

Hamiltonian theory of the composite-fermion Wigner crystal

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Experimental results indicating the existence of the high-magnetic-field Wigner crystal have been available for a number of years. While variational wave functions have demonstrated the instability of the Laughlin liquid to a Wigner crystal at sufficiently small filling, calculations of the excitation gaps have been hampered by the strong correlations. Recently a new Hamiltonian formulation of the fractional quantum-Hall problem has been developed. In this work we extend the Hamiltonian approach to include states of nonuniform density, and use it to compute the transport gaps of the Wigner crystal states. We find that the Wigner crystal states near $\nu = 1/5$ are quantitatively well described as crystals of composite fermions with four vortices attached. Predictions for gaps and the shear modulus of the crystal are presented, and found to be in reasonable agreement with experiments.

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I. INTRODUCTION AND PREVIEW

The fractional quantum-Hall (FQH) regime presents us with the quintessential problem of strong correlations. In a strong magnetic field B , the kinetic energy of the two-dimensional electron gas (2DEG) is quantized into Landau levels with energy $(n + \frac{1}{2})\omega_c$, where $\omega_c = eB/m$ is the cyclotron frequency. Each of these Landau levels (LL's) has a huge degeneracy equal to the number of quanta of flux penetrating the 2DEG. When the lowest Landau level (LLL) is partially full, it is seen that the kinetic energy is degenerate, and all the dynamics has to come from interactions. At certain special filling factors (recall that filling factor is $\nu = 2\pi n/eB$) the system reorganizes itself into new strongly correlated ground states¹ with fractionally charged excitations.²

The past decade has seen the development and acceptance of the composite-fermion (CF) concept as basic to the understanding of a variety of these electronic states.³ The CF is pictured as an electron bound to an even number l of quanta of statistical flux, which are opposed to the external field. At a mean-field level, each CF sees both the external field and the statistical field due to the other particles, and therefore moves in an effective field $B^* = B - 2\pi ln$, where n is the density of electrons. The principal fractions $\nu = p/2p + 1$ are seen to be exactly those fillings when the number of particles is exactly enough to fill an *integer* number of LL's of the effective field.

Thinking in terms of CF's greatly simplifies the description of different incompressible and compressible FQH states. CF's are believed to be the true quasiparticles in much the same way as Landau quasiparticles are for the normal Fermi liquid. The original CF theory was used to generate electronic wave functions.⁴ Contemporaneously, field-theoretic approaches⁵ were also developed to better understand the FQHE, and to compute response functions. Most of the field-theoretic approaches are based on the Chern-Simons (CS) transformation, a method of attaching flux to particles in two dimensions. Attaching an odd number of flux quanta to electrons transforms them into bosons and leads to the bosonic CS approach,⁵ while adding an even number leads to

the fermionic CS theory.⁶ In the mean-field approximation the fermionic CS theory recovers the picture of CF's in an effective magnetic field. Recently, based on the fermionic CS theory, a Hamiltonian approach was developed⁷ to describe liquid states in the FQH regime. In this approach, the CF representation is reached from the bare electronic coordinates by a series of canonical transformations. The end product is the electron density operator reexpressed in the CF coordinates suitable for further calculations and/or approximations. Physical quantities calculated in this approach seem to be in reasonable agreement with numerical results and experiments.⁸

The subject of this paper, however, is not the liquid FQH states, but the insulating states that have been detected experimentally at very low filling fractions.⁹ A natural candidate to exhibit such insulating behavior is the electronic Wigner crystal (WC). The simplest description of this state is the Hartree-Fock (HF) wave function¹⁰

$$\Psi_{HF}(\{\mathbf{r}_i\}) = \mathcal{A} \prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i), \quad (1)$$

where \mathcal{A} is the antisymmetrization operator, and $\phi_{\mathbf{R}_i}$ is a single-particle wave function that is localized at \mathbf{R}_i (lattice site) and belongs to the LLL. It is given by

$$\phi_{\mathbf{R}_i}(\mathbf{r}) = \exp[-|\mathbf{r} - \mathbf{R}_i|^2/4l_0^2 - i\mathbf{r} \times \mathbf{R}_i \cdot \hat{\mathbf{z}}/2l_0^2], \quad (2)$$

where $l_0 = (eB)^{-1/2}$ is the magnetic length. The wave function (1) has been improved by adding a Jastrow correlation factor,¹² and the energy of the resulting state has been shown to become lower than that of the liquid state at about the experimentally right filling fraction ($\nu \approx \frac{1}{7}$).¹⁰⁻¹² Thus, a very strong magnetic field favors crystalline order by localizing the electrons.

However, not all the experimental evidence supports the simple electronic WC picture. In particular, transport experiments^{13,14} suggest that the transport gap in this system is two orders of magnitude smaller than the theoretical estimate as calculated using the Hartree-Fock approximation.^{15,16} Moreover, close to the Laughlin fractions

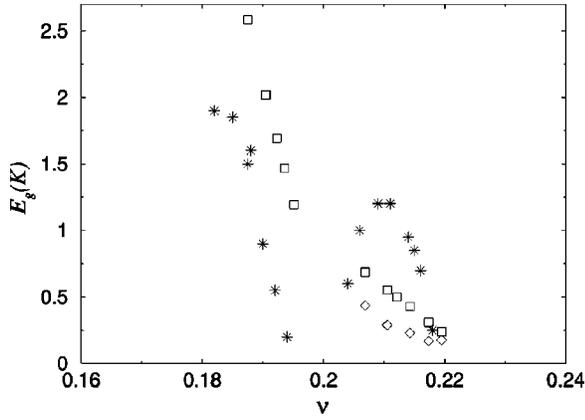


FIG. 1. The transport gap dependence on the filling factor around $\nu \approx 1/5$. Squares are our CF theory with four vortices attached for the hexagonal lattice. Diamonds represent our CF theory with four vortices attached for the oblique lattice (see the text). Stars are experimental results read off Fig. 3 of Ref. 14.

$\nu = 1/(2p+1)$, a dip in the longitudinal resistivity ρ_{xx} is observed,¹⁷ resembling the behavior of the correlated liquid state. The measurements of the Hall resistivity ρ_{xy} are surprising as well.¹⁷⁻¹⁹ The electronic WC is known to have a vanishing Hall conductance $\sigma_{xy} = 0$ (when pinned), which implies a vanishing Hall resistance $\rho_{xy} = 0$. On the contrary, experiments see Hall insulating behavior, that is, $\rho_{xy} \approx h/\nu e^2$. These problems led Yi and Fertig²⁰ to consider crystalline states with Laughlin-Jastrow correlations. The idea is to construct a trial wave function with correlation factors that keep electrons apart. Each electron is combined with l vortices to obtain the trial wave function

$$\Psi(\{\mathbf{r}_i\}) = \mathcal{A} \prod_{i \neq j} (z_i - z_j)^l \prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i). \quad (3)$$

The Coulomb energy for this wave function was then computed using Monte Carlo simulation. Yi and Fertig showed that the ground-state energy of the correlated WC is lower than that of the usual WC at experimentally relevant filling fractions.²⁰ Moreover, by introducing Laughlin-Jastrow correlations between the interstitials and the lattice electrons, the experimentally observed ρ_{xy} (Hall insulating behavior) can be explained.^{21,22} Unfortunately, the method becomes computationally too demanding to allow one to calculate other quantities of interest, such as the excitation spectrum.

Since the Laughlin-Jastrow correlations are precisely the ones that convert electrons into CF's, (Ref. 4) we are led to consider a crystal of composite fermions. The main advantage of the Hamiltonian approach is that one can easily compute the single-particle excitation gap (identical to the transport gap) along with the ground-state energy.

Our main result is that the CF theory with $l=4$ denoting four zeros attached to each particle gives the best description of the experimental phenomenology. Figure 1 shows the results of our calculations of the transport gap E_g as a function of the filling factor around $\nu = 1/5$ in the triangular lattices of CF's. Our theory reproduces the dependence of the excitation energies on the filling factors as measured in Ref. 14 reason-

ably well when $\nu < 1/5$. We also find that the shear modulus of the CF lattices goes down as the filling factor $\nu = 1/5$ is approached from below. This behavior is consistent with the experimentally observed increase of the threshold voltage for filling factors $\nu \rightarrow 1/5$ (Ref. 23) (if one interprets the results in terms of "weak pinning"²⁴).

For $\nu > 1/5$ we will show in Sec. V that the energy landscape becomes very flat, with many different lattice structures becoming nearly degenerate in energy. Not coincidentally, the convergence of our Hartree-Fock procedure is very poor in this region, and we are unable to identify the proper ground state in the clean limit. We have therefore presented two values for the gaps in this region, the upper one being for the triangular lattice, and the lower one being for a more oblique lattice. Neither gap follows the experimentally observed nonmonotonic dependence $E_g(\nu)$. (Note, however, the different slope below and above $\nu = 1/5$ in Fig. 1). We believe the main reason for this is the following: Since there are many local minima with different lattice structures that are very close in energy, disorder may play an important role in real samples. The experimental gaps may also be dominated in this region by disorder effects. Apart from this one region of discrepancy, our numbers for the gaps are in reasonable agreement with experiments.

The outline of the paper is as follows. In Sec. II we introduce the Hamiltonian formalism and show how the wave function (3) emerges naturally in this approach. In Sec. III we derive the expression of the electronic density operator in the CF representation and also calculate the Hall conductance for the CF lattice in the clean limit. In Sec. IV we formulate the HF theory using the electronic density derived in Sec. III. In Sec. V we present the results, discuss their physical import, and show that the HF solutions satisfy the conditions under which the formalism is derived (small gradients and small deviations of density around the mean). We end with some conclusions, caveats, and open questions. Some details of the calculations are relegated to the three appendices.

II. SETTING UP THE HAMILTONIAN FORMALISM

In this section we closely follow the exposition and notation of Murthy and Shankar.²⁵ Since most of the details are similar, we will mainly highlight the differences. We will follow Lopez and Fradkin⁶ in assuming that a good starting point for obtaining a perturbative solution of the N interacting electron problem moving in a uniform magnetic field $\mathbf{B} = -\hat{z}B$ is the noninteracting CS particle Hamiltonian ($\hbar = c = 1$, m is the mass of the electron, e its charge)

$$H_{CS} = \sum_i^N \frac{1}{2m} [-i\nabla_i + e\mathbf{A}^*(\mathbf{r}_i) + \mathbf{a}_{CS}(\mathbf{r}_i)]^2. \quad (4)$$

The CS particle is obtained from the electron by attaching l flux quanta to it. This flux attachment is the origin of the Chern-Simons gauge field $\mathbf{a}_{CS}(\mathbf{r}_i)$, which is defined by

$$\frac{\nabla \times \mathbf{a}_{CS}(\mathbf{r}_i)}{2\pi l} = \sum_i^N \delta^2(\mathbf{r} - \mathbf{r}_i) - n(\mathbf{r}), \quad (5)$$

where $n(\mathbf{r})$ is the expectation value of the electron density in the ground state (CS particle and electron densities are the same). Unlike the incompressible FQH state (liquid state) for which the electrons have a uniform density, we are concerned with the case where the density depends on the position. The vector potential $\mathbf{A}^*(\mathbf{r}_i)$ corresponds to the difference of the external magnetic field and the average field created by the attached flux tubes, or $B^*(\mathbf{r}) = B - 2\pi l n(\mathbf{r})/e$.

Our main assumption throughout this work is that for the appropriate field strength and electron density the electrons organize themselves into a density-wave state such that the ground-state expectation value of the density operator is a periodic function

$$n(\mathbf{r}) = n + \sum_{\mathbf{G} \neq 0} \delta n(\mathbf{G}) e^{i\mathbf{G} \cdot \mathbf{r}}, \quad (6)$$

where \mathbf{G} are the reciprocal lattice vectors. We also assume that the uniform component of the average density is much larger than any of the finite- \mathbf{G} modulations, i.e., $n \gg \delta n$. Effectively $\delta n/n$ is a small parameter in our theory. We will show in Sec. V that the HF ground state does satisfy the condition that $\delta n/n$ is small, thus *a posteriori* justifying our use of this approximation. The effects of disorder are ignored in what follows.

The complicated part of the CS Hamiltonian in Eq. (4) is the gauge field $\mathbf{a}_{CS}(\mathbf{r}_i)$. To get rid of it, one enlarges the Hilbert space⁷ by introducing a canonical pair of fields $a(\mathbf{q}), P(\mathbf{q})$ for every \mathbf{q}

$$[a(\mathbf{q}), P(\mathbf{q}')] = i(2\pi)^2 \delta^2(\mathbf{q} + \mathbf{q}'), \quad (7)$$

where $\mathbf{q} < Q = \sqrt{4\pi n}$. Instead of working with the CS Hamiltonian, we introduce an equivalent Hamiltonian

$$H = \sum_i^N \frac{1}{2m} [-i\nabla_i + e\mathbf{A}^*(\mathbf{r}_i) + \mathbf{a}(\mathbf{r}_i) + \mathbf{a}_{CS}(\mathbf{r}_i)]^2, \quad (8)$$

where $\mathbf{a}(\mathbf{r}_i) = -i\hat{z} \times \hat{\mathbf{q}} a(\mathbf{r}_i)$ is a transverse vector field. We also define a longitudinal vector field $\mathbf{P}(\mathbf{r}_i) = i\hat{\mathbf{q}} P(\mathbf{r}_i)$ ($\hat{\mathbf{q}}$ is a unit vector in the \mathbf{q} direction). This problem is equivalent to the original one provided we restrict our attention to states that are annihilated by the constraints $\chi(\mathbf{q}) = a(\mathbf{q})$ ($q < Q$). We will call states that are annihilated by these constraints physical states, that is,

$$\chi(\mathbf{q}) |\Psi_{phys}\rangle = 0. \quad (9)$$

We will continue to use notation $\chi(\mathbf{q})$ for the constraint operator in different representations.

Using the fields $a(\mathbf{q}), P(\mathbf{q})$, a unitary transformation is then constructed that shifts a to absorb the Fourier components of $\mathbf{a}_{CS}(\mathbf{r}_i)$ for $q < Q$. In the new representation, whose particles we call composite particles (CP), the Hamiltonian is

$$H_{CP} = \sum_i^N \frac{1}{2m} [-i\nabla_i + e\mathbf{A}^*(\mathbf{r}_i) + \mathbf{a}(\mathbf{r}_i) + 2\pi l \mathbf{P}(\mathbf{r}_i)]^2. \quad (10)$$

We neglected all the $q > Q$ Fourier components of the gauge field in deriving the above Hamiltonian, implying that our theory will not describe the motion correctly for large momenta (or short distances). In the following we will also be using the random phase approximation (RPA) generalized for the case of the inhomogeneous densities

$$\sum_j e^{i\mathbf{k} \cdot \mathbf{r}_j} \approx n(2\pi)^2 \delta^2(\mathbf{k}) + \delta n(-\mathbf{k}). \quad (11)$$

The constraints in the CP representation are given by

$$\chi(\mathbf{q}) = -\frac{qa(\mathbf{q})}{2\pi l} + \rho(\mathbf{q}) - n(\mathbf{q}), \quad (12)$$

where $\rho(\mathbf{q}) = \sum_j^N e^{-i\mathbf{q} \cdot \mathbf{r}_j}$ is the CP density operator and coincides with the corresponding electron operator.

Before proceeding with any further transformations on the Hamiltonian (10), we will show how the trial wave function used to compute the energy of the correlated WC in Ref. 20 emerges naturally within this approach.²⁵ The crudest approximation for the CP Hamiltonian is

$$H_{CP} \approx \sum_i^N \frac{1}{2m} [-i\nabla_i + e\mathbf{A}^*(\mathbf{r}_i)]^2 + \frac{n}{2m} \sum_{\mathbf{q}} [a(-\mathbf{q})a(\mathbf{q}) + (2\pi l)^2 P(-\mathbf{q})P(\mathbf{q})]. \quad (13)$$

In this expression we have assumed that the CP ground state can be regarded as largely homogeneous, so that the δn is neglected in the definitions of the RPA (11) and vector potential $\mathbf{A}^*(\mathbf{r}_i)$, which corresponds now to the uniform magnetic field. We have also neglected the coupling between the CP's and the oscillator fields $a(\mathbf{q}), P(\mathbf{q})$. Since this Hamiltonian (13) has been artificially made separable into a sum of the particle and the oscillator terms, we can write down the ground state as a product wave function. The particles are moving in a uniform magnetic field $B^* = B - 2\pi l n/e$, so there is a degeneracy in this problem, but we assume that their ground state is crystalline

$$\Psi_{CF}(\{\mathbf{r}_i\}) = \prod_i \phi_{\mathbf{R}_i}(\mathbf{r}_i), \quad (14)$$

where $\phi_{\mathbf{R}_i}(\mathbf{r}_i)$ are Gaussians centered on the lattice sites \mathbf{R}_i similar to Eq. (2), except that instead of magnetic length l_0 there is a new magnetic length $l_0^* = (eB^*)^{-1/2}$. CF stands for composite fermions. The oscillator term describes N independent harmonic oscillators with the ground state

$$\Psi_{osc}(\{\mathbf{q}\}) = \prod_{\mathbf{q}} \exp[-a^2(\mathbf{q})/4\pi l]. \quad (15)$$

Using the constraints (12) we can eliminate the field degrees of freedom $a(\mathbf{q})$ in favor of the \mathbf{r}_i in the expression for the oscillator wave function (15). Since the calculation is de-

scribed in Ref. 25 in great detail, we only give the final result for the projected oscillator wave function

$$\begin{aligned} \Psi_{osc}(\{\mathbf{r}_i\})_{[a(\mathbf{q})=2\pi l\rho(\mathbf{q}/q)]} &= \prod_{i<j} |z_i - z_j|^l \\ &\times \exp\left(-\sum_j l\nu |z_j|^2/4l_0^2\right). \end{aligned} \quad (16)$$

Here $z_i = x_i + iy_i$ is the complex coordinate and $\nu = 2\pi n/eB$ is the filling factor. The approximate CS wave function is a product of Eqs. (14) and (16) and agrees with Eq. (2.12) given in Ref. 20.

However, the fact remains that in this representation the oscillators and the particles remain strongly coupled. The oscillators are identified with the magnetoplasmons, which are high-energy degrees of freedom, while the particles will turn out to form the low-energy sector. Our next task will be to construct a canonical transformation to decouple the two sectors, so that we are left with a purely low-energy theory.

III. DECOUPLING AND THE ELECTRON DENSITY OPERATOR IN THE FINAL REPRESENTATION

Before we turn to the technical details of the decoupling transformation, it is worthwhile to articulate the philosophy of our approach. If one were able to find the exact canonical transformation, and implement it exactly, then one would be left with a final theory in which the fermions are purely low-energy objects, and the oscillators are purely high-energy objects. In particular, the oscillators should obey Kohn's theorem,²⁸ while all reference to the bare mass should have disappeared from the low-energy fermionic part of the Hamiltonian. In other words, the bare kinetic energy of the CF's should be quenched in the final representation. Finally, the projected electronic density when expressed in the final representation should obey the magnetic translation algebra appropriate to the LLL.²⁹ Unfortunately, this program cannot be implemented fully in practice. What can be implemented is a sequence of transformations that achieves some measure of the above at long-distance scales (small q). We will see that the oscillators do end up obeying Kohn's theorem, since this is a small- q property. Similarly, the magnetic translation algebra will be seen to occur in its small- q form. However, while the tendency for the quenching of the mass will be manifest, the mass depends on all distance scales, and its quenching cannot be shown within a long-distance approximation. Our approach will be to *assume* the exact quenching of bare mass, since we know this to be true in the LLL, and write the final Hamiltonian in the low-energy sector as a pure interaction term. Thus, while the proximate goal of the canonical transformation is to decouple the high- and low-energy parts, its ultimate goal is to obtain the electronic density operator in the final representation.

We return now to the CP Hamiltonian (10) and the set of CP constraints (12). Using the RPA approximation as given by Eq. (11), the CP Hamiltonian can be recast into the following form:

$$\begin{aligned} H_{CP} &= \sum_i^N \frac{\Pi_j^2}{2m} + \frac{\sqrt{\pi l}}{m} \sum_{\mathbf{q}}^Q [A(\mathbf{q})c^\dagger(\mathbf{q}) + \text{H.c.}] \\ &+ \frac{\pi l}{m} \sum_{\mathbf{q}}^Q \sum_{\mathbf{G}}^Q \left\{ [(n\delta_{\mathbf{G},0} + \delta n(\mathbf{G})) \right. \\ &\times [A^\dagger(\mathbf{q})A(\mathbf{q}-\mathbf{G})\hat{q}_-(\widehat{q-\mathbf{G}})_+] + \text{H.c.}] \left. \right\}. \end{aligned} \quad (17)$$

The first term in Eq. (17) is the CP kinetic energy, with $\Pi_j = -i\nabla_j + e\mathbf{A}^*(\mathbf{r}_j)$, which only depends on the particle degrees of freedom. The second is the coupling between the particle and the auxiliary field (oscillator) degrees of freedom, with $c(\mathbf{q}) = \hat{q}_- \sum_j \Pi_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}$ and the ‘‘destruction’’ operator $A(\mathbf{q}) = [a(\mathbf{q}) + i2\pi lP(\mathbf{q})]/\sqrt{4\pi l}$.²⁶ The last term describes the oscillators and does not depend on the particle degrees of freedom. In order to decouple the high-energy oscillators from the low-energy CP's we need to compute the commutators between the newly introduced operators. It is straightforward to deduce from Eq. (7) that

$$[A(\mathbf{q}), A^\dagger(\mathbf{q}')] = (2\pi)^2 \delta^2(\mathbf{q}-\mathbf{q}'). \quad (18)$$

The commutator for the operator $c(\mathbf{q})$ is found from the commutator of the canonical momenta Π_x and Π_y and then using the RPA (11). The result is

$$\begin{aligned} [c(\mathbf{q}), c^\dagger(\mathbf{q}')] &\approx \hat{q}_- \hat{q}'_+ \{ -4\pi l n \delta n(\mathbf{q}-\mathbf{q}') \\ &+ 2eB^* [n(2\pi)^2 \delta^2(\mathbf{q}-\mathbf{q}') + \delta n(\mathbf{q}-\mathbf{q}')] \}. \end{aligned} \quad (19)$$

According to our assumption the average density is a periodic function, as in Eq. (6), therefore the right-hand side of Eq. (19) differs from zero only if the difference $\mathbf{q}-\mathbf{q}'$ in Eq. (19) is equal to a reciprocal lattice vector \mathbf{G} .

Our task is to decouple the Hamiltonian (17) by eliminating the term that couples the particle and the oscillator degrees of freedom. We will show how one can construct a canonical transformation that accomplishes this decoupling. Once the canonical transformation is found we can derive the electron density operator in the ‘‘final’’ representation (FR). Operators in the FR are expressed in terms of the CF coordinates. We will show below that there are good reasons to believe that composite fermions are the true quasiparticles in the FQH regime and the FR density operator represents the physical charge density.

The calculation is a straightforward extension of the procedure given in Ref. 25 for the case of the homogeneous liquid, and we relegate the details of this calculation to Appendix A. In what follows, only the results of applying the transformation to the Hamiltonian (17), the density operator, and the set of constraints (12) are presented.

By construction, the term coupling the particles and the oscillators is not present in the FR Hamiltonian. Substituting Eqs. (A5), (A6), and (A9) into the expression for the CF Hamiltonian (17), we find that the oscillator term in the FR is equal to $\omega_c \sum_{\mathbf{q}} A^\dagger(\mathbf{q})A(\mathbf{q})$ with $\omega_c = eB/m$ exactly as in the liquid, to order $(\delta n/n)^2$. This is a physically correct re-

sult because according to the Kohn's theorem²⁸ the limit $\omega_c(\mathbf{q} \rightarrow 0)$ should not depend on the electron interactions in the lowest Landau level. The kinetic energy in the FR is

$$T = \sum_j^N \frac{\Pi_{j-} \Pi_{j+}}{2m} + \sum_j^N \left(\frac{eB^*}{2m} - \frac{\pi l}{m} \delta n(\mathbf{r}_j) \right) - \frac{1}{2mn} \sum_{\mathbf{q}}^Q c^\dagger(\mathbf{q}) c(\mathbf{q}) + \frac{1}{2mn^2} \sum_{\mathbf{q}}^Q \sum_{\mathbf{G}}^Q c^\dagger(\mathbf{q}) \times c(\mathbf{q} - \mathbf{G}) \hat{q}_-(\widehat{q - \mathbf{G}})_+ . \quad (20)$$

In an ideal calculation the particle kinetic energy should disappear in the FR; the electronic kinetic energy is subsumed into the oscillator term. As has been stated above, it is impossible to show this in a small- q approximation such as the one we are using.

The electron density operator in the FR is obtained by solving the flow equation that is derived in a way that follows closely the calculation for the kinetic energy T leading to Eq. (A8). The result of the integration of the flow equation is

$$\rho(\mathbf{q}, \lambda) = \rho(\mathbf{q}) + \rho_0(\mathbf{q}, \lambda) + \frac{q}{4n\sqrt{\pi l(1+\mu^2)}} \sum_{\mathbf{G}}^Q \frac{\delta n(\mathbf{G})}{N} \times [A(\mathbf{q} - \mathbf{G}) \hat{q}_-(\widehat{q - \mathbf{G}})_+ + \text{H.c.}] + \frac{q(2 + \mu^2 - 2\sqrt{1+\mu^2})}{8\pi l n^2 \mu^4 \sqrt{1+\mu^2}} \sum_{\mathbf{G}}^Q \frac{\delta n(\mathbf{G})}{N} \times [c(\mathbf{q} - \mathbf{G}) \hat{q}_-(\widehat{q - \mathbf{G}})_+ + \text{H.c.}] , \quad (21)$$

where $\mu^2 = 1/\nu - 1$. The FR operator $\rho_0(\mathbf{q}, \lambda)$ is the leading term in the perturbation expansion of the density in the parameter $\delta n/n$ and is formally identical²⁷ to the density operator in the case when the average electron density is uniform,²⁵

$$\rho_0(\mathbf{q}, \lambda) = \frac{q}{2\sqrt{\pi l(1+\mu^2)}} [A(\mathbf{q}) + \text{H.c.}] - \frac{q(\sqrt{1+\mu^2} - 1)}{4\pi l n \mu^2 \sqrt{1+\mu^2}} [c(\mathbf{q}) + \text{H.c.}] . \quad (22)$$

It is now straightforward to get the FR expression for the set of constraints (12), just by using the previously determined FR operators $A(\mathbf{q}, \lambda)$ [see Appendix A, Eq. (A5)] and $\rho(\mathbf{q}, \lambda)$ [Eqs. (21) and (22)]. The expression that results is

$$\chi(\mathbf{q}, \lambda) = \chi_0(\mathbf{q}, \lambda) - \delta n(\mathbf{q}) + \frac{q(2 + \mu^2 - \mu^4 - 2\sqrt{1+\mu^2})}{8\pi l n^2 \mu^4 \sqrt{1+\mu^2}} \times \sum_{\mathbf{G}}^Q \delta n(\mathbf{G}) [c(\mathbf{q} - \mathbf{G}) \hat{q}_-(\widehat{q - \mathbf{G}})_+ + \text{H.c.}] . \quad (23)$$

Here again $\chi_0(\mathbf{q}, \lambda)$ is the part of the constraint that corresponds to the case of the uniform average density and is given by

$$\chi_0(\mathbf{q}, \lambda) = \rho(\mathbf{q}) + \frac{q(\sqrt{1+\mu^2} - 1)}{4\pi l n \mu^2} [c(\mathbf{q}) + \text{H.c.}] . \quad (24)$$

The main observation about Eq. (23) is that oscillator degrees of freedom cancel out up to order $\delta n/n$, implying that the constraint acts only on particles. This reassures us of the self-consistency of the decoupling scheme, since there is no use decoupling the high- and low-energy modes in the Hamiltonian if the constraint still nontrivially couples them.

Because the particles are confined entirely to the lowest Landau level, one expects the physical charge-density operator to obey the magnetic translation algebra. However, this is not true for the density operator defined by Eq. (21). We are allowed to modify the definition of $\rho(\mathbf{q}, \lambda)$ in CP representation by adding to it any multiple of the constraint, since in an exact calculation in the physical states the constraint is equal to zero. Following the same approach as in the liquid states²⁵ we try the linear combination

$$\rho(\mathbf{q}) - 1/(\mu^2 + 1)\chi(\mathbf{q}) . \quad (25)$$

This operator has the virtue that its FR matrix elements are of order q^2 or higher, consistent with Kohn's theorem.²⁸ The FR expression of the density operator (25) (which we will call the preferred density) is

$$\tilde{\rho}(\mathbf{q}) = \frac{\mu^2}{\mu^2 + 1} \rho(\mathbf{q}) - \frac{q}{4\pi n l (1 + \mu^2)} [c(\mathbf{q}) + \text{H.c.}] + \frac{\delta n(\mathbf{q})}{\mu^2 + 1} - \frac{q(1 - \sqrt{1 + \mu^2})}{4\pi n l (1 + \mu^2) \mu^2} \sum_{\mathbf{G}}^Q \delta n(\mathbf{G}) [c(\mathbf{q} - \mathbf{G}) \times \hat{q}_-(\widehat{q - \mathbf{G}})_+ + \text{H.c.}] . \quad (26)$$

The calculation of the commutator of the preferred density operators to first order in $\delta n/n$ gives

$$[\tilde{\rho}(\mathbf{q}), \tilde{\rho}(\mathbf{q}')] = i l_0^2 (\mathbf{q} \times \mathbf{q}') \tilde{\rho}(\mathbf{q} + \mathbf{q}') + i l_0^2 (\mathbf{q} \times \mathbf{q}') \frac{1}{1 + \mu^2} \sum_{\mathbf{G}}^Q \frac{\delta n(\mathbf{G})}{n} \times \chi_0(\mathbf{q} + \mathbf{q}' - \mathbf{G}, \lambda) . \quad (27)$$

Here $\chi_0(\mathbf{q}, \lambda)$ is the constraint to zeroth order (in $\delta n/n$), Eq. (24). We conclude that the magnetic algebra is satisfied for physical states that are destroyed by the constraint. This is a weaker result than was obtained in the translationally invariant case, but nevertheless still preserves the equivalence of this theory to the original electronic theory in the LLL at long distances.

The preferred density encodes many nonperturbative features that are known to be true for the original electronic problem. It shows the correct fractional charge of the quasiparticles,² obeys the magnetic translation algebra in the small q limit,²⁹ and has matrix elements of order q^2 or higher

from the ground state. Thus it is plausible that simple HF calculations with this density will capture the essential physics in the FQH regime. Certainly, this expectation has been borne out in calculations for the liquid states.⁸

Thus all the features of the translationally invariant Hamiltonian theory, namely, compliance with Kohn's theorem, simultaneous decoupling of the Hamiltonian and the constraints, and a preferred density that obeys the algebra of magnetic translations, carry over for the nonuniform electronic density state.

It is of interest to determine the Hall conductance σ_{xy} for the nonuniform average density state. In the clean limit, when there is no external potential, translation invariance implies that $\sigma_{xy} = \nu e^2/h$. Our theory does indeed predict this in the clean limit. In order to see this we need the FR expression for the current. Starting with the electron current, and eliminating the CS vector potential we find the following CP current:

$$(J_{CP})_+(\mathbf{q}) = \frac{1}{m} \sum_j \Pi_{j+} e^{-i\mathbf{q} \cdot \mathbf{r}_j} + \frac{2n\sqrt{\pi}l}{m} A(\mathbf{q}) \hat{q}_+ + \frac{2\sqrt{\pi}l}{m} \sum_{\mathbf{G}} \delta n(\mathbf{G}) A(\mathbf{q}-\mathbf{G}) (\widehat{q-\mathbf{G}})_+. \quad (28)$$

The first term in Eq. (28) is just a definition of the operator $c(\mathbf{q})$ so to get the FR expression for the current we have to substitute the FR operators $A(\mathbf{q}, \lambda)$ and $c(\mathbf{q}, \lambda)$. The FR current consists of two terms, one that does not contain factors of δn and another proportional to $\delta n/n$. The first term is identical to the FR current for the uniform average density state and as shown in Ref. 25 depends only on the oscillator degrees of freedom. To calculate the first order contribution to the current we use expressions (A5) and (A6) that give the first order corrections in $\delta n/n$ for the operators $A(\mathbf{q}, \lambda)$ and $c(\mathbf{q}, \lambda)$, respectively. We find that it is also independent of the CF coordinates. Both terms add up to

$$J_+(\mathbf{q}) = \frac{2n\sqrt{\pi}l(1+\mu^2)}{m} A(\mathbf{q}) \hat{q}_+ + \frac{\sqrt{\pi}l(1+\mu^2)}{m} \sum_{\mathbf{G}} \delta n(\mathbf{G}) A(\mathbf{q}-\mathbf{G}) (\widehat{q-\mathbf{G}})_+. \quad (29)$$

Because the current in the FR depends only on the operators that represent the oscillators, we can ignore the particle sector in the conductance calculation. The argument in Ref. 25 for the uniform average density case goes through and gives $\sigma_{xy} = \nu e^2/h$ in the limit $\mathbf{q} \rightarrow 0$, which is the correct unquantized Hall conductance in the clean limit. Note, however, in the presence of disorder it is believed that due to the pinning of the Wigner crystal $\sigma_{xy} \rightarrow 0$, $\sigma_{xx} \rightarrow 0$ such that ρ_{xy} is its classical value. A complete theory including disorder effects is currently nonexistent, and we will confine ourselves to the clean limit in the sequel.

IV. HARTREE-FOCK APPROXIMATION

The goal of this section is to describe two methods to compute the transport gap of the Wigner crystal state. This transport gap can also be understood as the gap to creating a particle-hole excitation where the particle and hole are widely separated. It is important to note that generically there are *gapless collective* excitations in any crystalline state. However, these collective (magnetophonon) modes are neutral, and cannot contribute towards carrying electrical current. In the following, we focus on the energies of the single-particle charged excitations from which one can compute the transport gap.

Having determined the correct canonical transformation by decoupling the CS Hamiltonian in the previous section, we shift our focus to the Coulomb interaction that was hitherto ignored. At the small filling factors for which the Wigner crystal occurs, the lowest-Landau-level approximation is appropriate. Now the LLL electronic Hamiltonian is given by

$$H = \frac{1}{2S} \sum_{\mathbf{q}} [V(\mathbf{q}) \rho(\mathbf{q}) \rho(-\mathbf{q}) - V(\mathbf{q}) e^{-q^2 l_0^2/2} \rho(0)]. \quad (30)$$

Here S is the area of the system and the second term results from undoing the normal ordering of the interaction term in the original electronic Hamiltonian. This is necessary since we need to have the full operator ρ in order to write it in the final representation. The two-dimensional Fourier transform of the Coulomb potential is suppressed at large momenta \mathbf{q} by multiplying it with a Gaussian

$$V(\mathbf{q}) = \frac{2\pi e^2}{q\epsilon} e^{-q^2 \Lambda^2}, \quad (31)$$

where a parameter Λ may be used to interpolate between the pure Coulomb potential and the Coulomb potential that is effective only when the distance between two particles is larger than Λ . ϵ is the dielectric constant. We emphasize that the form of the Fourier transform of the potential given in Eq. (31) does not accurately describe the effect of the sample thickness, but is rather chosen for illustrative purposes, since it is computationally convenient. The second term in Eq. (30) is a constant that will be ignored in what follows. The first term of the electronic Coulomb interaction when transformed to the CF coordinates will serve as our model Hamiltonian

$$H = \frac{1}{2S} \sum_{\mathbf{q}} V(\mathbf{q}) \tilde{\rho}(\mathbf{q}) \tilde{\rho}(-\mathbf{q}). \quad (32)$$

The density operator $\tilde{\rho}(\mathbf{q})$ is given by Eq. (26). It is useful at this point to rewrite it so that the dependence on the modulated average density δn is explicit,³⁰

$$\begin{aligned} \tilde{\rho}(\mathbf{q}) &= (1 - c^2)\rho(\mathbf{q}) - il_0^2 \sum_j \mathbf{q} \times \Pi_j^* e^{-i\mathbf{q} \cdot \mathbf{r}_j} \\ &\quad - c^2 \sum_{\mathbf{G}, \mathbf{G} \neq \mathbf{q}} \frac{\delta n(\mathbf{G})}{N} \frac{\mathbf{q} \cdot \mathbf{G}}{\mathbf{G}^2} e^{i(\mathbf{G} - \mathbf{q}) \cdot \mathbf{r}_j} \\ &\quad + \frac{il_0^2 c}{c+1} \sum_{\mathbf{G}} \frac{\delta n(\mathbf{G})}{N} \sum_j \mathbf{q} \times \Pi_j^* e^{i(\mathbf{G} - \mathbf{q}) \cdot \mathbf{r}_j}. \end{aligned} \quad (33)$$

Here Π_j^* is the momentum operator that corresponds to the uniform average density case [$\Pi_j^* = \Pi_j$ ($\delta n = 0$)] and $c = \sqrt{l\nu}$. It will also be convenient to have separate symbols for the different orders of the δn in Eq. (33), so we write $\tilde{\rho}(\mathbf{q}) = \tilde{\rho}_0(\mathbf{q}) + \sum_{\mathbf{G}} \delta n(\mathbf{G}) \tilde{\rho}_1(\mathbf{q}, \mathbf{G})$.

The Hamiltonian (32) describes a many-body CF problem that we will treat within the Hartree-Fock approximation. We justify the use of the HF approximation by arguing that the CF is the true quasiparticle in the FQH regime. It will be assumed throughout this study that the average density modulation is small compared to the uniform background ($\delta n/n \ll 1$) so as a convenient basis we will choose the wave functions of the free CF moving in the uniform magnetic field $B^* = B - 2\pi l n/e$. The Landau gauge will be used in what follows. Wave functions will be denoted as $|n, X\rangle$, with n as a CF Landau level index and X as a kinetic momentum component in the y direction.

We will now derive the HF Hamiltonian. The model Hamiltonian (32) in the $|n, X\rangle$ basis may be written

$$\begin{aligned} H &= \frac{1}{2S} \sum_{\mathbf{q}} V(\mathbf{q}) \sum_{n_1 X_1, \dots, n_3 X_3} \langle n_1 X_1 | \tilde{\rho}(\mathbf{q}) | n_2 X_2 \rangle \\ &\quad \times \langle n_2 X_2 | \tilde{\rho}(-\mathbf{q}) | n_3 X_3 \rangle c_{n_1, X_1}^\dagger c_{n_3, X_3} \\ &\quad + \frac{1}{2S} \sum_{\mathbf{q}} V(\mathbf{q}) \sum_{n_1 X_1, \dots, n_4 X_4} \langle n_1 X_1 | \tilde{\rho}(\mathbf{q}) | n_4 X_4 \rangle \\ &\quad \times \langle n_2 X_2 | \tilde{\rho}(-\mathbf{q}) | n_3 X_3 \rangle c_{n_1, X_1}^\dagger c_{n_2, X_2}^\dagger c_{n_3, X_3} c_{n_4, X_4}, \end{aligned} \quad (34)$$

where $c_{n, X}^\dagger$ ($c_{n, X}$) is the CF creation (destruction) operator. The usual HF pairings are made in the two-body term of the Hamiltonian (34), giving two contributions—a direct and an exchange term. Because the X_i dependence of the density matrix elements in Eq. (34) is very simple,

$$\begin{aligned} \langle n_1 X_1 | \tilde{\rho}_0(\mathbf{q}) | n_2 X_2 \rangle &= \langle n_1 | \tilde{\rho}_0(\mathbf{q}) | n_2 \rangle \\ &\quad \times \exp[-iq_x(X_1 + X_2)/2] \delta_{X_1, X_2 - q_y l_0^{*2}} \end{aligned} \quad (35)$$

$$\begin{aligned} \langle n_1 X_1 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}) | n_2 X_2 \rangle &= \langle n_1 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}) | n_2 \rangle \\ &\quad \times \exp[-i(q_x - G_x)(X_1 + X_2)/2] \\ &\quad \times \delta_{X_1, X_2 - (q_y - G_y) l_0^{*2}} \end{aligned} \quad (36)$$

[here $l_0^* = (eB^*)^{-1/2}$ is the CF magnetic length], one can eliminate all the dependence on the momentum X_i in the Eq. (34). To this end we introduce the operator

$$\Delta_{nn'}(\mathbf{q}) = \frac{1}{g} \sum_{\mathbf{X}} \exp(-iq_x X - iq_x q_y l_0^{*2}/2) c_{n, X}^\dagger c_{n', X + q_y l_0^{*2}}, \quad (37)$$

where g is the degeneracy of a Landau level. $\Delta_{nn'}(\mathbf{G})$ is the order parameter of the density modulation corresponding to the wave vector \mathbf{G} . Note that $\Delta_{nn'}(\mathbf{q})$ has the property

$$\sum_n \Delta_{nn}(0) = \nu. \quad (38)$$

After doing sums over X_i we find the following contributions to the HF Hamiltonian.

- (1) one-body term, zeroth order in δn ,

$$\begin{aligned} H_{ob}^{00} &= \frac{g}{2S} \sum_{\mathbf{q}} V(\mathbf{q}) \sum_{n_1, n_2, n_3} \langle n_1 | \tilde{\rho}_0(\mathbf{q}) | n_2 \rangle \\ &\quad \times \langle n_2 | \tilde{\rho}_0(-\mathbf{q}) | n_3 \rangle \Delta_{n_1 n_3}(0). \end{aligned} \quad (39)$$

- (2) A one-body term, first order in δn ,

$$\begin{aligned} H_{ob}^{01} &= \frac{g}{2S} \sum_{\mathbf{q}, \mathbf{G}} V(\mathbf{q}) \sum_{n_1, n_2, n_3} (\langle n_1 | \tilde{\rho}_1(-\mathbf{q}, \mathbf{G}) | n_2 \rangle \\ &\quad \times \langle n_2 | \tilde{\rho}_0(\mathbf{q}) | n_3 \rangle + \langle n_1 | \tilde{\rho}_0(-\mathbf{q}) | n_2 \rangle \langle n_2 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}) | n_3 \rangle) \\ &\quad \times \delta n(\mathbf{G}) e^{il_0^{*2} \mathbf{q} \times \mathbf{G}/2} \Delta_{n_1 n_3}(-\mathbf{G}). \end{aligned} \quad (40)$$

- (3) A one-body term, second order in δn ,

$$\begin{aligned} H_{ob}^{11} &= \frac{g}{2S} \sum_{\mathbf{q}, \mathbf{G}, \mathbf{G}_1} V(\mathbf{q}) \sum_{n_1, n_2, n_3} \langle n_1 | \tilde{\rho}_1(-\mathbf{q}, \mathbf{G}) | n_2 \rangle \\ &\quad \times \langle n_2 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}_1) | n_3 \rangle \delta n(\mathbf{G}) \delta n(\mathbf{G}_1) \\ &\quad \times \exp\{il_0^{*2} [\mathbf{q} \times (\mathbf{G} + \mathbf{G}_1) - \mathbf{G} \times \mathbf{G}_1]/2\} \\ &\quad \times \Delta_{n_1 n_3}(-\mathbf{G} - \mathbf{G}_1). \end{aligned} \quad (41)$$

- (4) A two-body term, zeroth order in δn , direct and exchange contributions,

$$\begin{aligned} H_{tb}^{00} &= \frac{g^2}{S} \sum_{\mathbf{G}} V(\mathbf{G}) \sum_{n_1, \dots, n_4} \langle n_1 | \tilde{\rho}_0(-\mathbf{G}) | n_4 \rangle \langle n_2 | \tilde{\rho}_0(\mathbf{G}) | n_3 \rangle \\ &\quad \times \langle \Delta_{n_1 n_4}(-\mathbf{G}) \rangle \Delta_{n_2 n_3}(\mathbf{G}) \\ &\quad - \sum_{\mathbf{q}, \mathbf{G}} V(\mathbf{q}) \sum_{n_1, \dots, n_4} \langle n_1 | \tilde{\rho}_0(-\mathbf{q}) | n_4 \rangle \\ &\quad \times \frac{g}{S} \langle n_2 | \tilde{\rho}_0(\mathbf{q}) | n_3 \rangle \langle \Delta_{n_1 n_3}(\mathbf{G}) \rangle \Delta_{n_2 n_4}(-\mathbf{G}) e^{il_0^{*2} \mathbf{G} \times \mathbf{q}}. \end{aligned} \quad (42)$$

(5) A two-body term, first order in δn , direct contributions,

$$\begin{aligned}
H_{ibd}^{01} &= \frac{g^2}{S} \sum_{\mathbf{G}, \mathbf{G}_1} \sum_{n_1, \dots, n_4} [V(-\mathbf{G}-\mathbf{G}_1) \\
&\times \langle n_1 | \tilde{\rho}_1(\mathbf{G}+\mathbf{G}_1, \mathbf{G}) | n_4 \rangle \langle n_2 | \tilde{\rho}_0(-\mathbf{G}-\mathbf{G}_1) | n_3 \rangle \\
&+ V(-\mathbf{G}_1) \langle n_1 | \tilde{\rho}_0(\mathbf{G}_1) | n_4 \rangle \\
&\times \langle n_2 | \tilde{\rho}_1(-\mathbf{G}_1, \mathbf{G}) | n_3 \rangle] \delta n(\mathbf{G}) \\
&\times \langle \Delta_{n_1 n_4}(\mathbf{G}_1) \rangle \Delta_{n_2 n_3}(-\mathbf{G}-\mathbf{G}_1). \quad (43)
\end{aligned}$$

(6) A two-body term, first order in δn , exchange contributions,

$$\begin{aligned}
H_{ibe}^{01} &= -\frac{g}{S} \sum_{\mathbf{q}, \mathbf{G}, \mathbf{G}_1} \sum_{n_1, \dots, n_4} V(\mathbf{q}) (\langle n_1 | \tilde{\rho}_1(-\mathbf{q}, \mathbf{G}) | n_4 \rangle \\
&\times \langle n_2 | \tilde{\rho}_0(\mathbf{q}) | n_3 \rangle \exp\{i l_0^{*2} [\mathbf{G} \times (\mathbf{q}-\mathbf{G}_1)/2 \\
&+ \mathbf{G}_1 \times \mathbf{q}]\} + \langle n_1 | \tilde{\rho}_0(-\mathbf{q}) | n_4 \rangle \\
&\times \langle n_2 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}) | n_3 \rangle \exp\{i l_0^{*2} [\mathbf{G} \times (\mathbf{q}+\mathbf{G}_1)/2 \\
&+ \mathbf{G}_1 \times \mathbf{q}]\}) \times \delta n(\mathbf{G}) \langle \Delta_{n_1 n_3}(\mathbf{G}_1) \rangle \Delta_{n_2 n_4}(-\mathbf{G}-\mathbf{G}_1). \quad (44)
\end{aligned}$$

(7) A two-body term, second order in δn , direct contribution,

$$\begin{aligned}
H_{ibd}^{11} &= \frac{g^2}{S} \sum_{\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2} \sum_{n_1, \dots, n_4} V(-\mathbf{G}-\mathbf{G}_1) \\
&\times \langle n_1 | \tilde{\rho}_1(\mathbf{G}+\mathbf{G}_1, \mathbf{G}_1) | n_4 \rangle \langle n_2 | \tilde{\rho}_1(-\mathbf{G}-\mathbf{G}_1, \mathbf{G}_2) | n_3 \rangle \\
&\times \delta n(\mathbf{G}_1) \delta n(\mathbf{G}_2) \langle \Delta_{n_1 n_4}(\mathbf{G}) \rangle \Delta_{n_2 n_3}(-\mathbf{G}-\mathbf{G}_1-\mathbf{G}_2). \quad (45)
\end{aligned}$$

(8) A two-body term, second order in δn , exchange contribution,

$$\begin{aligned}
H_{ibe}^{11} &= -\frac{g}{S} \sum_{\mathbf{q}, \mathbf{G}, \mathbf{G}_1, \mathbf{G}_2} V(\mathbf{q}) \sum_{n_1, \dots, n_4} \langle n_1 | \tilde{\rho}_1(-\mathbf{q}, \mathbf{G}_1) | n_4 \rangle \\
&\times \langle n_2 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}_2) | n_3 \rangle \\
&\times \langle \Delta_{n_1 n_3}(\mathbf{G}) \rangle \delta n(\mathbf{G}_1) \delta n(\mathbf{G}_2) \Delta_{n_2 n_4}(-\mathbf{G}-\mathbf{G}_1-\mathbf{G}_2) \\
&\times \exp\{i l_0^{*2} [(\mathbf{G}_1+\mathbf{G}_2) \times \mathbf{q} + \mathbf{G} \times (\mathbf{G}_1-\mathbf{G}_2) - \mathbf{G}_1 \times \mathbf{G}_2 \\
&+ 2\mathbf{G} \times \mathbf{q}]/2\}. \quad (46)
\end{aligned}$$

The matrix elements of the density operator can be calculated using the formulas (B1), (B2) and (B3) that are given in the Appendix B. The momenta \mathbf{G}_i run over a discrete set of reciprocal lattice vectors. The momentum \mathbf{q} is a continuous variable. The summation over \mathbf{q} in those terms of the Hamiltonian where it appears can be done in a closed form as we

show in Appendix C because the potential in Eq. (31) was chosen so that these integrals could be performed analytically.

We will group all the nonoperator entries in Eqs. (39)–(46) under the notation $U_{n_1 n_2}(\mathbf{G})$ (renaming the dummy summation variables where necessary) and represent the HF Hamiltonian in a form convenient for further discussion,

$$H_{HF} = g \sum_{\mathbf{G}, n_1, n_2} U_{n_1 n_2}(\mathbf{G}) \Delta_{n_1 n_2}(\mathbf{G}). \quad (47)$$

Obviously, $U_{n_1 n_2}(\mathbf{G})$ depends on the expectation value of the order parameter operator $\Delta_{n_1 n_2}(\mathbf{G})$ both directly and through the density modulation δn , because

$$\begin{aligned}
\delta n(\mathbf{G}) &= g \sum_{n_1, n_2} \langle n_1 | \tilde{\rho}_0(\mathbf{G}) | n_2 \rangle \langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle \\
&+ g \sum_{\mathbf{G}_1, n_1, n_2} \langle n_1 | \tilde{\rho}_1(\mathbf{G}, \mathbf{G}_1) | n_2 \rangle \delta n(\mathbf{G}_1) \\
&\times \langle \Delta_{n_1 n_2}(\mathbf{G}-\mathbf{G}_1) \rangle. \quad (48)
\end{aligned}$$

In the i th iteration of the numerical HF procedure $\Delta_{n_1 n_2}(\mathbf{G})$ is calculated using the solution of the $(i-1)$ th (previous) iteration. The density modulation is then calculated as a numerical solution to the system of the linear equations defined in Eq. (48).

Having found the HF Hamiltonian, we can solve it to find the single-particle spectrum of the many-body system. We will assume that the CF form a Wigner lattice with one particle per unit cell. The reciprocal lattice constant for a triangular lattice is given by

$$G_0 = \frac{1}{l_0} \sqrt{\frac{4\pi\nu}{\sqrt{3}}}. \quad (49)$$

However, we will find that in some regions of filling factor the triangular lattice is not the ground state, and we will explore other lattice structures.

We have used two different schemes to perform the calculation, one due to Côté and MacDonald³¹ and the other due to Yoshioka and Lee.³² Below we will outline the essence of each of these methods.

The method by Côté and MacDonald (CM) starts from the single-particle Green's function that they define as

$$G_{n_1 n_2}(X_1, X_2, \tau) = -\langle T c_{n_1, X_1}(\tau) c_{n_2, X_2}^\dagger(0) \rangle, \quad (50)$$

where T is the time-ordering operator. The relationship of the Green's function Fourier transform to the physically relevant expectation value of the order parameter is

$$\begin{aligned}
 \langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle &= G_{n_2 n_1}(\mathbf{G}, \tau=0^-) \\
 &\equiv \frac{1}{g} \sum_{X_1, X_2} G_{n_2 n_1}(X_2, X_1, \tau=0^-) \\
 &\quad \times \exp(-iG_x(X_1+X_2)/2) \delta_{X_1, X_2 - G_y l_0^{*2}}.
 \end{aligned} \tag{51}$$

The set of crystal order parameters $\langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle$ is then used to find the ground-state energy

$$E_{HF} = \frac{\epsilon_1}{\nu} \sum_{\mathbf{G}, n_1, n_2} U_{n_1 n_2}(\mathbf{G}) \langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle, \tag{52}$$

where $\epsilon_1 = 1$ if it multiplies those terms of $U_{n_1 n_2}(\mathbf{G})$ that are given by Eqs. (39)–(41) (one-body terms) and $\epsilon_1 = 1/2$ if it multiplies terms that are given by Eqs. (42)–(46) (two-body terms). The transport gap E_g , also called activation energy, can be deduced from the chemical potential and the single-particle density of states $d(E)$ that is related to the Green's function through

$$d(E) = -\frac{1}{\pi} \sum_n \text{Im} G_{nn}(\mathbf{G}=0, i\omega_j \rightarrow E + i\delta), \tag{53}$$

here $\text{Im} G_{nn}$ is the imaginary part of the operator G_{nn} , ω_j are the Matsubara frequencies, δ is a small smoothing parameter.

All of the above is predicated on knowing the Green's function. We derive the Green's function equation of motion in the usual way by taking the commutator of the Hamiltonian (47) with a single-particle destruction operator c_{nX} ,

$$\begin{aligned}
 &\left(i\omega_j + \frac{\mu}{\hbar} \right) G_{n_1 n_2}(\mathbf{G}, \omega_j) - \sum_{\mathbf{G}, n_3} \frac{1}{\hbar} \\
 &\quad \times U_{n_1 n_3}(\mathbf{G}_1 - \mathbf{G}) G_{n_3 n_2}(\mathbf{G}_1, \omega_j) e^{i\mathbf{G} \times \mathbf{G}_1 l_0^{*2}} \\
 &= \delta_{n_1, n_2} \delta_{\mathbf{G}, 0},
 \end{aligned} \tag{54}$$

where μ is the the chemical potential. The system of Eqs. (54) is solved for the Green's function by diagonalizing its left-hand side with respect to the indices n_3 and \mathbf{G}_1 . One can find the expectation value of the order parameter and the density of states once the chemical potential is known. The chemical potential in turn is calculated by filling up the correct number of states, that is, by using Eq. (38).

The numerical iterative scheme starts by assuming a Gaussian form for the order parameters. (The exact expression depends on the filling factor and the state that is being constructed and will be discussed later). This initial set of $\langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle$ is then used to compute the effective potential $U_{n_1 n_2}(\mathbf{G})$. Next the equation of motion is solved to get a new set of order parameters and the process is repeated until the $\langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle$ converge with some prescribed accuracy. Another way to check the accuracy of the numerical solution is by using the following useful sum rule³¹

$$\sum_{\mathbf{G}, n_1, n_2} \langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle^2 = \nu. \tag{55}$$

The idea behind the second method, that of Yoshioka and Lee³² (YL), is to diagonalize the one-body HF Hamiltonian that can be rewritten in terms of the CF creation and destruction operators as

$$H = \sum_{\mathbf{G}, n_1, n_2, X} U_{n_1 n_2}(\mathbf{G}) e^{-iG_x X} c_{n_1, X - G_y l_0^{*2}/2}^\dagger c_{n_2, X + G_y l_0^{*2}/2}. \tag{56}$$

We assume that the CF form a periodic lattice with primitive translation vectors of the reciprocal lattice that are given by $\mathbf{Q}_1 = (Q_0, 0)$ and $\mathbf{Q}_2 = Q_0(p/q, \alpha)$. The first unitary transformation on the Hamiltonian (56) is defined by

$$a_{n_1, X, Y} = \frac{1}{\sqrt{s_m}} \sum_{s=0}^{s_m} e^{-is\alpha Q_0 Y} c_{n_1, X + s\alpha Q_0 l_0^{*2}}, \tag{57}$$

where $s_m = L/\alpha Q_0 l_0^{*2}$, L is the linear dimension of the system, $0 \leq X \leq \alpha Q_0 l_0^{*2}$ and $0 \leq Y < 2\pi/\alpha Q_0$. After making the transformation (57), the Hamiltonian is

$$\begin{aligned}
 H = &\sum_{\mathbf{G}, n_1, n_2, X, Y} U_{n_1 n_2}(\mathbf{G}) \exp(-iG_x X + iG_y Y \\
 &+ iG_x G_y l_0^{*2}/2) a_{n_1, X, Y}^\dagger a_{n_2, X, Y + G_x l_0^{*2}}.
 \end{aligned} \tag{58}$$

The variable Y in Eq. (58) is coupled through $G_x l_0^{*2}$. If this number is commensurate with $2\pi/\alpha Q_0$, which is the period of the variable Y , then we can simplify the Hamiltonian even further. Suppose then that the parameters are such that $NQ_0 l_0^{*2}/q = M2\pi/\alpha Q_0$, with M and N integers. We then introduce a new operator

$$b_{n, j, X, Y} = a_{n, X, Y + jQ_0 l_0^{*2}/q}, \tag{59}$$

where $1 \leq j \leq N$ and $0 \leq Y < l_0^{*2} Q_0/qM$. After inserting Eq. (59) into Eq. (58) the expression for the Hamiltonian is

$$\begin{aligned}
 H = &\sum_{\mathbf{G}} \sum_{X, Y} \sum_{n_1, j, n_2, k} U_{n_1 n_2}(\mathbf{G}) \exp(-iG_x X + iG_y Y) \\
 &\times \exp(iQ_0 Q_y j l_0^{*2}/q + iG_x G_y l_0^{*2}/2) \\
 &\times b_{n_1, j, X, Y}^\dagger b_{n_2, k, X, Y} \delta_{k, j + Q_x q/Q_0}.
 \end{aligned} \tag{60}$$

For every pair (X, Y) that takes values in the rectangular domain defined earlier, the Hamiltonian (60) can be diagonalized in the indices n_1, j , and n_2, k . The single-particle energies $E_{n_1, j}(X, Y)$ that thereby result are continuous in the variables X, Y and form energy bands. There are $n_m N$ energy bands (where $1 \leq n_1, n_2 \leq n_m$) and there is a large energy gap between the lower Mq th and $(Mq+1)$ th bands. The CF state that has the chemical potential in this large gap should have the lowest energy. Such a state occurs when the CF filling factor is $\nu^* = qM/N$, where we have defined the CF filling factor as $\nu^* = 2\pi l_0^{*2} n$. It is easy to derive the expres-

TABLE I. Electron WC ground state and activation energies for $\nu = 1/5$ and different values of Λ .

Λ	0	$l_0/2$	l_0	$3l_0/2$
E_{HF}	-0.3220	-0.3137	-0.2859	-0.2413
$E_g(e^2/\epsilon l_0)$	0.4728	0.5080	0.5080	0.4468

sion for the expectation value of the order parameter, by applying the transformations (57) and (59) to the definition (37) to get

$$\begin{aligned} \langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle &= \frac{2\pi l_0^{*2}}{L^2} \sum_{X,Y} \sum_{j,k} \exp(-iG_x X + iG_y Y) \\ &\quad \times \exp(iQ_0 Q_y j l_0^{*2}/q + iG_x G_y l_0^{*2}/2) \\ &\quad \times \langle b_{n_1, j, X, Y}^\dagger b_{n_2, k, X, Y} \rangle \delta_{k, j+Q_x q/Q_0}. \end{aligned} \quad (61)$$

As in the case of the previous method we find a solution to the HF problem by iterating until the order parameters converge. We calculate the ground-state energy by using Eq. (52) and the transport gap E_g as a smallest separation between the Mq th and $(Mq+1)$ th bands.³³

While the method of CM is numerically efficient, it is sometimes difficult to extract the transport gap from the smoothed density of states. There is no uncertainty in determining E_g when the method of YL is used.

V. RESULTS

Our experimental motivation is the work by Jiang *et al.*^{13,14} where the transport properties were measured around $\nu = 1/5$ Landau-level filling. In Ref. 13 an insulating phase was identified just above $\nu = 1/5$ at $\nu = 0.21$ by observing a large peak of the longitudinal resistance R_{xx} as a function of the external magnetic field. The activation energy was estimated from the Arrhenius plot at $E_g \sim 0.63$ K (with $B \approx 20$ T). The striking observation is that the magnitude of the transport gap compares very poorly with the results obtained from HF for the usual electron solid. The excitation energies for the triangular *electron* lattice with one particle per unit cell are given in Table I. We use the modified Coulomb potential given by Eq. (31) and present results for different values of the thickness parameter Λ . The calculation was done in the lowest-Landau-level approximation and for $\Lambda = 0$ it reproduces previous results.³² The energies are given in units of $e^2/\epsilon l_0$. In the same units the experimental result is $E_g \sim 2.8 \times 10^{-3} e^2/\epsilon l_0$, at least two orders of magnitude smaller than the theory.

We expect some reduction in the value of E_g when the relaxation of the lattice is accounted for,^{34,35} but it is difficult to believe that this correction would nearly exactly cancel the unrelaxed excitation energy. Besides, one would not expect the $E_g(\nu)$ for the electron WC to be nonmonotonic as observed experimentally.¹⁴

Now we proceed to carry out our program of considering crystals of CF's.

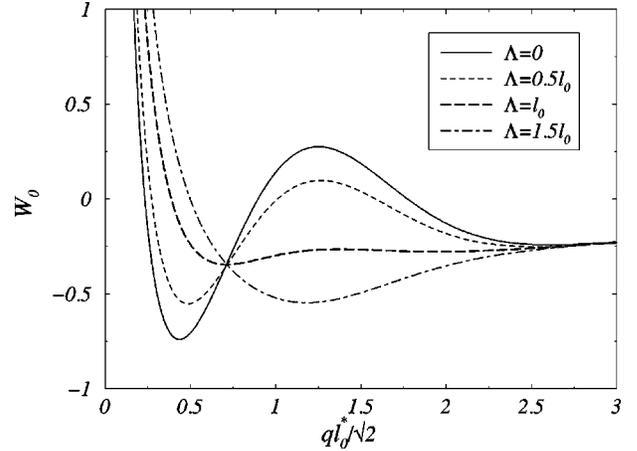


FIG. 2. Effective HF potential for different values of parameter Λ . The filling fraction specific factor $(1-l\nu)^2$ was omitted from the zeroth-order expression of the effective potential when generating these curves.

A. Crystals of composite fermions with two vortices attached

Let us examine how well CF's with $l=2$ describe the experimental situation. An electronic filling factor of $\nu = 1/5$ corresponds to a CF filling factor $\nu^* = \nu/(1-l\nu) = 1/3$. So the lowest CF Landau level is partially filled and it is reasonable to expect that the composite fermions form a lattice. As in the electron solid calculation only the lowest CF Landau level is kept ($n_m = 1$). Keeping two CF Landau levels ($n_m = 2$) we find similar results, indicating that including more Landau levels does not influence the calculation. Because the CF and electron effective potential $U_{00}(\mathbf{q})$ in Eq. (47) are different momentum functions it is not obvious that the CF lattice is triangular as is the case for the electron lattice. The functional form of $U_{00}(\mathbf{q})$ may be suggestive in that respect. One expects it to have a minimum at about the momentum \mathbf{q} equal to the shortest reciprocal vector. This argument cannot be exact in our theory because $U_{00}(\mathbf{G})$ depends on δn and is reevaluated self-consistently in every iteration. However, since the density modulations are small, we expect that a good approximation to $U_{00}(\mathbf{G})$ can be obtained by keeping only the δn -independent terms given by Eqs. (39) and (42) in the HF Hamiltonian [the term given by Eq. (39) is a constant]. In that case the approximate effective potential can be expressed as $U_{00}(\mathbf{G}) \equiv W_0(\mathbf{G}) \langle \Delta_{00}(\mathbf{G}) \rangle$, defining the effective interaction $W_0(\mathbf{G})$. We display the plot of this effective interaction in Fig. 2 for different values of the parameter Λ . Whereas for the $\nu^* = 1/3$ triangular lattice we expect a minimum at about $|\mathbf{q}| l_0^* \approx 1.56$, the minimum for CF effective potential is at much smaller wave vector, more so for a small Λ . This is why we do not limit ourselves to the triangular lattice but calculate the ground-state energies along with the E_g 's for three oblique (including triangular) lattices. The results together with the primitive reciprocal lattice vectors $\mathbf{b}_1, \mathbf{b}_2$ are given in Table II. Every lattice is rescaled by an overall factor that makes the volume of the unit cell a constant equal to $2\pi l_0^2/\nu$.

We find that for $\Lambda = 0$ composite fermions prefer the elongated lattices to the triangular one. For larger values of

TABLE II. $l=2$ CF lattice ground-state and activation energies for $\nu=1/5$, different values of Λ , and different unit cells.

	Λ	0	$l_0/2$	l_0	$3l_0/2$
$\mathbf{b}_1=(1,0)$	E_{HF}	-0.33	-0.34	-0.30	-0.24
$\mathbf{b}_2=(0.5,\sqrt{3}/2)$	$E_g(e^2/\epsilon l_0)$	0.08	0.13	0.13	0.14
$\mathbf{b}_1=(1,0)$	E_{HF}	-0.36	-0.33	-0.29	-0.24
$\mathbf{b}_2=(0.5,\sqrt{3})$	$E_g(e^2/\epsilon l_0)$	0.06	0.07	0.12	0.12
$\mathbf{b}_1=(1,0)$	E_{HF}	-0.37	-0.33	-0.29	-0.24
$\mathbf{b}_2=(0.5,3\sqrt{3}/2)$	$E_g(e^2/\epsilon l_0)$	0.04	0.05	0.09	0.10

Λ the triangular lattice is favored. The results for the excitation energy are somewhat closer to the experimental value but still too large.

The disagreement between the theory and the experiment is not only in the magnitude of the activation energy but also in its dependence on the filling factor. In our theory with $l=2$ the function $E_g(\nu^*)$ varies slowly and is monotonic around the CF filling factor $\nu^*=1/3$ ($\nu=1/5$). Figure 3 gives this dependence for the triangular lattice with $\Lambda=3l_0/2$. The experimental function (see Fig. 3 in Ref. 14) has a sharp peak between the filling factors $\nu\sim 0.22$ and $\nu\sim 0.21$ and for $\nu<1/5$ it rises sharply and saturates at lower filling factors.

Let us turn to CF's with four flux quanta to see how the results compare with experiments.

B. Crystals of composite fermions with four vortices attached

The behavior of the experimental gap with ν fits in more naturally within the CF model with $l=4$. When the filling factor $\nu<1/5$ the lowest CF Landau level is being populated and a CF quasiparticle lattice is assumed to be the stable state.³⁶ On the other hand, when $\nu>1/5$ the second CF Landau level is being populated, and one naturally expects some

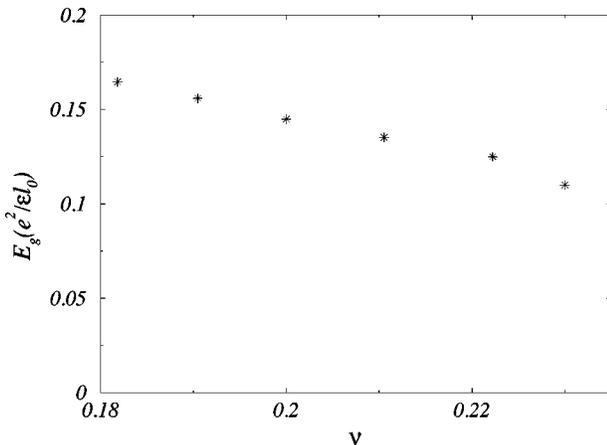


FIG. 3. The transport gap dependence on the filling factor around $\nu\approx 1/5$ ($l=2$) for $\Lambda=1.5l_0$.

difference in the behavior of the gap in the theory. We will see that this expectation is realized, but not in complete agreement with experiments.

Numerical constraints limited our HF basis to the two lowest CF Landau levels ($n_m=2$). The initial seed used in the HF procedure that converged to the correlated WC was

$$\langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle = \begin{cases} \nu^* e^{-\mathbf{G}^2 l_0^{*2}/4}, & \text{if } n_1, n_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $\nu>1/5$, the second CF Landau level is partially filled. Again we assume that the CF lattice is formed so the initial seed that we use in this case is

$$\langle \Delta_{n_1 n_2}(\mathbf{G}) \rangle = \begin{cases} e^{-\mathbf{G}^2 l_0^{*2}/4}, & \text{if } n_1, n_2 = 0 \\ (\nu^* - 1) e^{-\mathbf{G}^2 l_0^{*2}/4}, & \text{if } n_1, n_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Our results for the activation energy are presented in Fig. 1. The value of the parameter Λ is $3l_0/2$ (the results for $\Lambda=l_0$ are very similar) and we assume that the lattice is triangular. A magnetic field of $B=20$ T was used to convert the energy units to kelvin, in order to compare to the work of Jiang *et al.*¹⁴

First we will discuss the results for $\nu<1/5$. The transport gaps that we obtain are generally comparable to the experimental values. We also reproduce a correct $E_g(\nu)$ dependence here (see the left half of the Fig. 1). However, we do not observe saturation towards the lower filling factors. This may be an indication that perhaps CF's with $l=4$ are not the quasiparticles at very low fillings.

Another experimental probe supporting the crystalline nature of the insulating state is an I - V measurement.²³ Nonlinear I - V curves have a threshold voltage at which the differential resistance drops off that can be interpreted as a depinning of a weakly pinned Wigner crystal.²⁴ As the filling factor is varied the threshold voltages increase approaching the FQH state at $\nu=1/5$ both from above and below. This finding could be a consequence of a lattice getting less rigid as the FQH state is closer.^{24,37} We have calculated the shear modulus of the CF lattice for several fractions $\nu<1/5$. We first compute the ground-state energies of a triangular lattice

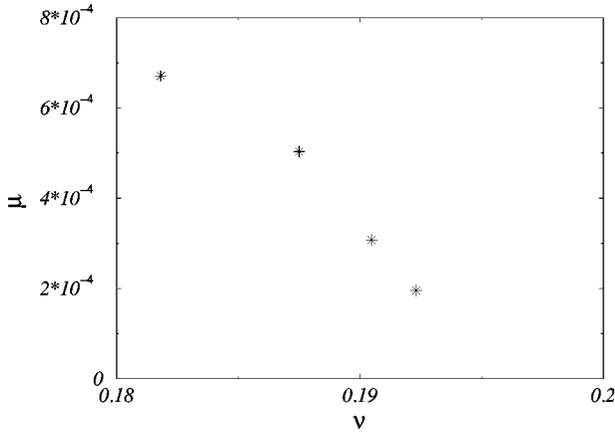


FIG. 4. The shear modulus μ for the triangular CF lattices as a function of filling factor ($\nu < 1/5$) for $\Lambda = 1.5l_0$.

with the primitive reciprocal lattice vectors $\mathbf{b}_1 = (1, 0)$, $\mathbf{b}_2 = (0.5, \sqrt{3}/2)$ and a deformed lattice such that its primitive reciprocal lattice vectors are $\mathbf{b}_1 = Q_0(1, 0)$, $\mathbf{b}_2 = Q_0(0.5, 3\sqrt{2}/4)$ (oblique lattice), with Q_0 chosen so that the area of the Brillouin zone is equal to that of the triangular lattice. Then the shear modulus μ is proportional to the difference of the ground-state energies. The results are presented in Fig. 4 for $\Lambda = 1.5l_0$. We observe that the lattice is indeed becoming softer as $\nu \rightarrow 1/5$. This conclusion is consistent with the experimental results²³ interpreted using the collective pinning theory.^{24,37}

For $\nu > 1/5$ the gaps for the triangular lattice, while being in the same range as their experimental counterparts, do not show the correct dependence on ν close to $\nu = 1/5$ (see the right half of the Fig. 1). We find that in this case the triangular lattice is not the lowest energy solution to the HF equations. Figure 5 gives the HF energies of several lattices for a fraction $\nu^* = 6/5$ (that corresponds to $\nu = 0.206 \dots$). The lattices that we consider are deformations of the triangular lattice obtained from it by changing the angle θ between the reciprocal lattice vectors \mathbf{b}_1 and \mathbf{b}_2 such that $|\mathbf{b}_1| = |\mathbf{b}_2|$ and the volume of the unit cell remains a constant. Figure 6 gives

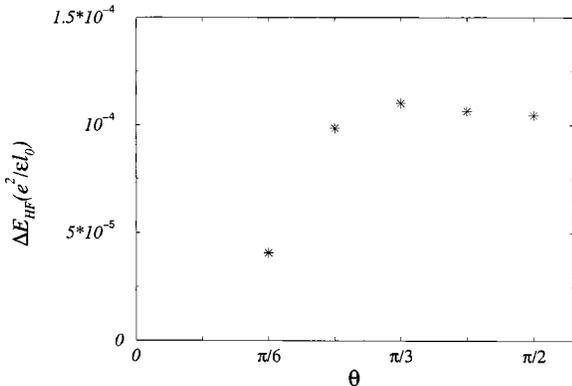


FIG. 5. The HF energies for CF lattices differing by an angle θ between the reciprocal lattice vectors for the filling factor $\nu \approx 0.206$ for $\Lambda = 1.5l_0$. The zero on the vertical axis corresponds to the energy $-0.265 \text{ } 15(e^2/\epsilon l_0)$.

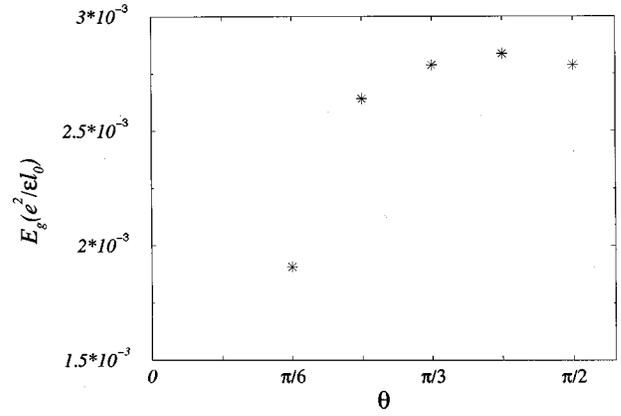


FIG. 6. The activation energies for CF lattices differing by an angle θ between the reciprocal lattice vectors for the filling factor $\nu \approx 0.206$ for $\Lambda = 1.5l_0$.

the corresponding activation energies. As is apparent from Fig. 5, the lattices with small angles θ are more stable within our HF scheme. The smallest angle for which the iterations reliably converge is $\theta = \pi/6$. The dependence of E_g on ν is presented for the triangular and the $\theta = \pi/6$ lattices in Fig. 1.

At this point we have run into an intrinsic limitation of the Hamiltonian theory: Since the exact transformation between the electronic coordinates and the CF coordinates is not known, the Hamiltonian itself is not known exactly. This means that we should not take the ground state energies that are predicted by our theory too seriously. Note also that the differences in ground-state energy between the different lattice structures are very tiny so any conclusion concerning the stability of one lattice compared to another should be taken with a grain of salt. We still can estimate the “shear modulus” for this class of lattices by looking at the difference in ground-state energy between the triangular and square lattices. This leads to an estimate of $\mu \approx 2 \times 10^{-5} e^2/\epsilon l_0$, an order of magnitude smaller than for $\nu < 1/5$. This means that the CF lattices are very soft for ν just above $1/5$, and disorder may potentially be very important in this case. As the filling factor increases the situation remains qualitatively similar but the differences in energy decrease. The HF energies for the triangular and square lattices are presented for several filling factors in Table III. The transport gaps for these two lattices are essentially the same.

A brief comparison of the ground-state energies of the different possible states is in order. At $\nu = 1/5$ and $\Lambda = 1.5l_0$ the CF-WC with four vortices has a ground-state energy of -0.265 per particle. The ground-state energies of the electronic WC and CF-WC with two vortices for the same parameters are equal to -0.24 and -0.241 as given in Tables I and II, respectively. It can be seen that the ground-state energy of the CF-WC with four vortices attached is lower than that of the CF-WC with two vortices and the electronic WC at $\nu = 1/5$. This reproduces the result found by Yi and Fertig²⁰ and reinforces our belief that CF’s with four vortices attached offer the best description of the state near $\nu = 1/5$.

C. Checking the consistency of the approximation scheme

Recall that the formalism that we have developed in Secs. II and III is based on two assumptions: (i) We have made a

TABLE III. Comparison of $l=4$ CF square and hexagonal lattice ground states for filling factors $\nu > 1/5$ and $\Lambda = 1.5l_0$. Energy in units of $e^2/\epsilon l_0$.

ν	0.2	0.2069	0.2105	0.2143	0.2174	0.2195
E_{HF} square	-0.264984	-0.265046	-0.265242	-0.265716	-0.266248	-0.266659
E_{HF} hexag.	-0.264984	-0.265040	-0.265232	-0.265708	-0.266247	-0.266658

small- q approximation, which implies that gradients are small, and (ii) We have assumed that $\delta n(\mathbf{G})/n \ll 1$. It is important to check that our HF solutions do satisfy both these assumptions.

The fact, that our formalism is accurate only for small q is mitigated by the fact that the potential is weak at large q due to finite thickness effects, so that density matrix elements for large q are not important. In previous gap calculations by using this formalism⁸ (with a similar though slightly different potential $2\pi e^{-\Lambda q/q}$), it was found that for $\Lambda > l_0$ the formalism reproduces the numerically computed gaps of the liquid states quite well. Since we are using $\Lambda = 1.5l_0$, we can expect that the small- q formalism is applicable.

Nevertheless, more explicit checks are possible. In Fig. 7 we plot the values of the charge-density wave (CDW) order parameter $\langle \Delta_{00}(\mathbf{G}) \rangle$ of Eq. (37) and $\delta n(\mathbf{G})$ as a function of $|\mathbf{G}|$ for the ground state of $\nu = 0.1875$, using CF's with four vortices attached. This value of ν corresponds to $\nu^* = 0.75$, and the plot is normalized with respect to ν^* . It is clear that the values of the order parameter rapidly go to zero as $|\mathbf{G}|$ increases. This suggests that large values of q do not matter. Another check is provided by a real-space plot of the density profile in Fig. 8 for the same ν . As can be seen, the maximum density deviations are of order $\pm 20\%$ around the mean density, and the density variations are on the length scale of the interparticle spacing, which is greater than the magnetic length.

One should also note from Fig. 7 that the value of $\delta n(\mathbf{G})/n$ never exceeds $1/15$, which is quite small. Note that our expansion is in powers of $\delta n(\mathbf{G})/n$, which should not be

confused with the real-space deviations of the density. We have checked the smallness of $\delta n/n$ by another method as well. We have recomputed the ground-state energies and transport gaps by discarding all terms in the HF-Hamiltonian that explicitly depend on δn . We find that the ground-state energies change by about 0.1% and that the gaps change by at most 5–10%. This indicates that the effects of the δn -dependent terms in the Hamiltonian are small, which is a more physical measure of the smallness of $\delta n/n$ than its numerical value.

We conclude that indeed both the approximations made in setting up the formalism are satisfied by the HF solutions.

VI. CONCLUSIONS, CAVEATS, AND OPEN QUESTIONS

Two-dimensional electron gases in high magnetic fields offer the best conditions for the realization of the Wigner crystal, since the magnetic field tends to localize the electrons. However, electronic correlations play a dominant role in the LLL because the kinetic energy is degenerate. Attempts at describing the Wigner crystal using uncorrelated, or weakly correlated states of electrons,¹² do capture some of the essential physics, such as the filling factor at which the Laughlin liquid becomes unstable to the Wigner crystal. However, these theories fail to capture the correct structure of the excitation spectrum, and predict gaps that are two orders of magnitude above experimental observations.

Since Laughlin-Jastrow correlations are the essence of the fractional quantum-Hall liquid states,^{2,4} it is natural to hy-

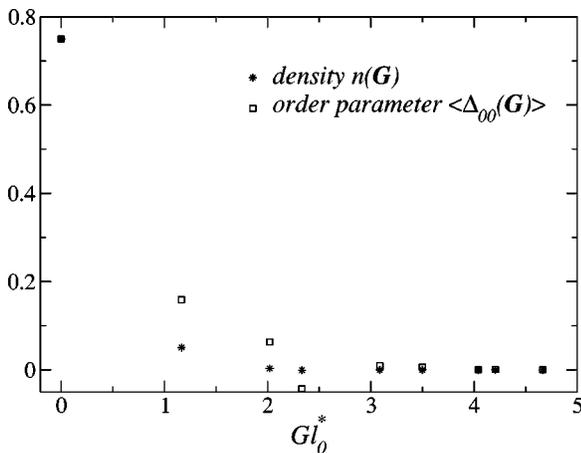


FIG. 7. The CDW order parameter $\langle \Delta_{00}(\mathbf{G}) \rangle$ and Fourier components of the density deviation $\delta n(\mathbf{G})$ as a function of $|\mathbf{G}|$ for $\nu = 0.1875$ and $\Lambda = 1.5l_0$. The y axis is normalized to $\nu^* = 0.75$.

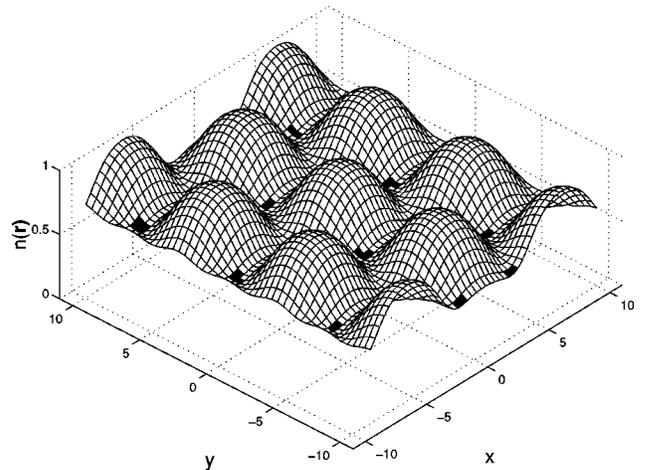


FIG. 8. Real-space density profile for the CF-WC at $\nu = 0.1875$ and $\Lambda = 1.5l_0$. Once again the density is normalized to $\nu^* = 0.75$. The crystal appears as small modulations around the mean density of 0.75.

pothesize that they are important in the Wigner crystal state as well. The first step in this direction was taken by Yi and Fertig,²⁰ who studied the ground-state energy as more and more vortices were attached to electrons forming a Wigner crystal. They found that indeed the Wigner crystals with vortices had better energies than the uncorrelated or weakly correlated crystals.²⁰

Unfortunately, ground-state energies cannot be probed directly in experiments. It is desirable to have predictions for observable physical properties that can distinguish between competing ground states. Calculating physical properties in a strongly correlated state is notoriously difficult. The composite fermion picture⁴ achieves the miracle of transforming a strongly correlated electronic problem into a weakly correlated problem of CF's. In the years since the discovery of the FQHE, much progress has been made in developing field-theoretic schemes that have predictive power.^{5,6} The latest in this long line of approaches is the Hamiltonian formalism,⁷ which has had reasonable success in computing gaps, magnetoexciton dispersions, and finite-temperature properties for the liquid states.⁸

In this paper, we have partially accomplished the goal of computing the physical properties of a strongly correlated Wigner crystal. Based on an extension of the Hamiltonian theory to account for the nonuniform density in the crystal state, we were able to compute gaps in the correlated crystal.

Our results show that qualitatively and semiquantitatively, a Wigner crystal state of CF's with four flux quanta attached offers the best description of the phenomenology of the high-field Wigner crystal near $\nu=1/5$. In particular, our predictions for gaps are within a factor of 2 of the experiments in the entire regime of interest. Our predictions show a different behavior for $\nu<1/5$ and $\nu>1/5$. While the theory has some discrepancies with the data¹⁴ for ν just above $1/5$, we believe we understand why this might be the case: Different lattice structures have very similar energies in this regime, and are very deformable. Consequently, disorder is expected to play a dominant role in determining the configuration, and hence the gaps, in this region of ν . Finally, we are able to estimate the shear modulus of the crystal above and below $1/5$, and we find them to become softer as $1/5$ is approached. This is consistent with the *increase* of the threshold voltage for nonlinear transport,²³ a standard feature of the theory of collective pinning.^{24,37}

Before we close, some caveats must be noted. An intrinsic limitation of the Hamiltonian theory⁷ is that the Hamiltonian is known only approximately. Thus the ground-state energies are not to be taken too seriously. This implies that this theory does not offer a trustworthy way to find the lowest-energy state. The strength of the Hamiltonian theory lies in the fact that if the nature of the state is known, the theory allows the computation of gaps, magnetoexcitons, and even finite temperature properties.⁸ With this in mind, let us note that we have not carried out an exhaustive search in the space of all possible states. We have confined ourselves to crystals with one CF per unit cell. While we did explore crystals other than triangular and square for $\nu>1/5$, we kept the two primitive reciprocal lattice vectors equal in magnitude. It is possible that some other crystal state that we have not explored

is the actual ground state in the clean limit. However, for the experimental observations this point is likely to be moot, because disorder probably plays a dominant role in this region of ν .

Many open questions remain. The most important, and the most intractable, is the influence of disorder. Disorder will cause lattice deformations, dislocations, and other defects. In a crystal of CF's, density variations are expected to produce a corresponding variation in the effective magnetic field. Thus, a random potential leads indirectly to a random magnetic field. In principle, the formalism we have developed here to deal with nonuniform density could be generalized to incorporate disorder, but the implementation appears difficult. In particular, it is difficult to visualize how the nonperturbative effects of disorder (localization of almost all states, changing σ_{xy} from $\nu e^2/h$ to 0, etc.) would emerge in a straightforward manner.

Another open question is the evolution of the Wigner crystal state with temperature, which could be explored in the clean limit along the line of reasoning laid out in this work. In particular, it would be of interest to obtain a prediction for the transition temperature between the Wigner crystal and the (presumably liquid) high-temperature state.

We hope to pursue these and other topics in future work.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we will construct the canonical transformation that diagonalizes the Hamiltonian (17) as discussed in the main text. First notice that putting every $\delta n(\mathbf{G})$ to zero takes us back to the uniform charge density case considered by Murthy and Shankar.²⁵ They showed that the canonical transformation in that case is

$$U_0(\lambda) \equiv e^{i\lambda S_0} = \exp \left\{ \lambda \theta \sum_{\mathbf{q}} \left[c^\dagger(\mathbf{q}) A(\mathbf{q}) - \text{H.c.} \right] \right\}, \quad (\text{A1})$$

where $\theta = 1/2n\sqrt{\pi l}$ and $\lambda = \arctan \mu/\mu$ with $\mu^2 = 1/l\nu - 1$. The value of the constant λ is fixed by requiring that the Hamiltonian in the FR does not have a term coupling the particle and the oscillator degrees of freedom.

The Hamiltonian (17) is different from the one considered in Ref. 25 by having terms proportional to $\delta n/n$. The same is true for the commutator of the kinetic momenta (19). We will assume that $\delta n/n$ is a small parameter and when diagonalizing the Hamiltonian we will only keep terms proportional to it. Consistent with this program, a reasonable guess for the canonical transformation is

$$U(\lambda) = \exp \left\{ i\lambda S_0 + \lambda \beta \sum_{\mathbf{q}} \sum_{\mathbf{G}}^{\varrho} [c^\dagger(\mathbf{q}) A(\mathbf{q}-\mathbf{G}) \delta n(\mathbf{G}) \hat{q}_- \times (\widehat{q-G})_+ - \text{H.c.}] \right\}, \quad (\text{A2})$$

where β is a constant that has to be determined later by requiring that the first order of the coupling term be zero. In Eq. (A1) $c(\mathbf{q})$ is the operator that corresponds to the uniform density, while in Eq. (A2) it depends on δn . Strictly speaking, these are two different operators.

We proceed as in Ref. 25 by determining the operators $A(\mathbf{q}, \lambda)$ and $c(\mathbf{q}, \lambda)$ in the new representation. Each of these operators is a sum of an unperturbed part that coincides formally with $\delta n=0$ result and a first order in $\delta n/n$ part. We introduce the notation $A_0(\mathbf{q}, \lambda) + A_1(\mathbf{q}, \lambda)$ for these parts [similarly for $c(\mathbf{q}, \lambda)$]. Using the canonical transformation (A2) and the commutation relations (18) and (19) we derive the following first-order flow equations for the operators

$$\frac{dA_1(\mathbf{q}, \lambda)}{d\lambda} = -\theta c_1(\mathbf{q}, \lambda) - \beta \sum_{\mathbf{G}}^{\varrho} c_0(\mathbf{q}-\mathbf{G}, \lambda) \delta n(\mathbf{G}) \times \hat{q}_- (\widehat{q-G})_+, \quad (\text{A3})$$

$$\frac{dc_1(\mathbf{q}, \lambda)}{d\lambda} = 2eB^* n \theta A_1(\mathbf{q}, \lambda) + \theta [2eB^*(\beta n + 1) - 4\pi l n] \times \sum_{\mathbf{G}}^{\varrho} A_0(\mathbf{q}-\mathbf{G}, \lambda) \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+. \quad (\text{A4})$$

Substituting Eq. (A4) into Eq. (A3) a second-order inhomogeneous ordinary differential equation is obtained for $A_1(\mathbf{q}, \lambda)$. The general solution depends on two arbitrary constants that are determined through the initial conditions $A_1(\mathbf{q}, 0) = c_1(\mathbf{q}, \lambda) = 0$. The result of the calculation is

$$A_1(\mathbf{q}, \lambda) = -\frac{\alpha \lambda \sin \mu \lambda}{2\mu} \sum_{\mathbf{G}}^{\varrho} A(\mathbf{q}-\mathbf{G}) \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+ + \left[\left(\frac{\theta}{2\mu n} - \frac{\pi l \theta}{\mu e B^*} \right) \sin \mu \lambda - \frac{\alpha \theta \lambda \cos \mu \lambda}{2\mu^2} \right] \sum_{\mathbf{G}}^{\varrho} c(\mathbf{q}-\mathbf{G}) \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+, \quad (\text{A5})$$

where a new constant $\alpha = 2\mu^2(\beta/\theta + \pi l/eB^* - 1/2n)$ was introduced. Having found $A_1(\mathbf{q}, \lambda)$, we can integrate $c_1(\mathbf{q}, \lambda)$ from the Eq. (A4) with the result

$$c_1(\mathbf{q}, \lambda) = -\frac{\alpha \lambda \sin \mu \lambda}{2\mu} \sum_{\mathbf{G}}^{\varrho} c(\mathbf{q}-\mathbf{G}) \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+ + \left[\left(\frac{\mu}{2\theta n} - \frac{\pi l \mu}{\theta e B^*} \right) \sin \mu \lambda + \frac{\alpha \lambda \cos \mu \lambda}{2\theta} \right] \times \sum_{\mathbf{G}}^{\varrho} A(\mathbf{q}-\mathbf{G}) \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+. \quad (\text{A6})$$

The only term in the Hamiltonian (17) that is not expressed through the operators $A(\mathbf{q})$ and $c(\mathbf{q})$ is the CP kinetic energy $T = \sum_i^N \Pi_j^2 / 2m$. We will compute this operator in FR by deriving first the flow equation for it. First though we have to rearrange T , by using the canonical momentum commutator, into

$$T = \sum_j^N \frac{\Pi_j - \Pi_{j+}}{2m} + \sum_j^N \left(\frac{eB^*}{2m} - \frac{\pi l}{m} \delta n(\mathbf{r}_j) \right). \quad (\text{A7})$$

The second term in Eq. (A7) will not contribute to the flow equation after applying the RPA to it. It turns out to describe the magnetic moment of the CP. After doing the appropriate commutators we find that to first order in $\delta n/n$ the kinetic energy operator T_1 obeys the flow equation

$$\frac{dT_1(\lambda)}{d\lambda} = \frac{eB^* \theta}{m} \sum_{\mathbf{q}}^{\varrho} [A_1^\dagger(\mathbf{q}, \lambda) c_0(\mathbf{q}, \lambda) + A_0^\dagger(\mathbf{q}, \lambda) c_1(\mathbf{q}, \lambda) + \text{H.c.}] + \frac{eB^* \beta - 2\pi l \theta}{m} \times \sum_{\mathbf{q}}^{\varrho} \sum_{\mathbf{G}}^{\varrho} [A_0^\dagger(\mathbf{q}, \lambda) c_0(\mathbf{q}-\mathbf{G}, \lambda) \times \delta n(\mathbf{G}) \hat{q}_- (\widehat{q-G})_+ + \text{H.c.}]. \quad (\text{A8})$$

We can integrate the kinetic energy from Eq. (A8) using the initial condition $T_1(0) = 0$. The resulting expression for the kinetic energy in FR is substituted together with the operators $A(\mathbf{q}, \lambda)$ and $c(\mathbf{q}, \lambda)$ in FR into the Hamiltonian (17). We fix the constant β by requiring that there be no coupling between the particle and the oscillator degrees of freedom. That way we get

$$\beta = -\frac{\mu + (\mu^2 - 1) \arctan \mu}{4n^2 \mu^2 \sqrt{\pi l} \arctan \mu}. \quad (\text{A9})$$

The other consequences of this transformation are given in the main text.

APPENDIX B

To calculate the matrix elements $\langle n_1 | \tilde{\rho}_0(\mathbf{q}) | n_2 \rangle$ and $\langle n_1 | \tilde{\rho}_1(\mathbf{q}, \mathbf{G}) | n_2 \rangle$ one needs to know what the corresponding

matrix elements for the operators $e^{i\mathbf{q}\cdot\mathbf{r}}$, $\mathbf{q}\times\Pi^*e^{i\mathbf{q}\cdot\mathbf{r}}$, and $\mathbf{q}\times\Pi^*e^{i(\mathbf{q}-\mathbf{G})\cdot\mathbf{r}}$ are. We will give here only the final expressions for these matrix elements, as the first two were derived in several papers (see, for example, Ref. 8) and the third can be found using a similar approach. The matrix elements are

$$\begin{aligned} \langle n_1|e^{i\mathbf{q}\cdot\mathbf{r}}|n_2\rangle &= \sqrt{\frac{n_2!}{n_1!}} \left(\frac{i}{\sqrt{2}}(q_x+iq_y)l_0^* \right)^{n_1-n_2} \\ &\times L_{n_2}^{n_1-n_2}(q^2l_0^{*2}/2)e^{-q^2l_0^{*2}/4}, \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} l_0^{*2}\langle n_1|\mathbf{q}\times\Pi^*e^{i\mathbf{q}\cdot\mathbf{r}}|n_2\rangle &= i\sqrt{\frac{n_2!}{n_1!}} \left(\frac{i}{\sqrt{2}}(q_x+iq_y)l_0^* \right)^{n_1-n_2} \\ &\times [-L_{n_2}^{n_1-n_2}(q^2l_0^{*2}/2) \\ &-n_1L_{n_2-1}^{n_1-n_2}(q^2l_0^{*2}/2) \\ &+(n_2+1)L_{n_2+1}^{n_1-n_2}(q^2l_0^{*2}/2)] \\ &\times e^{-q^2l_0^{*2}/4}, \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} l_0^{*2}\langle n_1|\mathbf{q}\times\Pi^*e^{i(\mathbf{q}-\mathbf{G})\cdot\mathbf{r}}|n_2\rangle &= i\sqrt{\frac{n_2!}{n_1!}} \left(\frac{i}{\sqrt{2}}[q_x-G_x+i(q_y-G_y)]l_0^* \right)^{n_1-n_2} \left(\frac{\mathbf{q}\cdot(\mathbf{q}-\mathbf{G})}{(\mathbf{q}-\mathbf{G})^2} \right)^{n_1-n_2} \\ &\times \{ -L_{n_2}^{n_1-n_2}[(\mathbf{q}-\mathbf{G})^2l_0^{*2}/2] - n_1L_{n_2-1}^{n_1-n_2}[(\mathbf{q}-\mathbf{G})^2l_0^{*2}/2] + (n_2+1)L_{n_2+1}^{n_1-n_2}[(\mathbf{q}-\mathbf{G})^2l_0^{*2}/2] \} \\ &+ i \frac{\mathbf{q}\times\mathbf{G}}{(\mathbf{q}-\mathbf{G})^2} (n_1-n_2)L_{n_2}^{n_1-n_2}[(\mathbf{q}-\mathbf{G})^2l_0^{*2}/2] \exp[-(\mathbf{q}-\mathbf{G})^2l_0^{*2}/4]. \end{aligned} \quad (\text{B3})$$

APPENDIX C

In this appendix we will illustrate the calculation of the integrals over the momentum \mathbf{q} that appear in the HF Hamiltonian Eq. (39)–(46). As an example, we will take the integral that appears in the exchange contribution of the two-body, first order in δn term. Other integrals are done in a similar way. We choose to integrate the following term that is part of Eq. (44),

$$\begin{aligned} U_{tbe} &= \frac{g}{2S} \sum_{\mathbf{q}} V(\mathbf{q}) \langle 0|\tilde{\rho}_0(-\mathbf{q})|1\rangle \langle 0|\tilde{\rho}_1(\mathbf{q},\mathbf{G})|0\rangle \\ &\times \exp[-il_0^{*2}\mathbf{q}\times(\mathbf{G}+2\mathbf{G}_1)/2]. \end{aligned} \quad (\text{C1})$$

Using the formulas given in the Appendix B for the density-operator matrix elements, we find

$$\begin{aligned} U_{tbe} &= \frac{2\pi e^2 g}{2\epsilon} \int_0^\infty \frac{dq}{(2\pi)^2} \int_0^{2\pi} d\theta e^{-q^2\Lambda^2} \\ &\times \left(il_0^* q e^{-i\theta} \frac{q^2 l_0^{*2}}{2\sqrt{2}} e^{-q^2 l_0^{*2}/4} \right) \\ &\times \left(c^2 \frac{\mathbf{q}\cdot\mathbf{G}}{G^2} - \frac{c}{2(c+1)} (q^2 l_0^{*2} + \mathbf{q}\cdot\mathbf{G} l_0^{*2}) \right) \\ &\times \exp[-(\mathbf{q}-\mathbf{G})^2 l_0^{*2}/4] \exp[-il_0^{*2}\mathbf{q}\times(\mathbf{G}+2\mathbf{G}_1)/2]. \end{aligned} \quad (\text{C2})$$

Taking into account that $\mathbf{q}\cdot\mathbf{G} = q(G_-e^{i\theta} + G_+e^{-i\theta})/2$ and $\mathbf{q}\times\mathbf{G} = q(G_+e^{-i\theta} - G_-e^{i\theta})/2i$, where $G_+ = G_x + iG_y$ and $G_- = G_x - iG_y$, we get

$$\begin{aligned} U_{tbe} &= \frac{e^2 g}{2\epsilon l_0^*} \int_0^\infty dx \int_0^{2\pi} \frac{d\theta}{2\pi} \exp[-x^2(\Lambda^2/l_0^{*2} + 1/2)] \\ &\times \left(ie^{-i\theta} \frac{x^3}{2\sqrt{2}} \right) \left(c^2 \frac{x(G_-e^{i\theta} + G_+e^{-i\theta})}{2G^2 l_0^*} \right. \\ &\left. - \frac{c}{2(c+1)} [x^2 + x(G_-e^{i\theta} + G_+e^{-i\theta})l_0^*] \right) \\ &\times \exp\{[(G_- + G_+)e^{i\theta} - G_+e^{-i\theta}]l_0^* x/2\} e^{-l_0^{*2}G^2/4}, \end{aligned} \quad (\text{C3})$$

where $x = ql_0^*$. First we will integrate with respect to the variable x . We notice that it is possible to extend the interval of the integration over the whole real axis. The integrand in Eq. (C3) is such that the odd/even powers of x are multiplied by the $\exp(i\theta n)$ with n odd/even. Then reversing the sign of x and making a transformation $\theta' = \theta + \pi$ does not change the integrand while shifting the integration with respect to x interval to $(-\infty, 0)$. The integral is then

$$\begin{aligned} U_{tbe} &= \frac{ie^2 g}{4\epsilon l_0^*} \int_0^{2\pi} \frac{d\theta}{2\pi} \left[I_4(\alpha, \beta) \frac{(G_- + G_+e^{-2i\theta})}{4\sqrt{2}} \right. \\ &\times \left(\frac{c^2}{G^2 l_0^*} - \frac{cl_0^*}{c+1} \right) - I_5(\alpha, \beta) \frac{ce^{-i\theta}}{4\sqrt{2}(c+1)} \left. \right] e^{-l_0^{*2}G^2/4}. \end{aligned} \quad (\text{C4})$$

We introduced a notation for the Gaussian integral $I_n(\alpha, \beta) = \int_{-\infty}^\infty dx \exp(-\alpha x^2 + 2\beta x)x^n$ with $\alpha = \Lambda^2/l_0^{*2} + 1/2$ and β

$=[(G_- + G_{1-})e^{i\theta} - G_{1+}e^{-i\theta}]_0^*/4$. An important observation is that after Eq. (C4) is expanded, the result is the sum of the integrals of the form (note the even powers of $e^{i\theta}$ that appear)

$$\int_0^{2\pi} \exp[(ae^{i\theta} + be^{-i\theta})^2] e^{i2n\theta} \frac{d\theta}{2\pi} = e^{2ab} \left(\frac{b}{a}\right)^n I_{|n|}(2ab),$$

$$\int_0^{2\pi} e^{(ae^{i\theta})^2} e^{i2n\theta} \frac{d\theta}{2\pi} = \begin{cases} 0 & \text{if } n > 0 \\ \frac{a^{|n|}}{|n|!} & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} e^{(be^{-i\theta})^2} e^{i2n\theta} \frac{d\theta}{2\pi} = \begin{cases} 0 & \text{if } n < 0 \\ \frac{b^n}{n!} & \text{otherwise.} \end{cases} \quad (\text{C5})$$

Here $a = (G_- + G_{1-})$ and $b = -G_{1+}$. The second and third lines in Eq. (C5) are given because they are used to calculate other integrals. They hold if either one of a or b are zero. The final answer is then a series of the modified Bessel functions multiplied by appropriate constants.

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