Chern-Simons matrix model: Coherent states and relation to Laughlin wave functions

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(Received 11 June 2001; published 4 December 2001)

Using a coherent state representation we derive many-body probability distributions and wave functions for the Chern-Simons matrix model proposed by Polychronakos and compare them to the Laughlin ones. We analyze two different coherent state representations, corresponding to different choices for electron coordinate bases. In both cases we find that the resulting probability distributions do not quite agree with the Laughlin ones. There is agreement on the long distance behavior, but the short distance behavior is different.

DOI: 10.1103/PhysRevB.64.245316 PACS number(s): 73.43.−f, 02.10.Yn, 02.40.Gh

I. INTRODUCTION

Recently an interesting connection between the quantum Hall effect (QHE) and noncommutative field theory appeared. In particular Susskind proposed in Ref. 1 that the Laughlin states at filling fractions $\nu=1/(2p+1)$ for a system of an infinite number of electrons confined in the lowest Landau level (LLL) can be described by a noncommutative U(1) Chern-Simons theory. The fields of this theory are infinite matrices that act on an infinite Hilbert space, appropriate to account for an infinite number of electrons. In the same spirit, Polychronakos later proposed a finite matrix model,[2] a regularized version of this noncommutative Chern-Simons theory in an effort to describe finite systems of limited spatial extent with a finite number of electrons. This matrix model was shown to reproduce the basic features of the quantum Hall droplets and their corresponding excitations[5], such as boundary and quasihole excitations at filling fraction $\nu=1/(2p+1)$.

In a subsequent paper,[3] Hellerman and Raamsdonk, trying to make the connection between the quantum Hall effect and the noncommutative matrix model more transparent, analyzed the states of the theory and concluded that the states of the matrix model are in one-to-one correspondence with the Laughlin states describing the QHE at filling fraction $\nu=1/2p+1)$. Similar arguments were put forward for the excited states. Although this is an interesting observation, the existence of a one-to-one mapping between states is not enough to prove the equivalence of the two theories.

Since the mapping in Ref. 3 is somewhat formal at the level of states, in this paper we try to go one step further and compare the two theories at the level of the wave functions. This requires a notion of coordinates, which is introduced via a coherent state representation for the states of the matrix model. We will present two different ways of arriving at a coherent state representation, which are not equivalent in general. In the special case of $\nu=1$ $(\theta=0)$, it turns out that both coherent state representations of the corresponding matrix model reproduce the $\nu=1$ Laughlin wave function. For $\nu=1/(2p+1)$ though, the two ways produce different probability distributions and neither agrees with the Laughlin one. The probability distributions emerging from the different coherent state representations have a long distance behavior similar to the corresponding Laughlin one, but quite a different short distance behavior.

This paper is organized as follows. In Sec. II we give a brief discussion of fermions in the lowest Landau level and the Laughlin wave functions. In Sec. III we briefly review the Chern-Simons finite matrix model proposed by Polychronakos. In Sec. IV we present two different approaches towards a coherent state representation of the matrix states. In Sec. V we comment on our results.

II. FERMIONS IN THE LOWEST LANDAU LEVEL

It is well known by now that the two-dimensional configuration space of charged particles in the LLL is equivalent to a one-dimensional phase space and therefore noncommutative.[6,7] To see this let us consider a charged particle in the presence of a strong uniform magnetic field. The Lagrangian is given by

$$L=\frac{m}{2}\dot{\vec{x}}^2+\vec{A}\cdot\dot{\vec{x}}-V(x_1,x_2), \tag{1}$$

where $V(x_1,x_2)$ is an abritrary confining potential. For convenience we shall consider V to be a harmonic oscillator potential of the form

$$V=\frac{1}{2}\omega\vec{x}^2. \tag{2}$$

The energy eigestates of this system lie on Landau levels and at the limit $m\rightarrow 0$ (or equivalently strong B) the system is confined to the LLL. In this case the kinetic energy term in (1) is negligible. We further use the radial gauge $A_i=\frac{1}{2}B\epsilon_{ij}x_j$. The canonical momentum of x_1 is then $p_1=Bx_2$ and as a result

$$[x_1,x_2]=\left[x_1,\frac{p_1}{B}\right]=\frac{i}{B}. \tag{3}$$

In the absence of a confining potential all the eigenstates at each Landau level are degenerate. The confining potential lifts the degeneracy. In the lowest Landau level the Hamiltonian is

0163-1829/2001/64(24)/245316(8)/$20.00

$$H=\frac{\omega}{B}(a^\dagger a+\tfrac{1}{2}), \tag{4}$$

where $a=\sqrt{B/2}(x_1+ix_2)$. The single-particle eigenstates are

$$|n\rangle=\frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle. \tag{5}$$

The corresponding normalized wave functions are

$$\Psi_n=\sqrt{\frac{B}{2\pi n!}}\quad \bar{z}^n e^{-|z|^2/2}, \tag{6}$$

where $z=\sqrt{B/2}(x_1+ix_2)$. These can be thought of as the coherent state representation of Eq. (5).

Consider now the case of N fermions. Since each state can be occupied by at most one fermion, the presence of the confining potential selects a unique ground state which is the minimum angular momentum state. This is the $\nu=1$ ground state wave function

$$\Psi_1(\vec{x}_1,\ldots,\vec{x}_N)=\prod_{i<j}(\bar{z}_i-\bar{z}_j)\exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right). \tag{7}$$

This corresponds to an incompressible circular droplet configuration of uniform density $\rho=B/2\pi$. This incompressibility is crucial in explaining the experimentally observed gap for $\nu=1$. The existence of a similar gap is less obvious in the case of noninteger filling fractions, where the LLL is only partially filled. In this case, it is believed that the repulsive Coulomb interactions among electrons are important in generating strong correlations which eventually produce a new ground state with a gap. Laughlin proposed that such a state, in the case $\nu=1/(2p+1)$, is described by a wave function[8]

$$\Psi_{2p+1}(\vec{x}_1,\ldots,\vec{x}_N)=\prod_{i<j}^{N}(\bar{z}_i-\bar{z}_j)^{2p+1}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right). \tag{8}$$

Using the connection to the one-component, two-dimensional plasma, Laughlin showed that this corresponds to an incompressible droplet of density $\rho=B/[2\pi(2p+1)]$. Although not an exact eigenfunction, Eq. (8) is quite close to the true solution, at least numerically. This equation vanishes quite rapidly if any two particles approach each other, and this helps minimize the expectation value of the Coulomb energy. Its success lies very much on the short distance behavior.

Our aim is to develop an appropriate coherent state representation for the states of the corresponding matrix model and to compare them directly to the Laughlin wave functions. Before we explain this in more detail, we shall give a brief review of the matrix Chern-Simons model proposed by Polychronakos.

III. CHERN-SIMONS MATRIX MODEL

The action describing the matrix Chern-Simons model is given by[2]

$$S=\int dt\,\frac{B}{2}\mathrm{Tr}\{\epsilon_{ab}(\dot{X}_a+i[A_0,\,X_a])X_b+2\theta A_0-\omega X_a^2\}$$
$$+\Psi^\dagger(i\dot{\Psi}-A_0\Psi), \tag{9}$$

where X_a, $a=1,2$ are $N\times N$ matrices and Ψ is a complex N-vector that transforms in the fundamental of the gauge group U(N),

$$X_a\to UX_aU^{-1},\quad \Psi\to U\Psi. \tag{10}$$

The A_0 equation of motion implies the constraint

$$G\equiv -iB[X_1,X_2]+\Psi\Psi^\dagger-B\theta=0 \tag{11}$$

The trace of this equation gives

$$\Psi^\dagger\Psi=NB\theta. \tag{12}$$

It is interesting to note that in the absence of Ψ's, the parameter θ has to be zero. In this case the action (9) is that of a one-dimensional Hermitian matrix model in a harmonic potential, which is known to be equivalent to N one-dimensional fermions in a confining potential.[9]

Upon quantization the matrix elements of X_a and the components of Ψ become operators, obeying the following commutation relations:

$$[\Psi_i,\Psi_j^\dagger]=\delta_{ij},$$

$$[(X_1)_{ij},(X_2)_{kl}]=\frac{i}{B}\delta_{il}\delta_{jk}. \tag{13}$$

The Hamiltonian is

$$H=\omega\left(\frac{N^2}{2}+\sum A_{ij}^\dagger A_{ji}\right), \tag{14}$$

where $A=\sqrt{B/2}(X_1+iX_2)$. The system contains $N(N+1)$ oscillators coupled by the constraint (11). As explained in Ref. 2, upon quantization, the operator G becomes the generator of unitary rotations of both X_a and Ψ. The trace part (12) demands that $NB\theta$, being the number operator for Ψ's, is quantized to an integer. The traceless part of the constraint demands the physical states to be singlets of SU(N).

Since the $A_{ij}^\dagger$ transform in the adjoint and the $\Psi_i^\dagger$ transform in the fundamental representation of SU(N), a purely group theoretical argument implies that a physical state being a singlet has to contain Nn $\Psi^\dagger$'s, where n is an integer. This leads to the quantization of $B\theta=k$. We shall see later that in identifying this matrix model with a fermionic system, k has to be an even integer.

Explicit expressions for the states were written down in Ref. 3. (Essentially equivalent results were also obtained by Polychronakos.[4]) The ground state, being an SU(N) singlet with the lowest number of $A^\dagger$'s, is of the form

$$|\Psi\rangle=[\epsilon^{i_1,\ldots,i_N}\quad \Psi_{i_1}^\dagger(\Psi^\dagger A^\dagger)_{i_2}\cdots(\Psi^\dagger A^{\dagger N-1})_{i_N}]^k|0\rangle, \tag{15}$$

where $|0\rangle$ is annihilated by A's and Ψ's, while the excited states can be written as

$$|\Psi_{\rm exc}\rangle=\prod_{i=1}^{N-1}(\mathrm{Tr}A^{\dagger i})^{c_i}\times[\epsilon^{i_1,\ldots,i_N}\Psi^\dagger_{i_1}(\Psi^\dagger A^\dagger)_{i_2}\cdots(\Psi^\dagger A^{\dagger N-1})_{i_N}]^k|0\rangle. \quad (16)$$

IV. COHERENT STATE REPRESENTATIONS

We now present a coherent state representation for the matrix states (15) and (16). In doing this one has to make a choice of coordinates that eventually parametrize the phase space of the underlying one-dimensional system.

From the matrix model point of view, a natural choice would be to diagonalize A and interpret its eigenvalues as phase space coordinates of particles. We refer to this as the A representation. This is essentially a generalized complex random matrix model.

Another choice would be to diagonalize X_1 and interpret its eigenvalues as one-dimensional coordinates. This is the X representation. In this, the elements of X_2 are canonically conjugate to X_1. Once the X representation of the matrix states (15) and (16) is derived, we can then use the usual coherent state representation to express the wave functions in terms of phase space coordinates.

Clearly in the two cases, the A and X representations, the notion of phase space coordinates is different; as a result we derive different expressions for the corresponding wave functions. Below we present in detail the two representations and compare the results to the Laughlin wave functions.

A. A representation

We define the coherent state $|Z,\phi\rangle$ such that

$$A_{mn}|Z,\phi\rangle=Z_{mn}|Z,\phi\rangle,$$
$$\Psi_n|Z,\phi\rangle=\phi_n|Z,\phi\rangle, \quad (17)$$

where Z is a complex $N\times N$ matrix and ϕ is a complex vector. Let us consider the matrix ground state of the form

$$|2p+1\rangle=[\epsilon^{i_1,\ldots,i_N}\Psi^\dagger_{i_1}(\Psi^\dagger A^\dagger)_{i_2}\cdots(\Psi^\dagger A^{\dagger N-1})_{i_N}]^{2p}|0\rangle \quad (18)$$

and reexpress its scalar product in terms of the coherent state wave functions using the completeness relation

$$\langle 2p+1|2p+1\rangle=\int\prod_{i,j,l}^{N}dZ_{ij}dZ^*_{ij}d\phi_l d\bar{\phi}_l\ \langle 2p+1|Z,\phi\rangle\times\langle Z,\phi|2p+1\rangle. \quad (19)$$

Before we talk about the general case, it is interesting to demonstrate how the coherent state representation works for $p=0$. The scalar product (19) can be written as

$$\langle 1|1\rangle=\int\prod_{i,j,l}^{N}dZ_{ij}dZ^*_{ij}e^{-\mathrm{Tr}\,Z^\dagger Z}\int d\phi_l d\bar{\phi}_l e^{-\bar{\phi}\phi}. \quad (20)$$

Since the ϕ integration is trivial, Eq. (20) defines essentially a complex random matrix model with probability distribution $e^{-\mathrm{Tr}\,Z^\dagger Z}$. A detailed analysis of this model has been given in Ref. 10.

Z is a complex matrix that can be diagonalized as (nondiagonalizable matrices form a set of measure zero)

$$Z=XEX^{-1}, \quad (21)$$

where E is diagonal with $E_{ii}=z_i$. Integration over the nondiagonal part of Z in Eq. (20) gives the following result,[10]

$$\langle 1|1\rangle=\int\prod_{i=1}^{N}dz_i d\bar{z}_i e^{-|z_i|^2}\prod_{i<j}|z_i-z_j|^2, \quad (22)$$

up to normalization factors. One can immediatelly recognize the integrand of Eq. (22) as the probability distribution corresponding to the $\nu=1$ Laughlin wave function. In the A representation we have identified the phase space coordinates of the fermions with the eigenvalues of the matrix Z in Eq. (17).

We would like now to extend this approach in the case $p\neq 0$ in the presence of the extra Ψ degrees of freedom. This can be viewed as a generalized complex random matrix model. (An attempt for a random matrix formulation of the Laughlin theory of the fractional QHE has been proposed by Callaway.[11]) As shown in Ref. 10 any complex regular $N\times N$ matrix X can be expressed in one and only one way as

$$X=UYV, \quad (23)$$

where U is a unitary matrix, Y is a triangular matrix with $Y_{ij}=0$ for $i>j$ and $Y_{ii}=1$, and V is a diagonal matrix with real positive diagonal elements. Using this particular parametrization we find

$$\begin{aligned}\langle 2p+1|Z,\phi\rangle&=[\epsilon^{i_1,\ldots,i_N}\phi_{i_1}(Z\phi)_{i_2}(Z^2\phi)_{i_2}\cdots(Z^{N-1}\phi)_{i_N}]^{2p}e^{-\mathrm{Tr}\,Z^\dagger Z/2}e^{-\bar{\phi}\phi/2}\\&=[\epsilon^{i_1,\ldots,i_N}\phi_{i_1}(UYEY^{-1}U^{-1}\phi)_{i_2}(UYE^2Y^{-1}U^{-1}\phi)_{i_2}\cdots(UYE^{N-1}Y^{-1}U^{-1}\phi)_{i_N}]^{2p}e^{-\mathrm{Tr}\,Z^\dagger Z/2}e^{-\bar{\phi}\phi/2}\\&=[\det(UY)]^{2p}\prod_{i<j}(z_i-z_j)^{2p}\prod_{i=1}^{N}\xi_i^{2p}e^{-\mathrm{Tr}\,Z^\dagger Z/2}e^{-\bar{\phi}\phi/2},\end{aligned} \quad (24)$$

where $\xi=(UY)^{-1}\phi$.

Further using Eqs. (21) and (23) we have

$$\mathrm{Tr}\ Z^{\dagger}Z=\mathrm{Tr}(\bar{E}HEH^{-1}), \tag{25}$$

where $H=Y^{\dagger}Y$ and $\det H=1$ ($\det Y=1$).

In performing the integration over ϕ's it is convenient to change variables from ϕ to ξ. Since U is unitary and $\det Y=1$,

$$\prod_{l=1}^{N} d\bar{\phi}_l d\phi_l e^{-\bar{\phi}\phi}=\prod_{l=1}^{N} d\bar{\xi}_l d\xi_l e^{-\bar{\xi}H\xi}. \tag{26}$$

Further following Ref. 10, one can show that

$$dZ_{ij}dZ^{*}_{ij}=\prod_{i} dz_i d\bar{z}_i\prod_{i<j} |z_i-z_j|^4\prod_{i\neq j} dH_{ij}$$
$$\equiv d\mu(z,H)\prod_{i<j} |z_i-z_j|^4 \tag{27}$$

up to normalization factors. Putting everything together we find

$$\langle 2p+1|2p+1\rangle$$
$$=\int d\mu(z,H)\prod_{i<j} |z_i-z_j|^{4p+4}e^{-\mathrm{Tr}(\bar{E}HEH^{-1})}d\bar{\xi}_i d\xi_i$$
$$\times\prod_{i=1}^{N} (\bar{\xi}_i\xi_i)^{2p}e^{-\xi^{\dagger}H\xi}. \tag{28}$$

Integration over ξ and H would produce a quantity that depends only on z_i and $\bar{z}_i$. It is clear from the above expression that in the absence of the ξ integration, the integration over the nondiagonal elements of Z would produce a probability distribution similar to the corresponding Laughlin one. Since, however, ξ's couple to H and H in turn couples to z_i and $\bar{z}_i$, this integration will not necessarily produce a probability distribution that agrees with the Laughlin one. The integration over ξ and H is much more involved now and it is hard to extract a closed expression for arbitrary N. However, as we shall see one can find general features of the probability distribution that do not agree with what one would expect from the Laughlin distribution.

We first perform the ξ integration by introducing a source term as follows:

$$\int d\xi_i d\bar{\xi}_i\prod_{i=1}^{N} (\xi_i\bar{\xi}_i)^{2p}e^{-\bar{\xi}H\xi}$$
$$=\prod_{i=1}^{N} \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial \bar{J}_i}\right)^{2p}\int d\xi_i d\bar{\xi}_i e^{-\bar{\xi}H\xi+\bar{J}\xi+J\bar{\xi}}\Big|_{J,\bar{J}=0}$$
$$=\prod_{i=1}^{N} \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial \bar{J}_i}\right)^{2p} e^{\bar{J}H^{-1}J}\Big|_{J,\bar{J}=0}. \tag{29}$$

Equation (28) can now be written as

$$\langle 2p+1|2p+1\rangle$$
$$=\int \prod_{i} dz_i d\bar{z}_i\prod_{i<j} |z_i-z_j|^{4p+4}\prod_{i=1}^{N} \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial \bar{J}_i}\right)^{2p}$$
$$\times\int \prod_{i\neq j} dH_{ij}e^{-\mathrm{Tr}(\bar{E}HEH^{-1})+\bar{J}H^{-1}J}\Big|_{J,\bar{J}=0}. \tag{30}$$

We are now left with the H integration

$$I=\int \prod_{1\leqslant i\neq j\leqslant N} dH_{ij}e^{-\mathrm{Tr}(\bar{E}HEH^{-1})}e^{\bar{J}H^{-1}J}. \tag{31}$$

To do this we follow an iterative procedure as in Ref. 10. At each step of the iteration we integrate over the variables of the last row and column of H and thus decrease by one the size of the matrix. In the absence of the source terms, the structure of the reduced matrix remains the same and this produces a simple recursion formula, which eventually leads to Eq. (22). This is not quite the case when there are source terms; as a result there is no simple recursion formula.

The iteration procedure is defined as follows: Let H', E', etc., be the relevant matrices of order n and let H, E, be those obtained from H', E' by removing the last row and last column. Greek (Latin) indices run from 1 to n ($n-1$). Let $\Delta'_{\alpha\beta}$ be the cofactor of $H'_{\alpha\beta}$ in H' and Δ_{ij} the cofactor of H_{ij} in H. Let $g_i=H'_{in}$. Because $\det H'=\det H=1$, the following relations are true:[10]

$$\Delta'_{\alpha\beta}=H'^{-1}_{\beta\alpha},\quad \Delta_{ij}=H^{-1}_{ji},$$

$$H'_{nn}=1+\sum_{i,j} g^{*}_i g_j\Delta_{ji},$$

$$\Delta'_{in}=-\sum_{l} \Delta_{il}g^{*}_l,$$

$$\Delta'_{ij}=H'_{nn}\Delta_{ij}-\sum_{l,k} g^{*}_k g_l\Delta^{lk}_{ij},$$

$$\Delta^{lk}_{ij}=\Delta_{ij}\Delta_{lk}-\Delta_{ik}\Delta_{lj}. \tag{32}$$

Let

$$\Phi^{J}_{n}=\mathrm{Tr}(\bar{E}'H'E'H'^{-1})-\bar{J}'H'^{-1}J'\equiv\Phi^{0}_{n}-\bar{J}'H'^{-1}J'. \tag{33}$$

Using Eq. (32) we find

$$\Phi^{J}_{N}=|z_N|^2+\Phi^{J}_{N-1}-J_N\bar{J}_N+\langle g|H^{-1}(\bar{E}-\bar{z}_N)$$
$$\times H(E-z_N)H^{-1}|g\rangle-\langle g|H^{-1}J\bar{J}H^{-1}|g\rangle$$
$$+g^{*}_l(H^{-1}J)_l\bar{J}_N+J_N(\bar{J}H^{-1})_l g_l. \tag{34}$$

Integration over g's produces the following result:

$$I \sim \int \prod_{1 \leq i \neq j \leq N-1} dH_{ij} e^{-|z_N|^2} e^{-\Phi^J_{N-1}} e^{J_N \bar{J}_N} \times \frac{\exp\left[\left(\sum \bar{J}_i X^{-1}_{ij} J_j\right) J_N \bar{J}_N\right]}{\det(X_{ij} - J_i \bar{J}_j)}, \tag{35}$$

where

$$X_{ij} = (H^{-1}(\bar{E} - \bar{z}_N) H (E - z_N) H^{-1})_{ij} - J_i \bar{J}_j = X^0_{ij} - J_i \bar{J}_j. \tag{36}$$

The expression (35) can be further simplified using the following relations:

$$\bar{J}_i (X^0 - J\bar{J})^{-1}_{ij} J_j = \bar{J}_i ((X^0)^{-1} + (X^0)^{-1} J \bar{J} (X^0)^{-1} + \cdots)_{ij} J_j = \frac{\bar{J}_i (X^0)^{-1}_{ij} J_j}{1 - \bar{J}_i (X^0)^{-1}_{ij} J_j}. \tag{37}$$

Furthermore

$$\begin{aligned}\det(X^0 - J\bar{J}) &= e^{\mathrm{Tr}\ln(X^0 - J\bar{J})} \\ &= e^{\mathrm{Tr}\{\ln X^0 + \ln[1 - (X^0)^{-1} J \bar{J}]\}} \\ &= \det(X^0) e^{\ln \bar{J}(X^0)^{-1} J} \\ &= \prod_{i<N} |z_i - z_N|^2 \sum_{1 \leq l,m \leq N-1} \bar{J}_l (X^0)^{-1}_{lm} J_m,\end{aligned} \tag{38}$$

where we have used the fact that

$$\det X^0 = \det[H^{-1}(\bar{E} - \bar{z}_N) H (E - z_N) H^{-1}] = \prod_{i<N} |z_i - z_N|^2. \tag{39}$$

Substituting Eqs. (37) and (38) in Eq. (35) we find

$$I \sim \frac{e^{-|z_N|^2}}{\prod_{i<N} |z_i - z_N|^2} \int \prod_{1 \leq i \neq j \leq N-1} dH_{ij} \frac{e^{-\Phi^J_{N-1}}}{(1-M)} e^{J_N \bar{J}_N/(1-M)}, \tag{40}$$

where

$$M = \sum_{1 \leq i,j \leq N-1} \bar{J}_i (X^0)^{-1}_{ij} J_j = \sum_{1 \leq i,j \leq N-1} \frac{\bar{J}_i}{(z_i - z_N)} H^{-1}_{ij} \frac{J_j}{(\bar{z}_j - \bar{z}_N)}. \tag{41}$$

Already this result can be used to evaluate Eq. (28) for the simple case of a 2×2 matrix model. Although this is a rather trivial case, it is worth presenting it, since it highlights properties of the probability distribution that are not in agreement with the Laughlin one.

To simplify the calculation let us further choose $p=1$. For the $N=2$ case after the first iteration we find from Eq. (40)

$$I \sim \frac{e^{-|z_1|^2} e^{-|z_2|^2}}{|z_1 - z_2|^2} \frac{e^{J_1 \bar{J}_1}}{1-M} e^{J_2 \bar{J}_2/(1-M)}, \tag{42}$$

where $M = 1 - J_1 \bar{J}_1 / |z_1 - z_2|^2$. The functional diferentiation with respect to J's, Eq. (29), gives

$$\begin{aligned}&\left(\frac{\partial}{\partial J_1} \frac{\partial}{\partial \bar{J}_1}\right)^2 \left(\frac{\partial}{\partial J_2} \frac{\partial}{\partial \bar{J}_2}\right)^2 I\Big|_{J,\bar{J}=0} \\ &= 2 \frac{e^{-|z_1|^2} e^{-|z_2|^2}}{|z_1 - z_2|^2} \left(\frac{\partial}{\partial J_1} \frac{\partial}{\partial \bar{J}_1}\right)^2 \times \frac{e^{J_1 \bar{J}_1}}{(1 - J_1 \bar{J}_1 / |z_1 - z_2|^2)^3}\Bigg|_{J_1, \bar{J}_1 = 0} \\ &= \frac{e^{-|z_1|^2} e^{-|z_2|^2}}{|z_1 - z_2|^2} \left[4 + \frac{24}{|z_1 - z_2|^2} + \frac{48}{|z_1 - z_2|^4}\right].\end{aligned} \tag{43}$$

Using Eqs. (28), (29), and (43) we find

$$\langle 3|3 \rangle \sim \int dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 e^{-|z_1|^2 - |z_2|^2} |z_1 - z_2|^6 \times \left[1 + \frac{6}{|z_1 - z_2|^2} + \frac{12}{|z_1 - z_2|^4}\right]. \tag{44}$$

The first term corresponds to the probability distribution for the $\nu = 1/3$ ground state Laughlin wave function. There are extra terms, though, that are dominant at short distances as $z_1 \to z_2$. In this simple case we find that the distribution emerging from the matrix model has a long distance behavior similar to the corresponding Laughlin one, but its short distance behavior is quite different. We shall now argue that this behavior prevails for any N.

It is clear from Eq. (40) that the first step of the iteration produces an expression that is not quite similar to the original one if $J's \neq 0$. As a result the integration over H'_{in} at each subsequent step of the iteration becomes quite involved. Although it is very hard to derive an exact expression for I as a function of z_i's, it is quite straightforward to explore its dependence on $|z_{N-1} - z_N|$. This is sufficient, for example, to get information about the short and long distance behavior of the probability distribution as $|z_{N-1} - z_N| \ll 1$ and $|z_{N-1} - z_N| \gg 1$, respectively.

The dependence of I on $|z_{N-1} - z_N|$ comes first from the overall factor $1/\Pi_{i<N} |z_i - z_N|^2$ and second from the factor M; see Eqs. (40) and (41). Expanding M we find

$$\begin{aligned}M = &\frac{\bar{J}_{N-1} \bar{J}_{N-1}}{|z_{N-1} - z_N|^2} + \frac{\bar{J}_{N-1}}{(z_{N-1} - z_N)} \sum_{j<N-1} H^{-1}_{N-1,j} \frac{J_j}{(\bar{z}_j - \bar{z}_N)} \\ &+ \sum_{i<N-1} \frac{\bar{J}_i}{(z_i - z_N)} H^{-1}_{i,N-1} \frac{J_{N-1}}{(\bar{z}_{N-1} - \bar{z}_N)} \\ &+ \sum_{1 \leq i,j \leq N-2} \frac{\bar{J}_i}{(z_i - z_N)} H^{-1}_{ij} \frac{J_j}{(\bar{z}_j - \bar{z}_N)}.\end{aligned} \tag{45}$$

It is clear now that in order to explicitly demonstrate the $|z_N - z_{N-1}|$ dependence we need to evaluate the functional derivatives with respect to J_N and J_{N-1}, as in Eq. (30). We find

$$\prod_i \left(\frac{\partial}{\partial J_i}\frac{\partial}{\partial \bar{J}_i}\right)^2 I|_{J,\bar{J}=0} = \frac{\exp\left(-\sum_i |z_i|^2\right)}{\prod_{i<N}|z_i - z_N|^2}\Bigg[A + \frac{B}{z_{N-1}-z_N} + \frac{B^*}{\bar{z}_{N-1}-\bar{z}_N} + \frac{C}{|z_{N-1}-z_N|^2} + \frac{D}{(z_{N-1}-z_N)^2} + \frac{D^*}{(\bar{z}_{N-1}-\bar{z}_N)^2} + \frac{1}{|z_{N-1}-z_N|^2}\times\left(\frac{E}{z_{N-1}-z_N} + \frac{\bar{E}}{\bar{z}_{N-1}-\bar{z}_N}\right) + \frac{F}{|z_{N-1}-z_N|^4}\Bigg], \tag{46}$$

where $A,\ B,\ C,\ D,\ E$, and F are in general functions of z's containing factors of the form $(z_i - z_N)$, $(z_i - z_{N-1})$, $(z_i - z_j)$, and their complex conjugates, where $i \leqslant N-2$. Substituting this expression in Eq. (30), we find a probability distribution that is dominated by a term proportional to $|z_N - z_{N-1}|^6$ when $|z_N - z_{N-1}| \gg 1$, similar to the $\nu = 1/3$ Laughlin distribution. However, when $|z_N - z_{N-1}| \ll 1$ the matrix distribution is dominated by a term proportional to $|z_N - z_{N-1}|^2$. The long and short distance behavior for any N is the same as the one found in the simple $N=2$ case, Eq. (44).

It is straightforward now to see that similar results can be derived for any $p \neq 0$. In particular, the long distance behavior of the probability distribution is that of the $\nu = 1/(2p+1)$ Laughlin distribution while the short distance behavior is that of a $\nu = 1$ Laughlin distribution.

Further the probability distribution cannot be factorized as $\Psi^*\Psi$, where Ψ is the corresponding many-particle wave function, which is both antisymmetric and holomorphic in z's. This indicates that the identification of the eigenvalues of the matrix Z with the actual holomorphic coordinates of fermions may not be appropriate.

B. *X* representation

Going to an X representation first, we derive one-dimensional fermionic wave functions, where the coordinates have been identified with the eigenvalues of the matrix X_1, then transform the wave functions to the coherent state representation. We define the state $|X,\phi\rangle$ such that

$$\hat{X}_1|X,\phi\rangle = X|X,\phi\rangle, \quad \Psi|X,\phi\rangle = \phi|X,\phi\rangle. \tag{47}$$

Using Eq. (47) we find

$$\langle 2p+1|X,\phi\rangle = [\epsilon^{i_1\cdots i_N}\phi_{i_1}(A\phi)_{i_2} \times\cdots(A^{N-1}\phi)_{i_N}]^{2p}\langle 0|X,\phi\rangle, \tag{48}$$

where

$$\langle 0|X,\phi\rangle = e^{-\mathrm{Tr}(BX^2/2)}e^{-\bar{\phi}\phi/2},$$

$$A_{ij} = \sqrt{\frac{B}{2}}\left(X_{ij} - \frac{1}{B}\frac{\partial}{\partial X_{ji}}\right). \tag{49}$$

Since Eq. (48) is completely antisymmetric in the i_n indices, the differential operator $\partial/\partial X_{ji}$ produces a nonzero contribution only if it acts on the ground state wave function $\langle 0|X,\phi\rangle$. We then find

$$\langle 2p+1|X,\phi\rangle = (\sqrt{2B})^{pN(N-1)}[\epsilon^{i_1,\ldots,i_N}\phi_{i_1}(X\phi)_{i_2} \times\cdots(X^{N-1}\phi)_{i_N}]^{2p}e^{-\mathrm{Tr}(BX^2/2)}e^{-\bar{\phi}\phi/2}. \tag{50}$$

X being a Hermitian matrix, it can be diagonalized by a unitary transformation

$$X = UxU^{-1}, \quad x_{ij} = x_i\delta_{ij}. \tag{51}$$

Using this in Eq. (50) we find

$$\begin{aligned}\langle 2p+1|X,\phi\rangle &= (\sqrt{2B})^{pN(N-1)}[\epsilon^{i_1,\ldots,i_N}\phi_{i_1}(UxU^{-1}\phi)_{i_2} \times\cdots(Ux^{N-1}U^{-1}\phi)_{i_N}]^{2p}e^{-\mathrm{Tr}(BX^2/2)}e^{-\bar{\phi}\phi/2}\\ &= (\sqrt{2B})^{pN(N-1)}[\det U]^{2p} \times\prod_{i<j}(x_i - x_j)^{2p}\prod_{i=1}^{N}(U^{-1}\phi)_i^{2p}e^{-\mathrm{Tr}(BX^2/2)}e^{-\bar{\phi}\phi/2}.\end{aligned} \tag{52}$$

Using Eq. (52) we can express the scalar product of the $|2p+1\rangle$ state as

$$\begin{aligned}\langle 2p+1|2p+1\rangle &= \int [dX_{ij}]d\bar{\phi}_l d\phi_l\langle 2p+1|X,\phi\rangle\langle X,\phi|2p+1\rangle\\ &\sim \int [dX_{ij}]\prod_{i<j}(x_i - x_j)^{4p}e^{-\mathrm{Tr}BX^2}d\bar{\phi}_l d\phi_l \times\prod_i [(U^{-1}\phi)_i(\bar{\phi}U)_i]^{2p}e^{-\bar{\phi}\phi}.\end{aligned} \tag{53}$$

Since U is a unitary matrix the integration over ϕ's completely decouples [unlike the case in Eq. (28)]. Further,[9]

$$[dX_{ij}] = dx_i\prod_{i<j}(x_i - x_j)^2[dU], \tag{54}$$

where $[dU]$ is the Haar measure. Integration over the nondiagonal elements of X gives

$$\langle 2p+1|2p+1\rangle \sim \int dx_i \prod_{i<j} (x_i-x_j)^{4p+2} \exp\left(-B\sum_i x_i^2\right). \tag{55}$$

This is the probability distribution for the one-dimensional Calogero ground state wave function

$$\prod_{i<j} (x_i-x_j)^{2p+1} \exp\left(-B\sum_i x_i^2/2\right).$$

This is not surprising; it was already indicated in Refs. 2 and 12 that the Chern-Simons matrix model is equivalent to the Calogero model.

Let us now use a coherent state representation for the Calogero wave function

$$\langle x_1, \ldots, x_N|\Psi\rangle = \prod_{i<j} (x_i-x_j)^{2p+1} \exp\left(-B\sum_i x_i^2/2\right).$$

The coherent state representation of any wave function $\langle x|\Psi\rangle$ can be written as

$$\langle z|\Psi\rangle = \int dx \langle z|x\rangle\langle x|\Psi\rangle, \tag{56}$$

where $\hat{z}|z\rangle = z|z\rangle$ and $\hat{z} = \sqrt{B/2}(\hat{x}+i\hat{y})$, $[\hat{x}, \hat{y}] = i/B$. Using

$$\langle z|x\rangle = e^{-Bx^2/2} e^{\sqrt{2B}\bar{z}x} e^{-\bar{z}^2/2} e^{-|z|^2/2}, \tag{57}$$

we find

$$\langle z|\Psi\rangle = \int dx\, e^{-B(x-\bar{z}/\sqrt{2B})^2} f(x), \tag{58}$$

where $f(x)$ is given by $\langle x|\Psi\rangle = f(x) e^{-Bx^2/2}$. In evaluating (58) we expand $f(x)$ around $\bar{z}/\sqrt{2B}$.

$$\begin{aligned}\langle z|\Psi\rangle &= e^{-|z|^2/2} \sum_{k=0}^{\infty} \int dx\, e^{-B(x-\bar{z}/\sqrt{2B})^2} \times \frac{1}{(2k)!}\left(x - \frac{\bar{z}}{\sqrt{2B}}\right)^{2k} \left.\frac{\partial^{2k} f}{\partial x^{2k}}\right|_{x=\bar{z}/\sqrt{2B}} \\ &= e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)}{(2k)!B^{k+1/2}} \left.\frac{\partial^{2k} f}{\partial x^{2k}}\right|_{x=\bar{z}/\sqrt{2B}} \\ &= \sqrt{\frac{\pi}{B}} e^{-|z|^2/2} \left[\exp\left(\frac{1}{4B}\frac{\partial^2}{\partial x^2}\right) f(x)\right]_{x=\bar{z}/\sqrt{2B}}.\end{aligned} \tag{59}$$

The coherent state representation of the many-body Calogero wavefunction is thus

$$\langle z_1, \ldots, z_N|\Psi\rangle \sim \exp\left(-\sum_i |z_i|^2/2\right)\left[\exp\left(\frac{1}{4B}\sum_i \frac{\partial^2}{\partial x_i^2}\right) \times \prod_{i<j} (x_i-x_j)^{2p+1}\right]_{x_i=\bar{z}_i/\sqrt{2B}}. \tag{60}$$

For $p=0$, Eq. (60) gives the $\nu=1$ Laughlin state. For $p \neq 0$ the coherent representation of the Calogero state has a long distance behavior similar to the $\nu=1/(2p+1)$ Laughlin state, but a different short distance behavior. This connection between the one-dimensional Calogero wave function and the Laughlin states has already been noted in Ref. 13.

V. SUMMARY AND DISCUSSION

In an attempt to clarify the exact correspondence between the Chern-Simons matrix model introduced by Polychronakos and the fractional QHE at filling fraction $\nu=1/m$, as described by Laughlin wave functions, we have derived the matrix model wave functions using a coherent state representation. We have presented two different coherent state representations, each one implementing a different choice for the phase space coordinates of the underlying one-dimensional fermionic system.

In the A representation, the eigenvalues of the matrix A are identified with the phase space coordinates z of the fermions, while in the X representation, the eigenvalues of the matrix X_1 are identified with the one-dimensional coordinates x of the fermions.

Both choices give identical results when $p=0$, or equivalently $\theta=0$ in Eq. (9). The corresponding wave function is identical to the $\nu=1$ Laughlin wave function.

For $p \neq 0$ the two representations give different results. Although the explicit expressions for the probability distributions are different, they share some common features. They both have the same long and short distance behavior. Comparing them to the corresponding $\nu=1/(2p+1)$ Laughlin distributions, we find that it is only the long distance behavior that is in agreement. The short distance behavior does not agree with the Laughlin one.

A noticeable difference between the two representations is that the X representation leads to a holomorphic wave function with antisymmetric properties, while this is not possible for the A representation.

As we mentioned earlier there is an ambiguity in introducing electron coordinates in the matrix model. This has to do with the choice of coherent state representation. In this paper we analyzed two particular coherent state representations, which seem to be the natural choices from the matrix model point of view. In both cases we find that the emerging wave functions do not quite agree with the Laughlin one. Although this by itself does not prove that the original matrix model is not equivalent to the Laughlin theory for the $\nu=1/m$ fractional QHE, it makes the precise correspondence between the two models less transparent. One can argue that there may exist another coherent state representation, corresponding to another choice for the electron coordinates, such that there is agreement with the Laughlin wave function. However, for the matrix model to be truly useful in the context of QHE, this new coordinate choice should be easily identifiable.

ACKNOWLEDGMENTS

We would like to thank A.P. Polychronakos for many useful discussions on the matrix model and its relation to the Calogero model. Discussions with V.P. Nair are also acknowledged. This work was supported in part by the NSF Grant No. PHY-9970724 and PSC-31 grant from CUNY.

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