Mesoscopic circuits with charge discreteness: Quantum transmission lines

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We propose a quantum Hamiltonian for a transmission line with charge discreteness. The periodic line is composed of an inductance and a capacitance per cell. In every cell the charge operator satisfies a nonlinear equation of motion because of the discreteness of the charge. In the basis of one energy per site, the spectrum can be calculated explicitly. The incorporation of electrical resistance in the line has been considered briefly.

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I. INTRODUCTION: *LC* QUANTUM CIRCUITS WITH CHARGE DISCRETENESS

Nowadays, technological miniaturization of circuits is increasing and their mesoscopic aspects become more relevant. Recently, a theory for mesoscopic circuits was proposed by Li and Chen[1] where charge discreteness was considered explicitly. This is very much related to miniaturization since the number of charges in these systems is expected to be more and more reduced. In the *LC* circuit studied in Ref. 1, the Hamiltonian operator was given by

$$\hat{H}=\hat{T}+\hat{V}, \tag{1}$$

where $V(\hat{Q})=\hat{Q}^2/2C$, with $\hat{Q}$ the charge operator, is the electrical energy in the capacitance C and the magnetic-energy operator term $\hat{T}$, related to the inductance L, was given by

$$\hat{T}=\frac{2\hbar^2}{Lq_e^2}\left[\sin^2\left(\frac{Lq_e}{2\hbar}k\right)\right], \tag{2}$$

in k representation (or pseudocurrent representation, $0<k<2\pi$). The above operator has some resemblance with a mechanical kinetic operator in the limit of small k. In Eq. (2) the constant q_e represents the elementary charge in the electric system. In this representation, the charge operator $\hat{Q}$ is given by

$$\hat{Q}=\frac{i\hbar}{L}\frac{\partial}{\partial k}, \tag{3}$$

and the corresponding eigenfunctions are $e^{inLq_e k/\hbar}$ with discrete eigenvalues nq_e where n is an integer. Since the current operator $\hat{I}$ is formally obtained from $\hat{I}=[1/i\hbar][\hat{H},\hat{Q}]$ then, in the k representation, it becomes

$$\hat{I}=\frac{\hbar}{Lq_e}\sin\left(\frac{Lq_e}{\hbar}k\right). \tag{4}$$

In the limit $q_e\to 0$ all these operators become the usual ones associated with a quantum *LC* circuit with continuous charge.[3] Moreover, the current operator (4) is bounded with extrema values $\pm(\hbar/Lq_e)$. After Ref. 1, the above Hamiltonian describes phenomena such as persistent current, Coulomb blockage and others.

As pointed out in Ref. 2, the above set of operators defines an algebra with some similitude to this related to space-time discreteness.[4–6] In fact, the charge-current commutator $[\hat{Q},\hat{I}]$, usually proportional to the identity, becomes modified with a kinetic term that is zero when $q_e=0$. So, space and charge discreteness could be described with the same mathematical tools.

On the other hand, dissipation is a subject very much related to electric circuits. This phenomenon defines a decoherence time related to mesoscopic aspect. Dissipation can be considered in many forms, the usual one is to connect the system to a bath with many degrees of freedom, and with some assumptions with respect to the decaying rate.[7] Phenomenological *RLC* circuits can be also considered using Caldirola-Kanai theory[8–10] in a direct way.[11]

In this paper we are interested in a quantum transmission line. Such transmission lines are usually used in mesoscopic physics. For instance, in Ref. 12, a quantum transmission line with continuous charge was considered and connected to a metal ring. It was quantized and used as an environment to study zero-point fluctuations influence on a metal ring. Particularly, we shall consider a quantum transmission line with charge discreteness. For this purpose we shall use the ideas discussed above. In this way, we consider a periodic transmission line composed of cells. Every cell has an inductance L and a capacitance C.

In Sec. II, we consider briefly the classical transmission line that will be quantized by standard procedure. In Sec. III, we present the quantized Hamiltonian that contains explicitly charge discreteness. Also in that section, we write the motion equation of the charge in the line using Heinsenberg equation of motion. In Sec. IV, in the one energy per site approximation, we find the spectral properties of the systems. In Sec. V, we study briefly the incorporation of electrical resistance in the quantum transmission line with charge discreteness. In the Sec. VI, conclusions are touched.

II. CLASSICAL TRANSMISSION LINE

In this section we shall consider a periodic transmission line composed in every cell of an inductance L and a capacitance C. Classically, the evolution equation for the continuous charge Q_l, at cell l $(l\in Z)$ in the transmission line, is given by the expression

$$L\frac{d^2}{dt^2}Q_l=\frac{1}{C}(Q_{l+1}+Q_{l-1}-2Q_l), \tag{5}$$

which can be obtained from the classical Hamiltonian given by

$$H_{\text{clas}}=\sum_l \frac{\phi_l^2}{2L}+\frac{1}{2C}(Q_{l+1}-Q_l)^2, \tag{6}$$

where the variable ϕ_l corresponds to the magnetic flux in the inductance at position l and it is proportional to the classical current in the cell. It is explicitly given by

$$\phi_l=L\frac{dQ_l}{dt}. \tag{7}$$

The integer l in Eq. (5) represents the index of the cell at position l. The first term depending on the current in Eq. (6) is the equivalent to the kinetic energy of a mechanical system and is related to the stored magnetic energy in the inductance. In the quadratic potential, the crossed term $(Q_{l+1}Q_l/C)$ represents the interaction term between cells. In fact, for continuous variables, the above Hamiltonian is equivalent to this one of mechanical vibrations and could be quantized directly. By canonical transformation, the classical Hamiltonian (6) could be transformed to normal modes and its spectral frequencies are well known. For us, the important fact is that the crossed term in Eq. (6) defines the interaction between two consecutive sites or cells. This interaction term will be preserved in the quantization process.

III. A HAMILTONIAN FOR QUANTUM TRANSMISSION LINES WITH CHARGE DISCRETENESS

Due to charge discreteness in the quantum case, the structural changes are only expected to occur in the kinetic part of Eq. (6) where the usual quadratic term is transformed in a trigonometric function (Sec. I). Then, the quantum transmission line with charge discreteness can be quantized directly. So, from the classical Hamiltonian (6), and Eq. (2) and (3), the quantum Hamiltonian for the transmission line with charge discreteness is

$$\hat{H}=\sum_l\left\{\frac{2\hbar^2}{Lq_e^2}\sin^2\left(\frac{Lq_e}{2\hbar}k_l\right)-\frac{\hbar^2}{2L^2C}\left(\frac{\partial}{\partial k_{l+1}}-\frac{\partial}{\partial k_l}\right)^2\right\}, \tag{8}$$

where the quantity k_l corresponds to the pseudocurrent in the cell at position l and varies between $0<k_l<2\pi$. As pointed before, the kinetic part is not quadratic in k_l, which is very much related to the charge discreteness assumptions ($q_e\neq 0$). The study of the Hamiltonian (8) is the purpose of this work. Remark that in the limit $q_e\to 0$ the kinetic part becomes proportional to k_l^2 with formal similitude to the case of mechanical vibrations (phonons). This corresponds to the continuous charge case and the name *circuitons* for these propagating modes is appropriate. Nevertheless, *circuitons* are the limit case with zero charge discreteness $q_e\to 0$. In a general context, since the above Hamiltonian describes the quantization of the line and the quantization of the charge we shall call *cirquitons* the normal modes (propagating modes) of the above Hamiltonian. This appellation seems appropriate because it remembers: (i) the quantization process and (ii) the discreteness of the charge (q_e). The existence of these propagating modes is ensured since the Hamiltonian (8) is invariant under (discrete) spatial traslation.

The motion equation related to the quantum transmission line can be found with the usual quantum mechanical evolution rules (Heisenberg equations). The pseudocurrent k_l has associated the canonical conjugate operator of the charge, namely, the operator $\hat{K}_l$ with eigenvalues k_l. It satisfies the canonical commutation relation

$$L[\hat{Q}_l,\hat{K}_l]=i\hbar. \tag{9}$$

As function of this operator, the Hamiltonian of the transmission line becomes

$$\hat{H}=\sum_l\left\{\frac{2\hbar^2}{Lq_e^2}\sin^2\left(\frac{Lq_e}{2\hbar}\hat{K}_l\right)+\frac{1}{2C}(\hat{Q}_{l+1}-\hat{Q}_l)^2\right\}, \tag{10}$$

and the evolution equation for the charge operator in the Heinsenberg representation $(d/dt)\hat{Q}_l=(i/\hbar)[\hat{H},\hat{Q}_l]$ can be computed explicitly

$$\frac{d}{dt}\hat{Q}_l=\frac{\hbar}{q_eL}\sin\left(\frac{Lq_e}{\hbar}\hat{K}_l\right). \tag{11}$$

Which is similar to the nonlinear expression (4) for every cell. In fact, Eq. (11) defines the current operator and it is bounded such as in the case mentioned in Sec. I. The motion equation for the pseudocurrent operator $(d/dt)\hat{K}_l=(i/\hbar)[\hat{H},\hat{K}_l]$ is

$$\frac{d}{dt}\hat{K}_l=\frac{1}{CL}(\hat{Q}_{l+1}+\hat{Q}_{l-1}-2\hat{Q}_l). \tag{12}$$

In Eqs. (11) and (12), we notice that the formal limit $q_e\to 0$ gives the usual linear motion equation for transmission lines with similitude to the classical one (5). Nevertheless, in the general case, the evolution equation is nonlinear.

IV. ONE ENERGY PER SITE: EXCITATIONS ON THE TRANSMISSION LINE

The study of the spectral properties of the Hamiltonian (8), or Eq. (10), is difficult because of the nonlinear term associated with the magnetic energy. Without charge discreteness ($q_e\to 0$), this operator is quadratic and the usual normal modes technique could be used.[13] In this section we solve the spectral properties on the vector basis of the system without interaction. We shall find a particular spectral solution in the basis of one energy per site or cell.

The Hamiltonian (10) can be written as

$$\hat{H}=\sum_l \hat{H}_l-\frac{1}{C}\hat{Q}_{l+1}\hat{Q}_l, \tag{13}$$

where the site-Hamiltonian $\hat{H}_l$ corresponds to this one of a LC circuit (1) with capacitance $C/2$. Since the total Hilbert space, where the Hamiltonian (13) acts, is the direct product of the spaces associated to every site l, then we consider the sub-basis $\{|l\rangle, l \in Z\}$, where every element $|l\rangle$ is an eigenstate of the Hamiltonian $\hat{H}_l$ with energy V. Namely,

$$\hat{H}_l|l\rangle = V|l\rangle. \tag{14}$$

Since $\langle n|l\rangle = \delta_{l,n}$ and the operator $\hat{Q}_l$ acts only in the site l, the complete Hamiltonian $\hat{H}$ is Hermitian and becomes tridiagonal in this basis. Explicitly, it is given by the matrix elements

$$\langle l|\hat{H}|l\rangle = V \quad \text{and} \quad \langle l|\hat{H}|l+1\rangle = -\alpha\frac{q_e^2}{C}, \tag{15}$$

where α is a dimensionless constant that we keep as one $(\alpha=1)$. Formally, this can be carried out by an adequate normalization. Noticed that the off-diagonal terms in Eq. (15) are related to the interacting term in Eq. (13), namely, $(1/C)\hat{Q}_{l+1}\hat{Q}_l$ and then it was expected to be proportional to q_e^2.

The LC energy stored in a cell is now spread into a band due to charge interaction with the two neighboring cells. The spectrum of the tridiagonal Hamiltonian is well known and corresponds to the so-called tight-binding approximations in solid state physics. In fact, assuming a general state of the form $|\psi\rangle = \Sigma\psi_l|l\rangle$, the Schrödinger equation related to the tridiagonal Hamiltonian (15) becomes

$$E\psi_l = V\psi_l - \frac{q_e^2}{C}(\psi_{l+1}+\psi_{l-1}), \tag{16}$$

with E the energy. The eigenstates are of the form $\psi_l = e^{i\theta l}$, where the phase θ is a real number $(0<\theta<2\pi)$, and the spectrum becomes

$$E = V - \frac{2q_e^2}{C}\cos\theta, \tag{17}$$

where we have considered an infinite transmission line (i.e., θ is a continuous variable). The above expression defines the density of states and the thermodynamics properties of these lines could be calculated.

The spectrum (17) defines an ensemble of quantum excitations on the transmission line, nevertheless, they are not the more general because they are only defined in the subspace spanned by $\{|l\rangle, l \in Z\}$ in the complete Hilbert space. For instance, others kind of excitations could be found if we consider two energies by site, or different energies by site.

V. TRANSMISSION LINES WITH RESISTANCE

The Hamiltonian (10) describes a quantum transmission line with charge discreteness, but it does not consider dissipation. On the other hand, electrical resistance (Ohm law) is an intrinsic phenomenon in electrical conduction. After Bateman's work,[14] classical linear dissipation could be studied in a Lagrangian way by consider a time-decaying exponential factor multiplying the Langrangian function. From this, the classical Hamiltonian could be written in the standard way. So, quantization becomes attainable from the usual correspondence between position momentum and its associated operators.[8–10] In our case, a similar procedure could be implemented and we obtain the time-dependent Hamiltonian

$$\hat{H}(t) = \sum_l \left\{ e^{(-R/L)t}\frac{2\hbar^2}{Lq_e^2}\sin^2\left(\frac{Lq_e}{2\hbar}\hat{K}_l\right) + e^{(R/L)t}\frac{1}{2C}(\hat{Q}_{l+1}-\hat{Q}_l)^2\right\}, \tag{18}$$

where the constant R represents the resistance. With the above Hamiltonian, and the Heinsenberg motion equations, we obtain for the charge operator

$$\frac{d}{dt}\hat{Q}_l = \frac{\hbar}{q_e L}e^{(-R/L)t}\sin\left(\frac{Lq_e}{\hbar}\hat{K}_l\right), \tag{19}$$

and for the pseudocurrent operator

$$\frac{d}{dt}\hat{K}_l = \frac{1}{CL}e^{(R/L)t}(\hat{Q}_{l+1}+\hat{Q}_{l-1}-2\hat{Q}_l). \tag{20}$$

From Eqs. (19) and (20), the equation for the variation of the charge becomes

$$L\frac{d^2}{dt^2}\hat{Q}_l = -R\frac{d}{dt}\hat{Q}_l + \frac{1}{C}\left[\cos\left(\frac{Lq_e}{\hbar}\hat{K}_l\right)\right](\hat{Q}_{l+1}+\hat{Q}_{l-1}-2\hat{Q}_l), \tag{21}$$

which in the formal limit $q_e \to 0$ becomes the usual one incorporating a resistance R in every cell of the transmission line. So, the time-dependent Hamiltonian (18) describes the dynamics of a quantum transmission line with charge discreteness and resistance.

VI. CONCLUSIONS AND DISCUSSIONS

We have proposed a quantum Hamiltonian for a transmission line composed of a periodic array of inductances and capacitances with charge discreteness [Eq. (8) or Eq. (10)]. To construct this Hamiltonian we have used the charge-discreteness procedure proposed in Ref. 1 for a LC circuit. In our case, the corresponding Hilbert space is given by the tensorial product of this one of every cell, corresponding to a LC circuit. In the particular basis of one energy per site, or cell, we have found the spectrum of the line (17).

Note that charge discreteness produces nonlinear terms in the equation of motion in the transmission line Eqs. [(11) and (12)]. The incorporation of electrical resistance was considered by using the Caldirola-Kanai theory (Sec. V).

As a final remark we note that disorder systems are usually studied in solid state physics[15] and it is well known that localization of states could exist. In our case, we believe that disorder can be incorporated in the line by considering for instance, every inductance as a random quantity. In this case, it seems interesting to study the role of disorder and the nonlinearity due to charge discreteness. On the other hand, it is known that decoherence effects break localization,[16,17] then the role of environment and resistance on the transmission line must also break localization.

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