(\lambda,\Omega)>\rho_l(\lambda,\Omega)$. This latter relation follows from the fact that the average length of a bond, $\overline{l_b}>0$, and that of a spin cluster, $\overline{l_s}>0$, are given by

$$\overline{l_b}=\lim_{\lambda\to0}\frac{1}{\lambda}\left[1-\frac{\rho_l(\lambda,\Omega)}{r_l(\lambda,\Omega)}\right], \quad \overline{l_s}=\lim_{\lambda\to0}\frac{1}{\lambda}\left[1-\frac{\pi_l(\lambda,\Omega)}{p_l(\lambda,\Omega)}\right]. \quad (3.8)$$

Inserting the functions in Eqs. (3.6) and (3.7) into Eqs. (3.4) and (3.5), we obtain a set of ordinary differential equations:

$$\frac{dp_l}{d\Gamma}=-\pi_l\rho_l, \quad \frac{d\pi_l}{d\Gamma}=-\pi_l(p_l-\pi_l+\rho_l),$$

$$\frac{dr_l}{d\Gamma}=-\pi_l\rho_l, \quad \frac{d\rho_l}{d\Gamma}=-\rho_l(r_l-\rho_l+\pi_l). \quad (3.9)$$

involving the functions p_l,π_l,r_l, and ρ_l.

These equations are solved in the Appendix. Here we consider only the scaling behavior of the average lengths, and for this it is enough to treat the small λ expansions up to linear order:

$$p_l(\lambda,\Omega)=p_0(\Omega)+\lambda p_1(\Omega),$$

$$\pi_l(\lambda,\Omega)=p_0(\Omega)+\lambda\pi_1(\Omega),$$

$$r_l(\lambda,\Omega)=r_0(\Omega)+\lambda r_1(\Omega), \quad \rho_l(\lambda,\Omega)=r_0(\Omega)+\lambda\rho_1(\Omega), \quad (3.10)$$

Inserting the expressions in Eqs. (3.10) into (3.9), for the correction terms we obtain:

$$\frac{dp_1}{d\Gamma}=-\pi_1 r_0-\rho_1 p_0, \quad \frac{d\pi_1}{d\Gamma}=-\pi_1 r_0-p_0(p_1-\pi_1+\rho_1),$$

$$\frac{dr_1}{d\Gamma}=-\pi_1 r_0-\rho_1 p_0, \quad \frac{d\rho_1}{d\Gamma}=-\rho_1 p_0-\rho_0(r_1-\rho_1+\pi_1). \tag{3.11}$$

Now, noting that

$$\frac{d\ln(p_1-\pi_1)}{d\Gamma}=p_0, \quad \frac{d\ln(r_1-\rho_1)}{d\Gamma}=r_0, \tag{3.12}$$

after integration we obtain, using Eqs. (2.9) and (2.10), that $p_1-\pi_1=A_p/r_0$ and $r_1-\rho_1=A_r/p_0$, where A_p and A_r are integration constants. Consequently the average lengths from Eq. (3.8) are given by

$$\overline{l_s}=\frac{p_1-\pi_1}{p_0}=\overline{l_s}(\Omega_0)\frac{r_0(\Omega_0)p_0(\Omega_0)}{r_0(\Omega)p_0(\Omega)}=\overline{l_s}(\Omega_0)\frac{\overline{y_0^2}-\Delta^2}{y_0^2-\Delta^2} \tag{3.13}$$

and

$$\overline{l_b}=\frac{r_1-\rho_1}{r_0}=\overline{l_b}(\Omega_0)\frac{r_0(\Omega_0)p_0(\Omega_0)}{r_0(\Omega)p_0(\Omega)}=\overline{l_b}(\Omega_0)\frac{\overline{y_0^2}-\Delta^2}{y_0^2-\Delta^2}. \tag{3.14}$$

At the line of fixed points $\Omega\to 0$, one can see that $\overline{l_s}\sim\overline{l_b}\sim L_\Omega$, consequently the previous interpretation of the dynamical exponent z in Eq. (2.28) is also justified with average length scales.

Now, to calculate the correlation length ξ in the paramagnetic phase $\Delta>0$, one should take into account that the ratio of (nondecimated transverse fields)/(nondecimated couplings) at an energy scale Ω, is given by p_0/r_0. Consequently the number of nondecimated spins in a cluster is given by $\sim\overline{l_s}p_0/r_0\sim 1/r_0^2\sim\Delta^{-2}$, which stays constant as the energy-scale is lowered. This quantity is actually the measure of the size of the average correlated domain in the system, where the couplings between the spins, being larger then the transverse fields, are decimated out. Therefore, in this way we have an estimate for the correlation length close to the critical point,

$$\xi\sim\Delta^{-2}\sim\delta^{-2}, \tag{3.15}$$

which is consistent with Fisher's result in Eq. (1.5).

B. Scaling of magnetization moments

In this subsection we perform a similar calculation for the joint distribution function $P_\mu(h,\mu,\Omega)$, and calculate the average size of a magnetic moment, $\overline{\mu}(\Omega)$, as a function of the energy cutoff. The joint distribution function $P_\mu(h,\mu,\Omega)$ satisfies the differential equation

$$\frac{dP_\mu(h,\mu,\Omega)}{d\Omega}=P_\mu(h,\mu,\Omega)[R_0(\Omega,\Omega)-P_0(\Omega,\Omega)] -R_0(\Omega,\Omega)\int_h^\Omega dh'\frac{\Omega}{h'}\int_0^\mu d\mu' P_\mu \times(h',\mu',\Omega)P\left(\frac{h}{h'}\Omega,\mu-\mu',\Omega\right), \tag{3.16}$$

which can be derived along the lines of Eqs. (2.4) and (3.1). The second term on the right-hand side of Eq. (3.16) is a convolution; therefore, we introduce the Laplace transform

$$\tilde{P}_\mu(h,s,\Omega)=\int_0^\infty e^{-\mu s}P_\mu(h,\mu,\Omega)d\mu, \tag{3.17}$$

which satisfies the relation

$$\frac{d\tilde{P}_\mu(h,s,\Omega)}{d\Omega} =\tilde{P}_\mu(h,s,\Omega)[R_0(\Omega,\Omega)-\tilde{P}_\mu(\Omega,0,\Omega)]-R_0(\Omega,\Omega) \times\int_h^\Omega dh'\tilde{P}_\mu(h',s,\Omega)\tilde{P}_\mu\left(\frac{h}{h'}\Omega,s,\Omega\right)\frac{\Omega}{h'}. \tag{3.18}$$

In Eq. (3.18) the different s components are separated for $s=0$; when $\tilde{P}_\mu(h,0,\Omega)=P_0(h,\Omega)$, the solution is given in Eqs. (2.6). As for the joint distribution of the lengths, $P_l(h,\lambda,\Omega)$, in Eq. (3.6) we are looking for the solution for general s in the form

$$\tilde{P}_\mu(h,s,\Omega)=\frac{\pi_\mu(s,\Omega)}{\Omega}\left(\frac{\Omega}{h}\right)^{1-p_\mu(s,\Omega)}. \tag{3.19}$$

Here again $p_\mu(0,\Omega)=\pi_\mu(0,\Omega)=p_0(\Omega)$, whereas $p_\mu(s,\Omega)>\pi_\mu(s,\Omega)$ for $s>0$, since the average cluster moment, $\overline{\mu}>0$, is given by

$$\overline{\mu}=\lim_{s\to 0}\frac{1}{s}\left[1-\frac{\pi_\mu(s,\Omega)}{p_\mu(s,\Omega)}\right]. \tag{3.20}$$

Putting Eq. (3.19) into Eq. (3.16), we find that the functions p_μ and π_μ satisfy the differential equations

$$\frac{dp_\mu}{d\Gamma}=-\pi_\mu r_0,$$

$$\frac{d\pi_\mu}{d\Gamma}=-\pi_\mu(r_0-p_0+p_\mu). \tag{3.21}$$

Keeping in mind that the average cluster moment $\overline{\mu}$, and thus the average magnetization, m are defined as

$$m=\frac{\overline{\mu}}{\overline{l_s}}, \tag{3.22}$$

and are related to the small s asymptotics of the distribution in Eq. (3.19), we perform the expansions up to linear order:

$$p_\mu(s,\Omega)=p_0(\Omega)+s\tilde{p}_1(\Omega),$$

$$\pi_\mu(s,\Omega)=p_0(\Omega)+s\tilde{\pi}_1(\Omega). \tag{3.23}$$

For the correction terms $\tilde{p}_1$ and $\tilde{\pi}_1$, we derive differential equations, in terms of the functions p_0 and r_0, as

$$\frac{d\tilde{p}_1}{dp_0}=\frac{\tilde{\pi}_1}{p_0},$$

$$\frac{d\tilde{\pi}_1}{dp_0}=\frac{\tilde{\pi}_1}{p_0}+\frac{\tilde{p}_1}{r_0}, \tag{3.24}$$

which leads to

$$(y_0^2-\Delta^2)\frac{d^2\tilde{p}_1}{dy_0^2}=\tilde{p}_1, \tag{3.25}$$

where $y_0=y_0(\Omega)$ is given in Eq. (2.16). We note that with the solution for $\tilde{p}_1$, for the average cluster moment we have

$$\overline{\mu}=\frac{\tilde{p}_1-\tilde{\pi}_1}{p_0}=-\frac{\int_{y_0}^{y_0}dy_0'\tilde{p}_1(y_0')/(y_0'-\Delta)}{y_0+\Delta}. \tag{3.26}$$

At the critical point with $\Delta=0$ the solution of Eq. (3.25) is given in simple power-law forms

$$\tilde{p}_1=y_0^{-\tau}, \quad \Delta=0, \tag{3.27}$$

where $\tau=(\sqrt{5}-1)/2$ is the positive root of the equation: $\tau(\tau+1)=1$. [The other linearly independent solution with $\tau=-(\sqrt{5}+1)/2$ is physically unacceptable, since the average cluster moment would be smaller than 1.] From Eq. (3.24) we have $\tilde{\pi}_1=-\tau y_0^{-\tau}$, and using Eq. (3.26) for the average cluster moment at the critical point we obtain

$$\overline{\mu}=\text{const}\, y_0^{-(1+\tau)}=\overline{\mu}_0\left[\ln\left(\frac{\Omega_0}{\Omega}\right)\right]^\Phi, \quad \Phi=\frac{1}{\tau}=\frac{1+\sqrt{5}}{2}. \tag{3.28}$$

In this way we have rederived Fisher's result[6] for the scaling behavior of the average cluster moment in a direct way.

In Griffiths phases with $\Delta\neq 0$, the differential equation in Eq. (3.25) in terms of the variable $y=y_0/\Delta$ is related to the Legendre differential equation and the physically acceptable solution can be expressed by the hypergeometric function,[11] $F(a,b;c;z)$, as

$$\tilde{p}_1=|\Delta|^{-\tau}y^{-\tau}F\left(\frac{\tau}{2},\frac{1}{2}+\frac{\tau}{2};\frac{3}{2}+\tau;\frac{1}{y^2}\right)=|\Delta|^{-\tau}f_1(y), \tag{3.29}$$

where, in the limit $\Delta\to 0$, we recover the solution at the critical point in Eq. (3.27). From Eq. (3.24) we obtain

$$\tilde{\pi}_1=-|\Delta|^{-\tau}(y-1)y^{-(\tau+1)}F\left(\frac{\tau}{2}+1,\frac{1}{2}+\frac{\tau}{2};\frac{3}{2}+\tau;\frac{1}{y^2}\right)$$
$$=|\Delta|^{-\tau}\phi_1(y), \tag{3.30}$$

and putting Eqs. (3.29) and (3.30) into Eq. (3.26), for the average cluster moment we obtain

$$\overline{\mu}=\text{const}|\Delta|^{-\tau-1}\frac{f(y)}{y+1}, \tag{3.31}$$

where $f(y)=f_1(y)-\phi_1(y)$. Here one should differentiate between the paramagnetic ($\Delta>0$, $y>0$) and ferromagnetic ($\Delta>0$, $y>0$) phases. In the former case the average cluster moment approaches a finite limiting value, as $\Omega/\Omega_0\to 0$, whereas in the ferromagnetic phase, where $y\to 1^-$ in the fixed point, $\overline{\mu}$ is divergent, as $\overline{\mu}(\Omega)\sim\Omega^{-2|\Delta|}$. For the average magnetization in Eq. (3.22), one then obtains

$$m=m_0\frac{(1-y)f(y)}{(1-\overline{y})f(\overline{y})}, \tag{3.32}$$

where m_0 is the average magnetization at $\Omega=\Omega_0$, and $\overline{y}$ denotes the value of the variable y at the same energy scale. The average magnetization in the paramagnetic phase is zero, whereas in the ferromagnetic phase one has to evaluate Eq. (3.32) along the lines of semicritical fixed points: $\Omega/\Omega_0\to 0$. Here, taking the limit $|\Delta|\ll 1$, i.e., close to the critical point, we have $(1-\overline{y})^{-1}\sim|\Delta|$ and $f(\overline{y})\sim|\Delta|^\tau$, so that[12]

$$m=\text{const}|\Delta|^{1-\tau}=\text{const}|\delta|^{1-\tau}. \tag{3.33}$$

From Eq. (3.33) one can read the critical exponent of the average magnetization as

$$\beta=1-\tau=2-\Phi, \tag{3.34}$$

which corresponds to Fisher's result in Eq. (1.4).

IV. SCALING OF THERMODYNAMICAL QUANTITIES

In the previous sections we presented solutions of the RG equations in the entire Griffiths region for the distribution of couplings, transverse fields, lengths, and magnetization moments. Then, with those distributions, average quantities, such as length scales, magnetization, etc. were calculated at zero temperature and in the absence of a longitudinal magnetic field. In this section we extend these calculations, and determine the scaling form of singular thermodynamic quantities as a function of a small, but finite, temperature $T>0$, or magnetic field $H>0$.

To treat the effect of a small finite temperature in the RG scheme, one should first note that the thermal energy sets an energy scale $\Omega_T\sim T$, and that the RG decimation should be stopped as Ω is lowered to Ω_T. At that energy scale a fraction of spin clusters, n_{Ω_T}, in Eq. (2.25) is not decimated out, and these spins are loosely coupled as compared with the temperature T. Consequently the entropy per spin, s, is given as the contribution of non-interacting spin clusters,

$$s \simeq n_{\Omega_T} \ln 2, \tag{4.1}$$

whereas the specific heat can be obtained through a derivation: $c_V = T(\partial s/\partial T)$. From Eqs. (4.1) and (2.25), for the singular behavior we obtain:

$$s(T) \sim c_V(T) \sim T^{2|\Delta|}, \tag{4.2}$$

which is valid both in ordered and disordered Griffiths phases.

Next we consider the effect of a small longitudinal field, $H>0$, at zero temperature. During renormalization the local longitudinal field H_l at site l is transformed as

$$\tilde{H}_l = H\mu_l, \tag{4.3}$$

so that the energy scale related to the longitudinal field is given by $\Omega_H = H\overline{\mu}(\Omega)$. As Ω is lowered to Ω_H, i.e., when the energy scale satisfies the equation

$$\Omega_H = H\overline{\mu}(\Omega_H), \tag{4.4}$$

the RG procedure is stopped, and the remaining spin clusters are practically uncoupled. Then the average magnetization and the average susceptibility satisfy the equations

$$m(H) = m(\Omega = \Omega_H), \quad \chi = \frac{\partial m}{\partial H}. \tag{4.5}$$

In the *disordered Griffiths phase*, where $\overline{\mu}(\Omega_H)$ has a Ω_H-independent limiting value, we have $\Omega_H \sim H$; consequently, from Eq. (3.32) the singular behavior is given by

$$m(H) \sim \left(\frac{H}{H_D}\right)^{2\Delta}, \quad \Delta > 0. \tag{4.6}$$

More generally the scaling form is given in Eq. (1.12), where the scaling function can be computed using Eqs. (3.32), (2.16), and (2.17). Similarly, for the scaling of the susceptibility in the disordered Griffiths phase one obtains

$$\chi(H) \sim \left(\frac{H}{H_D}\right)^{-1+2\Delta}, \quad \chi(T) \sim T^{-1+2\Delta}, \quad \Delta > 0, \tag{4.7}$$

where the temperature dependence follows from the scaling relation $\Omega_H \sim \Omega_T$.

In the *ordered Griffiths phase*, where $\overline{\mu}(\Omega_H) \sim \Omega_H^{-2|\Delta|}$, as given above Eq. (3.32) we have $\Omega_H \sim H^{1/(1+2|\Delta|)}$. Putting this result into Eq. (3.32), and using the asymptotic expansion for the hypergeometric functions[11] in Eqs. (3.29) and (3.30), for the leading field dependence of the magnetization we obtain

$$m(H) - m(0) \sim \left(\frac{H}{H_D}\right)^{2|\Delta|/(1+2|\Delta|)} \ln\left(\frac{H}{H_D}\right), \quad \Delta < 0, \tag{4.8}$$

and, similarly for the susceptibility,

$$\chi(H) \sim \left(\frac{H}{H_D}\right)^{-1/(1+2|\Delta|)} \ln\left(\frac{H}{H_D}\right), \quad \Delta < 0. \tag{4.9}$$

Note that in the ordered Griffiths phase the singularity exponent is different from that in the disordered Griffiths phase, and that there is a logarithmic correction term. The temperature dependence of the susceptibility, which follows from the relation $\Omega_H \sim \Omega_T$, is given by

$$\chi(T) \sim T^{-1+2|\Delta|} \ln T, \quad \Delta < 0. \tag{4.10}$$

We conclude this section by stating that all singularities of different physical quantities, both in (strongly) ordered and disordered Griffiths phases, can be expressed by the nonlinear quantum control parameter Δ.

V. DISCUSSION

In this paper the strong-disorder RG method is applied in the strongly disordered and strongly ordered Griffiths phases of the random transverse-field Ising spin chain. With this calculation we have demonstrated that the RG method leads to asymptotically exact results in the entire Griffiths region. The key concept of our solution is the introduction of a nonlinear quantum control parameter Δ, which remains invariant under the RG transformation, even if the renormalization is approximate in the starting decimation steps. Δ, which is a measure of the asymmetry in the renormalization between the couplings and the transverse fields, is simply related to the dynamical exponent z, and all the singularities of the different physical quantities in the Griffiths phases can be expressed by it. In this way we have presented an example for a RG transformation, where the relevant *nonlinear scaling field*[13] outside the critical fixed point is exactly constructed, and the off-critical singularities are calculated analytically. The line of fixed points controlling the singular behavior in the Griffiths phases are found to be *strongly attractive*: for any weak initial disorder, having the same asymmetry parameter Δ, the system scales into the same fixed point. This is a remarkable weak-universality property of the system. We note that previous numerical[8,14] and analytical[15] results about the RTIC are in accordance with our RG findings.

At this point one may ask how far these results are general, and how they could apply for other random quantum spin systems. The above scenario is certainly valid for those problems, which can be mapped to the RTIC, so that the RG equations can be transformed into an equivalent form. Free fermionic spin-1/2 models, such as the random XX model with dimerization or the random XY model with an X/Y anisotropy, are such examples.[16,17] Also, the one-dimensional Sinai-walk problem, i.e., a random walk in a random environment with a global bias,[18] can be mapped to the RTIC (Ref. 19), thus a renormalization analysis[20] also leads to asymptotically exact results for this problem.

Other, more general, random quantum spin systems could belong to two main classes: (i) systems having an IRFP and a line of semicritical fixed points which are *strongly attractive*, and ii) those models where a cross-over phenomena takes place when the strength of the disorder is increased.

Models belonging to the first class include, among others, the random spin-1/2 Heisenberg chain[16] and random $q \geqslant 2$ states quantum Potts chains.[21] For these systems we expect that a strong-disorder RG calculation also leads to asymptotically exact results, and the physical picture obtained in the analytical treatment of the RTIC stays qualitatively correct. Indeed, scaling arguments and numerical calculations about specific models are in favor of our conjecture.[10]

The second class of models includes, among others, the random quantum clock and quantum Ashkin-Teller chains,[22] and probably several higher-dimensional systems (two-dimensional random transverse-field Ising model,[23,24] etc.). For these problems the disorder should exceed a limiting strength, when the IRFP and the line of semicritical fixed points become attractive. Above this limiting disorder value the strong disorder RG could be asymptotically exact. To verify this possibility, however, one should perform detailed numerical investigations.

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APPENDIX: DISTRIBUTION FUNCTION OF LENGTHS

Here we present a general solution for the joint distribution functions $P_l(h,l,\Omega)$ and $R_l(J,l,\Omega)$, in the vicinity of the line of fixed points $\Omega/\Omega_0 \to 0$. This amounts to solving the set of differential equations in Eqs. (3.9) involving the functions $p_l(\lambda,\Omega), \pi_l(\lambda,\Omega), r_l(\lambda,\Omega)$ and $\rho_l(\lambda,\Omega)$, which appear in the Laplace transforms $\tilde{P}_l(h,\lambda,\Omega)$ and $\tilde{R}_l(h,\lambda,\Omega)$ in Eqs. (3.6) and (3.7).

We start with the solution *at the critical point* where the fixed-point distributions of the couplings and the transverse fields are identical, so that $p_l(\lambda,\Omega)=r_l(\lambda,\Omega)$ and $\pi_l(\lambda,\Omega)=\rho_l(\lambda,\Omega)$. Here we have just two differential equations

$$\frac{dp_l}{d\Gamma} = -\pi_l^2, \quad \frac{d\ln \pi_l}{d\Gamma} = -p_l \pi_l. \tag{A1}$$

From Eq. (A1) it follows that $d(p_l^2-\pi_l^2)/d\Gamma = 0$; therefore, p_l^2 and π_l^2 differ by an Ω-independent term

$$\pi_l^2(\lambda,\Omega) = p_l^2(\lambda,\Omega) - c^2(\lambda), \tag{A2}$$

and we can write a simple differential equation

$$\frac{dp_l}{d\Gamma} + p_l^2 = c^2. \tag{A3}$$

Since Eq. (A3) is equivalent to Eq. (2.13), for its solution, from Eq. (2.16), with the substitution $\Delta \to c$, we have

$$p_l = \frac{p_l^0 c + c^2 \tanh[c \ln(\Omega_0/\Omega)]}{c + p_l^0 \tanh[c \ln(\Omega_0/\Omega)]}, \tag{A4}$$

where now $p_l^0 = p(\lambda,\Omega_0)$ at a reference point, $\Omega = \Omega_0$.

Close to the line of fixed points, as $\Omega_0/\Omega \to 0$, we should have $c(\lambda) \to 0$, in order to have a finite scaling combination in Eq. (A4). In this small-λ limit from Eqs. (3.8) and (A2) we obtain that $c^2(\lambda) = a^2\lambda + O(\lambda^2)$, where the value of the constant a is connected to the average lengths at $\Omega = \Omega_0$ and $\overline{l_0} = \overline{l}(\Omega_0)$. As $\overline{l_0} = a^2/2(p_l^0)^2$. As $\Omega \to 0$, the average length of a cluster or bond is divergent as

$$\overline{l} = \overline{l}_0 p_0^2 \left(\ln \frac{\Omega_0}{\Omega} \right)^2, \tag{A5}$$

which is the same as the typical distance between remaining spins, L_Ω, as given in Eq. (2.27), if we make the identification $a^2 = p_0$. To obtain the joint distribution of the fields (couplings) and lengths, we use the fact that at the fixed point of the RG transformation the appropriate scaling variable in Eq. (A4) is $a\lambda^{1/2}\ln(\Omega_0/\Omega) = O(1)$; therefore,

$$p_l(\lambda,\Omega) = a\lambda^{1/2} \coth\left[a\lambda^{1/2} \left(\ln \frac{\Omega_0}{\Omega} \right) \right],$$

$$\pi_l(\lambda,\Omega) = a\lambda^{1/2} \sinh^{-1}\left[a\lambda^{1/2} \left(\ln \frac{\Omega_0}{\Omega} \right) \right]. \tag{A6}$$

Consequently at the fixed point, $P_l(h,l,\Omega)$ and $R_l(J,l,\Omega)$ can be obtained by an inverse Laplace transform of Eqs. (3.6) and (3.7), with Eqs. (A6).

In the Griffiths region, i.e., *outside the critical point* one should consider a system of four coupled differential equations in Eqs. (3.9), where one can construct two Ω-independent combinations of the variables

$$p_l(\lambda,\Omega) - r_l(\lambda,\Omega) = 2\Delta(\lambda) \tag{A7}$$

and

$$p_l(\lambda,\Omega) r_l(\lambda,\Omega) - \pi_l(\lambda,\Omega)\rho_l(\lambda,\Omega) = D(\lambda)^2. \tag{A8}$$

Thus there are two variables left, $y_l = (p_l + r_l)/2$ and $u_l = p_l - \pi_l$, which satisfy the differential equations

$$\frac{dy_l}{d\Gamma} + y_l^2 = d^2, \quad d^2 = \Delta(\lambda)^2 + D(\lambda)^2 \tag{A9}$$

and

$$\frac{du_l}{d\Gamma} + u_l^2 - (y_l + |\Delta(\lambda)|)u_l = 0. \tag{A10}$$

The solution of Eq. (A9) is analogous to that of Eq. (A3), and immediately given by

$$y_l = \frac{y_l^0 d + d^2 \tanh[d(\Gamma - \Gamma_0)]}{d + y_l^0 \tanh[d(\Gamma - \Gamma_0)]}, \tag{A11}$$

where $y_l^0 = y_l(\lambda, \Omega_0)$ at the reference energy, $\Omega = \Omega_0$.

To integrate Eq. (A10), we first note that it is a Bernoulli-type differential equation, and that its solution can be expressed as

$$\frac{1}{u_l} = E(\Gamma)\left(\int \frac{1}{E(\Gamma')} d\Gamma' + C \right), \tag{A12}$$

with

$$E(\Gamma) = \exp\left(- \int [y_l(\Gamma') + |\Delta(\lambda)|] d\Gamma' \right). \tag{A13}$$

Using the solution for $y_l(\Gamma)$ in Eq. (A11) we can perform the integration for $E(\Gamma)$ as

$$E(\Gamma) = e^{-\Gamma|\Delta|} \{ d \coth[d(\Gamma - \Gamma_0)] + y_l^0 \ \sinh[d(\Gamma - \Gamma_0)] \}^{-1}, \tag{A14}$$

putting this into Eq. (A12), one can integrate once more, giving

$$u_l = \frac{f_+(\Gamma) - f_-(\Gamma)}{f_+(\Gamma)(d + |\Delta|)^{-1} + f_-(\Gamma)(d - |\Delta|)^{-1} + C e^{-\Gamma|\Delta|}}, \tag{A15}$$

where $f_\pm(\Gamma) = (y_0 \pm d) \exp[\pm(\Gamma - \Gamma_0) d]$ and the value of the constant C follows from the boundary condition at $\Gamma = \Gamma_0$.

Having the solution at hand first, we check that at the critical point, where $\Delta(\lambda) = 0$ and thus $d^2 = D^2 = c^2$ we recover the previous solution. Indeed, the constant in Eq. (A15) at the critical point is given by $C = [(y_l^0)^2 - d^2]^{1/2}$, and than combining Eqs. (A11) and (A15) in the small-Ω and -λ scaling limits we recover the result in Eq. (A6).

In the Griffiths phases, $|\Delta(\lambda)| > 0$, keeping Eq. (3.8) in mind in the small-λ limit we have

$$d(\lambda) - |\Delta(\lambda)| = \frac{d(\lambda)^2 - \Delta(\lambda)^2}{d(\lambda) + |\Delta(\lambda)|} = A\lambda |\Delta(\lambda = 0)| + O(\lambda^2), \tag{A16}$$

so that the appropriate scaling combination in Eqs. (A11) and (A15), as $\Omega \to 0$ and $\lambda \to 0$, is $\lambda(\Omega_0/\Omega)^{2|\Delta|} = O(1)$. The constant A in Eq. (A16) is related to the average cluster size at $\Omega = \Omega_0$, $\overline{l}_s(\Omega_0)$, so that finally, along the line of semicritical points, we obtain

$$u_l = \frac{(y_l^0 + |\Delta|)^2 (\Omega_0/\Omega)^{2|\Delta|} \lambda / (2|\Delta|) \overline{l}_s(\Omega_0)}{(y_l^0 + |\Delta|)^2 (\Omega_0/\Omega)^{2|\Delta|} \lambda (1/4\Delta)^2 \overline{l}_s(\Omega_0) + 1}. \tag{A17}$$

Now the joint distribution of the fields (couplings) and lengths can be obtained through inverse Laplace transformation of Eqs. (3.3) using Eqs. (3.6) and (3.7) and the solutions in Eqs. (A7), (A8), (A11), and (A17).

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