Classical dynamics in a random magnetic field

Staffan Grundberg and Jørgen Rammer
Department of Theoretical Physics, Umeå University, S-901 87 Umeå, Sweden

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The classical dynamics in a quenched random magnetic field is investigated using both the Langevin and Boltzmann equations. In the former case, a field-theoretic method permits the average over the random magnetic field to be performed exactly. A self-consistent theory is constructed using the functional method for the effective action, allowing a determination of the particle response to external forces. In the Boltzmann approach the random magnetic field is treated as a random Lorentz driving force. Langevin dynamics always leads to negative magnetoresistance, whereas the Boltzmann description can lead to either positive or negative magnetoresistance. In both descriptions the resistivity decreases at low average magnetic fields as the temperature increases. In contrast to the Langevin result, the Boltzmann resistivity increases at moderate fields as a function of temperature, displaying a nonmonotonic temperature dependence. The transverse resistivity decreases as the temperature increases in the case of Langevin dynamics, whereas the opposite behavior occurs in the Boltzmann case.

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The dynamics of a charged particle constrained to move in two dimensions in a random magnetic field has received much recent experimental interest. Such a system can be realized by coating a heterostructure two-dimensional electron gas in a homogeneous magnetic field with a type-II superconductor.[1] superconducting grains,[2] or an overlayer of ferromagnets.[3]

The semiclassical dynamics of a degenerate electron gas in a random magnetic field has been studied using the Boltzmann equation, and the effect of the random magnetic field has been treated either as a source of scattering,[4] or as a random Lorentz driving force.[5] Here we shall study a two-dimensional classical gas, say a nondegenerate two-dimensional electron gas, in a random magnetic field using both the Langevin and Boltzmann approach. We first present a field-theoretic formulation of the Langevin dynamics, allowing the average over the random magnetic field to be performed exactly. A self-consistent theory is then constructed that makes it possible to calculate the nonequilibrium properties of the particle in a random magnetic field. Subsequently, using the Boltzmann equation with a random Lorentz driving force, a gas obeying Maxwell-Boltzmann statistics is studied.

The Langevin dynamics of a particle is described by the equation

$$m\ddot{\mathbf{x}}_t+\eta\dot{\mathbf{x}}_t=e\dot{\mathbf{x}}_t\times\mathbf{B}(\mathbf{x}_t)+\mathbf{F}_t+\boldsymbol{\xi}_t, \tag{1}$$

where $\mathbf{x}_t$ is the position of the particle in a plane perpendicular to the quenched random magnetic field $\mathbf{B}(\mathbf{x})$ (chosen along the $\hat{\mathbf{z}}$ axis). The mass and charge of the particle are m and e, respectively, and η is the friction coefficient, and $\mathbf{F}$ is an external force. The magnetic field, $\mathbf{B}(\mathbf{x})=\mathbf{B}+\delta\mathbf{B}(\mathbf{x})$, is assumed a Gaussian distributed stochastic variable and thus characterized by its mean, $\mathbf{B}=\langle\mathbf{B}(\mathbf{x})\rangle=B\hat{\mathbf{z}}$, and the correlation function $\nu(\mathbf{x}-\mathbf{x}')=e^2\langle\delta B(\mathbf{x})\,\delta B(\mathbf{x}')\rangle$ taken to be a Gaussian function $\nu(\mathbf{x}-\mathbf{x}')=\nu_0/(4\pi a^2)\exp\{-(\mathbf{x}-\mathbf{x}')^2/(4a^2)\}$, with range a and strength ν_0 in our numerical calculations. The thermal noise, $\boldsymbol{\xi}$, is the white-noise stochastic process with zero mean and correlation function specified according to the fluctuation-dissipation theorem $\langle\xi_\alpha(t)\xi_\beta(t')\rangle=2\eta k_B T\delta_{\alpha\beta}\delta(t-t')$, where the brackets now denote averaging with respect to the thermal noise.

The average particle motion is conveniently described by reformulating the stochastic problem in terms of the field theory of classical statistical dynamics.[6] The probability functional for a realization $\mathbf{x}_t$ of the motion of the particle is expressed as a functional integral over the auxiliary variable $\tilde{\mathbf{x}}_t$, and we are led to consider the generating functional (for details on the method and notation applied, we refer to Ref. 7, where the technique was applied to the case of a random scalar potential)

$$\mathcal{Z}[\mathbf{F},\mathbf{J}]=\int\mathcal{D}\mathbf{x}_t\int\mathcal{D}\tilde{\mathbf{x}}_t e^{i\mathcal{S}[\mathbf{x}\cdot\tilde{\mathbf{x}}]}, \tag{2}$$

where in the action

$$\mathcal{S}[\mathbf{x},\tilde{\mathbf{x}}]=\tilde{\mathbf{x}}[D_R^{-1}\mathbf{x}+\mathbf{F}+e\dot{\mathbf{x}}\times\delta\mathbf{B}(\mathbf{x})+\boldsymbol{\xi}]+\mathbf{J}\mathbf{x}, \tag{3}$$

matrix notation is used in order to suppress the integrations over time and summations over Cartesian indices. The inverse free retarded Green's function is $D_R^{-1}(t,t')=[(m\partial_t^2+\eta\partial_t)1-ieB\sigma^y\partial_t]\delta(t-t')$, where matrix notation is used for its Cartesian components, i.e., 1 and σ^y denote the unit and Pauli matrices in Cartesian space, respectively.

The Gaussian averages with respect to the thermal noise and the quenched random magnetic field are immediately performed and we obtain the averaged functional

$$Z[f]=\langle\langle\mathcal{Z}\rangle\rangle=\int\mathcal{D}\phi\exp\left(i\mathcal{S}[\phi]+if\phi+\frac{1}{2}\phi K\phi\right), \tag{4}$$

where the action obtained upon averaging, $S=S_0+S_B$, consists of a term originating from the random magnetic field

$$S_B[\phi]=\frac{i}{2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\,\nu(\mathbf{x}_t-\mathbf{x}_{t'})(\tilde{\mathbf{x}}_t\times\dot{\mathbf{x}}_t)\cdot(\tilde{\mathbf{x}}_{t'}\times\dot{\mathbf{x}}_{t'}), \tag{5}$$

and a quadratic term, $S_0[\phi]=\phi D^{-1}\phi/2$, specified in terms of the free inverse symmetric $[D_A^{-1\alpha\alpha'}(t,t')=D_R^{-1\alpha'\alpha}(t',t)]$ matrix Green's function, where the matrix D^{-1} in the dynamical, or Keldysh, indices in addition is a matrix in Cartesian indices and time

$$D^{-1}=\begin{pmatrix} 2i\eta k_B T\delta(t-t')1 & D_R^{-1} \\ D_A^{-1} & 0 \end{pmatrix}. \tag{6}$$

The notation $\phi(t)=(\tilde{\mathbf{x}}(t),\mathbf{x}(t))=(\phi_1(t),\phi_2(t))$ and $f=(\mathbf{F},\mathbf{J})$ is employed, and the source term, $\mathbf{J}(t)$, coupling to the particle position, $\mathbf{x}(t)$, allow us to generate the correlation functions. In order to obtain self-consistent equations involving the two-point Green's function in a two-particle irreducible fashion, we have added a two-particle source term K in Eq. (4).

Assuming that the external force is constant, the particle reaches a constant steady-state velocity, $\mathbf{v}=\langle\langle\dot{\mathbf{x}}_t\rangle\rangle$, and the average force on the particle vanishes, $\mathbf{F}-\eta\mathbf{v}+e\mathbf{v}\times\mathbf{B}+\mathbf{F}_B=\mathbf{0}$, i.e., there will be a balance between the external force, the average friction force, the Lorentz force due to the average magnetic field, and the average force due to the random part of the magnetic field, $\mathbf{F}_B=\langle\langle e\dot{\mathbf{x}}(t)\times\delta\mathbf{B}(\mathbf{x}(t))\rangle\rangle$. Pursuing an equation for the average force due to the random magnetic field, we consider the effective action $\Gamma[\overline{\phi},G]=W[f,K]-f\overline{\phi}-\overline{\phi}K\overline{\phi}/2-i\,\mathrm{Tr}(GK)/2$, the generator of two-particle irreducible vertex functions, i.e., the Legendre transform of the generator of connected Green's functions, $W[f,K]=-i\ln Z[f,K]$, which satisfies the equations[7]

$$\frac{\delta\Gamma}{\delta\overline{\phi}}=-f-K\overline{\phi},\quad \frac{\delta\Gamma}{\delta G}=-\frac{i}{2}K, \tag{7}$$

where $\overline{\phi}$ is the average field with respect to the action $S[\phi]+f\phi+\phi K\phi/2$, and G is the full connected two-point matrix Green's function, and Tr denotes the trace over all variables and indices.

In the physical problem of interest, the sources K and $\mathbf{J}$ vanish, $K=0$ and $\mathbf{J}=0$, and the full matrix Green's function has, due to the normalization of the generating functional, $Z[\mathbf{F},\mathbf{J}=0,K=\mathbf{0}]=1$, the structure in Keldysh space

$$G_{ij}=\begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix}=-i\begin{pmatrix} 0 & \langle\langle\tilde{x}_t^\alpha x_{t'}^{\alpha'}\rangle\rangle \\ \langle\langle x_t^\alpha\tilde{x}_{t'}^{\alpha'}\rangle\rangle & \langle\langle\delta x_t^\alpha\delta x_{t'}^{\alpha'}\rangle\rangle \end{pmatrix}, \tag{8}$$

where $\delta\mathbf{x}_t=\mathbf{x}_t-\langle\langle\mathbf{x}_t\rangle\rangle$. Similarly, in the absence of sources the expectation value of the auxiliary field vanishes, and the average field is, therefore, given by $\overline{\phi}=(\langle\langle\tilde{\mathbf{x}}_t\rangle\rangle,\langle\langle\mathbf{x}_t\rangle\rangle)=(\mathbf{0},\mathbf{v}t)$, where $\mathbf{v}$ is the average velocity of the particle. The retarded Green's function $G_{\alpha\beta}^R$ gives the linear response to the force F_β, and $G_{\alpha\beta}^K$ is the correlation function (both matrices in Cartesian indices as indicated).

We now employ the standard method for constructing a self-consistent theory following Cornwall *et al.*,[8] and employ the Hartree approximation as described in detail in Ref. 7. In the physical problem of interest, the two-particle source, K, vanishes, and the second equation in Eq. (7) yields the Dyson equation, $G^{-1}=D^{-1}-\Sigma[\overline{\phi},G]$, with the Keldysh matrix self-energy given by

$$\Sigma_{ij}=\begin{pmatrix} \Sigma^K & \Sigma^R \\ \Sigma^A & 0 \end{pmatrix}=2i\left.\frac{\delta\langle S_B[\overline{\phi}+\psi]\rangle_G}{\delta G_{ij}}\right|_{K=0,\ \mathbf{J}=\mathbf{0}}. \tag{9}$$

The matrix self-energy has two independent components [since $\Sigma_{\alpha\alpha'}^A(t,t')=\Sigma_{\alpha'\alpha}^R(t',t)$]

$$\begin{aligned}\Sigma_{tt'}^{K\mu\nu}=&\int\frac{d\mathbf{k}}{(2\pi)^2}\,\nu(\mathbf{k})\exp[\mathbf{k}\cdot\mathbf{v}(t-t')-\varphi_{\mathbf{k}}(t,t')]\\ &\times\epsilon_{3\mu\beta}\{\ddot{G}_{t't}^{K\beta\delta}i[v_\beta-(\dot{G}_{t't}^{K\beta\gamma}-\dot{G}_{tt}^{K\beta\gamma})k_\gamma]\\ &\times[v_\delta-(\dot{G}_{t't'}^{K\delta\sigma}-\dot{G}_{tt'}^{K\delta\sigma})k_\sigma]\}\epsilon_{3\delta\nu},\end{aligned} \tag{10}$$

$$\begin{aligned}\Sigma_{tt'}^{R\mu\nu}=&\epsilon_{3\mu\delta}\epsilon_{3\alpha\nu}\partial_t\int\frac{d\mathbf{k}}{(2\pi)^2}\,\nu(\mathbf{k})\exp[\mathbf{k}\cdot\mathbf{v}(t-t')-\varphi_{\mathbf{k}}(t,t')]\\ &\times[\dot{G}_{tt'}^{R\delta\alpha}+iv_\delta k_\rho G_{tt'}^{R\rho\alpha}-i(\dot{G}_{tt}^{K\delta\sigma}-\dot{G}_{tt'}^{K\delta\sigma})k_\sigma k_\rho G_{tt'}^{R\rho\alpha}].\end{aligned} \tag{11}$$

The influence of thermal and magnetic disorder induced fluctuations are described by the fluctuation exponent $\varphi_k(t,t')=i\mathbf{k}[G^K(t,t)-G^K(t,t')]\mathbf{k}$. For any given average velocity of the particle, $\mathbf{v}$, the Dyson equation and Eq. (9) constitute a set of self-consistent equations for the Green's functions and the self-energies.

The average force due to the random part of the magnetic field is determined by the averaged equation of motion and the first Keldysh component of the first equation in Eq. (7), which yields

$$\begin{aligned}F_{Bt}^\mu=&\left.\frac{\delta\langle S_B[\overline{\phi}+\psi]\rangle_G}{\delta\overline{\phi}_1^\mu(t)}\right|_{\overline{\phi}_t=(\mathbf{0},\mathbf{v}t)}\\ =&\int_{-\infty}^{\infty}dt'\int\frac{d\mathbf{k}}{(2\pi)^2}\,\nu(\mathbf{k})\epsilon_{3\alpha\beta}\epsilon_{3\mu\delta}\,\exp[\mathbf{k}\cdot\mathbf{v}(t-t')\\ &-\varphi_{\mathbf{k}}(t,t')]\{\dot{G}_{tt'}^{R\delta\alpha}[v_\beta-(\dot{G}_{t't}^{K\beta\sigma}-\dot{G}_{t't'}^{K\delta\sigma})k_\sigma]\\ &+\ddot{G}_{tt'}^{K\delta\beta}k_\sigma G_{tt'}^{R\sigma\alpha}+ik_\gamma G_{tt'}^{R\gamma\alpha}[v_\beta-(\dot{G}_{t't}^{K\beta\sigma}-\dot{G}_{t't'}^{K\delta\sigma})k_\sigma]\\ &\times[v_\delta-(\dot{G}_{tt}^{K\delta\rho}-\dot{G}_{tt'}^{K\delta\rho})k_\rho]\}\end{aligned} \tag{12}$$

The self-consistent Langevin theory is intractable to analytical treatment, except in limiting cases, but it is manageable numerically. For any given average velocity of the particle, the coupled equations of Green's functions and self-energies may be solved numerically by iteration. The force due to the random part of the magnetic field can then be evaluated numerically from Eq. (12).

Before displaying the numerical results obtained from the Langevin approach, we discuss the dynamics in a random magnetic field using the Boltzmann equation

$$\mathbf{v}\cdot\nabla_x f+e[\mathbf{E}+\mathbf{v}\times\mathbf{B}(\mathbf{x})]\cdot\nabla_p f=-\frac{f-f_{MB}}{\tau}, \qquad (13)$$

where elastic collisions are described by a relaxation time τ. In the absence of the random part of the magnetic field the Boltzmann equation yields the same average velocity as the Langevin equation since we identify $\tau=m/\eta$. Solving the Boltzmann equation, treating the random magnetic field in lowest-order perturbation theory, proceeds analogously to the treatment of the degenerate electron gas,[5] except that the particles obey Maxwell-Boltzmann instead of Fermi-Dirac statistics. The nondegeneracy results in an additional integration over velocity, giving for the current density

$$\mathbf{j}=\int d\mathbf{v}\frac{\partial f_{MB}}{\partial \mathbf{v}}\mathbf{E}\cdot[a_1(v)\mathbf{v}+a_2(v)\mathbf{v}\times\mathbf{B}], \qquad (14)$$

where the coefficients $a_1(v)=e^2m^{-1}|c(v)|^{-2}\,\mathrm{Im}\,c(v)$ and $a_2(v)=e^2m^{-1}|c(v)|^{-2}\,\mathrm{Re}\,c(v)$ are specified in terms of $c(v)=\omega_c+i/\tau-\Sigma(v)$, where $\omega_c=|e|B/m$ is the cyclotron frequency in the average magnetic field and the effect of the fluctuations in the magnetic field is described by

$$\Sigma(v)=\frac{i\nu_0(4\pi a^2\omega_c)^{-1}}{1-e^{(2\pi)/(\omega_c\tau)}}\int_0^{2\pi}d\theta \times\exp\left(-\frac{v^2\sin^2\frac{\theta}{2}}{a^2\omega_c^2}+\frac{\theta}{\omega_c\tau}-i\theta\right). \qquad (15)$$

The quantities of interest are the changes in the longitudinal, along the direction of the current (defining the x axis), and transverse resistivities due to the random part of the magnetic field, and just as in the Boltzmann theory we now consider in the Langevin approach a classical gas of density n instead of a single particle. The excess longitudinal resistivity is in the Langevin approach $\Delta\rho_{xx}=(\rho_{xx}-\rho_{xx}^{(0)})=-F_B^x/(ne^2v)$, where v is the magnitude of the average velocity, and $\rho_{xx}^{(0)}=m/(ne^2\tau)$ is the resistivity in the absence of the random magnetic field. Similarly we have for the excess transverse resistivity $\Delta\rho_{xy}=(\rho_{xy}-\rho_{xy}^{(0)})=F_B^y/(ne^2v)$, where $\rho_{xy}^{(0)}=-B/(ne)$. In the Boltzmann approach the resistivities are calculated numerically from Eq. (14).

In Fig. 1 is shown the Langevin and Boltzmann excess longitudinal resistivities as functions of the average magnetic field for two different temperatures. We note that the exhibited magnetoresistance is negative. However, at low average magnetic fields the Boltzmann theory yields a positive magnetoresistance when the ratio, $x=l/a$, between the mean free path, $l=v_{\rm th}\tau$, and the correlation length of the random magnetic field, a, exceeds the value $x\sim 6$. In this case the scattering of the electrons is mainly due to the random magnetic field, which acts as spatially separated skew-symmetric scattering centers giving rise to a positive magnetoresistance. Such a positive magnetoresistance can persist as long as the average of the magnetic field does not exceed its fluctuations, since as the magnetic field becomes more homogeneous the skew-symmetric character of the scattering is reduced. At high average magnetic field, the magnetic field becomes more homogeneous leading to a negative magnetoresistance in accordance with the absence of magnetoresistance in the case of a homogeneous magnetic field. In the opposite regime, $l\ll a$, the electrons experience an almost constant magnetic field in between collisions with impurities, and the electrons diffuse between areas of size a having different magnetic fields. The relative difference between the magnetic fields in these areas decreases as the average magnetic field increases thereby again giving rise to a negative magnetoresistance. In Langevin dynamics the thermal noise gives rise to erratic motion and the velocity changes direction many times as the particle traverses a correlation length of the random magnetic field. Langevin dynamics, therefore, always exhibits negative magnetoresistance in accordance with the motion resembling that of the Boltzmann description when $l\ll a$. At low temperatures, $k_BT<\eta^2a^2/m$, the excess longitudinal resistivity is larger for Langevin than Boltzmann dynamics. In support of the interpretation just given, we have observed that at lower temperatures, where the thermal velocity in the Boltzmann description becomes smaller and the Boltzmann dynamics, therefore, resembles the Langevin dynamics, the difference in magnetoresistance decreases. Increasing the temperature always decreases the excess longitudinal resistivity for Langevin dynamics. In contrast, the Boltzmann theory yields at low average magnetic fields, i.e., $\omega_c\tau<1$, that the excess longitudinal resistivity strongly decreases with increased temperature, whereas at moderate average magnetic fields, $\omega_c\tau\sim 1$, the excess longitudinal resistivity is increased. The Boltzmann longitudinal resistivity thus exhibits at moderate average magnetic field strengths a nonmonotonic behavior as a function of temperature, since eventually at high enough temperature the excess resistivity vanishes. The reason for the nonmonotonic behavior of the temperature dependence of the Boltzmann magne-

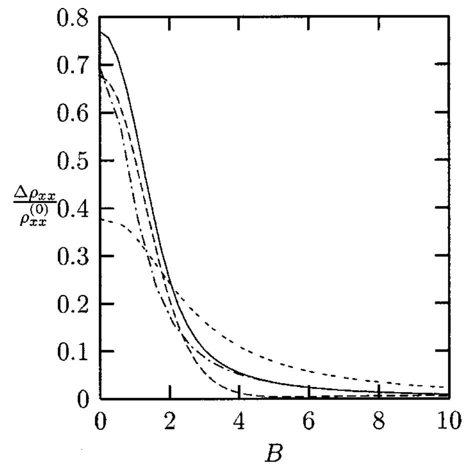

FIG. 1. Longitudinal excess resistivity due to the random part of the magnetic field as a function of the average magnetic field (in units of η/e). The solid curve in the figure represents the result of the Langevin theory, and the dash-dotted curve the result of the Boltzmann theory both at the temperature $k_BT=0.1\eta^2a^2m^{-1}$. The dashed curves are the behaviors at the temperature $k_BT=2\eta^2a^2m^{-1}$, the result of the Langevin theory being represented by the equidistantly spaced long dashes. The disorder strength is in all cases taken to be $\nu_0=10\eta^2a^2$.

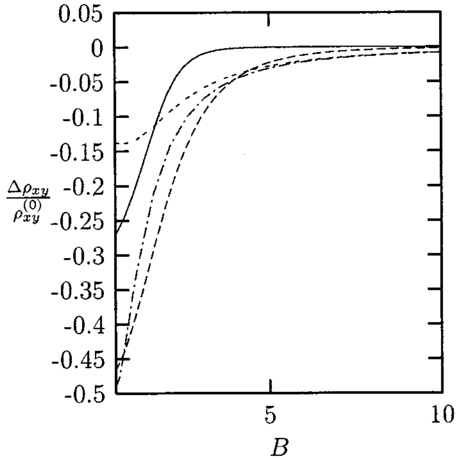

FIG. 2. Transverse excess resistivity due to the random part of the magnetic field as a function of the average magnetic field (in units of η/e). The solid curve in the figure represents the result of the Langevin theory, and the dash-dotted curve the result of the Boltzmann theory both at the temperature $k_BT=0.1\eta^2a^2m^{-1}$. The dashed curves are the behaviors at the temperature $k_BT=2\eta^2a^2m^{-1}$, the result of the Langevin theory being represented by the equidistantly spaced long dashes. The disorder strength is in all cases taken to be $\nu_0=10\eta^2a^2$.

toresistance is due to the competing factors in $\text{Im}\,\Sigma(v)$. Increasing the temperature broadens the Maxwell-Boltzmann distribution and the contribution from larger velocity values is strongly increased at moderate magnetic fields.

In Fig. 2 is shown the Langevin and Boltzmann excess transverse resistivities as functions of the average magnetic field for two different temperatures. The excess transverse resistivity has opposite sign compared to the transverse resistivity in the absence of the random magnetic field, indicating that the fluctuations in the magnetic field increase dissipation. Both descriptions exhibit a maximal absolute value at zero average magnetic field. As a function of temperature the Langevin dynamics exhibits the opposite feature compared to the longitudinal one: the magnitude of the excess transverse resistivity increases with increasing temperature, reflecting that increased velocity fluctuations, through the force due to the random part of the magnetic field, leads to increased transverse resistivity. In contrast, the magnitude of the Boltzmann excess transverse resistivity decreases when the temperature increases.

In conclusion, we have studied the transport properties of a classical gas in a quenched random magnetic field, using both the Langevin and Boltzmann descriptions. Langevin dynamics always exhibits negative magnetoresistance whereas in the Boltzmann description the magnetoresistance is positive or negative depending on the value of the mean free path. The temperature has at low average magnetic-field strengths the effect of reducing the excess longitudinal resistivities in both descriptions. However, the temperature dependence of the excess longitudinal resistivities behave at moderate average magnetic-field strengths oppositely in the two descriptions, showing in the Boltzmann case a nonmonotonic behavior with a maximum at a temperature $k_BT\simeq\eta^2a^2/m$. This behavior should be observable in a two-dimensional electron gas such as, e.g., the one studied in Ref. 2 by having a lower electron density.

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