Quasiparticle Hall transport of *d*-wave superconductors in the vortex state

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We present a theory of quasiparticle Hall transport in strongly type-II superconductors within their vortex state. We establish the existence of integer quantum spin Hall effect in clean unconventional $d_{x^2-y^2}$ superconductors in the vortex state from a general analysis of the Bogoliubov–de Gennes equation. The spin Hall conductivity σ^s_{xy} is shown to be quantized in units of $\hbar/8\pi$. This result does not rest on linearization of the BdG equations around Dirac nodes and therefore includes inter-nodal physics in its entirety. In addition, this result holds for a generic inversion-symmetric lattice of vortices as long as the magnetic field B satisfies $H_{c1}\ll B\ll H_{c2}$. We then derive the Wiedemann–Franz law for the spin and thermal Hall conductivity in the vortex state. In the limit of $T\rightarrow 0$, the thermal Hall conductivity satisfies $\kappa_{xy}=(4\pi^2/3)(k_B/\hbar)^2 T\sigma^s_{xy}$. The transitions between different quantized values of σ^s_{xy} as well as relation to conventional superconductors are discussed.

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I. INTRODUCTION

One of the fundamental characteristics of high temperature superconductors (HTS) is the apparent applicability of the *d*-wave[1] BCS based phenomenology to the broad range of quasiparticle properties in the superconducting state.[2] This is far from trivial property for materials known to exhibit strong electron correlations. Although the horizon is still not entirely clear and there remain few unresolved issues, examples being the temperature dependence of quasiparticle lifetimes or the penetration depth in the underdoped regime,[2,3] the cumulative weight of evidence indicates that the low energy properties of cuprate superconductors are indeed governed by nodal quasiparticles with Dirac-like dispersion, as seen in assorted spectroscopic[4] and transport measurements.[5] These and other experiments serve as the foundation for the "BCS-like *d*-wave paradigm" as it is currently used in both theory and interpretation of experiments.

A natural question is how does this picture hold together in a mixed phase, in the presence of an external magnetic field and an array of superconducting vortices and, if it does, are there some special features of the *d*-wave quasiparticle phenomenology which could be used to deepen our understanding of high temperature superconductivity? Recent activity on the experimental front appears most encouraging in this regard. In particular, measurements of the thermal Hall conductivity κ_{xy} in cuprate superconductors are currently viewed as especially informative probes of quasiparticle dynamics. These measurements provide a clean way of extracting quasiparticle contribution to κ_{xy} since phonons, the other source of significant thermal conduction, do not couple to the magnetic field by virtue of being neutral. In addition, contrary to what takes place in an ordinary electrical Hall conductivity experiment, vortices do not experience strong Lorenz force since there is only heat current and no net electrical supercurrent to which vortices are strongly coupled. Thus, vortices tend to remain stationary and their transport does not serve as a significant channel for heat conduction.

Recently, measurements of κ_{xy} were conducted by Ong's group[6] on YBCO samples with a very long mean free path. These experiments were carried out over a wide range of magnetic fields (up to $\sim$14 T) and at temperatures from $T\sim 12.5$ K to above the superconducting transition $T_c \sim 90$ K. Unfortunately, the experiments are resolution limited below 12.5 K as signal becomes too weak. At temperatures up to 25 K and for a (T dependent) range of magnetic fields $\sqrt{H/Tesla}<T/25\text{K}$ the experiments seem to suggest a rather simple scaling form[7] for κ_{xy}:

$$\kappa_{xy}(H,T)=\text{const.}\times\sqrt{H}T. \tag{1.1}$$

We will return later to this simple scaling form and its possible relation to the theory presented here.

On the theoretical front, the initial interest, largely inspired by Gorkov and Schrieffer[8] and, in a somewhat different context, by Anderson,[9] was directed at the formation of "Dirac Landau levels" and their signatures in the quasiparticle thermodynamics and transport.[10,11] This picture of Dirac Landau level quantization, while theoretically elegant and appealing, relies on the minimal coupling of the nodal BCS-like quasiparticles to the electromagnetic vector potential, **A**. The assumption of such minimal coupling seems innocent but it is not, on fundamental grounds. The physics behind the interaction of nodal quasiparticles with the external magnetic field and vortices was elucidated by Franz and Tešanović.[12] These authors devised a singular gauge transformation which allows one to recast the original problem of BCS-like quasiparticles, moving under the combined influence of an external magnetic field and a superflow arising from vortex array, into that of Bloch particles moving in an effective nonuniform and periodic magnetic field, the spatial average of which equals zero. This approach clearly demonstrates that the low energy portion of the quasiparticle spectrum can be described as that of a relativistic Dirac particle minimally coupled to a fictitious $U(1)$ gauge potential, i.e., the "Berry" gauge field **a**, which supplies the needed $\pm\pi$ winding in the quantum mechanical phase of a quasiparticle as it encircles a vortex. Such half-flux Aharonov–Bohm scattering arises entirely through interaction of quasiparticles with vortices and it does not involve the external magnetic field explicitly.

Thus, the cyclotron motion in a Dirac cone is caused exclusively by a *time-reversal invariant* "Berry" gauge field and cannot lead to any Dirac Landau level quantization. Further progress came through the work of Marinelli, Halperin, and Simon[13] who analyzed the quasiparticle excitation spectra of different vortex lattices and provided analytic symmetry arguments regarding the presence of nodal (zero energy) points in such spectra. These authors also devised a perturbation theory in the vicinity of nodal points which can be used to derive various results by analytic means. Various analytic results were also derived in the large anisotropy limit.[14,15] Finally, the intricacies of Dirac equation in presence of half-flux Bohm–Aharonov scattering were addressed in Ref. 16, where the tight-binding regularization was introduced to restore the exact singular gauge symmetry in numerical calculations using the low energy (linearized) theory. Such regularization describes a lattice *d*-wave superconductor and is a natural choice for cuprates: after all, most of the microscopic theories of HTS start from tight-binding effective Hamiltonians. We should note that similar results for the quasiparticle spectra were also found in fully self-consistent calculations on the original Bogoliubov–deGennes (BdG) equations, using the basis formed by eigenstates of the magnetic translation group.[17,18]

In this paper, we present a detailed analytical study of the quasiparticle Hall transport in a vortex state complemented by explicit numerical calculations.[19] We consider a lattice *d*-wave superconductor of Ref. 16. This is important and necessary since the straightforward linearization of BdG equations drops curvature terms and results in $\kappa_{xy}=0$.[7,20] We employ the Franz and Tešanović (FT) transformation so that we can use the familiar Bloch representation of the translation group in which the overall chirality of the problem vanishes. This should be contrasted with the original problem where the overall chirality is finite and the magnetic translation group states must be used instead. Naively, it might appear that after an FT singular gauge transformation the effects of the magnetic field have somehow been transformed away since the new problem is found to have zero average effective magnetic field. Of course, this is not true. The presence of magnetic field in the original problem reveals itself fully in the FT transformed quasiparticle wave functions. Alternatively, there is an "intrinsic" chirality imposed on the system which cannot be transformed away by a choice of the basis. One manifestation of this chirality is the Hall effect. The utility of the singular gauge transformation in the calculation of the electrical Hall conductivity in the *normal* 2D electron gas in a (nonuniform) magnetic field was realized by Nielsen and Hedegård.[21] They demonstrated that using singularly gauge transformed wave functions one still obtains the correct result, giving the electrical Hall conductance quantized in units of e^2/h if the chemical potential lies in the energy gap. In a superconductor, the question of Hall response becomes rather interesting as there is a strong mixing between particles and holes. Evidently, the electrical Hall response is very different from the normal state, since charge is not conserved in the state with broken $U(1)$ symmetry. Therefore, as pointed out in Ref. 22, charge cannot be transported by diffusion. On the other hand, the spin is still a good quantum number[22] and it is natural to ask what is the spin Hall conductivity in the vortex state of an extreme type-II superconductor.[23] Moreover, every channel of spin conduction simultaneously transports entropy[22,24,19] and we would expect some variation on Wiedemann–Franz law to hold between spin and thermal conductivity.

As one of our main results, we derive the Wiedemann–Franz law connecting the spin and thermal Hall transport in the *vortex state* of a *d*-wave superconductor. In the process, we show that the spin Hall conductivity, σ_{xy}^s, just like the electrical Hall conductivity of a normal state in a magnetic field, is topological in nature and can be explicitly evaluated as a first Chern number characterizing the eigenstates of our singularly gauge transformed problem.[25,23,26] Consequently, as $T\rightarrow 0$, the spin Hall conductivity is quantized in the units of $\hbar/8\pi$ when the energy spectrum is gapped, which, combined with the Wiedemann–Franz law, implies the quantization of κ_{xy}/T. We then explicitly compute the quantized values of σ_{xy}^s for a sequence of gapped states using our lattice *d*-wave superconductor model in the case of an inversion-symmetric vortex lattice. Within this model one is naturally led to consider the *BCS–Hofstadter* problem: the BCS pairing problem defined on a uniformly frustrated tight-binding lattice. We find a sequence of plateau transitions, separating gapped states characterized by different quantized values of σ_{xy}^s. At a plateau transition, level crossings take place and σ_{xy}^s changes by an even integer.[27] Both the origin of the gaps in quasiparticle spectra and the sequence of values for σ_{xy}^s are rather different than in the normal state, i.e., in the standard Hofstadter problem.[28] In a superconductor, the gaps are strongly affected by the pairing and the interactions of quasiparticles with a vortex array. The sequence of σ_{xy}^s changes as a function of the pairing strength (and therefore interactions), measured by the maximum value of the gap function Δ.[29] Finally, we discuss the relation of our results to those obtained within the continuum low energy (linearized) theory.

II. BOGOLIUBOV–DeGENNES HAMILTONIAN

The experimental evidence points toward well defined *d*-wave quasiparticles in cuprate superconductors in the absence of the external magnetic field. This suggests that to zeroth order fluctuations can be ignored and that one can think in terms of an effective BCS Hamiltonian, the simplest of which is written on the 2D tight-binding lattice with the nearest neighbor interaction thus naturally implementing $d_{x^2-y^2}$ pairing. In question is then the response of such a superconductor to an externally applied magnetic field **B**. All high temperature superconductors are extreme type-II forming a vortex state in a wide range of magnetic fields. This immediately sets up the contrast between $\mathbf{B}=0$ and $\mathbf{B}\neq 0$ situations: first, the problem is not spatially uniform and therefore momentum is not a good quantum number and second, the array of $(hc/2e)$ vortex fluxes poses topological constraint on the quasiparticles encircling the vortices. Therefore, despite ignoring any fluctuations, the problem is far from trivial and demands careful examination.

The natural starting point is therefore the mean-field BCS Hamiltonian written in second quantized form:[30]

$$H=\int d\mathbf{x}\psi_\alpha^\dagger(\mathbf{x})\left(\frac{1}{2m^*}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-\mu\right)\psi_\alpha(\mathbf{x})+\int d\mathbf{x}\int d\mathbf{y}[\Delta(\mathbf{x},\mathbf{y})\psi_\uparrow^\dagger(\mathbf{x})\psi_\downarrow^\dagger(\mathbf{y})+\Delta^*(\mathbf{x},\mathbf{y})\psi_\downarrow(\mathbf{y})\psi_\uparrow(\mathbf{x})], \tag{2.1}$$

where $\mathbf{A}(\mathbf{x})$ is the vector potential associated with the uniform external magnetic field $\mathbf{B}$, single electron energy is measured relative to the chemical potential μ, $\psi_\alpha(\mathbf{x})$ is the fermion field operator with spin index α, and $\Delta(\mathbf{x},\mathbf{y})$ is the pairing field. For convenience we will define an integral operator $\hat{\Delta}$ such that:

$$\hat{\Delta}\psi(\mathbf{x})=\int d\mathbf{y}\Delta(\mathbf{x},\mathbf{y})\psi(\mathbf{y}). \tag{2.2}$$

In the strictest sense, on the mean-field level this problem must be solved self-consistently which renders any analytical solution virtually intractable. On the other hand, in the case at hand the vortex lattice is dilute for a wide range of magnetic fields, and by the very nature of cuprate superconductors having short coherence length, the size of the vortex core can be ignored relative to the distance between the vortices. Thus, to the first approximation, all essential physics is captured by fixing the amplitude of the order parameter Δ while allowing vortex defects in its phase. Moreover, on a tight-binding lattice the vortex flux is concentrated inside the plaquette and thus the length-scale associated with the core is implicitly the lattice spacing $\boldsymbol{\delta}$ of the underlying tight-binding lattice. As shown in Ref. 16, under these approximations the *d*-wave pairing operator in the vortex state can be written as a differential operator:

$$\hat{\Delta}=\Delta_0\sum_{\boldsymbol{\delta}}\eta_{\boldsymbol{\delta}}\, e^{i\phi(\mathbf{x})/2}e^{i\boldsymbol{\delta}\cdot\mathbf{p}}e^{i\phi(\mathbf{x})/2}. \tag{2.3}$$

The sums are over nearest neighbors and on the square tight-binding lattice $\boldsymbol{\delta}=\pm\hat{x},\pm\hat{y}$; the vortex phase fields satisfy $\nabla\times\nabla\phi(\mathbf{x})=2\pi\hat{z}\Sigma_i\delta(\mathbf{x}-\mathbf{x}_i)$ with $\mathbf{x}_i$ denoting the vortex positions and $\delta(\mathbf{x}-\mathbf{x}_i)$ being a 2D Dirac delta function; $\mathbf{p}$ is a momentum operator, and

$$\eta_{\boldsymbol{\delta}}=\begin{cases}1 & \text{if } \boldsymbol{\delta}=\pm\hat{x}\\ -1 & \text{if } \boldsymbol{\delta}=\pm\hat{y}.\end{cases} \tag{2.4}$$

The operator $\eta_{\boldsymbol{\delta}}$ follows from the *d*-wave pairing: $\Delta=2\Delta_0[\cos(k_x\delta_x)-\cos(k_y\delta_y)]$. For notational convenience we will use units where $\hbar=1$ and return to the conventional units when necessary.

It is straightforward to derive the continuum version of the tight binding lattice operator $\hat{\Delta}$ (see Ref. 16):

$$\hat{\Delta}=\frac{1}{2p_F^2}\{\partial_x,\{\partial_x,\Delta(\mathbf{x})\}\}-\frac{1}{2p_F^2}\{\partial_y,\{\partial_y,\Delta(\mathbf{x})\}\}+\frac{i}{8p_F^2}\Delta(\mathbf{x})\times[(\partial_x^2\phi)-(\partial_y^2\phi)], \tag{2.5}$$

but for convenience we will keep the lattice definition Eq. (2.3) throughout. One can always define continuum as an appropriate limit of the tight-binding lattice theory. With the above definitions, the Hamiltonian Eq. (2.1) can now be written in the Nambu formalism as

$$H=\int d\mathbf{x}\Psi^\dagger(\mathbf{x})\hat{H}_0\Psi(\mathbf{x}), \tag{2.6}$$

where the Nambu spinor $\Psi^\dagger=(\psi_\uparrow^\dagger,\psi_\downarrow)$ and the matrix differential operator

$$\hat{H}_0=\begin{pmatrix}\hat{h} & \hat{\Delta}\\ \hat{\Delta}^* & -\hat{h}^*\end{pmatrix}. \tag{2.7}$$

In the continuum formulation $\hat{h}=(1/2m^*)(p-(e/c)\mathbf{A})^2-\mu$, while on the tight-binding lattice:

$$\hat{h}=-t\sum_{\boldsymbol{\delta}} e^{i\int_{\mathbf{x}}^{\mathbf{x}+\boldsymbol{\delta}}(\mathbf{p}-\frac{e}{c}\mathbf{A})\cdot d\mathbf{l}}-\mu. \tag{2.8}$$

t is the hopping constant and μ is the chemical potential. The equations of motion of the Nambu fields Ψ are then:

$$i\hbar\dot{\Psi}=[\Psi,H]=\hat{H}_0\Psi. \tag{2.9}$$

Note, that the Hamiltonian in Eq. (2.1) is our starting *unperturbed* Hamiltonian. In order to compute the linear response to externally applied perturbations we will have to add terms to Eq. (2.1). In particular, we will consider two types of perturbations in the later sections: First, partly for theoretical convenience, we will consider a weak gradient of magnetic field ($\nabla\mathbf{B}$) on top of the uniform $\mathbf{B}$ already taken into account fully by Eq. (2.1). The $\nabla\mathbf{B}$ term induces spin current in the superconductor.[31] The response is then characterized by spin conductivity tensor σ^s which in general has nonzero off-diagonal components. Second, we consider perturbing the system by pseudo-gravitational field, which formally induces flow of energy (see Refs. 32–35) and allows us to compute thermal conductivity κ_{xy} via linear response. The advantage of these formal considerations are made clear in Sec. IV.

A. Particle-hole symmetry

The equations of motion (2.9) for stationary states lead to Bogoliubov–de Gennes equations[16]

$$\hat{H}_0\Phi_n=\epsilon_n\Phi_n. \tag{2.10}$$

The solution of these coupled differential equations are quasi-particle wave functions that are rank two spinors in the Nambu space, $\Phi^T(\mathbf{r})=(u(\mathbf{r}),v(\mathbf{r}))$. The single particle excitations of the system are completely specified once the quasiparticle wave functions are given, and as discussed later, transverse transport coefficients can be computed solely on the basis of Φ's. It is a general symmetry of the BdG equations that if $(u_n(\mathbf{r}),v_n(\mathbf{r}))$ is a solution with energy ϵ_n, then there is always another solution $(-v_n^*(\mathbf{r}),u_n^*(\mathbf{r}))$ with energy $-\epsilon_n$ (see, for example, Ref. 30).

In addition, on the tight-binding lattice, if the chemical potential $\mu=0$ in the above BdG Hamiltonian Eq. (2.7), then there is a *particle-hole* symmetry in the following sense: if $(u_n(\mathbf{r}),v_n(\mathbf{r}))$ is a solution with energy ϵ_n, then there is always another solution $e^{i\pi(r_x+r_y)}(u_n(\mathbf{r}),v_n(\mathbf{r}))$ with energy $-\epsilon_n$. Thus we can choose:

$$\begin{pmatrix} u_n^{(-)}(\mathbf{r}) \\ v_n^{(-)}(\mathbf{r}) \end{pmatrix} = e^{i\pi(r_x+r_y)} \begin{pmatrix} u_n^{(+)}(\mathbf{r}) \\ v_n^{(+)}(\mathbf{r}) \end{pmatrix}, \tag{2.11}$$

where + $(-)$ corresponds to a solution with positive (negative) energy eigenvalue. We will refer to this as particle hole transformation $\hat{P}_H$.

B. Franz–Tešanović transformation and translation symmetry

In order to elucidate another important symmetry of the Hamiltonian Eq. (2.7), we follow FT[12,16] and perform a "bipartite" singular gauge transformation on the Bogoliubov–de Gennes Hamiltonian Eq. (2.10),

$$\hat{H}_0 \to U^{-1}\hat{H}_0 U, \quad U=\begin{pmatrix} e^{i\phi_e(\mathbf{r})} & 0 \\ 0 & e^{-i\phi_h(\mathbf{r})} \end{pmatrix}, \tag{2.12}$$

where $\phi_e(\mathbf{r})$ and $\phi_h(\mathbf{r})$ are two auxiliary vortex phase functions satisfying

$$\phi_e(\mathbf{r})+\phi_h(\mathbf{r})=\phi(\mathbf{r}). \tag{2.13}$$

This transformation eliminates the phase of the order parameter from the pairing term of the Hamiltonian. The phase fields $\phi_e(\mathbf{r})$ and $\phi_h(\mathbf{r})$ can be chosen in a way that avoids multiple valuedness of the wave functions. The way to accomplish this is to assign the singular part of the phase field generated by any given vortex to either $\phi_e(\mathbf{r})$ or $\phi_h(\mathbf{r})$, but not both. Physically, a vortex assigned to $\phi_e(\mathbf{r})$ will be seen by electrons and be invisible to holes, while vortex assigned to $\phi_h(\mathbf{r})$ will be seen by holes and be invisible to electrons. For periodic Abrikosov vortex array, we implement the above transformation by dividing vortices into two groups A and B, positioned at $\{\mathbf{r}_i^A\}$ and $\{\mathbf{r}_i^B\}$, respectively (see Fig. 1). We then define two phase fields $\phi^A(\mathbf{r})$ and $\phi^B(\mathbf{r})$ such that

$$\nabla\times\nabla\phi^\alpha(\mathbf{r})=2\pi\hat{z}\sum_i \delta(\mathbf{r}-\mathbf{r}_i^\alpha), \quad \alpha=A,B, \tag{2.14}$$

and identify $\phi_e=\phi^A$ and $\phi_h=\phi^B$. On the tight-binding lattice the transformed Hamiltonian becomes

$$\hat{H}_N=\sum_{\boldsymbol{\delta}} \{\sigma_3(-te^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}(\mathbf{a}-\sigma_3\mathbf{v})\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot\mathbf{p}}-\mu) + \sigma_1\Delta_0\eta_{\boldsymbol{\delta}}e^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}\mathbf{a}\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot\mathbf{p}}\}, \tag{2.15}$$

where

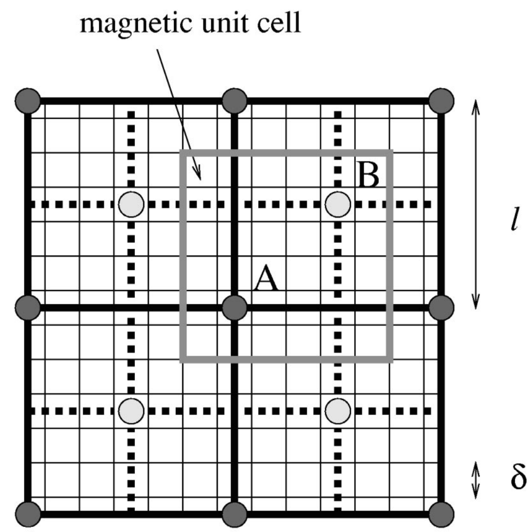

FIG. 1. Example of A and B sublattices for the square vortex arrangement. The underlying tight-binding lattice, on which the electrons and holes are allowed to move, is also indicated.

$$\mathbf{v}=\tfrac{1}{2}\nabla\phi-\frac{e}{c}\mathbf{A}; \quad \mathbf{a}=\tfrac{1}{2}(\nabla\phi^A-\nabla\phi^B), \tag{2.16}$$

σ_1 and σ_3 are Pauli matrices operating in Nambu space, and the sum is again over the nearest neighbors. Note that the integrand of Eq. (2.15) is proportional to the superfluid velocities

$$\mathbf{v}_s^\alpha=\frac{1}{m^*}\left(\nabla\phi^\alpha-\frac{e}{c}\mathbf{A}\right), \quad \alpha=A,B \tag{2.17}$$

and is therefore explicitly gauge invariant as are the off-diagonal pairing terms.

From the perspective of quasiparticles $\mathbf{v}_s^A$ and $\mathbf{v}_s^B$ can be thought of as *effective* vector potentials acting on electrons and holes, respectively. Corresponding effective magnetic field seen by the quasiparticles is $\mathbf{B}_{\rm eff}^\alpha=-(m^*c/e)(\boldsymbol{\nabla}\times\mathbf{v}_s^\alpha)$, and can be expressed using Eqs. (2.14) and (2.15) as

$$\mathbf{B}_{\rm eff}^\alpha=\mathbf{B}-\phi_0\hat{z}\sum_i \delta(\mathbf{r}-\mathbf{r}_i^\alpha), \quad \alpha=A,B, \tag{2.18}$$

where $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ is the physical magnetic field and $\phi_0=hc/e$ is the flux quantum. We observe that quasi-electrons and quasi-holes propagate in the effective field which consists of (almost) uniform physical magnetic field $\mathbf{B}$ and an array of opposing delta function "spikes" of unit fluxes associated with vortex singularities. The latter are different for electrons and holes. As discussed in Refs. 12,16 this choice guarantees that the effective magnetic field vanishes on average, i.e., $\langle\mathbf{B}_{\rm eff}^\alpha\rangle=0$ since we have precisely one flux spike (of A and B type) per flux quantum of the physical magnetic field. Flux quantization guarantees that the right hand side of Eq. (2.18) vanishes when averaged over a vortex lattice unit cell containing two physical vortices. It also implies that there must be equal numbers of A and B vortices in the system.

The essential advantage of the choice with vanishing $\langle \mathbf{B}_{\rm eff}^{\alpha}\rangle$ is that $\mathbf{v}_s^A$ and $\mathbf{v}_s^B$ can be chosen periodic in space with periodicity of the magnetic unit cell containing an integer number of electronic flux quanta hc/e. Notice that vector potential of a field that does not vanish on average can never be periodic in space. Condition $\langle \mathbf{B}_{\rm eff}^{\alpha}\rangle = 0$ is therefore crucial in this respect. The singular gauge transformation Eq. (2.12) thus maps the original Hamiltonian of fermionic quasiparticles in finite magnetic field onto a new Hamiltonian which is formally in zero average field and has only "neutralized" singular phase windings in the off-diagonal components.

The resulting new Hamiltonian now commutes with translations spanned by the magnetic unit cell, i.e.,

$$[\hat{T}_{\mathbf{R}},\hat{H}_N]=0, \tag{2.19}$$

where the translation operator $\hat{T}_{\mathbf{R}}=\exp(i\mathbf{R}\cdot\mathbf{p})$. We can therefore label eigenstates with a "vortex" crystal momentum quantum number $\mathbf{k}$ and use the familiar Bloch states as the natural basis for the eigenproblem. In particular we seek the eigensolution of the BdG equation $\hat{H}_N\psi=\epsilon\psi$ in the Bloch form

$$\psi_{n\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}\Phi_{n\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}\begin{pmatrix}U_{n\mathbf{k}}(\mathbf{r})\\ V_{n\mathbf{k}}(\mathbf{r})\end{pmatrix}, \tag{2.20}$$

where $(U_{n\mathbf{k}},V_{n\mathbf{k}})$ are periodic on the corresponding unit cell, n is a band index, and $\mathbf{k}$ is a crystal wave vector. Bloch wave function $\Phi_{n\mathbf{k}}(\mathbf{r})$ satisfies the "off-diagonal" Bloch equation $\hat{H}(\mathbf{k})\Phi_{n\mathbf{k}}=\epsilon_{n\mathbf{k}}\Phi_{n\mathbf{k}}$ with the Hamiltonian of the form

$$\hat{H}(\mathbf{k})=e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{H}_N e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{2.21}$$

Note that the dependence on $\mathbf{k}$, which is bounded to lie in the first Brillouin zone, is continuous. This will become important when topological properties of spin transport are discussed in the Sec. III C.

III. SPIN CONDUCTIVITY

Within the framework of linear-response theory,[36] spin dc conductivity can be related to the spin current–current retarded correlation function $D^R_{\mu\nu}$ through:

$$\sigma^s_{\mu\nu}=\lim_{\Omega\to 0}\ \lim_{q_1,q_2\to 0} -\frac{1}{i\Omega}\left(D^R_{\mu\nu}(q_1,q_2,\Omega)-D^R_{\mu\nu}(q_1,q_2,0)\right). \tag{3.1}$$

The retarded correlation function $D^R_{\mu\nu}(\Omega)$ can in turn be related to the Matsubara finite temperature correlation function

$$D_{\mu\nu}(i\Omega)=-\int_0^{\beta}e^{i\tau\Omega}\langle T_\tau j^s_\mu(\tau)j^s_\nu(0)\rangle d\tau \tag{3.2}$$

as

$$\lim_{q_1,q_2\to 0} D^R_{\mu\nu}(q_1,q_2,\Omega)=D_{\mu\nu}(i\Omega\to\Omega+i0). \tag{3.3}$$

In Eq. (3.2) the spatial average of the spin current $\mathbf{j}^s(\tau)$ is implicit, since we are looking for dc response of spatially inhomogeneous system. In the next section we derive the spin current and evaluate the above formulas.

A. Spin current

In order to find the dc spin conductivity, we must first find the spin current. More precisely, since we are looking only for the spatial average of the spin current $\mathbf{j}^s(\tau)$ we just need its $\mathbf{k}\to 0$ component. In direct analogy with the $\mathbf{B}=0$ situation,[24] we can define the spin current by the continuity equation:

$$\dot{\rho}^s+\nabla\cdot\mathbf{j}^s=0, \tag{3.4}$$

where $\rho^s=\hbar/2(\psi^\dagger_{\uparrow x}\psi_{\uparrow x}-\psi^\dagger_{\downarrow x}\psi_{\downarrow x})$ is the spin density projected onto z-axis. We can then use equations of motion for the ψ fields, Eq. (2.9), and compute the current density $\mathbf{j}^s$ from Eq. (3.4).

In the limit of $q\to 0$ the spin current can be written as (see Appendix B)

$$j^s_\mu=\frac{\hbar}{2}\Psi^\dagger V_\mu\Psi, \tag{3.5}$$

where the Nambu field $\Psi^\dagger=(\psi^\dagger_\uparrow,\psi_\downarrow)$ and the generalized velocity matrix operator V_μ satisfies the following commutator identity:

$$V_\mu=\frac{1}{i\hbar}[x_\mu,\hat{H}_0]. \tag{3.6}$$

Equation (3.6) is a direct restatement of the fact that spin can be transported by diffusion, i.e., it is a good quantum number in a superconductor, and that the average velocity of its propagation is just the group velocity of the quantum mechanical wave.

In the clean limit, the transverse spin conductivity σ^s_{xy} defined in Eq. (3.1) is

$$\sigma^s_{xy}(T)=\frac{\hbar^2}{4i}\sum_{m,n}(f_n-f_m)\frac{V_y^{mn}V_x^{nm}}{(\epsilon_n-\epsilon_m+i0)^2}, \tag{3.7}$$

where $f_m=(1+\exp(\beta\epsilon_n))^{-1}$ is the Fermi–Dirac distribution function evaluated at energy ϵ_m. For details of the derivation see Appendix B. The indices m and n label quantum numbers of particular states. The matrix elements V_μ^{mn} are

$$V_\mu^{mn}=\langle m|V_\mu|n\rangle=\int d\mathbf{x}(u_m^*,v_m^*)V_\mu\begin{pmatrix}u_n\\ v_n\end{pmatrix}, \tag{3.8}$$

where the particle-hole wave functions u_m,v_n satisfy the Bogoliubov–deGennes equation (2.10). Note that unlike the longitudinal dc conductivity, transverse conductivity is well defined even in the absence of impurity scattering. This demonstrates the fact that the transverse conductivity is *not* dissipative in origin. Rather, as will be discussed in Sec. III C, its nature is topological.

In the limit of $T\to 0$ the expression (3.7) for σ^s_{xy} becomes

$$\sigma_{xy}^{s}=\frac{\hbar^{2}}{4i}\sum_{\epsilon_m<0<\epsilon_n}\frac{V_x^{mn}V_y^{nm}-V_y^{mn}V_x^{nm}}{(\epsilon_m-\epsilon_n)^2}. \tag{3.9}$$

The summation extends over all states below and above the Fermi energy which, by the nature of the superconductor, is automatically set to zero.

B. Vanishing of the spin conductivity at half filling $(\mu=0)$

It is useful to contrast the *semiclassical* approach with the full quantum mechanical treatment of transverse spin conductivity. In semiclassical analysis the starting unperturbed Hamiltonian is usually defined in the *absence* of magnetic field **B**. One then assumes semiclassical dynamics and no interband transitions. In this picture, if there is particle-hole symmetry in the original ($\mathbf{B}=0$) Hamiltonian, then there will be no transverse spin (thermal) transport, since the number of carriers with a given spin (energy) will be the same in opposite directions. In this context, similar argument was put forth in Ref. 7. However, the problem of a *d*-wave superconductor is not so straightforward. As pointed out in Ref. 16, in the nodal (Dirac fermion) approximation, the vector potential is solely due to the superflow while the uniform magnetic field enters as a Doppler shift, i.e., Dirac scalar potential. Semiclassical analysis must then be started from this vantage point and the above conclusions are not straightforward, since the quasiparticle motion is irreducibly quantum mechanical.

Here we present an argument for the full quantum mechanical problem, without relying on the semiclassical analysis. We show that spin conductivity tensor Eq. (3.7) vanishes at $\mu=0$ due to particle-hole symmetry Eq. (2.11). First note that the Fermi–Dirac distribution function satisfies $f(\epsilon)=1-f(-\epsilon)$. Therefore, the factor f_m-f_n changes sign under the particle-hole transformation $\hat{P}_H$ Eq. (2.11), while the denominator $(\epsilon_m-\epsilon_n)^2$ clearly remains unchanged. In addition, each of the matrix elements V_μ^{mn} changes sign under $\hat{P}_H$. Thus the double summation over all states in Eq. (3.7) yields zero.

Consequently the spin transport vanishes for a clean strongly type-II BCS *d*-wave superconductor on a tight binding lattice at half filling. Due to Wiedemann–Franz law, which we derive in the next section, thermal Hall conductivity also vanishes at half filling at sufficiently low temperatures. Note that this result is independent of the vortex arrangement, i.e., it holds even for disordered vortex array and does not rely on any approximation regarding inter- or intranodal scattering.

C. Topological nature of spin Hall conductivity at $T=0$

In order to elucidate the topological nature of σ_{xy}^s, we make use of the translational symmetry discussed in Sec. II B and formally assume that the vortex arrangement is periodic. However, the detailed nature of the vortex lattice will not be specified and thus any vortex arrangement is allowed within the magnetic unit cell. The conclusions we reach are therefore quite general.

We will first rewrite the velocity matrix elements V_μ^{mn} using the singularly gauge transformed basis as discussed in Sec. II B. Inserting unity in the form of the FT gauge transformation Eq. (2.12)

$$V_\mu^{mn}=\langle m|V_\mu|n\rangle=\langle m|UU^{-1}V_\mu UU^{-1}|n\rangle. \tag{3.10}$$

The transformed basis states $U^{-1}|n\rangle$ can now be written in the Bloch form as $e^{i\mathbf{k}\cdot\mathbf{r}}|n_{\mathbf{k}}\rangle$ and therefore the matrix element becomes

$$V_\mu^{mn}=\langle m_{\mathbf{k}}|e^{-i\mathbf{k}\cdot\mathbf{r}}U^{-1}V_\mu Ue^{i\mathbf{k}\cdot\mathbf{r}}|n_{\mathbf{k}}\rangle=\langle m_{\mathbf{k}}|V_\mu(\mathbf{k})|n_{\mathbf{k}}\rangle. \tag{3.11}$$

We used the same symbol **k** for both bra and ket because the crystal momentum in the first Brillouin zone is conserved. The resulting velocity operator can now be simply expressed as

$$V_\mu(\mathbf{k})=\frac{1}{\hbar}\frac{\partial\hat{H}(\mathbf{k})}{\partial k_\mu}, \tag{3.12}$$

where $\hat{H}_0(\mathbf{k})$ was defined in Eq. (2.21). Furthermore the matrix elements of the partial derivatives of $\hat{H}(\mathbf{k})$ can be simplified according to

$$\left\langle m_{\mathbf{k}}\left|\frac{\partial\hat{H}(\mathbf{k})}{\partial k_\mu}\right|n_{\mathbf{k}}\right\rangle=(\epsilon_{\mathbf{k}}^n-\epsilon_{\mathbf{k}}^m)\left\langle m_{\mathbf{k}}\left|\frac{\partial n_{\mathbf{k}}}{\partial k_\mu}\right.\right\rangle$$
$$=-(\epsilon_{\mathbf{k}}^n-\epsilon_{\mathbf{k}}^m)\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_\mu}\right|n_{\mathbf{k}}\right\rangle, \tag{3.13}$$

for $m\neq n$. Utilizing Eqs. (3.12) and (3.13), Eq. (3.9) for σ_{xy}^s can now be written as

$$\sigma_{xy}^s=\frac{\hbar}{4i}\int\frac{d\mathbf{k}}{(2\pi)^2}\sum_{\epsilon^m<0<\epsilon^n}\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_x}\right|n_{\mathbf{k}}\right\rangle\left\langle n_{\mathbf{k}}\left|\frac{\partial m_{\mathbf{k}}}{\partial k_y}\right.\right\rangle$$
$$-\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_y}\right|n_{\mathbf{k}}\right\rangle\left\langle n_{\mathbf{k}}\left|\frac{\partial m_{\mathbf{k}}}{\partial k_x}\right.\right\rangle. \tag{3.14}$$

The identity $\Sigma_{\epsilon_{\mathbf{k}}^m<0<\epsilon_{\mathbf{k}}^n}(|m_{\mathbf{k}}\rangle\langle m_{\mathbf{k}}|+|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|)=1$, can be further used to simplify the above expression to read

$$\sigma_{xy}^{s,m}=\frac{\hbar}{8\pi}\frac{1}{2\pi i}\int d\mathbf{k}\left(\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_x}\right|\left.\frac{\partial m_{\mathbf{k}}}{\partial k_y}\right\rangle-\left\langle\frac{\partial m_{\mathbf{k}}}{\partial k_y}\right|\left.\frac{\partial m_{\mathbf{k}}}{\partial k_x}\right\rangle\right), \tag{3.15}$$

where $\sigma_{xy}^{s,m}$ is a contribution to the spin Hall conductance from a completely filled band *m*, well separated from the rest of the spectrum. Therefore the integral extends over the entire magnetic Brillouin zone that is topologically a two-torus T^2. Let us define a vector field $\hat{A}$ in the magnetic Brillouin zone as

$$\hat{A}(\mathbf{k})=\langle m_{\mathbf{k}}|\boldsymbol{\nabla}_{\mathbf{k}}|m_{\mathbf{k}}\rangle, \tag{3.16}$$

where $\boldsymbol{\nabla}_{\mathbf{k}}$ is a gradient operator in the **k** space. From Eq. (3.15) this contribution becomes

$$\sigma_{xy}^{s,m}=\frac{\hbar}{8\pi}\frac{1}{2\pi i}\int d\mathbf{k}[\nabla_{\mathbf{k}}\times\hat{A}(\mathbf{k})]_z, \tag{3.17}$$

where $[\ \ ]_z$ represents the third component of the vector. The topological aspects of the quantity in Eq. (3.17) were extensively studied in the context of integer quantum Hall effect (see e.g., Ref. 37) and it is a well known fact that

$$\frac{1}{2\pi i}\int d\mathbf{k}[\nabla_{\mathbf{k}}\times\hat{A}(\mathbf{k})]_z=C_1, \tag{3.18}$$

where C_1 is a first Chern number that is an integer. Therefore, a contribution of each filled band to σ_{xy}^s is

$$\sigma_{xy}^{s,m}=\frac{\hbar}{8\pi}N, \tag{3.19}$$

where N is an integer. The assumption that the band must be separated from the rest of the spectrum can be relaxed. If two or more fully filled bands cross each other the sum total of their contributions to spin Hall conductance is quantized even though nothing guarantees the quantization of the individual contributions. The quantization of the total spin Hall conductance requires a gap in the single particle spectrum at the Fermi energy. As discussed in Sec. V, the general single particle spectrum of the *d*-wave superconductor in the vortex state with inversion-symmetric vortex lattice is gapped and therefore the quantization of σ_{xy}^s is guaranteed.

IV. THERMAL CONDUCTIVITY

Before discussing the nature of the quasiparticle spectrum, we will establish a Wiedemann–Franz law between spin conductivity and thermal conductivity for a *d*-wave superconductor. This relation is naturally expected to hold for a very general system in which the quasiparticles form a degenerate assembly, i.e., it holds even in the presence of elastically scattering impurities.

Following Luttinger,[32] and Smrčka and Středa[34] we introduce a pseudo-gravitational potential $\chi=\mathbf{x}\cdot\mathbf{g}/c^2$ into the Hamiltonian Eq. (2.6) where $\mathbf{g}$ is a constant vector. The purpose is to include a coupling to the energy density on the Hamiltonian level. This formal trick allows us to equate statistical $(T\nabla(1/T))$ and mechanical $(\mathbf{g})$ forces so that the thermal current $\mathbf{j}^Q$, in the long wavelength limit given by

$$\mathbf{j}^Q=L^Q(T)\left(T\nabla\frac{1}{T}-\nabla\chi\right), \tag{4.1}$$

will vanish in equilibrium. Therefore it is enough to consider only the dynamical force $\mathbf{g}$ to calculate the phenomenological coefficient $L^Q_{\mu\nu}$. Note that thermal conductivity κ_{xy} is

$$\kappa_{\mu\nu}(T)=\frac{1}{T}L^Q_{\mu\nu}(T). \tag{4.2}$$

When the BCS Hamiltonian H introduced in Eq.(2.1) becomes perturbed by the pseudo-gravitational field, the resulting Hamiltonian H_T has the form

$$H_T=H+F, \tag{4.3}$$

where F incorporates the interaction with the perturbing field:

$$F=\frac{1}{2}\int d\mathbf{x}\Psi^\dagger(\mathbf{x})(\hat{H}_0\chi+\chi\hat{H}_0)\Psi(\mathbf{x}). \tag{4.4}$$

Since χ is a small perturbation, to the first order in χ the Hamiltonian H_T can be written as

$$H_T=\int d\mathbf{x}\left(1+\frac{\chi}{2}\right)\Psi^\dagger(\mathbf{x})\hat{H}_0\left(1+\frac{\chi}{2}\right)\Psi(\mathbf{x}), \tag{4.5}$$

i.e., the application of the pseudo-gravitational field results in rescaling of the fermion operators:

$$\Psi\to\tilde{\Psi}=\left(1+\frac{\chi}{2}\right)\Psi. \tag{4.6}$$

If we measure the energy relative to the Fermi level, the transport of heat is equivalent to the transport of energy. In analogy with the Sec. III A, we define the heat current $\mathbf{j}^Q$ through diffusion of the energy-density h_T. From conservation of the energy-density the continuity equation follows:

$$\dot{h}_T+\nabla\cdot\mathbf{j}^Q=0. \tag{4.7}$$

In the limit of $q\to 0$ the thermal current is

$$j^Q_\mu=\frac{i}{2}(\tilde{\Psi}^\dagger V_\mu\dot{\tilde{\Psi}}-\dot{\tilde{\Psi}}^\dagger V_\mu\tilde{\Psi}). \tag{4.8}$$

For details see Appendix C. Note that the quantum statistical average of the current has two contributions, both linear in χ,

$$\langle j^Q_\mu\rangle=\langle j^Q_{0\mu}\rangle+\langle j^Q_{1\mu}\rangle\equiv-(K^Q_{\mu\nu}+M^Q_{\mu\nu})\partial_\nu\chi. \tag{4.9}$$

The first term is the usual Kubo contribution to $L^Q_{\mu\nu}$ while the second term is related to magnetization of the sample[38] for transverse components of $\kappa_{\mu\nu}$ and vanishes for the longitudinal components. In Appendix C we show that at $T=0$ the term related to magnetization cancels the Kubo term and therefore the transverse component of $\kappa_{\mu\nu}$ is zero at $T=0$. To obtain finite temperature response, we perform Sommerfeld expansion and derive Wiedemann–Franz law for spin and thermal Hall conductivity.

As shown in Appendix C

$$L^Q_{\mu\nu}(T)=-\left(\frac{2}{\hbar}\right)^2\int d\xi\xi^2\frac{df(\xi)}{d\xi}\tilde{\sigma}^s_{\mu\nu}(\xi), \tag{4.10}$$

where

$$\tilde{\sigma}^s_{xy}(\xi)=\frac{\hbar^2}{4i}\sum_{\epsilon_m<\xi<\epsilon_n}\frac{V_x^{mn}V_y^{nm}-V_y^{mn}V_x^{nm}}{(\epsilon_m-\epsilon_n)^2}. \tag{4.11}$$

Note that $\tilde{\sigma}^s_{\mu\nu}(\xi=0)=\sigma^s_{\mu\nu}(T=0)$. For a superconductor at low temperature the derivative of the Fermi–Dirac distribution function is

$$-\frac{df(\xi)}{d\xi}=\delta(\xi)+\frac{\pi^2}{6}(k_BT)^2\frac{d^2}{d\xi^2}\delta(\xi)+\cdots \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.10) we obtain

$$L^{Q}_{\mu\nu}(T)=\frac{4\pi^2}{3\hbar^2}(k_BT)^2\sigma^s_{\mu\nu}, \tag{4.13}$$

where $\sigma^s_{\mu\nu}$ is evaluated at $T=0$. Finally, using Eq. (4.2), in the limit of $T\rightarrow 0$

$$\kappa_{\mu\nu}(T)=\frac{4\pi^2}{3}\left(\frac{k_B}{\hbar}\right)^2 T\sigma^s_{\mu\nu}. \tag{4.14}$$

We recognize the Wiedemann–Franz law for the spin and thermal conductivity in the above equation. As mentioned, this relation is quite general in that it is independent of the spatial arrangement of the vortex array or elastic impurities. Thus, quantization of the transverse spin conductivity σ^s_{xy} implies quantization of κ_{xy}/T in the limit of $T\rightarrow 0$.

V. QUASIPARTICLE SPECTRUM AND QUANTIZED CONDUCTIVITY

General features of the quasiparticle spectrum can be understood on the basis of symmetry alone. Since the time-reversal symmetry is broken, the Bogoliubov–de Gennes Hamiltonian H_0, Eq. (2.10), must be, in general, complex. According to the "noncrossing" theorem of von Neumann and Wigner,[39] a complex Hamiltonian can have degenerate eigenvalues unrelated to symmetry only if there are at least three parameters which can be varied simultaneously.

Since the system is two dimensional, with the vortices arranged on the lattice, there are two parameters in the Hamiltonian $\hat{H}(\mathbf{k})$ Eq. (2.21): vortex crystal momenta k_x and k_y which vary in the first Brillouin zone. Therefore, we should not expect any degeneracy to occur, *in general*, unless there is some symmetry which protects it. Away from half-filling $(\mu\neq 0)$ and with unspecified arrangement of vortices in the magnetic unit cell there is not enough symmetry to cause degeneracy. There is only *global* Bogoliubov–de Gennes symmetry relating quasiparticle energy $\epsilon_{\mathbf{k}}$ at some point $\mathbf{k}$ in the first Brillouin zone to $-\epsilon_{-\mathbf{k}}$.

In order for every quasiparticle band to be either completely below or completely above the Fermi energy, it is sufficient for the vortex lattice to have inversion symmetry. This can be readily seen by the following argument: Consider a vortex lattice with inversion symmetry. Then, by the very nature of the superconducting vortex carrying $(hc/2e)$ flux, there must be even number of vortices per magnetic unit cell and we are then free to choose Franz–Tesanovic labels A and B in such a way that $\mathbf{v}^A(-\mathbf{r})=-\mathbf{v}^B(\mathbf{r})$. To see this note that the explicit form of the superfluid velocities can be written as:[16]

$$\mathbf{v}^{\alpha}_s(\mathbf{r})=\frac{2\pi\hbar}{m^*}\int\frac{d^2k}{(2\pi)^2}\frac{i\mathbf{k}\times\hat{z}}{k^2}\sum_i e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}^{\alpha}_i)}, \tag{5.1}$$

where $\alpha=A$ or B and $\mathbf{r}^{\alpha}_i$ denotes the position of the vortex with label α. If the vortex lattice has inversion symmetry then for every $\mathbf{r}^A_i$ there is a corresponding $-\mathbf{r}^B_i$ such that $\mathbf{r}^A_i=-\mathbf{r}^B_i$. Therefore, under space inversion $\mathcal{I}$

$$\mathcal{I}\mathbf{v}^A(\mathbf{r})=\mathbf{v}^A(-\mathbf{r})=-\mathbf{v}^B(\mathbf{r}). \tag{5.2}$$

Recall that the tight-binding lattice Bogoliubov–de Gennes Hamiltonian written in the Bloch basis Eq. (2.21) reads:

$$\hat{H}(\mathbf{k})=\sum_{\boldsymbol{\delta}}\{\sigma_3(-te^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}(\mathbf{a}-\sigma_3\mathbf{v})\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot(\mathbf{k}+\mathbf{p})}-\mu)$$
$$+\sigma_1\Delta_0\eta_{\boldsymbol{\delta}}e^{i\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}\mathbf{a}\cdot d\mathbf{l}}e^{i\boldsymbol{\delta}\cdot(\mathbf{k}+\mathbf{p})}\}, \tag{5.3}$$

where

$$\mathbf{v}(\mathbf{r})\equiv\tfrac{1}{2}(\mathbf{v}^A(\mathbf{r})+\mathbf{v}^B(\mathbf{r}));\quad \mathbf{a}(\mathbf{r})\equiv\tfrac{1}{2}(\mathbf{v}^A(\mathbf{r})-\mathbf{v}^B(\mathbf{r})). \tag{5.4}$$

As before σ_1 and σ_3 are Pauli matrices operating in Nambu space and the sum is again over the nearest neighbors. It can be easily seen that upon applying the space inversion $\mathcal{I}$ to $\hat{H}(\mathbf{k})$ followed by complex conjugation $\mathcal{C}$ and $i\sigma_2$ we have a symmetry that for every $\epsilon_{\mathbf{k}}$ there is $-\epsilon_{\mathbf{k}}$, that is:

$$-i\sigma_2\mathcal{C}I\hat{H}(\mathbf{k})\mathcal{I}\mathcal{C}i\sigma_2=-\hat{H}(\mathbf{k}) \tag{5.5}$$

which holds for every point in the Brillouin zone. Therefore, in order for the spectrum *not* to be gapped, we would need band crossing at the Fermi level. But by the noncrossing theorem this cannot happen *in general*. Thus, the quasiparticle spectrum of inversion symmetric vortex lattice is gapped, unless an external parameter, other than k_x and k_y, is fine-tuned. As was established in the previous section, gapped quasiparticle spectrum implies quantization of the transverse spin conductivity σ^s_{xy} as well as κ_{xy}/T for T sufficiently low.

Precisely at half-filling $(\mu=0)$ σ^s_{xy} must vanish on the basis of particle-hole symmetry (see Sec. III B). We can then vary the chemical potential so that $\mu\neq 0$ and break particle-hole symmetry. Hence, the chemical potential μ can serve as the third parameter necessary for creating the accidental degeneracy, i.e., at some special values of μ^* the gap at the Fermi level will close (see Fig. 2). This results in a possibility of changing the quantized value of σ^s_{xy} by an integer in units of $\hbar/8\pi$ (Fig. 2).[27] In contrast to the integer quantum Hall effect in the normal state, the very nature of the superconductor forces the gap to be *always* centered at the Fermi level. This holds even when the effects of the tight-binding lattice are taken into account and consequently we achieve *plateau* dependence on the chemical potential.

Similarly, we can change the strength of the electron–electron attraction, which is proportional to the maximum value of the superconducting order parameter Δ_0 while keeping the chemical potential μ fixed. Again, as can be seen in Fig. 3, at some special values Δ_0^* the spectrum is gapless and the quantized Hall conductance undergoes a transition.

VI. SCALING FUNCTIONS

As shown in the previous section, the quasiparticle spectrum of a d-wave superconductor in vortex state is gapped, in general. Therefore, at temperatures T which are much less than the Δ_m (magnetic field $\mathbf{B}$ induced gap), T-dependence of thermal conductivity κ_{xy} can be determined uniquely. More-

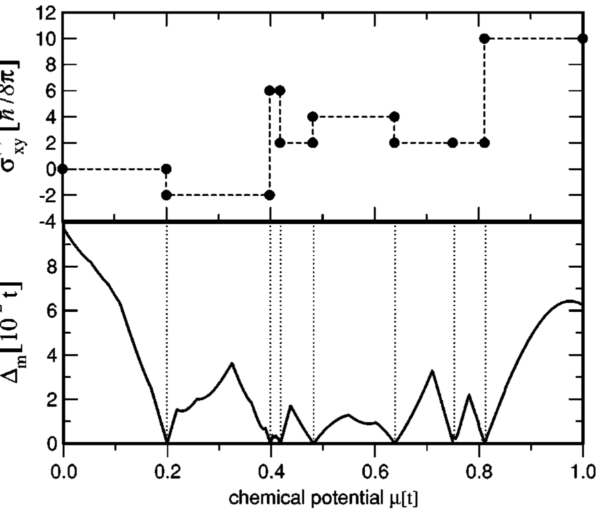

FIG. 2. The mechanism for changing the quantized spin Hall conductivity is through exchanging the topological quanta via (''accidental'') gap closing. The upper panel displays spin Hall conductivity σ^s_{xy} as a function of the chemical potential μ. The lower panel shows the magnetic field induced gap Δ_m in the quasiparticle spectrum. Note that the change in the spin Hall conductivity occurs precisely at those values of chemical potential at which the gap closes. Hence the mechanism behind the changes of σ^s_{xy} is the exchange of the topological quanta at the band crossings. The parameters for the above calculation were: square vortex lattice, magnetic length $l=4\delta$, $\Delta=0.1t$ or equivalently the Dirac anisotropy $\alpha_D=10$.

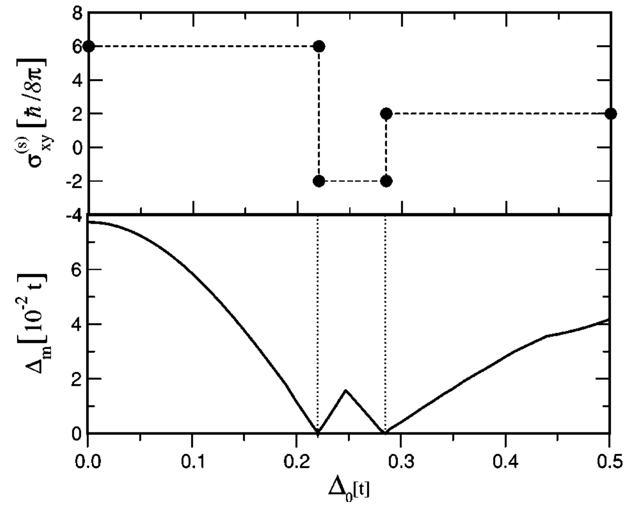

FIG. 3. The upper panel displays spin Hall conductivity σ^s_{xy} as a function of the maximum superconducting order parameter Δ_0. The lower panel shows the magnetic field induced gap in the quasiparticle spectrum. The change in the spin Hall conductivity occurs at those values of Δ_0 at which the gap closes. The parameters for the above calculation were: square vortex lattice, magnetic length $l=4\delta$, $\mu=2.2t$.

over, **B**-dependence of κ_{xy} comes entirely from the spin conductivity $\sigma^s_{xy}(\mathbf{B})$. That is:

$$\kappa_{xy}=\frac{4\pi^2}{3}\left(\frac{k_B}{\hbar}\right)^2 T\sigma^s_{\mu\nu}(B). \tag{6.1}$$

Curiously, if the above equation is naively combined with Simon and Lee scaling[7]

$$\kappa_{xy}(T,B)=T^2F_{xy}\left(\frac{\sqrt{B}}{T}\right), \tag{6.2}$$

the scaling function would be determined up to a proportionality constant C:

$$\kappa_{xy}(T,B)=CT\sqrt{B}. \tag{6.3}$$

In the recent experiments of Ong *et al.*[6] this is precisely the scaling seen in the temperature range up to $\sim$25 K.

While the above arguments are tempting in their simplicity, one must hesitate before proclaiming that they provide the explanation for the scaling observed by Ong *et al.*[6] First, the experiments are done at rather high temperatures and it is unlikely that the ultimate low temperature scaling regime, in which we expect our Eq. (6.1) to be rigorously satisfied, has been reached. Second, and even more glaring, is an intrinsic theoretical problem: once the low temperature scaling regime is reached and Eq. (6.1) holds we have argued that σ^s_{xy} (and therefore κ_{xy} as well) will be quantized as a function of magnetic field, rather than obeying $\sigma^s_{xy}\propto\sqrt{B}$ required for the scaling form Eq. (6.3) to hold. Furthermore, the Simon and Lee scaling form Eq. (6.2), derived under the assumption of linear, massless Dirac dispersion at the nodes, will itself become suspect in presence of a small mass gap necessary for the low temperature quantization. Still, one should never risk dismissing experimentally observed scaling laws, particularly not the one so simple as Eq. (6.3). It is conceivable that the ''staircase'' structure of quantized and oscillating $\sigma^s_{xy}(B)$ has a guiding ''envelope'' exhibiting an approximate $\sqrt{B}$ dependence and that the Simon and Lee scaling form Eq. (6.2) holds to a good approximation at temperatures low enough for linear dispersion at the nodes to become apparent while still high or comparable to the much lower energy scale of the Dirac mass gap. If this were the case, the experimentally observed scaling Eq. (6.3) would still hold to a good approximation. Further investigation of these issues is left for future study.

VII. CONTINUUM VERSUS LATTICE THEORY

The previous discussion concentrated on the tight-binding formulation of the problem which is important if the magnetic field is relatively large and if there is a strong interaction between the underlying ionic lattice and the quasiparticles. In usual experimental situations, however, the magnetic length is much longer than the inter-ionic spacing and we would expect that the length-scale associated with the ionic lattice becomes unimportant at low energies. This leads to a continuum formulation of the theory (see Sec. I).

A. Linearized continuum theory

On the basis of the $B=0$ problem, we expect that the low energy physics is confined in k-space around four nodal points on the Fermi surface. In order to treat the effect of the magnetic field and of the vortex lattice on the low energy properties of the spectrum, we would ideally like to construct a Hamiltonian which treats each of the four nodes independently. A natural way to do this was suggested by Simon and Lee[7] and later extensively used by others.[12,13,15,16]

As was pointed out in Ref. 16 there is a subtle problem associated with numerical implementation of the continuum linearized version of the theory. On the physical grounds any given problem involving charge e (quasi)particle interacting with a $+(hc/2e)$ vortex must lead to an identical eigenvalue spectrum as for a $-(hc/2e)$ vortex as long as the wave function of the particle vanishes precisely at the vortex. This is an exact singular gauge symmetry of our physical problem and therefore it should be present at all stages of any approximation. Extensive numerical study of the linearized approximation[16] pointed out that this symmetry is weakly violated in the quasiparticle spectra. The problem persists irrespective of the techniques chosen to diagonalize the linearized Hamiltonian, i.e., it is present in real space as well as momentum space representation. The issue was resolved in Ref. 16 by regularizing the problem on the tight-binding lattice which offered many advantages, some of which are utilized in this paper. On the other hand, the concept of single node physics is lost in a tight binding formulation as the latter naturally describes the physics of all four nodes. It would be desirable to formulate a properly regularized Hamiltonian describing single node physics only. From the work of Franz and Tesanovic,[12] the most natural candidate is a free, anisotropic, massless Dirac Hamiltonian

$$\mathcal{H}_0=\begin{pmatrix} v_F\hat{p}_x & v_\Delta\hat{p}_y \\ v_\Delta\hat{p}_y & -v_F\hat{p}_x \end{pmatrix} \tag{7.1}$$

perturbed by scalar and vector potentials defined in Eq. (2.17):

$$\mathcal{H}'=m\begin{pmatrix} v_F v^A_{sx} & \frac{1}{2}v_\Delta(v^A_{sy}-v^B_{sy}) \\ \frac{1}{2}v_\Delta(v^A_{sy}-v^B_{sy}) & v_F v^B_{sx} \end{pmatrix}. \tag{7.2}$$

v_F and v_Δ are Fermi and gap velocities, respectively.[12] We wish to argue that the above Hamiltonian is well defined and does not suffer from singular gauge asymmetry, provided that suitable boundary conditions and their transformations are also specified. The discrepancies encountered in the numerics are believed to result from neglecting the boundary conditions and the effect of the singular gauge transformation on them.

In the context of high energy physics, a similar problem was studied extensively. It was found that the Dirac-type equations in the presence of a single Aharonov–Bohm string require self-adjoint extensions which enlarge the Hilbert space by the wave functions with $r^{-\frac{1}{2}}$ radial behavior. In addition, the wave functions depend on a dimensionless parameter Θ which specifies the boundary conditions at the core. A consistent procedure enabling one to construct the needed self-adjoint extension involves the theory of von Neumann deficiency indices.[40,41]

The above considerations can be illustrated on a well studied problem of a Dirac particle in the field of a single string.[42,43] We will restrict ourselves to the case of a massless Dirac particle and $(hc/2e)$ flux carried by the string. The Hamiltonian

$$(i\partial\!\!\!/-eA\!\!\!/)\psi=0 \tag{7.3}$$

in this case is axially symmetric and after separation of variables

$$\psi(r,\phi)=\begin{pmatrix} \chi^1 \\ e^{i\phi}\chi^2 \end{pmatrix} e^{in\phi} \tag{7.4}$$

one obtains the equation for the radial wave functions:

$$H_r\chi\equiv\begin{pmatrix} 0 & -i\left(\frac{d}{dr}+\frac{\nu+1}{r}\right) \\ -i\left(\frac{d}{dr}-\frac{\nu}{r}\right) & 0 \end{pmatrix}\chi(r)=E\chi(r), \tag{7.5}$$

where $\nu=\frac{1}{2}+n$ is a half-integer. The solutions of this equation can be expressed through Bessel functions as

$$\chi_r\propto\begin{pmatrix} J_{\epsilon\nu}(|E|r) \\ iJ_{\epsilon(\nu+1)}(|E|r) \end{pmatrix}, \tag{7.6}$$

where $\epsilon=\pm1$. The square integrability condition determines sign of ϵ for all ν *except* $\nu=-1/2$. Both choices of ϵ result in a wave function diverging as $1/\sqrt{r}$ while still remaining square integrable. The requirement of regularity for all the solutions at $r=0$ turns out to be too restrictive and results in numerous pathologies such as incompleteness of the basis. The problem is solved by using von Neumann theory of self-adjoint extensions.[42,44] Since the radial H_r operator in Eq. (7.5) defined in the domain of regular functions has deficiency indices (1,1), one is forced to extend the Hilbert space by relaxing the condition of regularity and allowing for the wave functions[42]

$$\chi(r)\propto\frac{1}{r^{1/2}}\begin{pmatrix} i\sin\Theta \\ \cos\Theta \end{pmatrix}. \tag{7.7}$$

The angle $\Theta\in(\pi/4,\pi/4+\pi)$, determined by the details of the vortex core, parameterizes the self-adjoint extension.

The energy eigenstates for $\nu=-1/2$ are given by

$$\begin{pmatrix} \sin\mu J_{-1/2}(|E|r)+(-1)^n\cos\mu J_{1/2}(|E|r) \\ \sin\mu J_{1/2}(|E|r)-(-1)^n\cos\mu J_{-1/2}(|E|r) \end{pmatrix}, \tag{7.8}$$

where the parameter $\mu=\Theta$ if n is even, while for n odd $\mu=\pi-\Theta$. In addition, for some values of Θ there is a bound state.[42]

We encounter a peculiar situation in which the long distance physics governed by Eq. (7.3) still depends on boundary conditions imposed at the core. Linearization of the original BdG problem neglects terms "small" in conventional sense which, nevertheless, remain effectively present in Θ dependence of Eq. (7.8) . Although the linearized equation (7.3) has no length scale, the core physics, now effectively shrunk to a circle with vanishing radius around the singularity, manifests itself in the wave functions Eq. (7.7) with the long power law tails. In general, there are only two values of Θ corresponding to the pure δ-function magnetic field.[45] However, in the case of other contact interactions present in the core, the parameter Θ can be arbitrary.[45]

Various regularizations of the problem were studied leading to different choices of the self-adjoint extension.[46] One possibility is to put an impenetrable cylinder of small but finite radius ρ around a vortex thus forcing the wave functions to vanish on the surface of the cylinder. Such a boundary condition specification will immediately restore the $\pm(hc/2e)$ vortex invariance because the value of the wave function at the surface of the cylinder will not transform under singular gauge transformation. Of course, the true boundary conditions must descend from the original physical problem, in this case from the self-consistent solution of the full Bogoliubov–de Gennes equation including the core region. The quasiparticle wave functions do not, in general, vanish around the vortex.[47] Therefore, not only the quasiparticle wave functions but also the constraints imposed on them must be properly transformed under singular gauge transformation.

B. Beyond the linearized theory

As pointed out by Simon and Lee[7] and then later by Ye,[20] linearized theory predicts vanishing transverse conductivities. In order to include the description of Hall effects, the curvature terms must be included. Without regularization, the perturbative analysis of the curvature terms are far from straightforward. This has to do with the fast rise of both wave functions and perturbations around the core. If the regularization is taken into account and the wave functions are forced to vanish at some radius ρ around the core, the new length scale ρ will break the scale invariance of the Dirac equation. Moreover, the perturbative analysis of the continuum equation would predict that the contribution of each node to σ^s_{xy} can be either 0 or $\pm\frac{1}{2}$ in units of $\hbar/8\pi$.[19] Thus, if all four nodes are included, the only possible values of σ^s_{xy} are 0 or ± 2. On the other hand, explicit evaluation of σ^s_{xy} via the tight-binding lattice regularization indicates that a wide range of values for σ^s_{xy} is possible (see Figs. 2 and 3). An interesting question remaining for future study is how to implement the picture presented in Ref. 19, of individual Dirac nodes changing σ^s_{xy} by ± 1 through action of the perturbatively determined mass term as an "elementary building block" of more complicated band crossings.

One way to reach the continuum limit is to make the magnetic length l much longer than the tight-binding lattice constant δ. The drawback associated with the tight-binding regularization of the problem stems from the fact that the continuum limit is not smooth and unless an exact analytical solution is found it is complicated to analyze numerically. The similar situation occurs in a tight-binding Hofstadter problem of a 2D normal electron gas on the lattice.

VIII. CONVENTIONAL SUPERCONDUCTORS

Although the previous analysis was focused on unconventional d-wave superconductors, the main results can be directly generalized to conventional 2D s-wave superconductors in the vortex state: the transverse spin Hall conductivity σ^s_{xy} is quantized in units of $\hbar/8\pi$ and by Wiedemann–Franz law $\kappa_{xy}=(4\pi^2/3)(k_B/\hbar)^2 T\sigma^s_{xy}$ as $T\rightarrow 0$. At low magnetic fields, the quasiparticle spectrum is gapped by the virtue of the lowest Caroli–Matricon–de Gennes vortex core bound state having energy $\sim\Delta^2/\epsilon_F$. In the vortex lattice the CMdG bound states are extended and form bands. If the vortices are disordered, the states in the band tails will be localized and the effective quasiparticle gap will be increased because the localized states do not contribute to transport. As the magnetic field is increased, the spectrum becomes progressively altered and at some critical field $\mathbf{B}^*$ it develops nodal points (see Ref. 48). The transition between states with two different quantized values of σ^s_{xy} happens at the $\mathbf{B}$ induced gap closings. Therefore, a series of transitions between different spin Hall states is predicted to occur in the conventional superconductors as well.

IX. CONCLUSIONS

In conclusion, we examined a general problem of 2D type-II superconductors in the vortex state with inversion symmetric vortex lattice. The single particle excitation spectrum is typically gapped and results in quantization of transverse spin conductivity σ^s_{xy} in units of $\hbar/8\pi$.[27] The topological nature of this phenomenon is discussed. The size of the magnetic field induced gap Δ_m in unconventional d-wave superconductors is not universal and in principle can be as large as several percent of the maximum superconducting gap Δ_0. By virtue of the Wiedemann–Franz law, which we derive for the d-wave Bogoliubov–de Gennes equation in the vortex state, the thermal conductivity $\kappa_{xy}=(4\pi^2/3)(k_B/\hbar)^2 T\sigma^s_{xy}$ as $T\rightarrow 0$. Thus at $T\ll\Delta_m$ the quantization of κ_{xy}/T will be observable in clean samples with negligible Lande g factor and with well ordered Abrikosov vortex lattice. In conventional superconductors, the size of Δ_m is given by Matricon–Caroli-deGennes vortex bound states $\sim\Delta^2/\epsilon_F$.

In real experimental situations, Lande g factor is not necessarily small. In fact it is close to 2 in cuprates.[49] The Zeeman effect must therefore be included in the analysis. Nevertheless, detailed numerical examination of the quasiparticle spectra reveals that the spectrum remains gapped for a wide range of physically realizable parameters even if $g=2$. Thus, although the Zeeman splitting is a competing effect, in general it is not strong enough to prevent the quantization of σ^s_{xy} and consequently of κ_{xy}/T.

We have explicitly evaluated the quantized values of σ^s_{xy} on the tight-binding lattice model of $d_{x^2-y^2}$-wave BCS superconductor in the vortex state and showed that in principle a wide range of integer values can be obtained. This should be contrasted with the notion that the effect of a magnetic field on a d-wave superconductor is solely to generate a $d+id$ state for the order parameter, as in that case $\sigma^s_{xy}=\pm 2$ in units of $\hbar/8\pi$. In the presence of a vortex lattice, the situation appears to be more complex.

By fine-tuning some external parameter, for instance the strength of the electron–electron attraction which is proportional to Δ_0 or the chemical potential μ, the gap closing is achieved, i.e., $\Delta_m=0$. The transition between two different values of σ^s_{xy} occurs precisely when the gap closes and topological quanta are exchanged. The remarkable new feature is the *plateau* dependence of σ^s_{xy} on the Δ_0 or μ. It is qualitatively different from the plateaus in the ordinary integer quantum Hall effect. In superconductors, the physical origin of the plateaus is the gap in the quasiparticle spectrum generated by the superconducting pairing interactions and always automatically centered at the Fermi energy.

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APPENDIX A: GREEN'S FUNCTIONS: DEFINITIONS AND IDENTITIES

The one-particle Green's function matrix[36] is defined as

$$\hat{G}_{\alpha\beta}(\mathbf{r}_1\tau_1;\mathbf{r}_2\tau_2)\equiv-\langle T_\tau\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle, \tag{A1}$$

where T_τ denotes imaginary time ordering operator and $\alpha=1,2$ denotes components of a Nambu spinor $\Psi^\dagger(1)$ which is a shorthand for $\Psi^\dagger(\mathbf{r}_1,\tau_1)=(\psi^\dagger_\uparrow(\mathbf{r}_1,\tau_1),\psi_\downarrow(\mathbf{r}_1,\tau_1))$.

Due to the time independence of the Hamiltonian Eq. (2.1), the Green's function Eq. (A1) depends only on the imaginary time difference $\tau=\tau_1-\tau_2$. Therefore, its Fourier transform is given by

$$\hat{G}(\mathbf{r}_1,\mathbf{r}_2;i\omega)=\int_0^\beta e^{i\omega\tau}\hat{G}(\mathbf{r}_1,\mathbf{r}_2,\tau)d\tau, \tag{A2}$$

$$\hat{G}(\mathbf{r}_1,\mathbf{r}_2;\tau)=\frac{1}{\beta}\sum_{i\omega} e^{-i\omega\tau}\hat{G}(\mathbf{r}_1,\mathbf{r}_2;i\omega).$$

Here $\beta=1/(k_BT)$, T is temperature, and the fermionic frequency $\omega=(2l+1)\pi/\beta$, $l\in Z$.

Using the above relations, it is straightforward to derive the spectral representation of the following correlation functions between Ψ and its imaginary time derivatives $\partial_\tau\Psi\equiv\dot{\Psi}$:

$$\left\{\begin{array}{l}\langle T_\tau\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle\\ \langle T_\tau\dot{\Psi}_\alpha(1)\Psi^\dagger_\beta(2)\rangle\\ \langle T_\tau\Psi_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle\\ \langle T_\tau\dot{\Psi}_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle\end{array}\right.=-\frac{1}{\beta}\sum_{i\omega} e^{-iw\tau}\int d\epsilon\frac{\hat{A}(\mathbf{r}_1,\mathbf{r}_2;\epsilon)}{i\omega-\epsilon}\left\{\begin{array}{l}+1\\ -\epsilon\\ +\epsilon\\ -\epsilon^2\end{array}\right.. \tag{A3}$$

The spectral function $\hat{A}(\mathbf{r}_1,\mathbf{r}_2;\epsilon)$ can be written in terms of eigenfunctions $\Phi_n(\mathbf{r})=(u_n(\mathbf{r}),v_n(\mathbf{r}))^T$ of the Bogoliubov–deGennes Hamiltonian $\hat{H}_0$ in the form:

$$\hat{A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\epsilon)=\sum_n \delta(\epsilon-\epsilon_n)\Phi_{n\alpha}(\mathbf{r}_1)\Phi^\dagger_{n\beta}(\mathbf{r}_2), \tag{A4}$$

where ϵ_n is the eigen-energy associated with an eigenstate labeled by the quantum number n. Substituting Eq. (A4) into Eq. (A3) we can write the above correlation functions solely in terms of the eigenfunctions $\Phi_n(\mathbf{r})$:

$$\left\{\begin{array}{l}\langle T_\tau\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle\\ \langle T_\tau\dot{\Psi}_\alpha(1)\Psi^\dagger_\beta(2)\rangle\\ \langle T_\tau\Psi_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle\\ \langle T_\tau\dot{\Psi}_\alpha(1)\dot{\Psi}^\dagger_\beta(2)\rangle\end{array}\right.=-\frac{1}{\beta}\sum_{i\omega,n} e^{-iw\tau}\frac{\Phi_{n\alpha}(\mathbf{r}_1)\Phi^\dagger_{n\beta}(\mathbf{r}_2)}{i\omega-\epsilon_n}\left\{\begin{array}{l}+1\\ -\epsilon_n\\ +\epsilon_n\\ -\epsilon_n^2\end{array}\right.. \tag{A5}$$

In the calculations that follow we will also encounter Matsubara summations over the fermionic frequencies $\omega=(2l+1)\pi/\beta$, $l\in Z$, of the form:

$$S_{nm}(i\Omega)=\frac{1}{\beta}\sum_{i\omega}\frac{1}{(i\omega-\epsilon_n)(i\Omega+i\omega-\epsilon_m)}, \tag{A6}$$

where $\Omega=2\pi k/\beta$, $k\in Z$, is an outside bosonic frequency. The sum Eq. (A6) can be evaluated using standard techniques (see, e.g., Ref. 36) and yields:

$$S_{nm}(i\Omega)=\frac{f_n-f_m}{\epsilon_n-\epsilon_m+i\Omega}. \tag{A7}$$

Here f_n is a short hand for the Fermi–Dirac distribution function $f(\epsilon_n)=(1+\exp(\beta\epsilon_n))^{-1}$.

In order to derive spin and thermal currents we will need the explicit form of the generalized velocity operator $\mathbf{V}$ introduced in Eq. (3.6):

$$\mathbf{V}=\begin{pmatrix} \dfrac{\boldsymbol{\pi}}{m} & i\hat{\mathbf{v}}_\Delta \\ i\hat{\mathbf{v}}_\Delta^* & \dfrac{\boldsymbol{\pi}^*}{m} \end{pmatrix}, \tag{A8}$$

where the gap velocity operator is given by

$$i\hat{\mathbf{v}}_\Delta=i\Delta_0\,\eta_{\boldsymbol{\delta}}\boldsymbol{\delta}e^{i\phi(\mathbf{r})/2}(e^{i\boldsymbol{\delta}\cdot\mathbf{p}}-e^{-i\boldsymbol{\delta}\cdot\mathbf{p}})e^{i\phi(\mathbf{r})/2} \tag{A9}$$

and the canonical momentum equals

$$\boldsymbol{\pi}=-\frac{i}{2}\boldsymbol{\delta}e^{\frac{i}{\hbar}\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{\delta}}(\mathbf{p}-\frac{e}{c}\mathbf{A})\cdot d\mathbf{l}}+\text{h.c.} \tag{A10}$$

The following identities for operators $\hat{\mathbf{v}}_\Delta$ and $\boldsymbol{\pi}$ will be used in the next section:

$$\begin{cases} \boldsymbol{\pi}^*\Psi^\dagger\cdot\boldsymbol{\pi}\Psi &= i\hbar\nabla\cdot(\Psi^\dagger\boldsymbol{\pi}\Psi)+\Psi^\dagger\boldsymbol{\pi}^2\Psi \\ \boldsymbol{\pi}^*\Psi^\dagger\cdot\boldsymbol{\pi}\Psi &= -i\hbar\nabla\cdot(\boldsymbol{\pi}^*\Psi^\dagger\Psi)+(\boldsymbol{\pi}^*)^2\Psi^\dagger\Psi, \end{cases} \tag{A11}$$

$$\Psi^\dagger\Delta\Psi-\Delta\Psi^\dagger\Psi=\tfrac{1}{2}\boldsymbol{\nabla}\cdot(\Psi^\dagger\hat{\mathbf{v}}_\Delta\Psi-\hat{\mathbf{v}}_\Delta\Psi^\dagger\Psi), \tag{A12}$$

where all the differential operators act only on the adjacent functions unless otherwise specified. The above equations are straightforward to derive in continuum, while on the tight-binding lattice Eqs. (A11) and (A12) imply a symmetric definition of the lattice divergence operator.

The identities Eqs. (A11) and (A12) explicitly ensure that the generalized velocity operator V_μ is Hermitian, i.e., it satisfies the following identity:

$$\int d\mathbf{r}\mathbf{V}^*\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})\equiv\int d\mathbf{r}\mathbf{V}^*_{\alpha\beta}\Psi^\dagger_\beta(\mathbf{r})\Psi_\alpha(\mathbf{r})$$
$$=\int d\mathbf{r}\Psi^\dagger(\mathbf{r})\mathbf{V}\Psi(\mathbf{r}). \tag{A13}$$

APPENDIX B: SPIN CURRENT AND SPIN CONDUCTIVITY TENSOR

The time derivative of spin density $\rho_s=\hbar/2(\psi_\uparrow^\dagger\psi_\uparrow-\psi_\downarrow^\dagger\psi_\downarrow)$ can be written in Nambu formalism as

$$\dot{\rho}^s=\frac{i}{\hbar}[H,\rho^s]=\frac{\hbar}{2}(\dot{\Psi}^\dagger\Psi+\Psi^\dagger\dot{\Psi}). \tag{B1}$$

Using equations of motion (2.9) together with the explicit form of $\hat{H}_0$ operator Eq. (2.7) we obtain

$$\dot{\rho}^s=\frac{i}{2}\Bigg(\frac{(\pi^*)^2}{2m}\Psi_1^\dagger\Psi_1-\Psi_1^\dagger\frac{\pi^2}{2m}\Psi_1+\hat{\Delta}\Psi_1^\dagger\Psi_2-\Psi_1^\dagger\hat{\Delta}\Psi_2$$
$$+\hat{\Delta}^*\Psi_2^\dagger\Psi_1-\Psi_2^\dagger\hat{\Delta}^*\Psi_1-\frac{\pi^2}{2m}\Psi_2^\dagger\Psi_2+\Psi_2^\dagger\frac{(\pi^*)^2}{2m}\Psi_2\Bigg). \tag{B2}$$

Using the identities Eqs. (A11),(A12) it is easy to show that

$$\dot{\rho}^s=-\frac{\hbar}{4}\nabla_\mu(\Psi^\dagger V_\mu\Psi+V_\mu^*\Psi^\dagger\Psi)=-\nabla\cdot\mathbf{j}^s, \tag{B3}$$

where the generalized velocity operator V_μ is defined in Eq. (A8), and the last equality follows from the continuity equation (3.4) relating the spin density ρ^s and the spin current $\mathbf{j}^s$. Upon spatial averaging and utilizing Eq. (A13) we find the $q\to 0$ limit of the spin current

$$j_\mu^s=\frac{\hbar}{2}\Psi^\dagger V_\mu\Psi. \tag{B4}$$

The evaluation of the spin current–current correlation function Eq. (3.2) is straightforward and yields:

$$D_{\mu\nu}(i\Omega)=-\frac{\hbar^2}{4}\int_0^\beta e^{i\tau\Omega}[V_\mu(2)V_\nu(4)$$
$$\times\langle T_\tau\Psi^\dagger(1)\Psi(2)\Psi^\dagger(3)\Psi(4)\rangle_{3,4\to(\mathbf{y},0)}^{1,2\to(\mathbf{x},\tau)}]d\tau. \tag{B5}$$

Here $\Psi(1)\equiv\Psi(\mathbf{r}_1,\tau_1)$ and similarly the operator $V_\mu(2)$ acts only on functions of $\mathbf{r}_2$. Using Wick's theorem, identity Eq. (A5), and upon spatial averaging over $\mathbf{x}$ and $\mathbf{y}$ we obtain

$$D_{\mu\nu}(i\Omega)=\frac{\hbar^2}{4}\sum_{mn}\langle n|V_\mu|m\rangle\langle m|V_\nu|n\rangle S_{nm}(i\Omega). \tag{B6}$$

The double summation extends over the eigenstates $|m\rangle$ and $|n\rangle$ of the Hamiltonian Eq. (2.7), and $S_{mn}(i\Omega)$ is given in Eq. (A7). Analytically continuing $i\Omega\to\Omega+i0$ we finally obtain the expression for the retarded correlation function:

$$D_{\mu\nu}^R(\Omega)=\frac{\hbar^2}{4}\sum_{mn}\frac{V_\mu^{nm}V_\nu^{mn}}{\epsilon_n-\epsilon_m+\Omega+i0}(f_n-f_m), \tag{B7}$$

where V_{mn}^μ is defined in Eq. (3.8) and f_n is a short hand for the Fermi–Dirac distribution function $f(\epsilon_n)=(1+\exp(\beta\epsilon_n))^{-1}$. Finally, we substitute the last equation into Eq. (3.1) to obtain

$$\sigma_{\mu\nu}^s=\frac{\hbar^2}{4i}\sum_{mn}\frac{V_\mu^{nm}V_\nu^{mn}}{(\epsilon_n-\epsilon_m+i0)^2}(f_n-f_m). \tag{B8}$$

APPENDIX C: THERMAL CURRENT AND THERMAL CONDUCTIVITY TENSOR

In order to calculate thermal currents and thermal conductivity in the magnetic field we introduce a pseudo-gravitational potential $\chi=\mathbf{r}\cdot\mathbf{g}/c^2$.[32–35] This formal procedure is useful because it illustrates that the transverse thermal response is not given just by Kubo formula, but in addition it includes corrections related to magnetization. Throughout this section $\hbar=1$. The pseudo-gravitational potential enters the Hamiltonian, up to linear order in χ, as

$$H_T=\int d\mathbf{x}\left(1+\frac{\chi}{2}\right)\Psi^\dagger(\mathbf{x})\hat{H}_0\left(1+\frac{\chi}{2}\right)\Psi(\mathbf{x}), \tag{C1}$$

where H_0 is the Bogoliubov–deGennes Hamiltonian Eq. (2.7). The equations of motion for the fields Ψ thus become

$$\begin{aligned} i\dot{\Psi}=[\Psi,H_T]&=\left(1+\frac{\chi}{2}\right)\hat{H}_0\left(1+\frac{\chi}{2}\right)\Psi\\ &=(1+\chi)\hat{H}_0\Psi-i\nabla_\mu\chi V_\mu\Psi. \end{aligned}\tag{C2}$$

The last equality follows from the commutation relation Eq. (3.6). [Note that for $\chi\neq 0$ the Eq. (C2) differs from Eq. (2.9). Throughout this section $\dot{\Psi}$ will refer to Eq. (C2) unless explicitly stated otherwise.]

To find thermal current $\mathbf{j}^Q$ we start with the continuity equation

$$\dot{h}_T+\nabla\cdot\mathbf{j}^Q=0. \tag{C3}$$

The Hamiltonian density h_T follows from Eq. (C1) and reads

$$\begin{aligned} h_T=&\frac{1}{2m^*}(\pi_\mu^*\tilde{\Psi}_1^\dagger\pi_\mu\tilde{\Psi}_1-\pi_\mu\tilde{\Psi}_2^\dagger\pi_\mu^*\tilde{\Psi}_2)-\mu\tilde{\Psi}_\alpha^\dagger\sigma^3_{\alpha\beta}\tilde{\Psi}_\beta\\ &+\frac{1}{2}(\tilde{\Psi}_\alpha^\dagger\Delta_{\alpha\beta}\tilde{\Psi}_\beta+\Delta_{\alpha\beta}\tilde{\Psi}_\alpha^\dagger\tilde{\Psi}_\beta), \end{aligned}\tag{C4}$$

where $\tilde{\Psi}=(1+\frac{\chi}{2})\Psi$. Taking the time derivative of the Hamiltonian density h_T and using Eqs.(A11) and equations of motion (C2) we obtain

$$\begin{aligned} \dot{h}_T=i[H_T,h_T]=&\frac{i}{2m}\nabla_\mu(\dot{\tilde{\Psi}}^\dagger\Pi_\mu\tilde{\Psi}-\Pi_\mu^*\tilde{\Psi}^\dagger\dot{\tilde{\Psi}})\\ &+\frac{1}{2}(\tilde{\Psi}_\alpha\Delta_{\alpha\beta}\dot{\tilde{\Psi}}_\beta-\dot{\tilde{\Psi}}_\alpha^\dagger\Delta_{\alpha\beta}\tilde{\Psi}_\beta)\\ &+\frac{1}{2}(\Delta_{\alpha\beta}\dot{\tilde{\Psi}}_\alpha^\dagger\tilde{\Psi}_\beta-\Delta_{\alpha\beta}\tilde{\Psi}_\alpha^\dagger\dot{\tilde{\Psi}}_\beta), \end{aligned}\tag{C5}$$

where we introduced matrix operators

$$\Delta_{\alpha\beta}=\begin{pmatrix}0 & \hat{\Delta}\\ \hat{\Delta}^* & 0\end{pmatrix},\quad \Pi^\mu_{\alpha\beta}=\begin{pmatrix}\pi_\mu & 0\\ 0 & \pi_\mu^*\end{pmatrix}. \tag{C6}$$

Here $\dot{\tilde{\Psi}}=(1+\chi/2)\dot{\Psi}$ and π_μ is defined in Eq. (A10). Finally we use Eq. (A12) to extract $\mathbf{j}^Q$ from Eq. (C5). Upon spatial averaging and using the Hermiticity of V_μ Eq. (A13) the thermal current $\mathbf{j}^Q$ reads

$$j_\mu^Q=\frac{i}{2}(\tilde{\Psi}^\dagger V_\mu\dot{\tilde{\Psi}}-\dot{\tilde{\Psi}}^\dagger V_\mu\tilde{\Psi}). \tag{C7}$$

Note that the expression for $\mathbf{j}^Q$ contains two terms

$$\mathbf{j}^Q=\mathbf{j}_0^Q+\mathbf{j}_1^Q, \tag{C8}$$

where $\mathbf{j}_0^Q$ is independent of χ and $\mathbf{j}_1^Q$ is linear in χ. Explicitly:

$$\mathbf{j}_0^Q(\mathbf{r})=\tfrac{1}{2}\Psi^\dagger\{\mathbf{V},\hat{H}_0\}\Psi \tag{C9}$$

and

$$\begin{aligned} j_{\mu,1}^Q(\mathbf{r})=&-\frac{i}{4}\partial_\nu\chi\Psi^\dagger(V_\mu V_\nu-V_\nu V_\mu)\Psi+\frac{\partial_\nu\chi}{4}\Psi^\dagger((x_\nu V_\mu\\ &+3V_\mu x_\nu)\hat{H}_0+\hat{H}_0(3x_\nu V_\mu+V_\mu x_\nu))\Psi, \end{aligned}\tag{C10}$$

where $\{a,b\}=ab+ba$. Analogously to the situation in the normal metal, the thermal average of $\mathbf{j}_1^Q$ does not in general vanish in the presence of the magnetic field.[38]

The linear response of the system to the external perturbation can be described by

$$\langle j_\mu^Q\rangle=\langle j_{0\mu}^Q\rangle+\langle j_{1\mu}^Q\rangle=-(K_{\mu\nu}+M_{\mu\nu})\partial_\nu\chi\equiv-L^Q_{\mu\nu}\partial_\nu\chi, \tag{C11}$$

where

$$K_{\mu\nu}=-\frac{\delta\langle j_{0\mu}^Q\rangle}{\delta\partial_\nu\chi}=-\lim_{\Omega\to 0}\frac{P^R_{\mu\nu}(\Omega)-P^R_{\mu\nu}(0)}{i\Omega} \tag{C12}$$

is the standard Kubo formula for a dc response,[36] $P^R_{\mu\nu}(\Omega)$ being the retarded current–current correlation function, and

$$\begin{aligned} M_{\mu\nu}=-\frac{\delta\langle j_{1\mu}^Q\rangle}{\delta\partial_\nu\chi}=&-\sum_n\epsilon_n f_n\langle n|\{V_\mu,x_\nu\}|n\rangle\\ &+\sum_n\frac{i}{4}f_n\langle n|[V_\mu,V_\nu]|n\rangle \end{aligned}\tag{C13}$$

is a contribution from "diathermal" currents.[38] Note that the latter vanishes for the longitudinal response while it remains finite for the transverse response. As we will show later in this section, at $T=0$ there is an important cancellation between Eqs. (C12) and (C13) which renders the thermal conductivity $\kappa_{\mu\nu}$ well-behaved and prevents the singularity from the temperature denominator in Eq. (4.2).

The retarded thermal current–current correlation function $P^R_{\mu\nu}(\Omega)$ can be expressed in terms of the Matsubara finite temperature correlation function

$$\mathcal{P}_{\mu\nu}(i\Omega)=-\int_0^\beta d\tau e^{i\Omega\tau}\langle T_\tau j_\mu(\mathbf{r},\tau)j_\nu(\mathbf{r}',0)\rangle \tag{C14}$$

as

$$P^R_{\mu\nu}(\Omega)=\mathcal{P}_{\mu\nu}(i\Omega\to\Omega+i0). \tag{C15}$$

The current $j_\mu(\mathbf{r},\tau)$ in Eq. (C14) is given by

$$j_\mu(\tau)=\tfrac{1}{2}(\Psi^\dagger(\tau)V_\mu\partial_\tau\Psi(\tau)-\partial_\tau\Psi^\dagger(\tau)V_\mu\Psi(\tau)). \tag{C16}$$

As pointed out by Ambegaokar and Griffin[50] time ordering operator T_τ and time derivative operators ∂_τ do not commute and neglecting this subtlety can lead to formally divergent frequency summations.[24] Taking heed of this subtlety, substitution of Eq. (C16) into Eq. (C14) amounts to

$$\begin{aligned}\mathcal{P}_{\mu\nu}(i\Omega)=&-\frac{1}{4}\int_0^\beta d\tau e^{i\Omega\tau}V^\mu_{\alpha\beta}(2)V^\nu_{\gamma\delta}(4)\\ &\times\langle T_\tau\dot\Psi^\dagger_\alpha(1)\Psi_\beta(2)\dot\Psi^\dagger_\gamma(3)\Psi_\delta(4)\\ &-\dot\Psi^\dagger_\alpha(1)\Psi_\beta(2)\Psi^\dagger_\gamma(3)\dot\Psi_\delta(4)\\ &-\Psi^\dagger_\alpha(1)\dot\Psi_\beta(2)\dot\Psi^\dagger_\gamma(3)\Psi_\delta(4)\\ &+\Psi^\dagger_\alpha(1)\dot\Psi_\beta(2)\Psi^\dagger_\gamma(3)\dot\Psi_\delta(4)\rangle,\end{aligned} \tag{C17}$$

where the notation follows Eq. (B5) and Eq. (2.9). Upon utilizing the identities Eqs. (A5) and (A13) and performing the standard Matsubara summation we have

$$\mathcal{P}_{\mu\nu}(i\Omega)=\frac{1}{4}\sum_{nm}\langle n|V_\mu|m\rangle\langle m|V^\nu|n\rangle(\epsilon_n+\epsilon_m)^2S_{nm}(i\Omega), \tag{C18}$$

where S_{mn} is given by Eq. (A7). Analytically continuing $i\Omega\rightarrow\Omega+i0$ we obtain

$$\mathcal{P}^R_{\mu\nu}(\Omega)=\frac{1}{4}\sum_{nm}\frac{(\epsilon_n+\epsilon_m)^2V^{nm}_\mu V^{mn}_\nu}{\epsilon_n-\epsilon_m+\Omega+i0}(f_n-f_m). \tag{C19}$$

Note that the only difference between the Kubo contribution to the thermal response Eq. (C19) and the spin response Eq. (B7) is the value of the coupling constant. In the case of thermal response Eq. (C19), the coupling constant is $(\epsilon_n+\epsilon_m)/2$ which is eigenstate dependent, while in the case of spin response Eq. (B7) it is $\hbar/2$ and eigenstate independent.

Using Eq. (C12) we find that the Kubo contribution to the thermal transport coefficient is given by

$$K_{\mu\nu}=-\frac{i}{4}\sum_{nm}\frac{(\epsilon_n+\epsilon_m)^2}{(\epsilon_n-\epsilon_m+i0)^2}V^{nm}_\mu V^{mn}_\nu(f_n-f_m). \tag{C20}$$

This can be written as

$$K_{\mu\nu}=K^{(1)}_{\mu\nu}+K^{(2)}_{\mu\nu}, \tag{C21}$$

where

$$K^{(1)}_{\mu\nu}=-\frac{i}{4}\sum_{nm}\frac{4\epsilon_n\epsilon_m}{(\epsilon_n-\epsilon_m+i0)^2}V^{nm}_\mu V^{mn}_\nu(f_n-f_m) \tag{C22}$$

and

$$K^{(2)}_{\mu\nu}=-\frac{i}{4}\sum_{nm}V^{nm}_\mu V^{mn}_\nu(f_n-f_m). \tag{C23}$$

Similarly, we can separate the "diathermal" contribution Eq. (C13) as

$$M_{\mu\nu}=M^{(1)}_{\mu\nu}+M^{(2)}_{\mu\nu} \tag{C24}$$

where $M^{(1)}_{\mu\nu}$ and $M^{(2)}_{\mu\nu}$ refer to the first and second term in Eq. (C13), respectively. Using the completeness relation $\Sigma_m|m\rangle\langle m|=1$ it is easy to show that

$$M^{(2)}_{\mu\nu}=\frac{i}{4}\sum_{mn}(f_n-f_m)V^{nm}_\mu V^{mn}_\nu. \tag{C25}$$

Comparison of Eqs. (C23) and (C25) yields $M^{(2)}_{\mu\nu}+K^{(2)}_{\mu\nu}=0$. Therefore the thermal response coefficient is given by

$$L^Q_{\mu\nu}=K^{(1)}_{\mu\nu}+M^{(1)}_{\mu\nu}. \tag{C26}$$

Utilizing commutation relationships Eq. (3.6), $M^{(1)}_{\mu\nu}$ can be expressed in the form:

$$M^{(1)}_{\mu\nu}=\int\eta f(\eta)\mathrm{Tr}(\delta(\eta-\hat H_0)(x^\mu V^\nu-x^\nu V^\mu))d\eta, \tag{C27}$$

where the integral extends over the entire real line. The rest of the section follows closely Smrčka and Středa.[34] We define the resolvents $G^\pm$:

$$G^\pm\equiv(\eta\pm i0-\hat H_0)^{-1} \tag{C28}$$

and operators

$$A(\eta)=i\,\mathrm{Tr}\left(V_\mu\frac{dG^+}{d\eta}V_\nu\delta(\eta-\hat H_0)-V_\mu\delta(\eta-\hat H_0)V_\nu\frac{dG^-}{d\eta}\right), \tag{C29}$$

$$B(\eta)=i\,\mathrm{Tr}(V_\mu G^+V_\nu\delta(\eta-\hat H_0)-V_\mu\delta(\eta-\hat H_0)V_\nu G^-),$$

To facilitate Sommerfeld expansion we note that response coefficients $K^{(1)}_{\mu\nu}(T)$, $M^{(1)}_{\mu\nu}(T)$, $\sigma^s_{\mu\nu}(T)$ have generic form

$$L(T)=\int f(\eta)l(\eta)d\eta, \tag{C30}$$

and after integration by parts

$$L(T)=-\int\frac{df}{d\eta}\tilde L(\eta)d\eta. \tag{C31}$$

Here $\tilde L(\xi)$ is defined as

$$\tilde L(\xi)\equiv\int_{-\infty}^{\xi}l(\eta)d\eta. \tag{C32}$$

Note that $L(T=0)=\tilde L(\xi=0)$ and in particular the spin conductivity at $T=0$ satisfies $\sigma^s_{\mu\nu}(T=0)=\tilde\sigma^s_{\mu\nu}(\xi=0)$. Identities (C31) and (C32) will enable us to express the coefficients at finite temperature through the coefficients at zero temperature. For example, it follows from Eq. (C27) that

$$\tilde M^{(1)}(\xi)=\int_{-\infty}^{\xi}\eta\,\mathrm{Tr}(\delta(\eta-\hat H_0)(x^\mu V^\nu-x^\nu V^\mu)) \tag{C33}$$

and from Eqs. (B8), (C29)

$$\tilde{\sigma}^s_{\mu\nu}(\xi)=\frac{1}{4}\int_{-\infty}^{\xi}A(\eta)d\eta. \tag{C34}$$

Similarly, coefficient $K^{(1)}_{\mu\nu}$ from Eq. (C22) can be written as

$$K^{(1)}_{\mu\nu}=-i\int d\eta f(\eta)\eta\sum \delta(\eta-\epsilon_n)\epsilon_m \times\left(\frac{V^{nm}_{\mu}V^{mn}_{\nu}}{(\eta-\epsilon_m+i0)^2}-\frac{V^{mn}_{\mu}V^{nm}_{\nu}}{(\eta-\epsilon_m-i0)^2}\right) \tag{C35}$$

so that

$$\tilde{K}^{(1)}_{\mu\nu}(\xi)\equiv-i\int_{-\infty}^{\xi} d\eta\,\eta\sum \delta(\eta-\epsilon_n)\epsilon_m \times\left(\frac{V^{nm}_{\mu}V^{mn}_{\nu}}{(\eta-\epsilon_m+i0)^2}-\frac{V^{mn}_{\mu}V^{nm}_{\nu}}{(\eta-\epsilon_m-i0)^2}\right). \tag{C36}$$

Using definitions (C29), $\tilde{K}^{(1)}(\xi)$ can be expressed as

$$\tilde{K}^{(1)}_{\mu\nu}(\xi)=\int_{-\infty}^{\xi}\eta^2A(\eta)d\eta+\int_{-\infty}^{\xi}\eta B(\eta)d\eta. \tag{C37}$$

After integration by parts one obtains

$$\tilde{K}^{(1)}_{\mu\nu}(\xi)=\xi^2\int_{-\infty}^{\xi}A(\eta)d\eta+\int_{-\infty}^{\xi}(\eta^2-\xi^2) \times\left(A(\eta)-\frac{1}{2}\frac{dB}{d\eta}\right)d\eta. \tag{C38}$$

As shown in Ref. 34 the last term in this expression is exactly compensated by $\tilde{M}^{(1)}(\xi)$: This becomes evident after noting that definitions in Eq. (C29) imply

$$A-\frac{1}{2}\frac{dB(\eta)}{d\eta}=\frac{1}{2}\mathrm{Tr}\left[\frac{d\delta(\eta-\hat{H}_0)}{d\eta}(x^{\mu}V^{\nu}-x^{\nu}V^{\mu})\right]. \tag{C39}$$

Substituting the last identity into the Eq.(C38) and integrating the second term by parts we obtain

$$\tilde{K}^{(1)}_{\mu\nu}(\xi)=\xi^2\int_{-\infty}^{\xi}A(\eta)d\eta-\int_{-\infty}^{\xi}\eta\,\mathrm{Tr}(\delta(\eta-\hat{H}_0) \times(x^{\mu}V^{\nu}-x^{\nu}V^{\mu}))d\eta. \tag{C40}$$

The second term here is equal to $-M^{(1)}_{\mu\nu}$ from Eq. (C33). After the cancellation the result simply reads:

$$\tilde{L}^{Q}_{\mu\nu}(\xi)=\tilde{K}^{(1)}_{\mu\nu}(\xi)+\tilde{M}^{(1)}_{\mu\nu}(\xi)=\xi^2\int_{-\infty}^{\xi}A(\eta)d\eta. \tag{C41}$$

Or, using Eq. (C34)

$$\tilde{L}^{Q}_{\mu\nu}(\xi)=\left(\frac{2\xi}{\hbar}\right)^2\tilde{\sigma}^s_{\mu\nu}(\xi). \tag{C42}$$

Finally, from Eq. (C31) we find

$$L^{Q}_{\mu\nu}(T)=-\frac{4}{\hbar^2}\int\frac{df(\eta)}{d\eta}\eta^2\tilde{\sigma}^s_{\mu\nu}(\eta)d\eta. \tag{C43}$$

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